Stochastic Processes and their Applications 39 (1991) 221-237
North-Holland

Sticky Brownian motion as the strong limit of a sequence of random walks

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Received 16 May 1990
Revised 27 November 1990

We provide here a constructive definition of the sticky Brownian motion as we show that it is the almost sure uniform limit of path functions of a time changed random walk. The transition distribution of this process is also derived.

1. Preliminaries

1.1. Introduction

We provide here a constructive definition of the sticky Brownian motion, as we show that it is the almost sure uniform limit of path functions of a sequence of time-changed random walks. Let $X_1^{(n)}, X_2^{(n)}, X_3^{(n)}, \ldots$ be a sequence of independent identically distributed Bernoulli random variables defined on a probability space (Ω_n, F_n, P_n), $n = 1, 2, 3, \ldots$, that is $P_n(X_1^{(n)} = 1) = P_n(X_1^{(n)} = -1) = \frac{1}{2}$. Let $R_n(t)$ be a random walk defined on $[0, \infty)$ by

$$R_n(t) = \frac{1}{2^n} \sum_{i=1}^{k} X_i^{(n)} \quad \text{for } \frac{k}{2^{2n}} \leqslant t < \frac{k+1}{2^{2n}}, \quad k = 1, 2, 3, \ldots .$$

It is shown in [7] that $R_n(t)$ converges uniformly P_∞ a.s. on compact intervals of $[0, \infty)$ to a standard Brownian motion $W(t)$ defined on the projective limit space $(\Omega_\infty, F_\infty, P_\infty)$. That is

$$P_\infty\left\{ \lim_{n\to\infty} \sup_{t\in[0,1]} |R_n(t) - W(t)| = 0 \right\} = 1.$$

In this paper, we construct another random walk $R_n^*(t)$ on (Ω_n, F_n, P_n) with state space the lattice $\{k/2^n, k = 0, \pm 1, \pm 2, \ldots\}$ from $R_n(t)$ by a time change $T_n(t)$ defined also on (Ω_n, F_n, P_n). Thus $R_n^*(t) = R_n(T_n(t))$. The process $R_n^*(t)$ has same excursions as $R_n(t)$, the step duration off the state 0 is the same as the one of $R_n(t)$ (we know it is 2^{-2n}), but at zero $R_n^*(t)$ spends 2^n steps each of duration 2^{-2n}. In other words the step duration at 0 for $R_n^*(t)$ is 2^{-n}. We will show that $R_n^*(t)$ converges pathwise uniformly to a diffusion process $W^*(t)$ on $(\Omega_\infty, F_\infty, P_\infty)$ P_∞-a.s. $W^*(t)$ has speed measure m equal to the one of Brownian motion $W(t)$ on $\mathbb{R}-\{0\}$, but m has an atom at 0 and $m\{0\}=1$. The set $\{t: W^*(t)=0\}$ is a perfect set of positive Lebesgue measure, that is a generalized random Cantor set. The probability distribution of the Lebesgue measure of such a set in the interval $[0,1]$ will be derived. We have shown in [1] that as a diffusion process $W^*(t)$ satisfies a unique local property at 0.

We will generalize this result in the following way: Given any number $\rho>0$ we construct a sequence of random walks $R_n^{(\rho)}(t)$ in the same way as we did for $R_n^*(t)$ which converges in the same sense as above to a sticky Brownian motion with speed measure m such that $m(\mathrm{d}x)=2\,\mathrm{d}x+\rho I_{\{0\}}$.

Since the convergence is a uniform convergence of path functions in finite time intervals with probability 1, it follows that the random walks $R_n^*(t)$ provide a constructive definition of any sticky Brownian motion.

Moreover the transition distribution of $W^*(t)$ is derived along with the Laplace transform of its passage times.

Notes. (1) The excursions of $R_n^*(t)$ and $R_n(t)$ are the same but are shifted in time since $R_n^*(t)$ spends more time at 0 than $R_n(t)$.

(2) The processes $R_n^*(t)$ do not have the Markov property, yet they are shown to converge uniformly to a diffusion process (i.e. a process with continuous paths having the strong Markov property).

1.2. *The sticky Brownian motion*

Let $W(t)$ be a Brownain motion on a probability space (Ω, F, P) with local time $L(t,x)$. Let m be a locally finite and strictly positive measure on $(-\infty,+\infty)$ (i.e. m assigns a strictly positive number to any open interval). Then (see [3, p. 151; 5, p. 164; 6, p. 176]) we can construct a general diffusion in natural scale on $(-\infty,+\infty)$ with speed measure m by the following time change: Let $M(t)=\frac{1}{2}\int_{-\infty}^{+\infty} L(t,x)\,\mathrm{d}m(x)$, $M(t)$ is then a strictly increasing process P-a.s. Let $T(t)$ be the inverse of $M(t)$ that is $T(t,\omega)=\sup\{s: M(s,\omega)=t\}$. Then $X(t)=W(T(t))$ is a regular diffusion in natural scale on $(-\infty,+\infty)$ with speed measure m.

It is also shown in [3] that the occupancy time of the process $X(t)$ is absolutely continuous with respect to m and the Radon-Nikodym derivative is $L^*(t,x)=L(T(t),x)$ for all $x\in(-\infty,+\infty)$ that is

$$\int_0^t I_A(X(s))\,\mathrm{d}s=\int_A L(T(t),x)\,\mathrm{d}m(x)$$

for any Borel set A in $(-\infty,+\infty)$.

Now for any number $\rho>0$ let m be a measure such that $m(\mathrm{d}x)=2\,\mathrm{d}x+\rho I_{\{0\}}$. Then $M(t)=\frac{1}{2}\int_{-\infty}^{+\infty} L(t,x)\,\mathrm{d}m(x)=t+\rho L(t,0)$ and $T(t)=(t+\rho L(t,0))^{-1}$ the inverse function of $M(t)$. The process $W^*(t)=W(T(t))$ is a diffusion process with speed measure m equal to twice Lebesgue measure but has an atom at zero $-m\{0\}=\rho$. We will show in this section that $W^*(t)$ has same excursions as the standard

Brownian motion but the set $\{t\colon W^*(t)=0\}$ is a generalized (random) Cantor set of positive Lebesgue measure. $W^*(t)$ is called a sticky Brownian motion with *sticky coefficient* ρ.

We will define the local time $L^*(t,x)$ of $W^*(t)$ as the Radon Nikodym derivative of the occupancy time with respect to the speed measure $\mathrm{m}(\mathrm{d}x)=2\,\mathrm{d}x+\rho I_{\{0\}}$ then $L^*(t,x)=L(T(t),x)=L((t+\rho L(t,0))^{-1},x)$.

Note. In most texts, it is referred to the sticky Brownian motion, a Brownian motion 'slowly' reflecting at zero or with a sticky boundary at zero. Here we construct a sticky Brownian motion on $(-\infty,+\infty)$. The terminology 'sticky' is motivated by the fact that the set $\{t\colon W^*(t)=0\}$ has positive Lebesgue measure, (for the standard Brownian motion the measure is 0). We can speak of (and construct in a similar way) a Brownian motion 'sticky' at any other state than zero.

Harrison and Lemoine [4] defined in their paper the (reflected) sticky Brownian motion in the following way: Let $(W(t),\ t\geq 0)$ be standard Brownian motion defined on a probability space (Ω,F,P). Let $Y(t)=-\inf_{s\leq t}W(s)$, then it is well known that $Z(t)=W(t)+Y(t)$ is reflected Brownian motion (see Chapter 1). Now let $T(t)=t+Y(t)/\rho$ for some positive real number ρ. Then clearly $T(t)$ is strictly increasing continuous process P-a.s. Now let $U(t)$ be the inverse function of $T(t)$, that is $U(T(t))=t$. It is then shown in [4] that the process $W^*(t)=Z(U(t))$ has speed measure m such that $m(\mathrm{d}x)=2\,\mathrm{d}x+(1/\rho)I_{\{0\}}$. Therefore $W^*(t)$ is sticky Brownian motion on $[0,\infty)$ since $Y(t)=-\inf_{s\leq t}W(s)$ is the local time of the reflected process $Z(t)$. Using this fact, Harrison and Lemoine showed that the (reflected) sticky Brownian motion is the weak limit of a sequence 'modified storage processes'.

2. Construction and description of the process $R_n^*(t)$

The case of a simple symmetric random walk: Let $X_1, X_2, X_3, \ldots$ be a sequence of i.i.d. r.v.'s on a probability space (Ω,F,P) such that $P(X_1=1)=P(X_1=-1)=\frac{1}{2}$. Let also:

$S_n=X_0+X_1+X_2+X_3+\cdots+X_n$, where $X_0=0$ P-a.s.

$\nu_n=\#\{0<k\leq n\colon S_k=0\}$, the number of visits to 0 in n steps.

$M_n=n+k\nu_n$ for some fixed positive integer k.

$T_n=\inf\{j\colon M_j>n\}$.

Define S_n^* on (Ω,F,P) by $S_n^*=S_{T_n}$.

Fact 2.1. *S_n^* has the same excursions as S_n but remains at zero for k steps whenever 0 is hit.*

Proof. Let $N_1=\inf\{n\geq 1\colon S_n=0\}$, the first time S_n visits zero,

$$N_m=\inf\{n\geq N_{m-1}\colon S_n=0\},\quad m\geq 2.$$

Then $S_{N_m}=0$ and $M_{N_m}=N_m+km$ for all $m=1,2,3,\dots$ but $M_{N_m-1}=N_m-1+k(m-1)=N_m+km-(k+1)$.

$T_{N_m+km-j}=\inf\{i\colon M_i>N_m+km-j\}=N_m$ for all $j=1,2,3,\dots,k$. Therefore

$$S^*(N_m+km-j)=S(N_m)=0 \quad \text{for } j=1,2,\dots,k.$$

That is S^* remains at zero for k steps. Also note that

$$S^*(N_m+km+j)=S(N_m+j+1) \quad \text{for } j=0,1,2,3,\dots,N_{m+1}-(N_m+1).$$

This explains the fact that (S_n) and (S_n^*) have same excursions. □

Now let $R_n(t)$ be the random walk defined on $[0,1]$ as in [7]. Let $\nu_k^{(n)}$ be the number of visits to 0 in the first k steps of the process $R_n(t)$. So $\nu_{[2^{2n}t]}^{(n)}$ is the number of visits to 0 for the process $R_n(t)$ in $[0,t]$. Now define the following two processes on (Ω_n, F_n, P_n): $M_n(t)=t+\nu_{[2^{2n}t]}/2^n$ and $T_n(t)=\inf\{s\in[0,1]\colon M_n(s)>t\}$. $M_n(t)$ is a strictly increasing process having right continuous paths with left limits. $T_n(t)$ is a non-decreasing process with continuous paths. Let $R_n^*(t)=R_n(T_n(t))$.

Proposition 2.1. *$R_n^*(t)$ is a random walk with same excursions as $R_n(t)$, same jump duration off the state zero, but remains at* 0 *for* 2^n *steps so the sojourn length at* 0 *is* 2^{-n}.

Proof.

$$M_n\left(\frac{k}{2^{2n}}\right)=\frac{k}{2^{2n}}+\frac{\nu_k}{2^n}=\frac{k+2^n\nu_k}{2^{2n}}$$

and

$$T_n\left(\frac{k}{2^{2n}}\right)=\inf\left\{s\colon M_n(s)>\frac{k}{2^{2n}}\right\}.$$

Claim:

$$T_n\left(\frac{k}{2^{2n}}\right)=\frac{1}{2n}\inf\{j\colon j+2^n\nu_j>k\}.$$

Reason: The function $M_n(\cdot)$ is right continuous with left limits existing at each point and the jumps occur only at points of the form $j/2^{2n}$ at which the value of $M_n(\cdot)$ is also of that form.

Therefore

$$T_n\left(\frac{k}{2^{2n}}\right)=\inf\left\{\frac{j}{2^{2n}}\colon \frac{j+2^n\nu_j}{2^{2n}}>\frac{k}{2^{2n}}\right\}.$$

That is

$$T_n\left(\frac{k}{2^{2n}}\right)=\frac{1}{2^{2n}}\inf\{j\colon j+2^n\nu_j>k\}.$$

Using Fact 2.1, it is clear that $R_n^*(t)$ is the process described, because

$$R_n^*\left(\frac{j}{2^{2n}}\right)=R_n\left(T_n\left(\frac{j}{2^{2n}}\right)\right)=\frac{1}{2^{2n}}\sum_{i=1}^{j^*} r_i \quad \text{where } j^*=\inf\{k\colon k+2^n\nu_k>j\}. \qquad \square$$

3. Main result and generalization

Theorem.

$$P_\infty\left\{\lim_{n\to\infty}\sup_{t\in[0,1]}|R_n^*(t)-W^*(t)|=0\right\}=1$$

where $W^*(t)$ is sticky Brownian motion defined on the probability space $(\Omega_\infty, F_\infty, P_\infty)$ with speed measure m such that $m\{0\}=1$.

For the proof, we need the following proposition and lemmas:

Proposition 3.1.

$$P_\infty\left\{\lim_{n\to\infty}\sup_{t\in[0,1]}\left|\frac{\nu_{[2^{2n}t]}}{2^n}-L(t,0)\right|=0\right\}=1$$

where $L(t,0)$ is the local time at $x=0$ of the Brownian motion $W(t)$ such that

$$P_\infty\left\{\lim_{n\to\infty}\sup_{t\in[0,1]}|R_n(t)-W(t)|=0\right\}=1.$$

Proof. Let $u^+(t,x,\varepsilon)$ ($u^-(t,x,\varepsilon)$ respectively) be the number of upcrossings (downcrossings respectively) of the interval $[x, x+\varepsilon]$ completed by $W(t)$ in the interval $[0,t]$. It is shown in [2] that

$$\lim_{\varepsilon\to 0}\sup_{x}\sup_{t\in[0,1]}\left|\frac{2u^+(t,x,\varepsilon)}{\varepsilon}-L(t,x)\right|=0 \quad P_\infty\text{-a.s.}$$

So for $x=0$ and $\varepsilon=2^{-n}$ we obtain

$$\lim_{n\to\infty}\sup_{t\in[0,1]}\left|\frac{2u^+(t,0,2^{-n})}{2^{-n}}-L(t,0)\right|=0 \quad P_\infty\text{-a.s.}$$

Let $\tau_n=\inf\{t\geqslant 0\colon W(t)=2^{-n} \text{ or } -2^{-n}\}$, so $R_n(2^{-n})=W(\tau_n)$ for all $n=1,2,\ldots$.

It is shown in [6] that $R_m(j\alpha_m^2)=R_N(j\alpha_m^2+L_{mN}(j\alpha_m^2))$ for $m<N$. We then have $R_m(t)=R_N(t+L_{mN}(t))$ $\forall t\in[0,1]$. Making $N\to\infty$ we get $R_m(t)=W(t+\lim_{N\to\infty}L_{mN}(t))$. Let $K_m(t)=\lim_{N\to\infty}L_{mN}(t)$.

We know from [6] that this limit exists and is finite for each m P_∞-a.s. We then have $R_m(t)=W(t+K_m(t))$. Therefore

$$\frac{\nu_{[2^{2n}t]}}{2^n}=u^+(t+K_n(t),0,2^{-n})+u^-(t+K_n(t),0,2^{-n}).$$

It is also shown in [6] that $\lim_{n\to\infty}\sup_{t\in[0,1]}|K_n(t)|=0$. Now

$$\left|\frac{\nu_{[2^{2n}t]}}{2^n}-L(t,0)\right| \leqslant \left|\frac{u^+(t,0,2^{-n})}{2^{-n}}-\frac{L(t,0)}{2}\right| + \left|\frac{u^-(t,0,2^{-n})}{2^{-n}}-\frac{L(t,0)}{2}\right|.$$

Now we show that both terms of the right-hand side converge uniformly to 0 in $[0,1]$ as $n\to\infty$.

$$\left|2\frac{u^+(t,0,2^{-n})}{2^{-n}}-L(t,0)\right|$$
$$\leqslant \left|2\frac{u^+(t,0,2^{-n})}{2^{-n}}-L(t+K_n(t),0)\right| + |L(t+K_n(t),0)+L(t,0)|.$$

The first term of the right-hand side tends to 0 uniformly on $[0,1]$ as $n\to\infty$ by the uniform convergence of $2u^+(t,0,2^{-n})/2^{-n}$ to $L(t,0)$ and $K_n(t)$ to 0. The second term also tends to 0 because $L(t,0)$ is uniformly continuous on the interval $[0,1]$.

We shown the uniform convergence of $|2u^-(t,0,2^{-n})/2^{-n}-L(t,0)|$ to 0 in exactly the same way. And this completes the proof of the proposition. □

From now on let

$$A=\left\{\omega\in\Omega\colon \lim_{n\to\infty}\sup_{t\in[0,1]}|R_n(t)-W(t)|=0\right\}.$$

So that $P_\infty(A)=1$. Also note that the set

$$A'=\left\{\omega\in\Omega\colon \lim_{n\to\infty}\sup_{t\in[0,1]}\left|\frac{\nu_{[2^{2n}t]}}{2^n}-L(t,0)\right|=0\right\}$$

is equivalent to A. By Proposition 3.1,

$$A'=\left\{\omega\in\Omega\colon \lim_{n\to\infty}\sup_{t\in[0,1]}|M_n(t)-(t+L(t,0))|=0\right\}.$$

For each fixed $\omega\in A'$, $(M_n(t))$ is as a sequence of strictly increasing, right continuous with left limits (r.c.l.l.) functions on $[0,1]$ converging uniformly to the continuous function $(t+L(t,0))$ as $n\to\infty$.

Now $T_n(t)=\inf\{s\in[0,1]: M_n(s)>t\}$ is the generalized inverse of $M_n(t)$. $T_n(t)$ is then continuous and non-decreasing.

Lemma 3.1. *Let (f_n) be a sequence of strictly increasing functions such that f_n converges uniformly to a function f. Let f^{-1} be the generalized inverse of f. If f^{-1} is uniformly continuous and strictly increasing then f_n^{-1} converges uniformly to f^{-1} on the interval $(\inf f, \sup f)$.*

Proof. Let $\varepsilon > 0$ and $y \in (\inf f, \sup f)$ be arbitrary.

We want to show the existence of N_ε such that $\forall n > N_\varepsilon |f_n^{-1}(y) - f^{-1}(y)| < \varepsilon$. Since f^{-1} is uniformly continuous then $\exists \delta > 0$ and y_1, y_2 such that $y_1 < y < y_2$, $y_2 - y_1 < \delta$ and $|f^{-1}(y_2) - f^{-1}(y_1)| < \varepsilon$. Now since f^{-1} is strictly increasing then

$$f^{-1}(y_1) < f^{-1}(y) < f^{-1}(y_2). \tag{3.1}$$

Let $\delta_1 = \delta_1(\varepsilon)$ be such that $f^{-1}(y_1) + \delta_1 = f^{-1}(y)$ and $\delta_2 = \delta_2(\varepsilon)$ be such that $f^{-1}(y_2) - \delta_2 = f^{-1}(y)$. Since (f_n) converges uniformly to f then

$$\exists N_{\delta_1}: \quad |f_n(f^{-1}(y_1) - f(f^{-1}(y_1)| < \delta_1 \quad \forall n > N(\delta_1)$$

and

$$\exists N_{\delta_2}: \quad |f_n(f^{-1}(y_2) - f(f^{-1}(y_2)| < \delta_2 \quad \forall n > N(\delta_2).$$

So $\forall n \geq \max\{N_{\delta_1} N_{\delta_2}\} f_n(f^{-1}(y_1) < y < f_n(f^{-1}(y_2)$. Now apply f_n^{-1} to each member to the inequality above to get

$$f^{-1}(y_1) < f_n^{-1}(y) < f^{-1}(y_2). \tag{3.2}$$

Combining (3.1) and (3.2) with the fact that $|f^{-1}(y_2) - f^{-1}(y_1)| < \varepsilon$ will yield the desired result, that is $|f_n^{-1}(y) - f^{-1}(y)| < \varepsilon$ $\forall n \geq \max\{N_{\delta_1}, N_{\delta_2}\}$.

Note that $\max\{N_{\delta_1}, N_{\delta_2}\}$ is independent of y. And that completes the proof of the lemma. □

Lemma 3.1 will then imply that $T_n(t)$ converges uniformly to the process $(t + L(t, 0))^{-1}$ on the interval $[0, \sup_{t \in [0,1]}(t + L(t, 0)))$ for each fixed $\omega \in A'$.

Note. Since $(t + L(t, 0)) \geq t$, we have $[0, \sup_{t \in [0,1]}(t + L(t, 0))) \supseteq [0, 1]$. Therefore $T_n(t)$ converges uniformly to $(t + L(t, 0))^{-1}$ in $[0, 1]$ a.s.

Lemma 3.2. *Let (f_n), (g_n) be sequences of functions such that (f_n) converges uniformly to a function f and (g_n) converges uniformly to a function g. Assume that f_n is defined on the range of g_n for each n and that f is uniformly continuous on the range of g. Then $f_n(g_n)$ converges uniformly to $f(g)$ as $n \to \infty$.*

Proof. Follows by standard arguments. □

In our case f is a Brownian path. We know the paths of $W(t)$ are uniformly continuous functions on compact intervals of $[0, \infty)$, P_∞-a.s.

Combining Lemma 3.1 and Lemma 3.2 we get that

$$A \cap A' \in \left\{ \omega \in \Omega: \lim_{n \to \infty} \sup_{t \in [0,1]} |R_n(T_n(t)) - W(T(t))| = 0 \right\}$$

and we have already seen that $P_\infty(A \cap A') = 1$.

Now we return to the proof of the main theorem. Recall that the process $M(t) = t + L(t, 0)$ can be written as $\int_{-\infty}^{+\infty} L(t, x) m(\mathrm{d}x)$ where $m(\mathrm{d}x) = 2\,\mathrm{d}x + I_{\{0\}}$, therefore $W^*(t) = W((t + L(t, 0))^{-1})$ is sticky Browian motion described earlier in this section with speed measure m. □

Generalization: We have shown here how to approximate a sticky Brownian motion whose speed measure m is such that $m\{0\}=1$, by a sequence of well defined and well described random walks.

We will see now that given any positive number ρ we can construct a random walk $R_n^{(\rho)}(t)$ that converges uniformly to a sticky Brownian motion whose speed measure m is such that $m\{0\}=\rho$. ($R_n^*(t)$ corresponds then to $R_n^{(1)}(t)$.)

To this end we only need to modify our processes $M_n(t)$ in the following way: Let

$$M_n^\rho(t)=t+\rho\frac{\nu_{[2^{2n}t]}}{2^n}.$$

Then $M_n^\rho(t)$ will converge uniformly to $t+\rho L(t,0)$ as $n\to\infty$.

As before $T_n^\rho(t)$ (the generalized inverse of $M_n^\rho(t)$) will converge uniformly to $(t+\rho L(t,0))^{-1}$ a $n\to\infty$.

Now let $R_n^{(\rho)}(t)=R_n(T_n^\rho(t))$. By the same reasoning as before $R_n^{(\rho)}(t)$ is a random walk which spends $[\rho 2^n]$ steps at 0 each of duration 2^{-2n}. Equivalently stated the step duration at 0 of $R_n^{(\rho)}(t)$ is $\rho 2^{-n}$, while off 0 it is always 2^{-2n}.

$R_n^{(\rho)}(t)$ converges uniformly in t and in ρ to the time changed Brownian motion $W((t+\rho L(t,0))^{-1})$ which is sticky Brownian motion with speed measure m such that $m\{0\}=\rho$.

Note. When $\rho=0$, the random walk $R_n^{(\rho)}(t)$ is the same one as in [6] which we know converges to a standard Brownian motion. Therefore a standard Brownian motion can be thought of as a sticky Brownian motion with sticky coefficient 0. Also as $\rho\to\infty$, notice that $M_n^\rho(t)\to\infty$ after the first visit to 0, so that $T_n^\rho(t)$ tends to become a constant. Then it is plausible that $R_n^{(\rho)}(t)$ as $\rho\to\infty$ is absorbed at 0. In Section 5 it will become clear why the absorbed Brownian motion can be seen as a sticky Brownian motion with an infinite sticky coefficient.

Without any loss of generality, in the coming sections, we consider the case $\rho=1$ for convenience, that is $W^*(t)=W((t+L(t,0))^{-1})$ where $W(t)$ is a standard Brownian motion. We let P be the distribution induced on the function space $C_{[0,\infty)}$ by the process $W^*(t)$ constructed in the previous chapter. For $x\neq 0$, we let P^x be the distribution induced on $C_{[0,\infty)}$ by the process which begins in state x, has increments equal to those of the standard Brownian motion $W(t)$ up until the first time τ_0 that state 0 is hit, and then is defined from τ_0 onward by $W^*(t)$ starting at 0.

4. Local time and sojourn time at zero

First we are going to show, using the processes $R_n^*(t)$, that the occupancy measure $\int_0^t I_A(W^*(s))\,ds$ is absolutely continuous with respect to the speed measure m, and the Radon–Nikodym derivative is $L^*(t,x)=L(T(t),x)$ ($L^*(t,x)$ will be called the local time of $W^*(t)$), i.e. we have:

Proposition 4.1.

$$\int_0^t I_A(W^*(s))\,\mathrm{d}s=\int_A L^*(t,x)\,\mathrm{d}m(x).$$

We first prove a lemma.

Lemma 4.1. *Let (f_n) be a sequence of measurable functions on $[0,1]$ such that $f_n\to f$ pointwise. Suppose that $F_t(x)=\int_0^t I_{(-\infty,x]}(f(s))\,\mathrm{d}s$ is continuous in x for each $t\in[0,1]$. Then $F_t^n(x)=\int_0^t I_{(-\infty,x]}(f_n(s))\,\mathrm{d}s\to F_t(x)$ as $n\to\infty$ uniformly in t and in x.*

Proof. Notice that $F_t^n(x)=\mu\{s\leqslant t\colon f_n(s)\leqslant x\}$, where μ is the Lebesgue measure on $[0,1]$, is the distribution function of f_n after viewing each f_n as a random variable on the space $([0,1], B[0,1],\mu)$. The conclusion of the lemma is nothing else than the weak (or vague) convergence of f_n to f which is implied by the pointwise convergence. The continuity of $F_t(x)$ on $[0,1]$ is necessary to ensure the convergence of $F_t^n(x)$ on all of $[0,1]$. Now the uniform convergence follows because $\sup_{x\in R,\,t\in[0,1]}F_t^n(x)\leqslant 1$ and $F_t^n(x)$ is monotone in t and x. □

Proof of Proposition 4.1. Using this lemma, we note that $\int_0^t I_{(-\infty,x]}(R_n(s))\,\mathrm{d}s$ converges uniformly in $t\in[0,1]$ to $\int_0^t I_{(-\infty,x]}(W(s))\,\mathrm{d}s$. This is because we know $R_n(t)\to W(t)$ (uniformly therefore pointwise) and $\int_0^t I_x(W(s))\,\mathrm{d}s=0\ \ \forall x\in\mathbb{R}$ makes $\int_0^t I_{(-\infty,x]}(W(s))\,\mathrm{d}s$ be continuous in x. Likewise $\int_0^t I_{(-\infty,x]}(R_n^*(s))\,\mathrm{d}s\to\int_0^t I_{(-\infty,x]}(W^*(s))\,\mathrm{d}s\ \forall x\neq 0$. For any interval $A=(a,b)$ for which 0 is not an endpoint, note that, by the construction of $R_n^*(t)$, we have the following:

$$\int_0^t I_A(R_n^*(s))\,\mathrm{d}s=\int_0^{T_n(t)} I_A(R_n(s))\,\mathrm{d}s\quad\forall t\in[0,1],\ n=1,2,\ldots.$$

By making $n\to\infty$, the first integral tends to $\int_0^t I_A(W^*(s))\,\mathrm{d}s$, and the second integral, being the composition of two functions tends to $\int_0^{T(t)} I_A(W(s))\,\mathrm{d}s=\int_A L(T(t),x)\,\mathrm{d}x$ (by using Lemma 3.2 of the previous section). Therefore

$$\int_0^t I_A(W^*(s))\,\mathrm{d}s=\int_A L(T(t),x)\,\mathrm{d}x \tag{4.1}$$

where $T(t)=(t+L(t,0))^{-1}$. Now for any interval $A=(a,b)$ containing 0 and any $n=1,2,\ldots$,

$$\int_0^t I_A(R_n^*(s))\,\mathrm{d}s=\int_0^{T_n(t)} I_A(R_n(s))\,\mathrm{d}s+\frac{\nu_{[2^{2n}\cdot T_n(t)]}}{2^n}\quad\forall t\in[0,1].$$

By making $n\to\infty$, we get

$$\int_0^t I_A(W^*(s))\,\mathrm{d}s=\int_0^{T(t)} I_A(W(s))\,\mathrm{d}s+L(T(t),0)$$

$$=\int_A L(T(t),x)\,\mathrm{d}x+L(T(t),0)\quad\forall t\in[0,1].\qquad\square$$

The sojourn time at zero: It is clear from the way the random walks $R_n^*(t)$ are constructed that

$$\int_0^{M_n(t)} I_{\{0\}}(R_n^*(t))\,\mathrm{d}s = \frac{\nu_{[2^{2n}t]}}{2^n}.$$

This true because the left-hand side is the number of visits $R_n^*(t)$ mkes to zero in $[0, M_n(t)]$ (which is the same as the number of visits $R_n(t)$ makes to zero in $[0, t]$) times the jump duration for $R_n^*(t)$ at zero which we know is $1/2^n$. Therefore (using Proposition 3.1),

$$\lim_{n\to\infty} \int_0^{M_n(t)} I_{\{0\}}(R_n^*(t))\,\mathrm{d}s = \lim_{n\to\infty} \frac{\nu_{[2^{2n}t]}}{2^n} = L(t, 0).$$

Now since $T_n(t)$ is the generalized inverse of $M_n(t)$ (so that $T_n(M_n(t)) = t$), we have

$$\int_0^t I_{\{0\}}(R_n^*(t))\,\mathrm{d}s = \frac{\nu_{[2^{2n}T_n(t)]}}{2^n}.$$

Since the right-hand side is the composition of $\nu_{[2^{2n}t]}$ with $T_n(t)$, taking the limit as $n\to\infty$, the right-hand side converges to $L(T(t), 0)$ (by using Lemma 3.2). Therefore

$$\int_0^t I_{\{0\}}(W^*(s))\,\mathrm{d}s = L(T(t), 0) = L^*(t, 0).$$

From now on, since we will only be considering the local time at time $t = 0$, $L(t)$ (resp. $(L^*(t))$ will stand for $L(t, 0)$ (resp. $L^*(t, 0)$). In what follows, the fact that both $M(t)$ and $T(t)$ are strictly increasing continuous processes a.s. is crucial. It follows from it that $M(T(t)) = T(M(t)) = t$.

Now we are going to derive the probability densities of $T(t)$ and $L^*(t)$. First recall the density of $L(t)$: $L(t)$ is distributed like $\max_{s\leq t} W(s)$, itself distributed like $2|W(t)|$, therefore

$$P\{L(t)\leq x\} = \begin{cases} 0 & \text{if } x\leq 0,\\ 2\Phi(x/\sqrt{t}) - 1 & \text{if } x > 0, \end{cases}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\frac{-t^2}{2}\,\mathrm{d}t.$$

Then $E(L(t)) = \sqrt{2t/\pi}$.

The distribution of $T(t)$:

$$P\{T(t)\leq x\} = P\{M(x)\geq t\} = P\{x + L(x)\geq t\}$$
$$= P\{L(x)\geq t - x\} = 1 - P\{L(x) < t - x\}$$

therefore

$$P\{T(t)\leqslant x\}=\begin{cases}1 & \text{if } x\geqslant t,\\ 2-2\Phi\left(\dfrac{t-x}{\sqrt{x}}\right) & \text{if } 0<x<t.\end{cases}$$

The density of $T(t)$ is then (by using the chain rule)

$$f_{T(t)}(x)=\frac{\mathrm{d}}{\mathrm{d}x}P\{T(t)\leqslant x\}=\begin{cases}0 & \text{if } x\leqslant 0 \text{ or } x\geqslant t,\\ \dfrac{1}{\sqrt{2\pi}}\dfrac{x+t}{x\sqrt{x}}\exp-\dfrac{(t-x)^2}{2x} & \text{if } 0<x<t.\end{cases}$$

Distribution of $L^*(t)$: Recall that $L^*(t)=L((t+L(t))^{-1})$ so the left continuous inverse of $L^*(t)$ is $L^{*-1}(t)=(t+L(t))(L^{-1}(t))=L^{-1}(t)+t$, and then

$$\begin{aligned}P\{L^{*-1}(t)\leqslant x\}&=P\{L^{-1}(t)+t\leqslant x\}=P\{L^{-1}(t)\leqslant x-t\}\\&=P\{L(x-t)\geqslant t\}=1-P\{L(x-t)<t\}.\end{aligned}$$

Therefore

$$P\{L^{*-1}(t)\leqslant x\}=\begin{cases}0 & \text{if } x\leqslant t,\\ 2-2\Phi\left(\dfrac{t}{\sqrt{x-t}}\right) & \text{if } x>t.\end{cases}$$

Now $P\{L^{*-1}(x)\leqslant t\}=P\{L^*\geqslant x\}=1-P\{L^*(t)<x\}$. Finally

$$P\{L^*(t)\leqslant x\}=1-P\{L^{*-1}(x)\leqslant t\}=\begin{cases}1 & \text{if } t\leqslant x,\\ 2\Phi\left(\dfrac{x}{\sqrt{t-x}}\right)-1 & \text{if } t>x.\end{cases}$$

Note. When doing these computations, we notice that $P\{L^*(t)\leqslant x\}=P\{t-T(t)\leqslant x\}$ i.e. $T(t)+L^*(t)=t$ in distribution. But this actually holds a.s., because $M(t)=t+L(t)$ implies that $t=M(T(t))=T(t)+L(T(t))$. Therefore $E(L^*(t))=t-E(T(t))$.

5. The transition density of $W^*(t)$

Existence: Recall that $W^*(t)$ can be defined from a standard Brownian motion $W(t)$ by a time change $T(t)=M^{-1}(t)$ where $M(t)=\int_{-\infty}^{+\infty}L(t,x)\,\mathrm{d}m(x)$. We then have (see [3, p. 161]):

Theorem 5.1. *The transition distribution* $P^x\{W^*(t)\leqslant y\}$ *has a density* $p(t,x,y)$ *with respect to the speed measure* m, *that is*

$$P^x\{W^*(t)\in A\}=\int_A p(t,x,y)\,\mathrm{d}m(y)$$

such that $(t, x, y) \to p(t, x, y)$ *is continuous,* $p(t, x, y) = p(t, y, x) > 0$ *and*

$$\frac{\partial}{\partial t} p(t, x, y) = \Gamma p(t, x, \cdot)(y) \tag{5.1}$$

and

$$\frac{\partial}{\partial t} E^x[L^*(t, y)] = p(t, x, y) \tag{5.2}$$

where Γ *is the generator of the process* $W^*(t)$. □

To derive $p(t, x, y)$, we will not be needing equation (5.1), the generator of $W^*(t)$ is needed in the next section.

Since $\int_0^t I_{\{0\}}(W^*(s))\,\mathrm{d}s = L(T(t), 0) = L^*(t, 0) > 0$, we get by taking the expected value, $\int_0^t P(W^*(s) = 0)\,\mathrm{d}s = E(L^*(t, 0)) > 0$. Therefore

$$P(W^*(t) = 0) = \frac{\mathrm{d}}{\mathrm{d}t} E(L^*(t, 0)) = 1 - \frac{\mathrm{d}}{\mathrm{d}t} E(T(t)).$$

The first equality is in accordance with equation (5.2). Let us then compute $(\mathrm{d}/\mathrm{d}t)E(T(t))$,

$$E(T(t)) = \frac{1}{\sqrt{2\pi}} \int_0^t \frac{x+t}{\sqrt{x}} \exp - \frac{(t-x)^2}{2x}\,\mathrm{d}x.$$

Making $u = (t-x)/\sqrt{x}$ and taking into account that x belongs to the interval $[0, t]$, we have

$$\begin{aligned} E(T(t)) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{u^2 + 2t - u\sqrt{u^2+4t}}{2}\, 2 \exp - \frac{u^2}{2}\,\mathrm{d}u \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty (u^2 + 2t - u\sqrt{u^2+4t}) \exp - \frac{u^2}{2}\,\mathrm{d}u. \end{aligned}$$

By differentiating inside the integral (as we may here), we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E(T(t)) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \left(2 - \frac{2u}{\sqrt{u^2+4t}}\right) \exp - \frac{u^2}{2}\,\mathrm{d}u \\ &= 1 - \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{u}{\sqrt{u^2+4t}} \exp - \frac{u^2}{2}\,\mathrm{d}u. \end{aligned}$$

Therefore

$$P(W^*(t) = 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{u}{\sqrt{u^2+4t}} \exp - \frac{u^2}{2}\,\mathrm{d}u.$$

Let $v=\sqrt{u^2+4t}$ then $u^2=v^2-4t$ and $\mathrm{d}v=(u/\sqrt{u^2+4t})\,\mathrm{d}u$. Therefore

$$\begin{aligned}P(W^*(t)=0)&=\sqrt{\frac{2}{\pi}}\int_{2\sqrt{t}}^{\infty}\exp-\frac{v^2+4t}{2}\,\mathrm{d}v\\&=\sqrt{\frac{2}{\pi}}\int_{2\sqrt{t}}^{\infty}\exp-\frac{v^2+4t}{2}\,\mathrm{d}v\\&=\sqrt{\frac{2}{\pi}}\exp(2t)\int_{2\sqrt{t}}^{\infty}\exp-\frac{v^2}{2}\,\mathrm{d}v.\end{aligned}$$

That is

$$p(t,0,0)=P(W^*(t)=0)=2\exp(2t)(1-\Phi(2\sqrt{t})).$$

Note that this is a positive, decreasing function of t (as it ought to be), taking the value 1 at $t=0$. Because of symmetry, we have

$$P(W^*(t)<0)=\frac{1-P(W^*(t)=0)}{2}=\frac{1-2\exp(2t)(1-\Phi(2\sqrt{t}))}{2}.$$

In what follows we are going to compute the distribution $P^x\{W^*(t)\leqslant y\}$ of $W^*(t)$ starting at some $x\neq 0$, and the transition density $p(t,x,y)$. W.l.o.g. assume $x>0$. To this end we need the following proposition and its corollary.

Proposition 5.1.

$$P^x\{W^*(t)\geqslant y\}$$
$$=\begin{cases}\Phi\left(\dfrac{x-y}{\sqrt{t}}\right)-\Phi\left(\dfrac{-x-y}{\sqrt{t}}\right)+\displaystyle\int_0^t P\{W^*(t-s)\geqslant y\}\frac{2x\,\mathrm{e}^{-x^2/(2s)}}{s\sqrt{2\pi s}}\,\mathrm{d}s & \text{if } y>0,\\ \displaystyle\int_0^t P\{W^*(t-s)\geqslant y\}\frac{2x\,\mathrm{e}^{-x^2/(2s)}}{s\sqrt{2\pi s}}\,\mathrm{d}s & \text{if } y\leqslant 0.\end{cases}$$

Before proving this proposition, we state this immediate consequence:

Corollary 5.1.

$$P^x\{W^*(t)=0\}=\int_0^t P\{W^*(t-s)=0\}\frac{2x}{s\sqrt{2\pi s}}\mathrm{e}^{-x^2/(2s)}\,\mathrm{d}s\quad\forall x.$$

That is

$$p(t,x,0)=\int_0^t p(t-s,0,0)\frac{2x}{s\sqrt{2\pi s}}\mathrm{e}^{-x^2/(2s)}\,\mathrm{d}s\quad\forall y.\qquad\square$$

Proof of Proposition 5.1.

$$P^x\{W^*(t)\geq y\}=P^x\{W^*(t)\geq y,\ \tau_0>t\}+P^x\{W^*(t)\geq y,\ \tau_0\leq t\}$$
$$=\begin{cases}P^x\{W^0(t)\geq y\}+P\{W^*(t)\geq y,\ \tau_0\leq t\} & \text{if } y\geq 0,\\ P\{W^*(t)\geq y,\ \tau_0\leq t\} & \text{if } y<0,\end{cases}$$

where $W^0(t)$ is Brownian motion starting at x absorbed at 0. The second equality is true because the events $\{W^*(t)\geq y,\ \tau_0>t\}$ and $\{W^0(t)\geq y\}$ are equal since the increments of $W^*(t)$ are equal to those of $W(t)$ up to when state 0 is hit. Now

$$P^x\{W^*(t)\geq y,\ \tau_0<t\}=\int_0^t P^x\{W^*(t+\tau_0-s)\geq y/\tau_0=s\}\,\mathrm{d}F_{\tau_0}(s)$$
$$=\int_0^t P^0\{W^*(t-s)\geq y\}\,\mathrm{d}F_{\tau_0}(s).$$

The last equation follows by the strong Markov property.

Finding $\mathrm{d}F_{\tau_0}(s)$: $P^x\{\tau_0\leq s\}=P^0\{\tau_x\leq s\}$ where $\tau_x=\inf\{t\colon W(t)=x\}$, $W(t)$ is now a standard Brownian motion starting at 0. Now

$$P^0\{\tau_x\leq s\}=P^0\Big\{\max_{t\leq s} W(t)\geq x\Big\}=2-2\Phi\Big(\frac{x}{\sqrt{s}}\Big)$$

where

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\mathrm{e}^{-t^2/2}\,\mathrm{d}t.$$

Therefore

$$\mathrm{d}F_{\tau_0}(s)=\frac{\mathrm{d}}{\mathrm{d}s}P^0\{\tau_x\leq s\}=\frac{2x}{\sqrt{2\pi}s^{3/2}}\mathrm{e}^{-x^2/(2s)}.$$

Also

$$P^x\{W^0(t)\geq y\}=P^x\Big\{W(t)\geq y,\ \min_{u\leq t}W(u)>0\Big\}$$
$$=P\Big\{W(t)\geq y-x,\ \min_{u\leq t}W(u)>-x\Big\}$$
$$=\Phi\Big(\frac{x-y}{\sqrt{t}}\Big)-\Phi\Big(\frac{-x-y}{\sqrt{t}}\Big)$$

by the reflection principle. This ends the proof of the proposition. □

The proof of the corollary is immediate because $P^x\{W^0(t)=y\}=0$ for any $y\neq 0$. The reason why this corollary is enough to compute $p(t,x,y)$ for any x and y is that $p(t,x,y)$ is given by the formula

$$p(t,x,y)=\int_0^t p(t-s,0,y)\frac{2x}{s\sqrt{2\pi s}}\mathrm{e}^{-x^2/(2s)}\,\mathrm{d}s.$$

Now $p(t-s,0,y)=p(t-s,y,0)$ (by Theorem 5.1), and then can be computed in terms of $p(t,0,0)$ by applying the corollary once more, namely,

$$p(t-s,y,0)=\int_0^{t-s} p(t-s-u,0,0)\,\frac{2y}{u\sqrt{2\pi u}}\,e^{-y^2/(2u)}\,du.$$

The Laplace transform of $p(t,x,0)$: Note that $p(t,x,0)$ given in Corollary 5.1 is the convolution of $p(t,0,0)$ and the density of the distribution $F_{\tau_0}(s)=P^x\{\tau_0\leq s\}$. Computing $p(t,x,0)$ from that formula is tedious and complicated. However, in what follows, we are going to compute its Laplace transform, which will then be the product of the Laplace transforms of $p(t,0,0)$ and $F_{\tau_0}(x)$. It is well known (e.g. see [6, Lemma 2.11, p. 28]) that $E^x(e^{-\lambda\tau_0})=e^{-\sqrt{2\lambda x}}$ that is $\int_0^\infty e^{-\lambda t}\,dF_{\tau_0}(t)=e^{-\sqrt{2\lambda x}}$. Now to compute $\int_0^\infty e^{-\lambda t}p(t,0,0)\,dt$, it is easier to use the form $(\sqrt{2/\pi})\exp(2t)\int_{2\sqrt{t}}^\infty \exp(-v^2/2)\,dv$ for $p(t,0,0)$. Namely, let

$$I=\int_0^\infty e^{-\lambda t}e^{2t}\left(\int_{2\sqrt{t}}^\infty e^{-v^2/2}\,dv\right)dt,$$

we do an integration by parts by letting $u=\int_{2\sqrt{t}}^\infty e^{-v^2/2}\,dv$ and $dw=e^{-(\lambda-2)t}$. So

$$I=\sqrt{\frac{\pi}{2}}\frac{1}{\lambda-2}-\frac{1}{\lambda-2}\sqrt{\frac{\pi}{\lambda}}.$$

Therefore

$$\int_0^\infty e^{-\lambda t}p(t,0,0)\,dt=\sqrt{\frac{2}{\pi}}\,I=\frac{\sqrt{\lambda}-\sqrt{2}}{\sqrt{\lambda}(\lambda-2)}. \tag{5.3}$$

Now the Laplace transform of $p(t,x,0)$ is then

$$\int_0^\infty e^{-\lambda t}p(t,x,0)\,dt=\frac{\sqrt{\lambda}-\sqrt{2}}{\sqrt{\lambda}(\lambda-2)}\cdot e^{-\sqrt{2\lambda x}}. \tag{5.4}$$

Note. Knight (see [6, Theorem 6.2, p. 157]) considered a Brownian motion W on $[0,\infty)$ whose generator has domain

$$D=\{f\in C_\Delta^0[0,\infty)\colon f''\in C_{[0,\infty)}^0 \text{ and } c_1f(0)+\tfrac{1}{2}c_3f''(0)-c_2f'(0)=0\}$$

for some $c_i\geq 0$, $c_1+c_2+c_3=1$ and $c_1\neq 1$. ($C_\Delta^0[0,\infty)$ and $C_{[0,\infty)}^0$ are defined in [6, Definition 6.1, p. 153].) This boundary condition at 0 is the most general in the case of a Brownian motion. Using the resolvent operator, Knight computed the Laplace transform of the event $P\{W(t)=0\}$ (see [6, p. 160]), namely

$$\int_0^\infty e^{-\lambda t}P\{W(t)=0\}=\frac{c_3}{c_1+c_2\sqrt{2\lambda}+c_3\lambda}. \tag{5.5}$$

In order to compare this result with the Laplace transform found in equation (5.3), note that the sticky Brownian motion with speed measure m such that $m\{0\}=1$ corresponds to the boundary condition at 0, $\frac{1}{2}f''(0)-f'(0)=0$. Therefore in this case we have $c_1=0$, $c_2=c_3=\frac{1}{2}$. When we substitute these values in (3) we find that the Laplace transform is $1/(\sqrt{2\lambda}+\lambda)$. Note that this is the same as the Laplace transform we found in (5.3) (by direct computation from $p(t,0,0)$).

6. Passage time and maximum

We are going to compute the Laplace transform of a passage time of a sticky Brownian motion with any sticky coefficient ρ, and study its limit as ρ tends to 0 and to infinity. Following [5], $E_\rho^x\{e^{-\alpha\tau_b}\}=u_\rho(x)/u_\rho(b)$, where u_ρ satisfies the following differential equation:

$$\alpha u-\tfrac{1}{2}u''=0,\quad u'(0)=\tfrac{1}{2}\rho u''(0),\qquad \text{on } [0,b].$$

So

$$u_\rho(x)=c((\alpha\rho+\sqrt{2\alpha})\,e^{\sqrt{2\alpha}x}+(-\alpha\rho+\sqrt{2\alpha})\,e^{-\sqrt{2\alpha}x}).$$

Therefore

$$E_\rho^x\{e^{-\alpha\tau_b}\}=\frac{(\alpha\rho+\sqrt{2\alpha})\,e^{\sqrt{2\alpha}x}+(-\alpha\rho+\sqrt{2\alpha})\,e^{-\sqrt{2\alpha}x}}{(\alpha\rho+\sqrt{2\alpha})\,e^{\sqrt{2\alpha}b}+(-\alpha\rho+\sqrt{2\alpha})\,e^{-\sqrt{2\alpha}b}}\quad \text{for any } x\in[0,b]$$

(the constant c simplifies).

Note. Inverting $E^x\{e^{-\alpha\tau_b}\}$ will yield the distribution of the maximum of the sticky Brownian motion starting at any point x.

We could compute $E^0(\tau_b)$ by using fact #87 in [3, p. 136],

$$E^0(\tau_b)=\int_0^b m[0,z]\,dz=\int_0^b (2z+\rho)\,dz=b^2+\rho b$$

(recall that we have assumed that $m\{0\}=\rho$ so that $m[0,z]=2z+\rho$). This corresponds to $E^0(\tau_b)$ that can be computed from our $E_\rho^0\{e^{-\alpha\tau_b}\}$ by taking the limit of its derivative as α tends to 0.

Also as it is to be expected note that

$$\lim_{\rho\to 0} E_\rho^x\{e^{-\alpha\tau_b}\}=\frac{e^{\sqrt{2\alpha x}}+e^{-\sqrt{2\alpha x}}}{e^{\sqrt{2\alpha b}}+e^{-\sqrt{2\alpha b}}}$$

which is the Laplace transform of a passage time of the standard Brownian motion reflected at 0. Likewise

$$\lim_{\rho\to\infty} E_\rho^x\{e^{-\alpha\tau_b}\}=\frac{e^{\sqrt{2\alpha x}}-e^{-\sqrt{2\alpha x}}}{e^{\sqrt{2\alpha b}}-e^{-\sqrt{2\alpha b}}}$$

which now is the Laplace transform of a passage time of the standard Brownian motion absorbed at 0.

Acknowledgement

This paper is part of my Ph.D. thesis written at the University of Illinois under the supervision and guidance of Professor Frank Knight. I would like to thank him for many helpful and enlightening discussions. I am also grateful to the referee for some corrections and useful suggestions.

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