Bifurcation for a free boundary problem with surface tension modeling the growth of multi-layer tumors

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Abstract

This paper is devoted to the study of the bifurcation of a free boundary problem modeling the growth of tumors with the effect of surface tension being considered. The existence of infinitely many branches of bifurcation solutions is proved. The method of analysis is based on reducing the problem to an operator equation in certain Hölder space with a nonlinear Fredholm operator of index 0. The desired result then follows from the Crandall–Rabinowitz bifurcation theorem.

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1. Introduction

In this paper we study bifurcations of the following free boundary problem:

\[
\begin{align*}
\Delta \sigma &= \sigma, & & \text{in } \Omega_{\rho}, \\
\Delta p &= -\mu(\sigma - \tilde{\sigma}), & & \text{in } \Omega_{\rho}, \\
\frac{\partial \sigma}{\partial y} &= 0, & & \frac{\partial p}{\partial y} = 0, & & \text{on } \Gamma_0, \\
\sigma &= \tilde{\sigma}, & & \frac{\partial p}{\partial \nu} = 0, & & \text{on } \Gamma_{\rho}, \\
p &= \gamma \kappa & & \text{on } \Gamma_{\rho}.
\end{align*}
\]

(1.1)

Here \( \sigma = \sigma(x, y) \) and \( p = p(x, y) \) are unknown functions defined on the unknown domain

\[\Omega_{\rho} := \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}; 0 < y < \rho(x)\},\]

where \( \rho = \rho(x) \) is an unknown function, \( \Delta \) denotes the Laplacian in the \((x, y)\)-variables, \( \Gamma_0 \) denotes the lower boundary \( y = 0 \) of \( \Omega_{\rho} \), \( \Gamma_{\rho} \) denotes the upper boundary \( y = \rho(x) \) of \( \Omega_{\rho} \), \( \frac{\partial}{\partial \nu} \) is the derivative in the outward normal direction \( \vec{\nu} = (-\nabla \rho, 1)/\sqrt{|\nabla \rho|^2 + 1} \) of the boundary \( \Gamma_{\rho} \), \( \kappa \) denotes the mean curvature of the boundary \( \Gamma_{\rho} \), and \( \mu, \tilde{\sigma}, \gamma, \kappa \) are positive constants.

The above problem is a mathematical model for the growth of multi-layers, a kind of in vitro tumors cultivated in laboratory by using the recently developed tissue culture technique [14–16]. This model was recently investigated by Cui and Escher in [6]. In this model \( \sigma \) represents the nutrient concentration in the tumor, \( p \) represents the internal pressure within the tumor that causes the motion of cellular material, and \( \tilde{\sigma} \) is a threshold value for tumor cell proliferation: In a region where \( \sigma > \tilde{\sigma} \) nutrient is sufficient to sustain tumor cells alive and proliferating, so that local tumor volume there increases, while in a region where \( \sigma < \tilde{\sigma} \) nutrient is not enough to maintain tumor cells alive, so that local tumor volume there decreases (cf. [1,4,5]). The condition \( \sigma = \tilde{\sigma} \) on the upper boundary \( \Gamma_{\rho} \) means that the tumor receives constant nutrient supply from this boundary, the conditions \( \frac{\partial \sigma}{\partial y} = 0 \) and \( \frac{\partial p}{\partial y} = 0 \) on the lower boundary \( \Gamma_0 \) reflect the fact that neither nutrient nor tumor cells can pass through this part of the boundary. The condition \( \frac{\partial p}{\partial \nu} = 0 \) on the upper boundary has a similar meaning. The term \( \gamma \kappa \) in the last equation takes surface tension effects of the free boundary \( y = \rho(x) \) into account.

In [6] the authors considered well-posedness and asymptotic behavior of solutions of the evolutionary problem related to the above system of equations. Among other contributions, they proved that under the condition

\[\tilde{\sigma} > \hat{\sigma},\]

(1.2)

the problem (1.1) has a unique flat solution—a solution \((\sigma, p, \rho)\) such that each component does not depend on the \( x \)-variable, i.e., \( \sigma = \sigma(y) \), \( p = p(y) \) and \( \rho = \text{const} \). Moreover, they also proved that this unique flat solution is a locally asymptotically stable equilibrium of the corresponding evolutionary problem, provided the surface tension coefficient is large enough.

In this paper we consider nonflat solutions of the problem (1.1). To study such solutions, similarly as [3,10–13], we shall regard (1.1) as a bifurcation problem, with \( \gamma \) as a bifurcation parameter, and consider nonflat solutions bifurcating from the unique flat solution. Note that in [10–13] the bifurcation solutions are constructed by using the power series method. In this paper we shall use a different method that is inspired by the works of [2] and [7]. By this method, we shall convert the problem (1.1) into an abstract bifurcation problem in suitable Banach spaces.
For the sake of simplicity we impose the additional condition that $\rho(x)$, $\sigma(x, y)$ and $p(x, y)$ are $2\pi$-periodic in every component of $x$. Moreover it is no essential to consider the case $n = 2$. Higher dimensional periodic cases can be treated similarly. Thus $x \in \mathbb{R}$ and we have the following additional conditions:

$$\rho(x), \sigma(x, y) \text{ and } p(x, y) \text{ are } 2\pi \text{-periodic in } x.$$ 

Moreover we identify $2\pi$-periodic functions with functions over the unit circle $S^1 = \mathbb{R}/2\pi \mathbb{Z}$, and accordingly identify the function spaces $C_{\text{per}}(\mathbb{R})$, etc. with correspondingly the function spaces $C(S^1)$, etc. Hence, instead of (1.1), we shall study the following problem:

$$\begin{align*}
\Delta \sigma &= \sigma \quad \text{in } \Omega_{\rho}, \\
\Delta p &= -\mu(\sigma - \tilde{\sigma}) \quad \text{in } \Omega_{\rho}, \\
\frac{\partial \sigma}{\partial y} \bigg|_{y=0} &= 0, \\
\frac{\partial p}{\partial y} \bigg|_{y=0} &= 0 \quad \text{for } x \in S^1, \\
\sigma \big|_{y=\rho(x)} &= \tilde{\sigma}, \\
\frac{\partial p}{\partial y} \bigg|_{y=\rho(x)} &= 0 \quad \text{for } x \in S^1, \\
p \big|_{y=\rho(x)} &= \gamma_k \quad \text{for } x \in S^1. 
\end{align*}$$

(1.3)

Recall that, under the condition (1.2), the unique flat solution $(\sigma, p, \rho) = (\sigma_*, p_*, \rho_*)$ of (1.3) is given by (see [6])

$$\begin{align*}
\sigma_*(y) &= \tilde{\sigma} \frac{\cosh y}{\cosh \rho_*}, \\
p_*(y) &= \frac{1}{2} \mu \tilde{\sigma} \left( y^2 - \rho_*^2 \right) + \mu \tilde{\sigma} \left( 1 - \frac{\cosh y}{\cosh \rho_*} \right),
\end{align*}$$

(1.4)

$$\tanh \rho_* = \frac{\tilde{\sigma}}{\tilde{\sigma}}.$$  

(1.5)

We denote

$$\gamma_k = \frac{\mu(\tilde{\sigma} - \tilde{\sigma}) - \mu \tilde{\sigma} \rho_* [\sqrt{k^2 + 1} \tanh(\sqrt{k^2 + 1} \rho_*) - k \tanh(k \rho_*)]}{k^3 \tanh(k \rho_*)}, \quad k = 1, 2, \ldots.$$  

(1.6)

Then the main result of this paper is the following theorem:

**Theorem 1.1.** Assume that the condition (1.2) is satisfied. Then there is a positive integer $k_0$ such that $\gamma_k$, defined as (1.6), is a bifurcation point of the flat solution $(\rho_*, \sigma_*, p_*)$ of (1.3) is given by (see [6])

$$\begin{align*}
\sigma_*(y) &= \tilde{\sigma} \frac{\cosh y}{\cosh \rho_*}, \\
p_*(y) &= \frac{1}{2} \mu \tilde{\sigma} \left( y^2 - \rho_*^2 \right) + \mu \tilde{\sigma} \left( 1 - \frac{\cosh y}{\cosh \rho_*} \right),
\end{align*}$$

(1.4)

$$\tanh \rho_* = \frac{\tilde{\sigma}}{\tilde{\sigma}}.$$  

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$$\gamma_k = \frac{\mu(\tilde{\sigma} - \tilde{\sigma}) - \mu \tilde{\sigma} \rho_* [\sqrt{k^2 + 1} \tanh(\sqrt{k^2 + 1} \rho_*) - k \tanh(k \rho_*)]}{k^3 \tanh(k \rho_*)}, \quad k = 1, 2, \ldots.$$  

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Then the main result of this paper is the following theorem:

**Theorem 1.1.** Assume that the condition (1.2) is satisfied. Then there is a positive integer $k_0$ such that $\gamma_k$, defined as (1.6), is a bifurcation point of the flat solution $(\rho_*, \sigma_*, p_*)$ of (1.3), having the following asymptotic expressions:

$$\gamma_k = \gamma_k + O(\varepsilon),$$

$$\rho_*(x) = \rho_* + \varepsilon \cos kx + O(\varepsilon^2),$$

$$\sigma_*(x, y) = \sigma_*(y) + \varepsilon D_k(y) \cos kx + O(\varepsilon^2),$$

and

$$p_*(x, y) = p_*(y) + \varepsilon L_k(y) \cos kx + O(\varepsilon^2),$$

where $D_k(y)$ and $L_k(y)$ are some analytic functions depending only on $k$, $\tilde{\sigma}$, $\tilde{\sigma}$, $\mu$ and $\rho_*$ (see (2.12) and (2.14) for explicit expressions). Moreover, any other $\gamma$ is not a bifurcation point.
The layout of this paper is as follows. In the next section we shall study the linearized problem of (1.1) and compute the possible bifurcation points. In Section 3 we give the proof of the existence of bifurcation solutions.

2. The linearized problem and bifurcation points

In this section we consider linearization of the problem (1.3), and look for possible bifurcation points by studying the spectrum of the linearized problem.

We first introduce some notations. Given $m \in \mathbb{N}^+$ and $\alpha \in (0, 1)$, we denote by $h^{m+\alpha}(\mathbb{S})$ (respectively $h^{m+\alpha}(\tilde{\Omega})$) the so-called little Hölder space on $\mathbb{S}$ (respectively $\tilde{\Omega}$), i.e., the closure of $C^\infty(\mathbb{S})$ (respectively $C^\infty(\tilde{\Omega})$) in the usual Hölder space $C^{m+\alpha}(\mathbb{S})$ (respectively $C^{m+\alpha}(\tilde{\Omega})$). Besides, $C_+(\mathbb{S}^1)$ (respectively $h_+^{m+\alpha}(\tilde{\Omega}), h_+^{m+\alpha}(\mathbb{S}^1)$) stands for the cone of all positive functions in $C(\mathbb{S}^1)$ (respectively $h^{m+\alpha}(\tilde{\Omega}), h^{m+\alpha}(\mathbb{S}^1)$). Hereafter we shall fix $m \geq 2$ and $\alpha \in (0, 1)$.

By making the change of variables

$$x' = x, \quad y' = \frac{y}{\rho(x)},$$

the problem (1.3) is transformed into the following system of equations on the fixed domain $\Omega = \mathbb{S}^1 \times (0, 1)$:

$$\begin{cases}
A(\rho) \sigma = \sigma & \text{in } \Omega, \\
A(\rho) p = -\mu (\sigma - \tilde{\sigma}) & \text{in } \Omega, \\
\frac{\partial \sigma}{\partial y'} |_{y'=0} = 0, & \frac{\partial p}{\partial y'} |_{y'=0} = 0 & \text{on } \mathbb{S}^1, \\
|\sigma|_{y'=1} = \tilde{\sigma}, & B(\rho) p |_{y'=1} = 0 & \text{on } \mathbb{S}^1, \\
p |_{y'=1} = \gamma \kappa & \text{on } \mathbb{S}^1,
\end{cases} \quad (2.1)$$

where $A(\rho)$ and $B(\rho)$ are partial differential operators with coefficients depending on the unknown function $\rho = \rho(x)$, having respectively the following expressions:

$$A(\rho) v = \frac{\partial^2 v}{\partial x'^2} - \frac{2 y' \rho'}{\rho} \frac{\partial^2 v}{\partial x' \partial y'} + \frac{1 + \gamma^2 \rho^2 v^2}{\rho^2} \frac{\partial^2 v}{\partial y'^2} + \frac{y'(2\rho_0^2 - \rho \rho_{x'x'})}{\rho^2} \frac{\partial v}{\partial y'},$$

$$B(\rho) v = -\left( \frac{\partial v}{\partial x'} - \frac{\rho'}{\rho} \frac{\partial v}{\partial y'} \right) \bigg|_{y'=1} \frac{\partial \rho}{\partial x'} + \frac{1}{\rho} \frac{\partial v}{\partial y'} \bigg|_{y'=1}.$$ 

By these expressions, it is clear that

$$(A, B) \in C^\infty(h_+^{m+\alpha}(\mathbb{S}^1), \mathcal{L}(h^{m+\alpha}(\tilde{\Omega}), h^{m-2+\alpha}(\tilde{\Omega}) \times h^{m+\alpha}(\mathbb{S}^1))). \quad (2.2)$$

cf. [8,9]. Recall that the curvature $\kappa$ of the curve $y = \rho(x)$ is given by

$$\kappa(\rho) = -\frac{\partial^2 \rho}{\partial x'^2} \left[ 1 + \left( \frac{\partial \rho}{\partial x'} \right)^2 \right]^{-\frac{3}{2}}.$$ 

Besides, it is easy to see that $(\rho_\sigma, \sigma(\rho_{x}y'), p_{\sigma}(\rho_{x}y'))$ is a flat solution of the system (2.1).

Hereafter, we write $x, y$ instead of $x', y'$ for the sake of simplicity.

In order to linearize (2.1) in a neighborhood of $(\rho_\sigma, \sigma(\rho_{x}y), p_{\sigma}(\rho_{x}y))$, we put

$$\rho = \rho_\sigma + \epsilon \eta(x), \quad \sigma = \sigma_\sigma(\rho_{x}y) + \epsilon \Sigma(x, y), \quad p = p_{\sigma}(\rho_{x}y) + \epsilon P(x, y),$$
where $\eta(x)$, $\Sigma(x, y)$, $P(x, y)$ denote new unknown functions. Substituting these expressions into (2.1), dividing all equations by $\varepsilon$ for every $k$, and letting $\varepsilon \to 0$, we get the following system:

$$\frac{\partial^2 \Sigma}{\partial x^2} + \rho_+^{-2} \frac{\partial^2 \Sigma}{\partial y^2} = \Sigma + 2\rho_+^{-1} \sigma''(\rho_+ y)\eta + y\sigma''(\rho_+ y)\eta'' \quad \text{in } \Omega, \quad (2.3)$$

$$\left. \frac{\partial \Sigma}{\partial y} \right|_{y=0} = 0, \quad \Sigma|_{y=1} = 0 \quad \text{on } S^1, \quad (2.4)$$

$$\frac{\partial^2 P}{\partial x^2} + \rho_+^{-2} \frac{\partial^2 P}{\partial y^2} = -\mu \Sigma + 2\rho_+^{-1} p''(\rho_+ y)\eta + y p''(\rho_+ y)\eta'' \quad \text{in } \Omega, \quad (2.5)$$

$$\left. \frac{\partial P}{\partial y} \right|_{y=0} = 0, \quad \left. \frac{\partial P}{\partial y} \right|_{y=1} = 0 \quad \text{on } S^1, \quad (2.6)$$

$$P|_{y=1} = -y\eta'' \quad \text{on } S^1. \quad (2.7)$$

In the sequel we study nontrivial solutions of the above system. For this purpose we consider Fourier expansions of $\eta$, $\Sigma$ and $P$:

$$\eta(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

$$\Sigma(x, y) = A_0(y) + \sum_{k=1}^{\infty} (A_k(y) \cos kx + B_k(y) \sin kx),$$

$$P(x, y) = M_0(y) + \sum_{k=1}^{\infty} (M_k(y) \cos kx + N_k(y) \sin kx).$$

Substituting these expressions into Eqs. (2.3)–(2.6), and comparing coefficients of $\cos kx$, $\sin kx$ for every $k$, we get the following equations for $A_k(y)$, $B_k(y)$, $M_k(y)$ and $N_k(y)$:

$$\begin{cases}
-k^2 A_k(y) + \rho_+^{-2} A''_k(y) = A_k(y) + a_k f_k(y), \\
A'_k(0) = 0, \quad A(1) = 0, \quad k = 0, 1, 2, \ldots, \quad (2.8)
\end{cases}$$

$$\begin{cases}
-k^2 B_k(y) + \rho_+^{-2} B''_k(y) = B_k(y) + b_k f_k(y), \\
B'_k(0) = 0, \quad B(1) = 0, \quad k = 1, 2, \ldots, \quad (2.9)
\end{cases}$$

$$\begin{cases}
-k^2 M_k(y) + \rho_+^{-2} M''_k(y) = -\mu A_k(y) + a_k g_k(y), \\
M'_k(0) = 0, \quad M'(1) = 0, \quad k = 0, 1, 2, \ldots, \quad (2.10)
\end{cases}$$

$$\begin{cases}
-k^2 N_k(y) + \rho_+^{-2} N''_k(y) = -\mu B_k(y) + b_k g_k(y), \\
N'_k(0) = 0, \quad N'(1) = 0, \quad k = 1, 2, \ldots, \quad (2.11)
\end{cases}$$

where

$$f_k(y) = 2\rho_+^{-1} \sigma \frac{\cosh(\rho_+ y)}{\cosh \rho_+} - \rho_+^{-2} \frac{\gamma \sinh(\rho_+ y)}{\cosh \rho_+},$$

$$g_k(y) = 2\rho_+^{-1} \left( \mu \bar{\sigma} - \mu \bar{\sigma} \frac{\cosh(\rho_+ y)}{\cosh \rho_+} \right) - k^2 \left( \mu \bar{\sigma} \rho_+^2 - \mu \bar{\sigma} \frac{\gamma \sinh(\rho_+ y)}{\cosh \rho_+} \right).$$

One can easily verify that solutions of (2.8) and (2.9) are respectively given by

$$A_k(y) = a_k D_k(y), \quad k = 0, 1, 2, \ldots,$$
\[ B_k(y) = b_kD_k(y), \quad k = 1, 2, \ldots, \]

where
\[
D_k(y) = \frac{-\tilde{\sigma} \tanh(\rho_s)}{\cosh(\sqrt{k^2 + 1}\rho_s)} \cosh(\sqrt{k^2 + 1}\rho_*y) + \frac{\tilde{\sigma}}{\cosh \rho_*} \rho_* \cosh \rho_* y \sinh(\rho_*y),
\]
\[ k = 0, 1, 2, \ldots. \quad (2.12) \]

Substituting the above expressions of \( A_k(y) \) and \( B_k(y) \) into (2.10) and (2.11), we get the following solutions:
\[
M_k(y) = a_kL_k(y), \quad N_k(y) = b_kL_k(y), \quad k = 1, 2, \ldots, \quad (2.13)
\]

where
\[
L_k(y) = \mu \frac{\tilde{\sigma} \rho_\ast y^2}{\cosh(\sqrt{k^2 + 1}\rho_\ast)} - \frac{\mu \tilde{\sigma}}{\cosh(\sqrt{k^2 + 1}\rho_\ast)} y \sinh(\rho_*y) + \mu \frac{\tilde{\sigma} \rho_\ast}{\cosh(\sqrt{k^2 + 1}\rho_\ast)} \cosh(\sqrt{k^2 + 1}\rho_*y), \quad k = 1, 2, \ldots. \quad (2.14)
\]

A simple calculation shows that for \( k = 0 \) (2.10) has a solution if and only if \( a_0 = 0 \), and in this situation we have \( M_0 = c \), where \( c \) is an arbitrary constant. Thus we get the following expression for \( P \):
\[
P = c + M_k(y) \cos kx + N_k(y) \sin kx, \quad (2.15)
\]

where \( M_k(y) \) and \( N_k(y) \) are given by (2.13).

Substituting the expressions of \( \eta \) and \( P \) into (2.7) we get
\[
c = 0 \quad \text{and} \quad \mu(\bar{\sigma} - \tilde{\sigma} - \tilde{\sigma} \rho_\ast \sqrt{k^2 + 1} \tanh(\sqrt{k^2 + 1}\rho_\ast) - k \tanh(k\rho_\ast)) = \gamma k^2(a_k \cos kx + b_k \sin kx), \quad k = 1, 2, \ldots. \quad (2.16)
\]

Hence, the linearized problem (2.3)–(2.7) has a nontrivial solution if and only if \( \gamma = \gamma_k \), where
\[
\gamma_k = \frac{\mu(\bar{\sigma} - \tilde{\sigma}) - \mu \tilde{\sigma} \rho_\ast \sqrt{k^2 + 1} \tanh(\sqrt{k^2 + 1}\rho_\ast) - k \tanh(k\rho_\ast)}{k^3 \tanh(k\rho_\ast)}, \quad k = 1, 2, \ldots. \quad (2.17)
\]

We denote by \( h(k\rho_\ast) \) the numerator of the above expression, i.e.,
\[
h(k\rho_\ast) = \mu(\bar{\sigma} - \tilde{\sigma}) - \mu \tilde{\sigma} \rho_\ast \sqrt{k^2 + 1} \tanh(\sqrt{k^2 + 1}\rho_\ast) - k \tanh(k\rho_\ast)).
\]

Since \( h(k) \) is monotonically increasing (see [6]), from (1.5) we know that
\[
h(0) = \mu \bar{\sigma} - \frac{\mu \bar{\sigma} \rho_\ast \tanh \rho_\ast}{\rho_\ast} - \mu \tilde{\sigma} \rho_\ast \tanh^2 \rho_\ast = \mu \bar{\sigma} \cdot \frac{\rho_* - \sinh \rho_* \cosh \rho_*}{\rho_* \cosh^2 \rho_*} < 0,
\]
\[
\lim_{k \to \infty} h(k) = \mu(\bar{\sigma} - \tilde{\sigma}) > 0,
\]
which implies that there exists a positive integer \( k_1 \) such that \( h(k) \leq 0 \) for \( 0 \leq k < k_1 \), and \( h(k) > 0 \) for \( k \geq k_1 \). Let \( g(z) := \frac{\rho_*^2}{z^3 \tanh z} h(z) \). Obviously, there holds \( \gamma_k = g(k\rho_\ast) \). A simple calculation shows that
\[ g'(z) = \frac{\rho^3}{z^4 \tanh z} \left[ -3(\tilde{\sigma} - \bar{\sigma}) - z h'(z) + \left( 3 + \frac{z}{\cosh^2 z \tanh z} \right) h(z) - \frac{\tilde{\sigma} - \bar{\sigma}}{\cosh^2 z \tanh z} \right]. \]

(2.18)

Besides, by a standard limitation argument we know that \( \lim_{k \to \infty} zh'(z) = 0 \), which implies that for any given \( 0 < \varepsilon < \frac{3}{2}(\bar{\sigma} - \tilde{\sigma}) \), there exists \( k \geq k_1 \) such that \( |h(k\rho_*)| < \varepsilon \) for all \( k \geq k_0 \). This combined with (2.18) implies that \( g'(k\rho_*) < 0 \) for all \( k \geq k^* \), i.e., \( \gamma_k \) is strictly monotone decreasing on \( k \geq k_0 \). Summarizing, we have proved the following result:

**Theorem 2.1.** The linearized problem (2.3)–(2.7) has a nontrivial solution \((\eta, \Sigma, P)\) if and only if \( \gamma = \gamma_k \), \( k = 1, 2, \ldots \). In this case the nontrivial solution \((\eta, \Sigma, P)\) is given by

\[ \eta(x) = a_k \cos kx + b_k \sin kx, \]

\[ \Sigma(x, y) = D_k(y)(a_k \cos kx + b_k \sin kx), \]

\[ P(x, y) = L_k(y)(a_k \cos kx + b_k \sin kx), \]

where \( a_k \) and \( b_k \) are arbitrary constants satisfying \( a_k^2 + b_k^2 \neq 0 \), and \( D_k(y), L_k(y) \) are given by (2.12) and (2.14). Moreover, \( \lim_{k \to \infty} \gamma_k = 0 \), and there exists \( k_0 > 0 \) such that

\[ \gamma_k > 0 \quad \text{and} \quad \gamma_k \text{ is strictly monotone decreasing on } k \geq k_0. \]

### 3. The proof of the main result

In this section we prove that every \( \gamma_k \) with \( k \geq k_0 \) is a bifurcation point of the problem (2.1), and all other \( \gamma > 0 \) are not bifurcation points.

We first reduce the problem (2.1) into an abstract bifurcation problem with a single unknown function. Clearly, for a given \( \rho \in h^{m+\alpha}_+(S^1) \), the problem

\[
\begin{cases}
A(\rho)\sigma = \sigma & \text{in } \Omega, \\
\frac{\partial \sigma}{\partial y} \mid_{y=0} = 0 & \text{on } S^1, \\
\sigma \mid_{y=1} = \bar{\sigma} & \text{on } S^1
\end{cases}
\]

(3.1)

has a unique solution \( \sigma \in h^{m+\alpha}(\bar{\Omega}) \), which we denote by \( \mathcal{R}(\rho)\bar{\sigma} \). By (2.2) and the regularity theory for elliptic equations we see that

\[ \mathcal{R}(\cdot)\bar{\sigma} \in C^\infty(h^{m+\alpha}_+(S^1), h^{m+\alpha}(\bar{\Omega})). \]

(3.2)

Next we consider the following problem:

\[
\begin{cases}
A(\rho) p = -\mu(\mathcal{R}(\rho)\bar{\sigma} - \tilde{\sigma}) & \text{in } \Omega, \\
\frac{\partial p}{\partial y} \mid_{y=0} = 0 & \text{on } S^1, \\
B(\rho) p \mid_{y=1} = 0 & \text{on } S^1,
\end{cases}
\]

(3.3)

where we replaced \( \sigma \) with \( \mathcal{R}(\rho)\bar{\sigma} \). By a standard argument, (3.3) has a solution if and only if

\[ \Phi(\rho) \equiv \int_{\Omega} (\mathcal{R}(\rho)\bar{\sigma} - \tilde{\sigma}) \rho \, dx \, dy = 0. \]

(3.4)

Clearly,

\[ \Phi(\cdot) \in C^\infty(h^{m+\alpha}_+(S^1), \mathbb{R}). \]

(3.5)
Since $\mathcal{R}(\rho_\ast)\tilde{\sigma} = \sigma_\ast(y\rho_\ast) = \tilde{\sigma}/\cosh(y\rho_\ast)$ (by (1.4)) and $\tilde{\sigma} = \tilde{\sigma}/\cosh\rho_\ast$ (by (1.5)), we have

$$\Phi(\rho_\ast) = \tilde{\sigma} \int_0^1 \int_{\mathbb{S}} \left( \frac{\cosh(y\rho_\ast)}{\cosh\rho_\ast} - \frac{\sinh\rho_\ast}{\rho_\ast \cosh\rho_\ast} \right) \rho_\ast \, dx \, dy = 0.$$ 

Next we note that if $\rho = \rho_\ast + \varepsilon > 0$ then the problem (3.1) can be solved similarly as (1.2) to get

$$\sigma_\varepsilon = \mathcal{R}(\rho_\ast + \varepsilon)\tilde{\sigma} = \tilde{\sigma}/\cosh(\rho_\ast + \varepsilon).$$

Hence, denoting by $1$ the function on $\mathbb{S}$ which takes the value 1 identically, we have

$$\Phi'(\rho_\ast) 1 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \Phi(\rho_\ast + \varepsilon) - \Phi(\rho_\ast) \right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{S}} \left[ \mathcal{R}(\rho_\ast + \varepsilon)\tilde{\sigma} - \tilde{\sigma} \right] (\rho_\ast + \varepsilon) \, dx \, dy$$

$$- \lim_{\varepsilon \to 0} \int_{\mathbb{S}} \left[ \mathcal{R}(\rho_\ast + \varepsilon)\tilde{\sigma} - \tilde{\sigma} \right] \, dx \, dy$$

$$+ \lim_{\varepsilon \to 0} \int_{\mathbb{S}} \left[ \mathcal{R}(\rho_\ast + \varepsilon)\tilde{\sigma} - \mathcal{R}(\rho_\ast)\tilde{\sigma} \right] \, dx \, dy$$

$$= \int_{\mathbb{S}} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \mathcal{R}(\rho_\ast + \varepsilon)\tilde{\sigma} - \mathcal{R}(\rho_\ast)\tilde{\sigma} \right] \rho_\ast \, dx \, dy$$

$$= \int_{\mathbb{S}} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \frac{\cosh(\rho_\ast y + \varepsilon y)}{\cosh(\rho_\ast + \varepsilon)} - \frac{\cosh(\rho_\ast y)}{\cosh\rho_\ast} \right] \tilde{\sigma} \rho_\ast \, dx \, dy$$

$$= \int_{\mathbb{S}} \left( \frac{\cosh(\rho_\ast y + \varepsilon y)}{\cosh(\rho_\ast + \varepsilon)} \right)' \left|_{\varepsilon = 0} \right. \tilde{\sigma} \rho_\ast \, dx \, dy$$

$$= \frac{\tilde{\sigma} \rho_\ast}{\cosh^2 \rho_\ast} \int_{\Omega} \left[ y \sinh(\rho_\ast y) \cosh\rho_\ast - \cosh(\rho_\ast y) \sinh\rho_\ast \right] \, dx \, dy$$

$$= \frac{\pi \tilde{\sigma} \rho_\ast}{\cosh^2 \rho_\ast} \left( 2\rho_\ast - \sinh(2\rho_\ast) \right) \neq 0.$$
Clearly, 
\[ \mathcal{T} \in \mathcal{C}^\infty \left( h^{m+\alpha}_r (\mathbb{S}^1), \mathcal{L}(h^{m+\alpha}(\tilde{\Omega}), h^{m+2+\alpha}(\tilde{\Omega})) \right). \]

We now define 
\[ S(\rho) = \mu \Upsilon_0 \circ T(\rho)(R(\rho)\tilde{\sigma} - \hat{\sigma}), \]
where \( \Upsilon_0 \) denotes the trace operator, i.e., \( \Upsilon_0 \circ u = u|_{y=1} \) for \( u \in C(\tilde{\Omega}) \). Recall the curvature operator is given by 
\[ \kappa(\rho) = -\frac{\partial^2 \rho}{\partial x^2} \left[ 1 + \left( \frac{\partial \rho}{\partial x} \right)^2 \right]^{-\frac{3}{2}}. \] (3.7)

The above deduction shows that the following result holds:

**Lemma 3.1.** The system of equations (2.1) is equivalent to the following problem: Find \( \rho \in \mathcal{M} \) and \( c \in \mathbb{R} \) such that 
\[
\begin{cases}
S(\rho) + c = \gamma \kappa(\rho), \\
\Phi(\rho) = 0.
\end{cases}
\] (3.8)

Moreover, the operators \( S \) and \( \kappa \) satisfy 
\[ S \in \mathcal{C}^\infty \left( \mathcal{M}, h^{m+\alpha}(\mathbb{S}^1) \right), \]
\[ \kappa \in \mathcal{C}^\infty \left( \mathcal{M}, h^{m-2+\alpha}(\mathbb{S}^1) \right). \]

Now we define the following spaces: 
\[ X_k^{m+\alpha}(\mathbb{S}^1) = \text{the closure of the span } \{ \cos lx : l = 0, 1, 2, \ldots \} \text{ in } C^{m+\alpha}(\mathbb{S}^1), \]
\[ Y_k^{m+\alpha}(\tilde{\Omega}) = \text{the closure of } C^\infty \left( [0, 1], X_k^{\infty}(\mathbb{S}^1) \right) \cap C^\infty (\tilde{\Omega}) \text{ in } C^{m+\alpha}(\tilde{\Omega}), \]
where \( X_k^{\infty}(\mathbb{S}^1) = \bigcap_{l=0}^{\infty} X_k^{m+\alpha}(\mathbb{S}^1) \). It is obvious that \( X_k^{m+\alpha}(\mathbb{S}^1) \) and \( Y_k^{m+\alpha}(\tilde{\Omega}) \) are subspaces of the corresponding little Hölder spaces \( h^{m+\alpha}(\mathbb{S}^1) \) and \( h^{m+\alpha}(\tilde{\Omega}) \), and they are also Banach algebras.

**Lemma 3.2.** There holds the following assertions:

1. The operator \( \frac{\partial^2}{\partial x^2} \) maps \( X_k^{m+\alpha}(\mathbb{S}^1) \) into \( X_k^{m-2+\alpha}(\mathbb{S}^1) \) continuously, and maps \( Y_k^{m+\alpha}(\tilde{\Omega}) \) into \( Y_k^{m-2+\alpha}(\tilde{\Omega}) \) continuously.
2. If \( \rho, \eta \in X_k^{m+\alpha}(\mathbb{S}^1) \) then \( \rho \eta \in X_k^{m+\alpha}(\mathbb{S}^1) \) and \( \rho_x \eta \in X_k^{m-1+\alpha}(\mathbb{S}^1) \).
3. If \( \rho \in X_k^{m+\alpha}(\mathbb{S}^1) \) and \( v \in Y_k^{m+\alpha}(\tilde{\Omega}) \), then \( \rho v \in Y_k^{m+\alpha}(\tilde{\Omega}) \) and \( \rho_x v_x \in Y_k^{m-1+\alpha}(\tilde{\Omega}) \).

**Proof.** (1) is obvious by using the fact that \( X_k^{\infty}(\mathbb{S}^1) \) is dense in \( X_k^{m+\alpha}(\mathbb{S}^1) \). Assertions (2) and (3) follow from the trigonometric formulas like 
\[
2 \cos \alpha \cos \beta = \cos(\alpha - \beta) + \cos(\alpha + \beta),
2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta). \] \[\square\]
Lemma 3.3. For any integer \( k \geq 1 \), the following assertions hold:
\[
\mathcal{S} \in C^\infty(\mathcal{M} \cap X_k^{m+\alpha}(\mathbb{S}^1), X_k^{m+\alpha}(\mathbb{S}^1)),
\]
\[
\kappa \in C^\infty(\mathcal{M} \cap X_k^{m+\alpha}(\mathbb{S}^1), X_k^{m-2+\alpha}(\mathbb{S}^1)).
\]

Proof. By (3.7), the assertion for \( \kappa \) follows immediately from Lemma 3.2. In the following we prove the assertion for \( \mathcal{S} \).

For any \( \rho \in \mathcal{M} \cap X_k^{m+\alpha}(\mathbb{S}^1) \), we can find a sequence \( \{\rho_j\}_{j=1}^\infty \in \mathcal{M} \cap X_k^{\infty}(\mathbb{S}^1) \) such that \( \rho_j \to \rho \) in \( C^{m+\alpha}(\mathbb{S}^1) \). By the continuity of \( \mathcal{S} : \mathcal{M} \to C^{m+\alpha}(\mathbb{S}^1) \), we have \( \mathcal{S}(\rho_j) \to \mathcal{S}(\rho) \) in \( C^{m+\alpha}(\mathbb{S}^1) \). If we can prove \( \mathcal{S}(\rho_j) \in X_k^{\infty}(\mathbb{S}^1) \), then by definition of \( X_k^{m+\alpha}(\mathbb{S}^1) \) and \( X_k^{\infty}(\mathbb{S}^1) \) we obtain \( \mathcal{S}(\rho) \in \mathcal{M} \cap X_k^{m+\alpha}(\mathbb{S}^1) \). Hence, it suffices to prove that if \( \rho \in \mathcal{M} \cap X_k^{\infty}(\mathbb{S}^1) \), then \( \mathcal{S}(\rho) \in X_k^{\infty}(\mathbb{S}^1) \).

Let \( \rho \in \mathcal{M} \cap X_k^{\infty}(\mathbb{S}^1) \). We first prove that
\[
\mathcal{R}(\rho)\tilde{\sigma} \in C^{\infty}([0, 1], X_k^{\infty}(\mathbb{S}^1)) \cap C^\infty(\bar{\Omega}). \tag{3.9}
\]
Indeed, for any given positive function \( \rho \in \mathcal{M} \cap X_k^{\infty}(\mathbb{S}^1) \), the problem (3.1) is equivalent to the following problem:
\[
\begin{aligned}
\rho^2 \frac{\partial^2 \sigma}{\partial x^2} - 2y\rho \frac{\partial^2 \sigma}{\partial x \partial y} + (1 + \frac{\rho^2}{4}) \frac{\partial^2 \sigma}{\partial y^2} &+ y(2\rho^2 - \rho \rho_x) \frac{\partial \sigma}{\partial y} = \rho^2 \sigma & \quad & \text{in } \Omega, \\
\frac{\partial \sigma}{\partial y} &|_{y=0} = 0 & \quad & \text{on } \mathbb{S}^1, \\
\sigma &|_{\partial \mathbb{S}^1} = \tilde{\sigma} & \quad & \text{on } \mathbb{S}^1.
\end{aligned} \tag{3.10}
\]

Since \( \rho \in C^{\infty}(\mathbb{S}^1) \), by the well-known regularity theory for elliptic equations we see that \( \sigma = \mathcal{R}(\rho)\tilde{\sigma} \in C^{\infty}(\bar{\Omega}) \). Hence \( \sigma(x, y) \) has the Fourier expansion:
\[
\sigma(x, y) = a_0(y) + \sum_{j=1}^{\infty} \left( a_j(y) \cos jx + b_j(y) \sin jx \right),
\]
where \( a_0, a_j, b_j \in C^{\infty}[0, 1] \) (\( j = 1, 2, \ldots \)). We now prove that all \( b_j \)'s are zero, and if \( j \) is not proportional to \( k \), then \( a_j \) is also zero. Let \( H^1(\Omega) \) and \( H_0^1(\Omega) \) be respectively the usual \( H^1 \) and \( H_0^1 \) Sobolev spaces on \( \Omega \), and let \( W_k(\Omega) \) be the closure of \( Y_k(\bar{\Omega}) \) in \( H^1(\Omega) \). Observe that given \( w \in W_k(\Omega) \), then there exist \( a_0, a_j, b_j \in C^{\infty}([0, 1]) \) such that
\[
w(x, y) = a_0(y) + \sum_{j=1}^{\infty} a_j(y) \cos jkx \quad \text{for } (x, y) \in \Omega.
\]
We also denote
\[
Y_k(\Omega) = \left\{ w = w(x, y) \in H^1(\Omega): w(x, y) = \sum_{j=1}^{\infty} a_j(y) \cos jx \right\},
\]
\[
Z_k(\Omega) = \left\{ w = w(x, y) \in H^1(\Omega): w(x, y) = \sum_{j=1}^{\infty} b_j(y) \sin jkx \right\},
\]
\[
U_k(\Omega) = \left\{ w = w(x, y) \in H^1(\Omega): w(x, y) = \sum_{j=1}^{\infty} b_j(y) \sin jx \right\}.
\]
It is obvious that

\[
H^1(\Omega) = W_k(\Omega) \oplus Y_k(\Omega) \oplus Z_k(\Omega) \oplus U_k(\Omega).
\]

We now consider the functional \( J \) on \( W_k(\Omega) \cap H^1_0(\Omega) \) defined by

\[
J(u) = \frac{1}{2} \int_\Omega \left[ \rho^2 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] \rho \, dx \, dy + \int_\Omega (u + \bar{\sigma})^2 \rho \, dx \, dy
\]

for \( u \in W_k(\Omega) \cap H^1_0(\Omega) \).

A standard argument shows that \( J \) has a unique local minimum in \( W_k(\Omega) \cap H^1_0(\Omega) \), which we denote by \( \sigma_0 \). Note that since \( \rho \) is a \( C^\infty \) function, we actually have \( \sigma_0 \in C^\infty([0, 1], X^\infty_k(S^1)) \cap C^\infty(\overline{\Omega}) \). In the sequel we prove that \( \sigma = \sigma_0 + \bar{\sigma} \).

Since \( \sigma_0 \) is the minimum point of \( J \), we have, for any \( v \in W_k(\Omega) \cap H^1_0(\Omega) \),

\[
0 = J'(\sigma_0)v = \int_\Omega \left[ \rho^3 \frac{\partial \sigma_0}{\partial x} \frac{\partial v}{\partial x} + \rho \frac{\partial \sigma_0}{\partial y} \frac{\partial v}{\partial y} \right] \, dx \, dy + 2 \int_\Omega (\sigma_0 + \bar{\sigma})v \rho^3 \, dx \, dy. \tag{3.11}
\]

Noticing that \( \rho \in \mathcal{M} \cap X^\infty_k(S^1) \), similarly as in Lemma 3.2, one can easily deduce that

\[
\rho^3, \rho \frac{\partial \sigma_0}{\partial y}, (\sigma_0 + \bar{\sigma})\rho^3 \in W_k(\Omega), \quad \frac{\partial \sigma_0}{\partial x} \in Z_k(\Omega).
\]

Besides, if \( v \in Y_k(\Omega) \oplus Z_k(\Omega) \oplus U_k(\Omega) \), then there holds

\[
\frac{\partial v}{\partial y} \in Y_k(\Omega) \oplus Z_k(\Omega) \oplus U_k(\Omega).
\]

It follows that for any \( v \in Y_k(\Omega) \oplus Z_k(\Omega) \oplus U_k(\Omega) \) we have

\[
\int_\Omega \rho \frac{\partial \sigma_0}{\partial y} \frac{\partial v}{\partial y} \, dx \, dy = 0 \tag{3.12}
\]

and

\[
\int_\Omega (\sigma_0 + \bar{\sigma})v \rho^3 \, dx \, dy = 0. \tag{3.13}
\]

Next we consider the first term on the right-hand side of Eq. (3.11). Firstly, if \( v \in Y_k(\Omega) \), then from the properties of trigonometric functions we can see that

\[
\frac{\partial v}{\partial x} \in U_k(\Omega), \quad \frac{\partial \sigma_0}{\partial x} \frac{\partial v}{\partial x} \in Y_k(\Omega).
\]

So we have

\[
\int_\Omega \rho^3 \frac{\partial \sigma_0}{\partial x} \frac{\partial v}{\partial x} \, dx \, dy = 0 \quad \forall v \in Y_k(\Omega). \tag{3.14}
\]

Secondly, if \( v \in Z_k(\Omega) \) then a similar argument shows that

\[
\frac{\partial v}{\partial x} \in W_k(\Omega), \quad \frac{\partial \sigma_0}{\partial x} \frac{\partial v}{\partial x} \in Z_k(\Omega).
\]
It follows that
\[ \int_{\Omega} \rho^{3} \frac{\partial \sigma_{0}}{\partial x} \frac{\partial v}{\partial x} \, dx \, dy = 0 \quad \forall v \in Z_{k}(\Omega). \] (3.15)

Finally, if \( v \in U_{k}(\Omega) \), for the same reason there holds
\[ \frac{\partial v}{\partial x} \in Y_{k}(\Omega), \quad \frac{\partial \sigma_{0}}{\partial x} \frac{\partial v}{\partial x} \in U_{k}(\Omega), \]

which implies
\[ \int_{\Omega} \rho^{3} \frac{\partial \sigma_{0}}{\partial x} \frac{\partial v}{\partial x} \, dx \, dy = 0 \quad \forall v \in U_{k}(\Omega). \] (3.16)

Combining (3.12)–(3.16) we have, for any \( v \in Y_{k}(\Omega) \oplus Z_{k}(\Omega) \oplus U_{k}(\Omega) = (W_{k}(\Omega))^{\perp} \),
\[ \int_{\Omega} \rho^{3} \frac{\partial \sigma_{0}}{\partial x} \frac{\partial v}{\partial x} \, dx \, dy + 2 \int_{\Omega} (\sigma_{0} + \bar{\sigma}) v \rho^{3} \, dx \, dy = 0. \] (3.17)

Then (3.11) and (3.17) imply that for any \( v \in H^{1}_{0}(\Omega) \) there holds
\[ \int_{\Omega} \rho^{3} \frac{\partial \sigma_{0}}{\partial x} \frac{\partial v}{\partial x} \, dx \, dy + 2 \int_{\Omega} (\sigma_{0} + \bar{\sigma}) v \rho^{3} \, dx \, dy = 0. \] (3.18)

It follows by a standard argument that \( \sigma_{0} + \bar{\sigma} \) is a solution of the problem (3.10), and thus also a solution of (3.1). By uniqueness, we have \( \sigma = \sigma_{0} + \bar{\sigma} \). Hence, \( \sigma \in C^{\infty}([0, 1], X^{\infty}_{k}(S^{1})) \cap C^{\infty}(\bar{\Omega}) \) and (3.9) holds true.

Substituting \( \sigma = R(\rho)\bar{\sigma} \) into (3.3) and arguing similarly as above, we can show that \( p = T(\rho)R(\rho)\bar{\sigma} \in C^{\infty}([0, 1], X^{\infty}_{k}(S^{1})) \cap C^{\infty}(\bar{\Omega}) \). This readily implies that \( S(\rho) \in X^{\infty}_{k}(S^{1}) \), so that the assertion for \( S \) holds.

Let \( S_{k}, \kappa_{k} \) be respectively the restrictions of \( S, \kappa \) to \( \mathcal{M} \cap X^{m_{+}+\alpha}(S^{1}) \), and let \( \Phi_{k} \) be the restriction of the functional \( \Phi \) to \( X^{m_{+}+\alpha}_{k}(S^{1}) \). Instead of (3.8), in the following we consider the following problem:
\[ \begin{cases} S_{k}(\rho) + c = \gamma \kappa_{k}(\rho), \\ \Phi_{k}(\rho) = 0. \end{cases} \] (3.19)

Obviously, (3.19) is not equivalent to (3.8), but it is easy to see that a solution of (3.19) is clearly also a solution of (3.8).

For a sufficiently small \( \delta > 0 \), consider the set
\[ \mathcal{N}_{k,\delta}^{m} = \{ \tilde{\rho} \in X^{m_{+}+\alpha}_{k}(S^{1}) : \tilde{\rho} = \rho + c, \ \Phi_{k}(\rho) = 0, \ \| \rho - \rho_{\ast} \|_{C^{m_{+}+\alpha}(S^{1})} < \delta, \ |c| < \delta \}. \]

For the same reason as before, \( \mathcal{M} \cap X^{m_{+}+\alpha}_{k}(S^{1}) \) is also a \( C^{\infty} \)-Banach submanifold of codimension 1 in \( X^{m_{+}+\alpha}_{k}(S^{1}) \), and for any \( \rho \in \mathcal{M} \cap X^{m_{+}+\alpha}_{k}(S^{1}) \), the curve in \( X^{m_{+}+\alpha}_{k}(S^{1}) : t \mapsto \rho + t, -\delta < t < \delta, \) is transverse to \( \mathcal{M} \cap X^{m_{+}+\alpha}_{k}(S^{1}) \). Hence, if we choose \( \delta \) small enough, the set \( \mathcal{N}_{k,\delta}^{m} \) is open in \( X^{m_{+}+\alpha}_{k}(S^{1}) \), and the mapping \( (\rho, c) \mapsto \rho + c \) is a \( C^{\infty} \)-diffeomorphism of \( \mathcal{M} \cap X^{m_{+}+\alpha}_{k}(S^{1}) \times (\delta, \delta) \) onto \( \mathcal{N}_{k,\delta}^{m} \).
Now we define the mappings \( \bar{S}_k : \mathbb{N}^{m_{k,\delta}} \to X^{m+\alpha}(S^1) \) and \( \bar{\kappa}_k : \mathbb{N}^{m_{k,\delta}} \to X^{m-2+\alpha}(S^1) \) as follows:

For \( \bar{\rho} \in \mathbb{N}^{m_{k,\delta}} \), let \( \bar{\rho} = \rho + c \), where \( \rho \in \mathbb{M} \cap X^{m+\alpha}(S^1) \) and \( c \in \mathbb{R} \), then let

\[
\bar{S}_k(\bar{\rho}) = S_k(\rho) + c,
\]

\[
\bar{\kappa}_k(\bar{\rho}) = \kappa_k(\rho).
\]

It follows that the system (3.19) reduces to the following equation:

\[
\bar{S}_k(\bar{\rho}) = \gamma \bar{\kappa}_k(\bar{\rho}). \tag{3.20}
\]

We now further denote

\[
F(\bar{\rho}, \gamma) = \bar{S}_k(\bar{\rho}) - \gamma \bar{\kappa}_k(\bar{\rho}).
\]

Then Eq. (3.20) can be rewritten as

\[
F(\bar{\rho}, \gamma) = 0. \tag{3.21}
\]

We shall solve the bifurcation problem (3.21) by using the following theorem due to Crandall–Rabinowitz.

**Theorem 3.4.** (See [3].) Let \( X, Y \) be real Banach spaces and let \( G(u, \lambda) \) be a \( C^q \) \( (q \geq 3) \) map from a neighborhood of a point \((u_0, \lambda_0) \in X \times \mathbb{R} \) into \( Y \). Let the following assumptions hold:

1. \( G(u_0, \lambda_0) = 0 \), \( G'_{\lambda}(u_0, \lambda_0) = 0 \),
2. \( \text{Ker} \ G'_{\lambda}(u_0, \lambda_0) \) is one dimensional, spanned by \( u_0 \),
3. \( \text{Im} \ G'_{\lambda}(u_0, \lambda_0) \) has codimension 1,
4. \( G''_{u\lambda}(u_0, \lambda_0) \in \text{Im} \ G'_{\lambda}(u_0, \lambda_0) \), \( G''_{u}(u_0, \lambda_0)u_0 \not\in \text{Im} \ G'_{u}(u_0, \lambda_0) \).

Then \((u_0, \lambda_0) \) is a bifurcation point of the equation

\[
G(u, \lambda) = 0 \tag{3.22}
\]

in the following sense: In a neighborhood of \((u_0, \lambda_0) \) the set of solutions of Eq. (3.22) consists of two \( C^{q-2} \) smooth curves \( \Gamma_1 \) and \( \Gamma_2 \), which intersect only at the point \((u_0, \lambda_0) \). Furthermore, \( \Gamma_1, \Gamma_2 \) can be parameterized as follows:

\[
\Gamma_1: \ (u(\lambda), \lambda), \quad |\lambda - \lambda_0| \text{ is small, } u(\lambda_0) = u_0, \quad u'(\lambda_0) = 0,
\]

\[
\Gamma_2: \ (u(\varepsilon), \lambda(\varepsilon)) , \quad |\varepsilon| \text{ is small, } \ (u(0), \lambda(0)) = (u_0, \lambda_0), \quad u'(0) = u_0.
\]

In the sequel, we verify that the map \( F(\bar{\rho}, \gamma) \) satisfies all the assumptions in Theorem 3.4. Clearly, \( F \in C^\infty(\mathbb{N}^{m_{k,\delta}} \times \mathbb{R}^+, X^{m-2+\alpha}(S^1)) \) and

\[
F(\bar{\rho}_s, \gamma) = 0 \quad \text{for all } \gamma \in \mathbb{R}^+, \tag{3.23}
\]

where \( \bar{\rho}_s = \rho_s + 0 \), because \( (\rho_s, \sigma_s(\rho_s y), p_s(\rho_s y)) \) is a solution of (2.1) for all \( \gamma \in \mathbb{R}^+ \). From (3.23) we can easily see that

\[
\frac{d}{d\gamma} F(\bar{\rho}_s, \gamma) = 0, \quad \frac{d^2}{d\gamma^2} F(\bar{\rho}_s, \gamma) = 0. \tag{3.24}
\]

Besides, it is not difficult to see that the linearized equation of (3.21) at \( \bar{\rho}_s \), denoted as

\[
D_{\bar{\rho}} F(\bar{\rho}_s, \gamma) \tilde{\eta} = 0,
\]
is equivalent to the equation
\[ \bar{S}^{'}(\bar{\rho}_s) \bar{\eta} - \gamma \bar{\kappa}^{'}(\bar{\rho}_s) \bar{\eta} = 0, \]
which is further equivalent to the following system of equations
\[ \begin{cases} S^{'}(\rho_*) \eta + c = \gamma \kappa^{'}(\rho_*) \eta, \\
\Phi^{'}(\rho_*) \eta = 0, \end{cases} \quad (3.25) \]
where \( \bar{\eta} = \eta + c \). Since (3.25) is the linearization of (3.19), deducing similarly as in Section 2, we can see that (3.25) is equivalent to the system of Eqs. (2.3)–(2.7) with unknown \((\eta, \Sigma, P)\) in \( X_k^{m+\alpha}(\mathbb{Z}^1) \times Y_k^{m+\alpha}(\mathbb{Z}) \times Y_k^{m+\alpha}(\mathbb{Z}) \) (see also Theorem 2.1). From discussion of Section 2 one easily sees that (3.25) has nontrivial solutions if and only if \( \gamma = \gamma_k \) \((k \geq k_0)\), and all solutions of (3.25) are given by
\[ \eta(x) = C \cos kx, \quad c = C c_k, \]
where \( C \) is an arbitrary constant, and \( c_k \) is a real constant uniquely determined by \( k \). Thus we get
\[ \text{Ker} D_{\bar{\rho}} F(\bar{\rho}_s, \gamma_k) \text{ is a one-dimensional space spanned by } \bar{\eta}_k = \cos kx + c_k. \quad (3.26) \]
Calculations in Section 2 also shows that for any \( \bar{\eta}_l = \cos lx + c_l \) there holds
\[ D_{\bar{\rho}} F(\bar{\rho}_s, \gamma_k) \bar{\eta}_l = C \cos lx, \]
where the constant \( c_l \) is uniquely determined by \( l \), and \( C = (\gamma_l - \gamma_k)^2 \) with \( \gamma_l \) being the expression in (2.18) with \( k \) replaced by \( l \). One easily sees that \( C \neq 0 \) if \( l \neq k \). Hence we have
\[ \text{codim Im} D_{\bar{\rho}} F(\bar{\rho}_s, \gamma_k) = 1. \quad (3.27) \]
Furthermore, noticing that the linearized operator of \( \kappa(\rho) \) at \( \rho_* \) is a second order differential operator (see also (3.7)), we can easily deduce
\[ \frac{d}{d\gamma} D_{\bar{\rho}} F(\bar{\rho}_s, \gamma) \bar{\eta}_k \big|_{\gamma = \gamma_k} = k^2 \cos kx. \quad (3.28) \]
On the other hand, the conditions \( D_{\bar{\rho}} F(\bar{\rho}_s, \gamma_k) \bar{\eta}_k = 0 \) and
\[ D_{\bar{\rho}} F(\bar{\rho}_s, \gamma_k) \bar{\eta}_l = C \cos lx \quad \text{for any } l \in \mathbb{Z}^+ \]
imply that
\[ k^2 \cos kx \text{ is orthogonal to } D_{\bar{\rho}} F(\bar{\rho}_s, \gamma_k) \tilde{c} \quad \text{for any } \tilde{c} = \sum_{l=0}^{\infty} (\cos lx + \tilde{c}_l), \quad (3.29) \]
where \( \tilde{c}_l \) is a real constant depending on \( l \). It follows from (3.28) and (3.29) that
\[ \frac{d}{d\gamma} D_{\bar{\rho}} F(\bar{\rho}_s, \gamma) \bar{\eta}_k \big|_{\gamma = \gamma_k} \notin \text{Im} D_{\bar{\rho}} F(\bar{\rho}_s, \gamma_k). \quad (3.30) \]
Combining (3.21), (3.23), (3.24), (3.26), (3.27) and (3.30), thanks to Theorem 3.4 we have established the existence of bifurcating solutions of the problem (2.1). Returning to the original problem (1.3), we get the desired result. This completes the proof of Theorem 1.1. \qed

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