JOURNAL OF FUNCTIONAL ANALYSIS 126, 26–35 (1994)

Stochastic Equations in Hilbert Space with Application to Navier–Stokes Equations in Any Dimension

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Communicated by the Editors

Received July 7, 1992

We give an existence theorem for an abstract nonlinear stochastic evolution equation in a Hilbert space. The result is applicable to the stochastic Navier–Stokes equation in any dimension with a nonlinear noise term. © 1994 Academic Press, Inc.

1. INTRODUCTION

Our main goal is the following system of stochastic Navier–Stokes equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\, \Delta u + \langle u, \nabla \rangle u + \nabla p = F(u) + G(u)\, \dfrac{dw(t)}{dt} \\ \operatorname{div} u = 0. \end{cases} \tag{1}$$

This system was first considered in [1] with $G(u) = 1$, F independent of u, and a one-dimensional Wiener process. Note that this system is not covered by the existing theory of stochastic partial differential equations because of the quadratic type of nonlinearity. It has been investigated since then by many authors (see, for example, [8, 15, 16]) but the first existence results covering the case of G depending on u appeared in 1991: in [3] for dimension $n = 1$, [2] for $n = 2$, and [5] for $n \leqslant 4$. Next, we have a preprint [13] which contains an existence result for $n = 2$ under more restrictive conditions (the goal of the paper is the study of attractors so the conditions imposed ensure uniqueness as well). The proof in [5] uses the methods of nonstandard analysis. Here we give a standard proof of a similar result.

0022-1236/94 $6.00

The usual proof of existence of a Navier–Stokes equation (deterministic) is via finite-dimensional Galerkin approximations which are shown to have a cluster point and next, using some compactness argument, this cluster point is shown to solve the equation. The compactness theorem used here requires some regularity in time of the trajectories. This is precisely the main difficulty in obtaining solutions in the stochastic case since the regularity in time of solutions to stochastic Galerkin equations is poor. The nonstandard technique used in [5] benefits from the very construction of a nonstandard universe with some compactness built into it. In [13] the difficulty is overcome by estimates of the Hölder norms of Galerkin approximations, whereas the paper [2] uses discretization in space. Both standard proofs cover the two-dimensional case only and are quite laborious.

We propose a simple functional analytic method which works in any dimension. It is based on the use of the regularization technique which allows us to make use of some suitable compactness theorems. They provide the convergence of the probability distribution of the approximating sequence, and the Skorokhod imbedding theorem allows us to draw the desired conclusion.

In dimension 2 we show that there exists a solution which is more regular than the one obtained in the general case. In [5] some further regularity of solutions is shown for the special case of periodic boundary conditions for $n=2$. What of course remains open is the existence of strong solutions for $n=3$ (the problem has been open for many years in the deterministic case $G=0$ which is easier to handle).

2. Preliminaries

Let H be a separable Hilbert space with norm $|\cdot|$ and scalar product $(\cdot, \cdot)$ and let $(A, D(A))$ be a selfadjoint, positive definite operator on H such that $A^{-1}: H \to H$ is a compact operator. For any $s \geq 0$ we define the space H^{2s} as the domain of the operator A^s with the usual graph norm $|A^s \cdot|$. Let H^{-s} be the space dual to H^s. Denote by $|\cdot|_s$ the norm in H^s and by $(\cdot, \cdot)$ the antidual product between H^s and H^{-s}. Obviously $H^s \subseteq H \subseteq H^{-s}$ for any $s \geq 0$ provided we identify H with its dual. Note that $-A$ generates an analytic semi-group on H^s for any $s \in \mathbb{R}$. Denote by $S(t)$ the semi-group generated by $-A$. Consider a $B \in C(H; H^{-s})$ for some $s \geq 1$ such that for any $u \in H^s$

$$(B(u), u) = 0, \tag{2}$$

$$|B(u)|_{-s} \leq |u|^2. \tag{3}$$

Let U be another separable Hilbert space. We denote by $L_{\mathrm{HS}}(U; H)$ the space of Hilbert–Schmidt operators from U to H with the Hilbert–Schmidt norm $|\cdot|_{\mathrm{HS}}$.

We now recall some facts concerning the factorization method.

Let $h \in L^p(0, T; H^s)$. For any numbers $s \in \mathbb{R}$ and $\alpha \in (0, T]$ we define an operator R_α by

$$R_\alpha h(t) = \int_0^t (t-s)^{\alpha-1} S(t-s)\, h(s)\, ds.$$

LEMMA 2.1 ([7]). *Let W be a cylindrical Wiener process on U. Then for any $0 < p^{-1} < \alpha < \frac{1}{2}$ and for any predictable $\phi \in L^p(\Omega \times [0, T]; L_{\mathrm{HS}}(U; H))$ we have*

$$\pi \sin(\pi\alpha)^{-1} \int_0^t S(t-s)\, \phi(s)\, dW(s) = R_\alpha Y(t),$$

where

$$Y(t) = \int_0^t (t-s)^{-\alpha} S(t-s)\, \phi(s)\, dW(s).$$

LEMMA 2.2 ([9]). *Let $p > 0$, $p^{-1} + \gamma < 1$, and $s \in \mathbb{R}$. The operator R_α: $L^p(0, T; H^s) \to C(0, T; H^{s+2\gamma})$ is compact for any $\alpha \in (p^{-1} + \gamma, 1]$.*

LEMMA 2.3 ([16, Theorem IV.4.1]). *Suppose that M is a bounded set in $L^2(0, T; H^1)$ and a compact set in $C(0, T; H^{-\gamma})$ for a certain $\gamma \geq 0$. Then M is a compact set in $L^2(0, T; H)$.*

3. ABSTRACT EQUATIONS OF NAVIER–STOKES TYPE

We consider the right-hand-side term with nonlinear feedback F: $H \to H$ satisfying

F1. $F(\cdot) \in C(H, H^{-1})$,

F2. $|F(u)|_{-1} \leq C(1 + |u|)$ for a certain constant C.

Let G: $H \to C(H; L_{\mathrm{HS}}(U; H))$ be such that

G1. $G(\cdot) \in C(H; L_{\mathrm{HS}}(U; H))$,

G2. $|G(u)|_{\mathrm{HS}} \leq K(1 + |u|)$, where K is a constant.

Let W be a cylindrical Wiener process on U defined on a probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$. We study the equation

$$\begin{cases} dX(t) = [-AX(t) + B(X(t)) + F(X(t))]\,dt + G(X(t))\,dW(t), \\ X(0) = x_0 \in H. \end{cases} \tag{4}$$

THEOREM 3.1. *Suppose that* $x_0 \in H$ *and Conditions* F1, F2, G1, G2 *are satisfied. Then there exists a martingale solution*

$$X \in L^2(\Omega \times [0, T]; H^1) \cap L^p(\Omega; C(0, T; H^\gamma)) \cap L^p(\Omega; L^\infty(0, T; H))$$

for any $T, \gamma < 2 - s$, *and* $p \geqslant 1$.

Proof. Let $e_1, e_2, e_3, ...$ be the orthonormal basis of H consisting of eigenvectors of the operator A. Let Π_n be the orthogonal projection on the space $\operatorname{span}\{e_1, e_2, ..., e_n\}$. Consider the sequence of equations

$$\begin{cases} dX_n(t) = [-AX_n(t) + B_n(X_n(t)) + F_n(X_n)]\,dt + G_n(X_n(t))\,dW(t), \\ X_n(0) = \Pi_n x_0 \in H, \end{cases} \tag{5}$$

where $B_n(u) = \Pi_n B(\Pi_n u)$, $F_n(u) = \Pi_n F(\Pi_n u)$, and $G_n(u) = \Pi_n G(\Pi_n u)$ for $n = 1, 2, 3, ...$

Note that the equation (5) is a finite-dimensional stochastic equation. Hence there exists a martingale solution (possibly exploding) of (5) for any $n = 1, 2, 3, ...$ By applying the Itô formula to the function $|\cdot|^2$ we have

$$\begin{aligned} d\,|X_n(t)|^2 = (-2\,|X_n(t)|_1^2 + (F_n(X_n(t)), X_n(t)) + |G_n(X_n(t))|_{\mathrm{HS}}^2)\,dt \\ + 2(X_n(t), G_n(X_n(t))\,dW(t)), \end{aligned} \tag{6}$$

where we have taken account of $(B(X), X) = 0$. The mathematical expectation of the stochastic integral vanishes, therefore by using F2 and G2 and employing the Young inequality we have

$$\begin{aligned} \frac{d}{dt} E\,|X_n(t)|^2 &= -2E\,|X_n(t)|_1^2 + E\,|F(t, X_n(t))|_{-1}\,|X_n(t)|_1 + E\,|G_n(X_n(t))|_{\mathrm{HS}}^2 \\ &\leqslant K_1(1 + E\,|X_n(t)|^2), \end{aligned}$$

where the constant K_1 is independent of n. By the Gronwall lemma $E\,|X_n(t)|^2 \leqslant K_2 < +\infty$ for all $t \in (0, T]$ and all n, so the processes X_n do not explode. Simultaneously, we obtain

$$E \int_0^T |X_n(t)|_1^2\,dt \leqslant K_3 < +\infty \tag{7}$$

for all n.

Fix a $\gamma < 1$ and take a $p \geqslant 4$ such that $2p^{-1} + \gamma < 1$. By the Itô formula for the function $|\cdot|^{p/2}$ we have

$$d\,|X_n(t)|^{p/2} \leqslant \frac{p}{2}\,(-X_n(t)|_1^2 + (F_n(X_n), X_n(t)) + K_3\,|G_n(X_n(t))|_{\mathrm{HS}}^2)|\,X_n(t)|^{p/2-2}\,dt + \mu_n(t),$$

where

$$\mu_n(t) = \frac{p}{2}\,|X_n(t)|^{p/2-2}\,(X_n(t), G_n(X_n(t))\,dW(t)).$$

By the Schwartz inequality, using F2 and G2 we obtain the estimate

$$\sup_{t \in [0,\tau]} |X_n|^p \leqslant K_4\,|X_n(0)|^p + K_4 \int_0^T (1 + |X_n(t)|^p)\,dt + K_4 \sup_{t \in [0,\tau]} |\mu_n(t)|^2, \tag{8}$$

where the constant K_4 is independent of n. By the maximal inequality for martingales, see [11], for example, we have

$$E(\sup_{t \leqslant T} |X_n(t)|^p) \leqslant 2E\mu_n(T)^2 \leqslant K_5 E \int_0^T |X_n(t)|^{p-2}\,|G_n(X_n(t))|_{\mathrm{HS}}^2\,dt.$$

Now, by G2,

$$E(\sup_{t \in [0,\tau]} |X_n|^p) \leqslant K_6 E\,|X_n(0)|^p + K_7 \int_0^T (1 + E\,|X_n(t)|^p)\,dt,$$

hence using Gronwall's lemma we obtain

$$\sup_n E \sup_{t \leqslant T} |X_n(t)|^p < \infty. \tag{9}$$

By (3), F2, and G2 we have

$$E \int_0^T |B(X_n(t))|_{-s}^{p/2}\,dt \leqslant K_8 < \infty \tag{10}$$

$$E \int_0^T |F(X_n(t))|_{-1}^p\,dt \leqslant K_9 < \infty, \tag{11}$$

$$E \int_0^T |G(X_n(t))|^p\,dt \leqslant K_{10} < \infty, \tag{12}$$

for all n.

Fix an $\alpha \in (p^{-1}, \frac{1}{2})$. Denote $Z_n(t) = B_n(X_n(t))$, $V_n(t) = F_n(X_n(t))$, and

$$Y_n(t) = \int_0^t (t-s)^{-\alpha} S(t-s) G(X_n(t)) \, dW(s).$$

By the Young inequality and (12) for all n,

$$E \int_0^1 |Y_n(t)|^p \, dt \leqslant K_{11} < \infty. \tag{13}$$

Denote

$$\Lambda_1(\rho) = \left\{ w \in C(0, T; H) : w = R_\alpha u, \ u \in L^p(0, T; H), \right.$$
$$\left. \int_0^T |u(t)|^p \, dt \leqslant \rho \right\},$$

$$\Lambda_2(\rho) = \left\{ w \in C(0, T; H^{2\gamma - s}) : w = R_1 u, \ u \in L^{p/2}(0, T; H^{-s}), \right.$$
$$\left. \int_0^T |u(t)|_{-s}^{p/2} \, dt \leqslant \rho \right\},$$

$$\Lambda_3(\rho) = \left\{ w \in C(0, T; H^{2\gamma - 1}) : w = R_1 u, u \in L^p(0, T; H^{-1}), \right.$$
$$\left. \int_0^T |u(t)|_{-1}^{p} \, dt \leqslant \rho \right\},$$

and

$$\Xi(\rho) = \{ w \in C(0, T; H^{2\gamma - s}) : w(t) = S(t)x + w_1(t) + w_2(t) + w_3(t), \ |x| \leqslant \rho,$$
$$w_1 \in \Lambda_1(\rho), \ w_2 \in \Lambda_2(\rho) \ w_3 \in \Lambda_3(\rho), \}.$$

By Lemma 2.2 the sets $\Lambda_i(\rho)$, $i = 1, 2$, and $\Xi(\rho)$ are relatively compact in the space $C(0, T; H^{-s+2\gamma})$. Recall that

$$X_n(t) = S(t) \, x_n + R_1 Z_n(t) + R_1 V_n(t) + \pi^{-1} \sin(\pi\alpha) \, R_\alpha Y_n(t)$$

and, by (10), (11), (13), and the Chebyshev inequality, for any $n \geqslant 2$ and any $\varepsilon > 0$ we have $P(X_n \in \Xi(\rho)) \geqslant 1 - \varepsilon$ for sufficiently large ρ. By the Prokhorov theorem the sequence $\mathscr{L}(X_n)$ is tight on $C(0, T; H^{-s+2\gamma})$. By (7) and Lemma 2.3, the sequence $\mathscr{L}(X_n)$ is also tight on $L^2(0, T; H)$. Hence there is probability P^0 on $C(0, T; H^{-s+2\gamma}) \cap L^2(0, T; H)$ such that $\mathscr{L}(X_{n_k}) \to P^0$ weakly for a subsequence n_k. We denote

$$M_n(t) = \int_0^t G_n(X_n(s)) \, dW(s).$$

By [12, Theorem II.4.6], the sequence M_n is tight on $C(0, T; H)$. So there is a measure $\tilde{P}$ such that $\mathscr{L}(X_n, M_n) \to \tilde{P}$ weakly on $C(0, T; H^{2\gamma-2}) \times C(0, T; H)$ for some subsequence of n, still denoted by n. By the Skorokhod imbedding, see [10], there exists a probability space $(\Omega^1, \mathscr{F}^1, \mathscr{F}^1_t, Q)$ and a sequence of random variables (X^1_n, M^1_n) on Ω^1 such that the distributions of (X^1_n, M^1_n) and (X_n, M_m) are the same and $(X^1_n, M^1_n) \to (X, M)$ a.s. in $C(0, T; H^{2\gamma-s}) \times C(0, H)$ for a certain $(X, M) \in C(0, T; H^{2\gamma-s}) \times C(0, T; H)$.

Recall that

$$X^1_n(t) = \Pi_n x_0 + \int_0^t AX^1_n(s)\, ds + \int_0^t B_n(X^1_n(s))\, ds + \int_0^t F_n(X^1_n(s))\, ds + M^1_n(t), \tag{14}$$

where

$$\langle\langle M^1_n \rangle\rangle(t) = \int_0^t G_n(X^1_n(s))^* G_n(X^1_n(s))\, ds. \tag{15}$$

Passing to the limit in (14) and (15) we obtain that the X is a martingale solution of Eq. (4). Applying Fatou's lemma to (7) and (10) we obtain $X \in L^2(\Omega^1 \times [0, T]; H^1) \cap L^p(\Omega^1; L^\infty(0, T; H))$. ∎

We have in fact better regularity of solution.

COROLLARY 3.2. *The solution of the previous theorem is for a.a.* ω *in* $C(0, T; H^{-\alpha})$ *for* $\alpha > 0$.

Proof. Since for a.a. ω the function $t \mapsto X(t, \omega)$ is bounded in H, it is also weakly continuous in H ([14, p. 263]). Now let $\alpha > 0$ and consider the systems of eigenvalues λ_n and eigenfunctions e_n of A. We clearly have

$$\begin{aligned} |X(t) - X(s)|_{-\alpha} &= \sum_{k=1}^{\infty} \lambda_k^{-\alpha}(X(t) - X(s), e_k) \\ &\leqslant \sum_{k=1}^{n} \lambda_k^{-\alpha}(X(t) - X(s), e_k) + \lambda_n^{-\alpha} 2 \sup_{t \in [0, T]} |X(t)| \\ &\to 0, \end{aligned}$$

as $n \to \infty$ and $|t - s| \to 0$. ∎

4. APPLICATIONS TO NAVIER–STOKES EQUATIONS

Let $\mathcal{O} \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded open set with boundary of class C^2. We denote $\mathscr{V} = \{w \in C_0^\infty(\mathcal{O})^n : \operatorname{div} w = 0\}$, and we denote by H the closure of

$\mathscr{V}$ in the norm of $L^2(\mathcal{O})^n$. Let A be the Stokes operator in H which is a selfadjoint extension of the operator $-\Delta$ on $\mathscr{V}$.

We define the spaces H^s as in the previous section ($H^s = \text{dom}\, A^{s/2}$). Fix $s > n/2 + 1$. Note that H^{-s} is a closed subspace of the Sobolev space $H_{-s}(\mathcal{O})^n$. Let π be the orthogonal projection in $H_{-s}(\mathcal{O})^n$ on the space H^{-s}. Consider the operator $B(u) = \pi \langle u, \nabla \rangle u$ for any $u \in H$, where ∇ is the gradient operator. Since $\text{div}\, u = 0$, $(B(u), u) = 0$ for any $u \in H^1$. For $u \in H^s$ we have the well-known inequality

$$|B(u) v| \leqslant C\, |u|^2\, |v|_s,$$

hence $B \in C(H; H^{-s})$. As a corollary to Theorem 3.1 we have the following result.

THEOREM 4.1. *Let F, G, and W be as in the previous section. For any* $x_0 \in H$ *there exists a solution of the Navier–Stokes equation* (1),

$$\frac{\partial}{\partial t} X(x, t) = \nu A X(x, t) + \pi(X(x, t), \nabla)\, X(x, t) + G(X(t))\, \frac{\partial}{\partial t}\, W(x, t),$$

$$X(x, 0) = x_0.$$

Remark. Clearly all results remain intact if we replace $X(0)$ with a random variable independent of the Wiener process with $E\,|X(0)|^p < \infty$ for all p.

We now briefly dwell on the case $n = 2$. We can obtain some better regularity of solution which is due to the following estimate on B:

$$|(B(u), v)| \leqslant |u|_1\, |Au|^{1/2}\, |u|^{1/2}\, |v|. \tag{16}$$

THEOREM 4.2. *Suppose that F and G satisfy*

F2′. $|F(u)| \leqslant C(1 + |u|)$ *for some constant C.*

G2′. $|G(u)|_{\mathrm{HS}(U, H_1)} \leqslant K(1 + |u|)$ *where K is a constant.*

Then for $x_0 \in H_1$ *there is a solution to the Navier–Stokes equations in dimension* $n = 2$ *such that*

$$\sup_{t \in [0, T]} |X(t)|_1 + \int_0^T |AX(t)|^2\, dt < \infty \qquad a.e. \tag{17}$$

Proof. We show (17) for the Galerkin approximation X_n of the Proof of Theorem 3.1. Then (17) for X constructed in Theorem 3.1 follows by Fatou's lemma by a limit passage.

We apply the Itô formula to $|X_n|_1^2$ and we get

$$d\,|X_n(t)|_1^2 = (-2\nu\,|AX_n(t)|_1^2 + (F_n(X_n(t)), AX_n(t)) + |G_n(X_n(t))|^2_{\mathrm{HS}(U,H_1)})\,dt + 2(AX_n(t), G_n(X_n(t))\,dW(t)).$$

Now by employing F2′, G2′, and (16) and using the Young inequality we obtain

$$\begin{aligned} &|X_n(\tau)|_1^2 + \int_0^\tau \nu\,|AX_n(t)|_1^2\,dt \\ &= |X_n(0)|_1^2 + \frac{K_{12}}{\nu}\int_0^\tau K^2(t)(1+|X_n(t)|_1^2)\,dt \\ &+ \frac{K_{13}}{\nu}\int_0^\tau |X_n(t)|^2\,|X_n(t)|_1^4\,dt + \int_0^\tau (AX_n(t), G_n(X_n(t))\,dW(t)) \\ &+ \int_0^\tau \eta(t)\,dt, \end{aligned}$$

where $\eta \leqslant 0$. Now, following the idea of the uniqueness proof in [14, p. 295] (see also [13]), we consider the function

$$H(\tau) = \exp\left\{-\frac{K_{13}}{\nu}\int_0^\tau |X_n(t)|^2\,|X_n(t)|_1^2\,dt\right\}$$

and we compute $d(|X_n(t)|_1^2 H(t))$, obtaining

$$\begin{aligned} &H(\tau)\left(|X_n(\tau)|_1^2 + \int_0^\tau \nu\,|AX_n(t)|_1^2\,dt\right) \\ &= |X_n(0)|_1^2 + \frac{K_{12}}{\nu}\int_0^\tau H(t)\,K^2(t)(1+|X_n(t)|_1^2)\,dt \\ &+ \int_0^\tau H(t)(AX_n(t), G_n(X_n(t))\,dW(t)) + \int_0^\tau H(t)\,\eta(t)\,dt. \end{aligned}$$

Taking mathematical expectation we obtain

$$\begin{aligned} &EH(\tau)\left(|X_n(\tau)|_1^2 + \int_0^\tau \nu\,|AX_n(t)|_1^2\,dt\right) \\ &\leqslant |X_n(0)|_1^2 + K_{14} + K_{15}\int_0^\tau E(H(t)\,|X_n(t)|_1^2)\,dt. \end{aligned}$$

hence by the Gronwall lemma we have

$$EH(\tau)\left(|X_n(\tau)|_1^2+\int_0^\tau \nu\,|AX_n(t)|_1^2\,dt\right)\leqslant K_{16}.$$

We can estimate $H(\tau)$ from below on sets of measure arbitrarily close to one, whence the result. ∎

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