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An extension of the univalent condition for a family of integral operators

Daniel Breaz [a], Nicoleta Breaz [a], H.M. Srivastava [b,*]

[a] *Department of Mathematics and Informatics, "1 Decembrie 1918" University of Alba Iulia, RO-510009 Alba Iulia, Romania*

[b] *Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada*

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ABSTRACT

The main object of this work is to extend the univalent condition for a family of integral operators. Several other closely-related results are also considered. A number of known univalent conditions would follow upon specializing the parameters involved in our main result.

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and satisfy the following usual normalization condition:

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$. For some recent investigations of various interesting subclasses of the univalent function class $\mathcal{S}$, see the works by (for example) Altıntaş et al. [1], Gao et al. [4], and Owa et al. [6].

The following univalent condition was proven by Pescar [8].

Theorem 1 (*Pescar [8]*). *Let*

$$\alpha \in \mathbb{C} \quad (\Re(\alpha) > 0)$$

and

$$c \in \mathbb{C} \quad (|c| \leqq 1;\ c \neq -1).$$

* Corresponding author.
E-mail addresses: dbreaz@uab.ro (D. Breaz), nbreaz@uab.ro (N. Breaz), harimsri@math.uvic.ca (H.M. Srivastava).

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Suppose also that the function $f(z)$ *given by* (1.1) *is analytic in* $\mathbb{U}$. *If*

$$\left|c\,|z|^{2\alpha}+\left(1-|z|^{2\alpha}\right)\frac{zf''(z)}{\alpha f'(z)}\right|\leqq 1\quad (z\in\mathbb{U}),$$

then the function $F_\alpha(z)$ *defined by*

$$F_\alpha(z):=\left(\alpha\int_0^z t^{\alpha-1}f'(t)\mathrm{d}t\right)^{\frac{1}{\alpha}}=z+\cdots$$

is analytic and univalent in $\mathbb{U}$.

Ozaki and Nunokawa [7], on the other hand, proved another univalent condition asserted by Theorem 2.

Theorem 2 (*Ozaki and Nunokawa [7]*). *Let* $f\in\mathcal{A}$ *satisfy the following inequality:*

$$\left|\frac{z^2f'(z)}{[f(z)]^2}-1\right|\leqq 1\quad (z\in\mathbb{U}). \tag{1.2}$$

Then f *is univalent in* $\mathbb{U}$.

Yet another univalent condition was given by Pescar [9] as follows.

Theorem 3 (*Pescar [9]*). *Let the function* $g\in\mathcal{A}$ *satisfy the inequality* (1.2). *Also let*

$$\alpha\in\mathbb{R}\quad\left(\alpha\in\left[1,\frac{3}{2}\right]\right)\quad\text{and}\quad c\in\mathbb{C}.$$

If

$$|c|\leqq\frac{3-2\alpha}{\alpha}\quad (c\neq-1)$$

and

$$|g(z)|\leqq 1\quad (z\in\mathbb{U}),$$

then the function $G_\alpha(z)$ *defined by*

$$G_\alpha(z):=\left(\alpha\int_0^z[g(t)]^{\alpha-1}\,\mathrm{d}t\right)^{\frac{1}{\alpha}} \tag{1.3}$$

is in the univalent function class $\mathcal{S}$.

Finally, Breaz and Breaz [3] (see also [2]) considered the following family of integral operators and proved that the function $G_{n,\alpha}(z)$ defined by

$$G_{n,\alpha}(z):=\left([n(\alpha-1)+1]\int_0^z[g_1(t)]^{\alpha-1}\cdots[g_n(t)]^{\alpha-1}\,\mathrm{d}t\right)^{\frac{1}{n(\alpha-1)+1}}\quad (g_1,\ldots,g_n\in\mathcal{A}) \tag{1.4}$$

is univalent in $\mathbb{U}$.

Remark 1. In its special case when $n=1$, the integral operator in (1.4) would obviously reduce to the integral operator in (1.3).

In view of Remark 1, we propose to investigate further univalent conditions involving the general family of integral operators defined by (1.4). The following familiar result is of fundamental importance in our investigation.

Schwarz Lemma (*See, for Example, [5]*). *Let the analytic function* f *be regular in the open unit disk* $\mathbb{U}$ *and let* $f(0)=0$. *If*

$$|f(z)|\leqq 1\quad (z\in\mathbb{U}),$$

then

$$|f(z)|\leqq|z|\quad (z\in\mathbb{U}), \tag{1.5}$$

where the equality holds true only if

$$f(z)=Kz\quad (z\in\mathbb{U})\quad\text{and}\quad |K|=1. \tag{1.6}$$

2. The main univalent condition

In this section, we first state the main univalent condition involving the general integral operator given by (1.4).

Theorem 4. *Let $M \geqq 1$ and suppose that each of the functions $g_j \in \mathcal{A}$ $(j \in \{1, \ldots, n\})$ satisfies the inequality (1.2). Also let*

$$\alpha \in \mathbb{R} \quad \left(\alpha \in \left[1, \frac{(2M+1)n}{(2M+1)n-1}\right]\right) \quad \text{and} \quad c \in \mathbb{C}.$$

If

$$|c| \leqq 1 + \left(\frac{1-\alpha}{\alpha}\right)(2M+1)n \tag{2.1}$$

and

$$|g_j(z)| \leqq M \quad (z \in \mathbb{U};\ j \in \{1, \ldots, n\}),$$

then the function $G_{n,\alpha}(z)$ defined by (1.4) is in the univalent function class $\mathcal{S}$.

Proof. We begin by setting

$$f(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{g_j(t)}{t}\right)^{\alpha-1} \mathrm{d}t,$$

so that, obviously,

$$f'(z) = \prod_{j=1}^{n} \left(\frac{g_j(z)}{z}\right)^{\alpha-1} \tag{2.2}$$

and

$$f''(z) = (\alpha-1) \cdot \sum_{j=1}^{n} \left(\frac{g_j(z)}{z}\right)^{\alpha-2} \left(\frac{zg_j'(z) - g_j(z)}{z^2}\right) \cdot \prod_{k=1\ (k \neq j)}^{n} \left(\frac{g_k(z)}{z}\right)^{\alpha-1}. \tag{2.3}$$

We thus find from (2.2) and (2.3) that

$$\frac{zf''(z)}{f'(z)} = (\alpha-1) \sum_{j=1}^{n} \left(\frac{zg_j'(z)}{g_j(z)} - 1\right),$$

which readily shows that

$$\begin{aligned} \left| c\,|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right) \frac{zf''(z)}{\alpha f'(z)} \right| &= \left| c\,|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right) \frac{\alpha-1}{\alpha} \sum_{j=1}^{n} \left(\frac{zg_j'(z)}{g_j(z)} - 1\right) \right| \\ &\leqq |c| + \left(\frac{\alpha-1}{\alpha}\right) \cdot \sum_{j=1}^{n} \left(\left| \frac{z^2 g_j'(z)}{[g_j(z)]^2} \right| \cdot \frac{|g_j(z)|}{|z|} + 1 \right). \end{aligned}$$

Since

$$|g_j(z)| \leqq M \quad (z \in \mathbb{U};\ j \in \{1, \ldots, n\}),$$

by using the inequality (1.2), we obtain

$$\left| c\,|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right) \frac{zf''(z)}{\alpha f'(z)} \right| \leqq |c| + \left(\frac{\alpha-1}{\alpha}\right)(2M+1)\,n \quad (z \in \mathbb{U}),$$

which, in the light of the hypothesis (2.1), yields

$$\left| c\,|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right) \frac{zf''(z)}{\alpha f'(z)} \right| \leqq 1 \quad (z \in \mathbb{U}).$$

Finally, by applying Theorem 1, we conclude that the function $G_{n,\alpha}(z)$ defined by (1.4) is in the univalent function class $\mathcal{S}$. This evidently completes the proof of Theorem 4. □

3. Applications of Theorem 4

First of all, upon setting $M = 1$ in Theorem 4, we immediately arrive at the following application of Theorem 4.

Corollary 1. *Let each of the functions $g_j \in \mathcal{A}$ $(j \in \{1, \ldots, n\})$ satisfy the condition (1.2). Suppose also that*

$$\alpha \in \mathbb{R} \quad \left(\alpha \in \left[1, \frac{3n}{3n-1}\right]\right) \quad \text{and} \quad c \in \mathbb{C}.$$

If

$$|c| \leqq 1 + 3\left(\frac{1-\alpha}{\alpha}\right) n$$

and

$$|g_j(z)| \leqq 1 \quad (z \in \mathbb{U};\ j \in \{1, \ldots, n\}),$$

then the function $G_{n,\alpha}(z)$ defined by (1.4) is in the univalent function class $\mathcal{S}$.

Next we set $n = 1$ in Theorem 4. We thus obtain the following interesting consequence of Theorem 4.

Corollary 2. *Let $M \geqq 1$ and suppose that the function $g \in \mathcal{A}$ satisfies the condition (1.2). Also let*

$$\alpha \in \mathbb{R} \quad \left(\alpha \in \left[1, \frac{2M+1}{2M}\right]\right) \quad \text{and} \quad c \in \mathbb{C}.$$

If

$$|c| \leqq 1 + \left(\frac{1-\alpha}{\alpha}\right)(2M+1)$$

and

$$|g(z)| \leqq M \quad (z \in \mathbb{U}),$$

then the function $G_\alpha(z)$ defined by (1.3) is in the univalent function class $\mathcal{S}$.

Remark 2. Corollary 2 provides an extension of Theorem 3 due of Pescar [9].

Remark 3. If, in Theorem 4, we set $M = n = 1$, we obtain Theorem 3 due of Pescar [9].

Many other interesting corollaries and consequences of Theorem 4 can be deduced from Theorem 4 in a similar manner.

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