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On two questions about quaternion matrices

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Abstract

This paper answers two questions proposed in [Fuzhen Zhang, Linear Algebra Appl. 251 (1997) 21–57], showing that there exists a 2×2 quaternion matrix with only two non-similar left eigenvalues that is not diagonalizable, and giving some necessary and sufficient conditions for

$$\begin{vmatrix} A & B \\ -\overline{B} & \overline{A} \end{vmatrix} = 0.$$

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1. Introduction

The aim of this paper is to answer two of the questions proposed in [1].

Let R be the real number field, $C = R \oplus Ri$ be the complex number field, and $H = C \oplus Cj = R \oplus Ri \oplus Rj \oplus Rk$ be the quaternion division ring over R, where $k := ij = -ji$, $i^2 = j^2 = k^2 = -1$. If $\alpha = a_1 + a_2 i + a_3 j + a_4 k \in H$, where $a_i \in R$, then let $\overline{\alpha} = a_1 - a_2 i - a_3 j - a_4 k$ be the conjugate of α, $|\alpha| = \sqrt{\overline{\alpha}\alpha} = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}$. Let $M_{m\times n}(\Omega)$ be the set of all $m \times n$ matrices over a ring Ω with identity, and $M_n(\Omega) = M_{n\times n}(\Omega)$, $\Omega^n = M_{n\times 1}(\Omega)$. Suppose $A = (a_{ij}) \in$

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$M_{n\times n}(H)$. Let A^{T} be the transpose matrix of A, $\overline{A}$ be the conjugate matrix of A, and $A^* = (\overline{a}_{ij})^{\mathrm{T}}$ be the transpose conjugate matrix of A. Let $N(A) = \{X \in H^n \mid AX = 0\}$ be the null space of A. We write $A \sim B$ if A is similar to B. For $A(\lambda), B(\lambda) \in M_n(H[\lambda])$, we write $A(\lambda) \cong B(\lambda)$ if $A(\lambda)$ is congruence to $B(\lambda)$, i.e., there exist unimodular matrices $P(\lambda), Q(\lambda) \in M_n(H[\lambda])$ such that $P(\lambda)A(\lambda)Q(\lambda) = B(\lambda)$.

Let $A \in M_n(H)$, $\lambda \in H$. If there exists $0 \neq X \in H^n$ such that

$$AX = \lambda X \quad (\text{or } AX = X\lambda), \tag{1}$$

then λ is said to be a left (or right) eigenvalue of A, and X is said to be an eigenvector of A corresponding to the left (right) eigenvalue λ. The set of left (or right) eigenvalues of A is called the left (or right) spectrum, denoted by $\sigma_l(A)$ (or $\sigma_r(A)$). Let

$$\rho_l(A) = \sup\{|\lambda| \mid \lambda \in \sigma_l(A)\}$$

be the right spectral radius, and

$$\rho_r(A) = \sup\{|\lambda| \mid \lambda \in \sigma_r(A)\}$$

be the left (left) spectral radius.

If $A = A_1 + A_2 j \in M_{n\times n}(H)$, where $A_1, A_2 \in M_{n\times n}(C)$, then the complex representation matrix (or adjoint matrix [1]) of A is defined by

$$\chi_A = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix}. \tag{2}$$

A g-inverse of $A \in M_{m\times n}(C)$ will be denoted by $A^- \in M_{n\times m}(C)$ and understood as a complex matrix for which $AA^-A = A$. A Moore–Penrose inverse of $A \in M_{m\times n}(C)$ will be denoted by $A^+ \in M_{n\times m}(C)$ and understood as the unique complex matrix for which $AA^+A = A$, $A^+AA^+ = A^+$, $AA^+ = (AA^+)^*$ and $A^+A = (A^+A)^*$.

We shall study two questions, Questions 7.1 and 5.4 in [1], which are cited, respectively as:

Question 1. Suppose $A \in M_n(H)$ has n distinct left eigenvalues, any two of which are not similar. Is A diagonalizable?

Question 2. For $A, B \in M_n(C)$, what conditions can be imposed on A and B when

$$\begin{vmatrix} A & B \\ -\overline{B} & \overline{A} \end{vmatrix} = 0? \tag{3}$$

2. The answer to Question 1 is negative

Lemma 1. *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(H).$$

If $c \neq 0$, *then* $\lambda \in \sigma_l(A)$ *if and only if*

$$(\lambda - a)c^{-1}(\lambda - d) - b = 0. \tag{4}$$

Proof. $\lambda \in \sigma_l(A)$ if and only if $\lambda I - A$ is a singular matrix. Since

$$\lambda I - A = \begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix} \cong \begin{pmatrix} 0 & (\lambda - a)c^{-1}(\lambda - d) - b \\ -c & \lambda - d \end{pmatrix}, \tag{5}$$

if $c \neq 0$, then it is clear that $\lambda \in \sigma_l(A)$ if and only if $(\lambda - a)c^{-1}(\lambda - d) - b = 0$. □

The counterexample for Question 1 is as follows:

Let

$$A = \begin{pmatrix} -i - j & 1 - 2k \\ 1 & -i + j \end{pmatrix}. \tag{6}$$

Then A has only two distinct left eigenvalues

$$\lambda_1 = \sqrt{\frac{\sqrt{5} - 1}{2}} + \frac{\sqrt{5} - 3}{2} i - \left(\frac{\sqrt{5} - 1}{2}\right)^{3/2} k, \tag{7}$$

$$\lambda_2 = -\sqrt{\frac{\sqrt{5} - 1}{2}} + \frac{\sqrt{5} - 3}{2} i + \left(\frac{\sqrt{5} - 1}{2}\right)^{3/2} k. \tag{8}$$

We show that λ_1 and λ_2 are not similar and A is not diagonalizable.

Since

$$A = \begin{pmatrix} 1 & -j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -i & 0 \\ 1 & -i \end{pmatrix} \begin{pmatrix} 1 & -j \\ 0 & 1 \end{pmatrix}^{-1}, \tag{9}$$

by the Jordan canonical form of quaternion matrix (cf. [1, Theorem 6.4]), it is clear that A is not diagonalizable.

Since Re $\lambda_1 \neq$ Re λ_2, by Theorem 2.2 of [1], λ_1 and λ_2 are not similar. We now prove that A has only two left eigenvalues λ_1 and λ_2.

By Lemma 1, λ is a left eigenvalue of A if and only if

$$(\lambda + i + j)(\lambda + i - j) - (1 - 2k) = 0. \tag{10}$$

Write $\lambda = x - i$. Then $\lambda = x - i \in \sigma_l(A)$ if and only if

$$x^2 - xj + jx + 2k = 0. \tag{11}$$

Let $x = \lambda + i = y_1 + y_2 i + y_3 j + y_4 k$, where $y_i \in R$. Then (11) can be written as

$$\begin{aligned} &y_1^2 - y_2^2 - y_3^2 - y_4^2 + 2(y_1 y_2 + y_4)i \\ &\quad + 2y_1 y_3 j + 2(y_1 y_4 - y_2 + 1)k = 0. \end{aligned} \tag{12}$$

Thus, $\lambda = y_1 + (y_2 - 1)i + y_3 j + y_4 k \in \sigma_l(A)$ if and only if

$$\begin{aligned} &y_1^2 - y_2^2 - y_3^2 - y_4^2 = 0, \\ &y_1 y_2 + y_4 = 0, \\ &y_1 y_3 = 0, \\ &y_1 y_4 - y_2 + 1 = 0. \end{aligned} \tag{13}$$

By $y_1 y_3 = 0$, it is easy to see that $y_3 = 0$ and $y_1 \neq 0$. Thus, any $\lambda \in \sigma_l(A)$ can be written as

$$\lambda = y_1 + (y_2 - 1)i - y_1 y_2 k, \tag{14}$$

where

$$\begin{aligned} &y_1^2 - y_2^2 - y_1^2 y_2^2 = 0, \\ &y_1^2 y_2 + y_2 - 1 = 0. \end{aligned} \tag{15}$$

Since $y_2 = 1/(y_1^2 + 1) > 0$, it is easy to see that Eqs. (15) have only two solutions

$$(y_1, y_2) = \left(\sqrt{\frac{\sqrt{5} - 1}{2}}, \frac{\sqrt{5} - 1}{2} \right) \quad \text{or} \quad \left(-\sqrt{\frac{\sqrt{5} - 1}{2}}, \frac{\sqrt{5} - 1}{2} \right). \tag{16}$$

Thus A has only two left eigenvalues λ_1, λ_2. We further have

$$\sigma_r(A) = \{q^{-1} i q \mid 0 \neq q \in H\}, \tag{17}$$

$$\rho_r(A) = \rho_l(A) = 1. \tag{18}$$

3. Invertibility of the complex representation matrix

By Lemmas 5 and 6 of [4], clearly we have:

Lemma 2. *Let $A \in M_{m \times n}(C)$, $D \in M_{m \times q}(C)$. Then the following statements are equivalent*:

(i) *The matrix equation $AX = D$ has a solution $X \in M_{n \times q}(C)$*;

(ii) $\operatorname{rank}(A, D) = \operatorname{rank}(A)$;

(iii) $AA^- D = D$.

Moreover, if matrix equation $AX = D$ has a solution, then its general solution can be written as

$$X = A^- D + (I - A^- A)W, \tag{19}$$

with arbitrary $W \in M_{n \times q}(C)$, where all g-inverses involved are arbitrary but fixed.

Theorem 1. *If A, $B \in M_n(C)$, then the following statements are equivalent*:

(i) $\begin{pmatrix} \overline{A} & \overline{B} \\ -B & A \end{pmatrix}$ *is invertible*;

(ii) $\operatorname{rank}(A, B) = n$ *and*

$$\operatorname{rank}[(A, B) - (BA^{\mathrm{T}} - AB^{\mathrm{T}})(\overline{A}A^{\mathrm{T}} + \overline{B}B^{\mathrm{T}})^{-1}(-\overline{B}, \overline{A})] = n; \tag{20}$$

(iii) $\operatorname{rank}(A, B) = n$ *and*

$$\dim\{N[(A, B)^{-}(A, B)] \cap N(-\overline{B}, \overline{A})]\} = 0; \tag{21}$$

(iv)

$$\operatorname{rank}\{(A, B)[I_{2n} - (-\overline{B}, \overline{A})^{-}(-\overline{B}, \overline{A})]\} = n; \tag{22}$$

(v)

$$\operatorname{rank}\{(-\overline{B}, \overline{A})[I_{2n} - (A, B)^{-}(A, B)]\} = n, \tag{23}$$

where all g-inverses are arbitrary but fixed. Moreover, if condition (iv) *holds, letting* $S = I_{2n} - (-\overline{B}, \overline{A})^{-}(-\overline{B}, \overline{A})$, $D = S[(A, B)S]^{-} = \begin{pmatrix} D_1 & -\overline{D}_2 \end{pmatrix}^{\mathrm{T}}$, *then*

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}^{-1} = \begin{pmatrix} D_1 & D_2 \\ -\overline{D}_2 & \overline{D}_1 \end{pmatrix}. \tag{24}$$

Proof. "(i)$\Longleftrightarrow$(iv)": If

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$$

is invertible, then $\operatorname{rank}(A, B) = n$. Let $Q = A + Bj \in M_n(H)$. Then

$$\chi_A = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}.$$

By Theorem 4.3 of [1], Q is invertible. Let $Q^{-1} = X_1 + X_2 j$, where $X_1, X_2 \in M_n(C)$. Then by Theorem 4.2 of [1], we have

$$\chi_Q \chi_{Q^{-1}} = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ -\overline{X}_2 & \overline{X}_1 \end{pmatrix} = I_{2n}. \tag{25}$$

Let

$$Y = \begin{pmatrix} X_1 \\ -\overline{X}_2 \end{pmatrix}.$$

We have

$$(A, B)Y = I \quad \text{and} \quad (-\overline{B}, \overline{A})Y = 0. \tag{26}$$

Thus, the matrix equations

$$(A, B)X = I \quad \text{and} \quad (-\overline{B}, \overline{A})X = 0 \tag{27}$$

have a common solution $X = Y \in M_{2n\times n}(C)$.

Let

$$S = I_{2n} - (-\overline{B}, \overline{A})^{-}(-\overline{B}, \overline{A}), \tag{28}$$

where the g-inverse $(-\overline{B}, \overline{A})^{-}$ is arbitrary but fixed. By Lemma 2, the general solution of the matrix equation $(-\overline{B}, \overline{A})X = 0$ can be written as

$$X = SW, \tag{29}$$

with arbitrary $W \in M_{2n\times n}(C)$. Thus, there exists W_0 such that $Y = SW_0$. Using it to replace X in the equation $(A, B)X = I$, we have

$$(A, B)SW_0 = I_n. \tag{30}$$

By (30)

$$n \geqslant \text{rank}[(A, B)S] \geqslant \text{rank} I_n = n. \tag{31}$$

Thus we have (22).

Conversely, if condition (22) holds, then by Lemma 2, the matrix equation

$$((A, B)S)X = I_n \tag{32}$$

has a solution $X = [(A, B)S]^{-}$. Suppose

$$D = S[(A, B)S]^{-} \in M_{2n\times n}(C). \tag{33}$$

Then we have

$$(A, B)D = I \quad \text{and} \quad (-\overline{B}, \overline{A})D = 0. \tag{34}$$

Let

$$D = \begin{pmatrix} D_1 \\ -\overline{D}_2 \end{pmatrix}.$$

Then it is easy to see that

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \begin{pmatrix} D_1 & D_2 \\ -\overline{D}_2 & \overline{D}_1 \end{pmatrix} = I_{2n}. \tag{35}$$

Thus

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$$

is invertible, and

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}^{-1} = \begin{pmatrix} D_1 & D_2 \\ -\overline{D}_2 & \overline{D}_1 \end{pmatrix}. \tag{36}$$

"(iv)$\Longleftrightarrow$(ii)": If (iv) holds, then (i) holds and

$$\text{rank}(A, B) = \text{rank}(-\overline{B}, \overline{A}) = n.$$

Thus

$$\begin{aligned}(-\overline{B}, \overline{A})^{+} &= (-\overline{B}, \overline{A})^{*}((-\overline{B}, \overline{A})(-\overline{B}, \overline{A})^{*})^{-1} \\ &= \begin{pmatrix} -B^{\mathrm{T}} \\ A^{\mathrm{T}} \end{pmatrix} (\overline{A}A^{\mathrm{T}} + \overline{B}B^{\mathrm{T}})^{-1},\end{aligned}$$

and

$$\begin{aligned}&(A, B)(I_{2n} - (-\overline{B}, \overline{A})^{+}(-\overline{B}, \overline{A}))\\&\quad = (A, B) - (A, B)\begin{pmatrix}-B^{\mathrm{T}}\\ A^{\mathrm{T}}\end{pmatrix}(\overline{A}A^{\mathrm{T}} + \overline{B}B^{\mathrm{T}})^{-1}(-\overline{B}, \overline{A})\\&\quad = (A, B) - (BA^{\mathrm{T}} - AB^{\mathrm{T}})(\overline{A}A^{\mathrm{T}} + \overline{B}B^{\mathrm{T}})^{-1}(-\overline{B}, \overline{A}).\end{aligned}$$

Thus, (ii) holds.

Conversely, if (ii) holds, then by the above argument, we have

$$\operatorname{rank}\{(A, B)[I_{2n} - (-\overline{B}, \overline{A})^{+}(-\overline{B}, \overline{A})]\} = n, \tag{37}$$

for the g-inverses $(-\overline{B}, \overline{A})^{+}$. By "(iv)$\Longrightarrow$(i)", (i) holds. Thus by "(i)$\Longrightarrow$(iv)", condition (iv) holds.

"(i)$\Longleftrightarrow$(v)": Since

$$\begin{pmatrix}A & B\\ -\overline{B} & \overline{A}\end{pmatrix}$$

is invertible if and only if

$$\begin{pmatrix}-\overline{B} & \overline{A}\\ A & B\end{pmatrix}$$

is invertible, by "(i)$\Longleftrightarrow$(iv)", we have "(i)$\Longleftrightarrow$(v)".

"(v)$\Longleftrightarrow$(iii)": Since $(A, B)^{-}(A, B)$ is idempotent matrices,

$$R[I_{2n} - (A, B)^{-}(A, B)] = N[(A, B)^{-}(A, B)], \tag{38}$$

and

$$\operatorname{rank}[I_{2n} - (A, B)^{-}(A, B)] = 2n - \operatorname{rank}[(A, B)^{-}(A, B)]. \tag{39}$$

Thus by Corollary 6.2 of [2], we have

$$\begin{aligned}&\operatorname{rank}\{(-\overline{B}, \overline{A})[I_{2n} - (A, B)^{-}(A, B)]\}\\&\quad = \operatorname{rank}[I_{2n} - (A, B)^{-}(A, B)]\\&\qquad -\dim\{R[I_{2n} - (A, B)^{-}(A, B)] \cap N(-\overline{B}, \overline{A})\}\\&\quad = 2n - \operatorname{rank}[(A, B)^{-}(A, B)]\\&\qquad -\dim\{N[(A, B)^{-}(A, B)] \cap N(-\overline{B}, \overline{A})\}.\end{aligned}$$

Since $\operatorname{rank}[(A, B)^{-}(A, B)] = \operatorname{rank}(A, B)$, it is easy to see that $\operatorname{rank}(A, B) = n$ and $\dim\{N[(A, B)^{-}(A, B)] \cap N(-\overline{B}, \overline{A})\} = 0$ if and only if $\operatorname{rank}\{(-\overline{B}, \overline{A})[I_{2n} - (A, B)^{-}(A, B)]\} = n$. Thus we have "(v)$\Longleftrightarrow$(iii)". □

Corollary 1. *If* $AB^{\mathrm{T}} = BA^{\mathrm{T}}$, *then*

$$\begin{pmatrix}A & B\\ -\overline{B} & \overline{A}\end{pmatrix}$$

is invertible if and only if $\operatorname{rank}(A, B) = n$.

Corollary 2. *If* $A + (AB^{\mathrm{T}} - BA^{\mathrm{T}})(\overline{A}A^{\mathrm{T}} + \overline{B}B^{\mathrm{T}})^{-1}\overline{B}$ *is invertible, then*

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$$

is invertible.

Now, we can answer Question 2 as follows:

Theorem 2. *If* $A, B \in M_n(C)$, *then the following statements are equivalent*:

(i) $\begin{vmatrix} A & B \\ -\overline{B} & \overline{A} \end{vmatrix} = 0$;

(ii) $\operatorname{rank}(A, B) < n$ *or*

$$\operatorname{rank}[(A, B) - (BA^{\mathrm{T}} - AB^{\mathrm{T}})(\overline{A}A^{\mathrm{T}} + \overline{B}B^{\mathrm{T}})^{-1}(-\overline{B}, \overline{A})] < n; \tag{40}$$

(iii) $\operatorname{rank}(A, B) < n$ *or*

$$\dim\{N[(A, B)^{-}(A, B)] \cap N(-\overline{B}, \overline{A})\} \neq 0; \tag{41}$$

(iv)

$$\operatorname{rank}\{(A, B)[I_{2n} - (-\overline{B}, \overline{A})^{-}(-\overline{B}, \overline{A})]\} < n; \tag{42}$$

(v)

$$\operatorname{rank}\{(-\overline{B}, \overline{A})[I_{2n} - (A, B)^{-}(A, B)]\} < n, \tag{43}$$

where all g-inverses are arbitrary but fixed.

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