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# Vector Space Categories, Right Peak Rings and Their Socle Projective Modules

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## 1. INTRODUCTION

The concepts of a vector space category and a subspace category were introduced by Nazarova and Rojter [13]. They use it in their solution of the second Brauer–Thrall conjecture [13]. Vector space categories are also successfully applied by Ringel [17] in the investigation of one-relation tame algebras and by the author in [23].

Let $F$ be a division ring. We recall from [13, 16] that a *vector space category* $\mathbb{K}_F$ is an additive category $\mathbb{K}$ together with a faithful additive functor $|-|: \mathbb{K} \to \operatorname{mod}(F)$ from $\mathbb{K}$ to the category of finite-dimensional right vector spaces over $F$. The *subspace category* $\mathscr{U}(\mathbb{K}_F)$ of $\mathbb{K}_F$ is defined as follows. The objects of $\mathscr{U}(\mathbb{K}_F)$ are triples $(U, X, \varphi)$ where $U$ is a finite-dimensional right vector space over $F$, $X$ is an object in $\mathbb{K}$ and $\varphi: U_F \to |X|_F$ is an $F$-linear map. The map from $(U, X, \varphi)$ into $(U', X', \varphi')$ is a pair $(u, h)$ where $u \in \operatorname{Hom}_F(U, U')$ and $h: X \to X'$ is a map in $\mathbb{K}$ such that $|h|\varphi = \varphi' u$. The *factor space category* $\mathscr{F}(\mathbb{K}_F)$ of $\mathbb{K}_F$ is defined analogously. There is a pair of additive functors

$$\mathscr{U}(\mathbb{K}_F) \underset{S^+}{\overset{S^-}{\rightleftarrows}} \mathscr{F}(\mathbb{K}_F)$$

defined by taking the cokernel and the kernel, respectively.

Subspace categories of vector space categories play an important role in the representation theory of artinian rings. A principal motivation for them is the following useful observation of Nazarova and Rojter in [13] (see also [16, 24]). To any simple ideal $S$ in an artinian ring $\Lambda$ one can associate a vector space category $\mathbb{K}_F$ with $F = \operatorname{End}(S)^{\text{op}}$, an additive functor

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$T: \mathrm{mod}(\Lambda) \to \mathscr{U}(\mathbb{K}_F)^{\mathrm{op}}$ and a proper ring epimorphism $\varepsilon: \Lambda \to \Lambda'$. Under some assumptions any indecomposable $\Lambda$-module which is not a $\Lambda'$-module via $\varepsilon$ can be reconstructed from an indecomposable object in $\mathscr{U}(\mathbb{K}_F)$ by the functor $T$. If $\mathrm{Ext}^1_\Lambda(S, S) = 0$ then the functor $T$ is full (see [16, Proposition 3.2] and [24, Theorem 1.1] for a more detailed discussion). It follows that the classification of indecomposable $\Lambda$-modules is reduced to the classification of the indecomposable objects in $\mathscr{U}(\mathbb{K}_F)$ because we can suppose by induction that $\Lambda'$-modules are known.

The aim of this paper is to present methods for the computation of the indecomposable subspaces of arbitrary vector space categories.

Following an idea of Drozd [8] we define right peak rings and two additive functors from $\mathscr{U}(\mathbb{K}_F)$ and $\mathscr{V}(\mathbb{K}_F)$ to the category $\mathrm{mod}_{sp}(R)$ of finitely generated right modules with essential projective socles over appropriate right peak ring $R$. The functors are used for the computation of the indecomposable objects in $\mathscr{U}(\mathbb{K}_F)$ as well as in $\mathrm{mod}_{sp}(R)$. They play a role of the Coxeter functors [5, 21].

By a *right peak ring* we mean a semiperfect ring $R$ whose $\mathrm{soc}(R_R)$ is essential and it is a finite direct sum of a copy of a simple projective right module. Elementary properties of the category $\mathrm{mod}_{sp}(R)$ we need in this paper are included in Section 2.

In Section 3 we study Krull–Schmidt vector space categories with a finite number of pairwise nonisomorphic indecomposable objects. If $\mathbb{K}_F$ is such a vector space category and $K_1, ..., K_n$ are all pairwise nonisomorphic indecomposable objects in $\mathbb{K}$ we associate to $\mathbb{K}_F$ the right peak ring

$$\mathbf{R}_{\mathbb{K}} = \begin{pmatrix} E & {}_E K_F \\ 0 & F \end{pmatrix}$$

with $E = \mathrm{End}(K_1 \oplus \cdots \oplus K_n)$ and ${}_E K_F = {}_E|K_1 \oplus \cdots \oplus K_n|_F$. We suppose in this paper that $\mathbf{R}_{\mathbb{K}}$ is either schurian with the constant dimension property (see Section 2) or that $\mathbf{R}_{\mathbb{K}}$ is an artinian PI-ring (i.e., $\mathbf{R}_{\mathbb{K}}$ satisfies a polynomial identity). In both cases $\mathbf{R}_{\mathbb{K}}$ has a Morita duality. Our basic tool we use in this paper are the functors

$$\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}}) \xleftarrow{H} \mathscr{V}(\mathbb{K}_F) \underset{S^-}{\overset{S^+}{\rightleftarrows}} \mathscr{U}(\mathbb{K}_F) \xrightarrow{G} \mathrm{mod}_{sp}((\mathbf{R}_{\mathbb{K}})_*)$$

as well as the Coxeter scheme of $\mathbb{K}_F$ defined in Section 3. They give a constructive method for the study of the indecomposable objects in $\mathscr{U}(\mathbb{K}_F)$ provided it is of finite type and $\mathbf{R}_{\mathbb{K}}$ is a schurian finite-dimensional algebra over a field (Theorem 3.11). They also yield a useful formula for the calculation of the almost split sequences in $\mathrm{mod}_{sp}(R)$ provided $R$ is a right peak artin algebra (Corollary 3.7).

In Section 4 we study modules over artinian PI-rings of the form

$$R=\begin{pmatrix} G & {}_GM_S \\ 0 & S \end{pmatrix}$$

where $G$ is a division ring. Using the properties of the functor $H$ proved in Theorem 3.3 together with [17, Sect. 2.5] we prove that if $S$ is of finite representation type then there is an equivalence of categories

$$\mathrm{mod}(R)/[\mathrm{mod}(S)] \cong \mathrm{mod}_{sp}((\mathbf{R}_{\mathbb{K}})_*)$$

where $\mathbb{K}_G = \mathrm{Hom}_S({}_GM_S, \mathrm{mod}(S))$ and $[\mathrm{mod}(S)]$ is the two-sided ideal in $\mathrm{mod}(R)$ generated by $\mathrm{mod}(S)$ considered as the full subcategory of $\mathrm{mod}(R)$ via the ring epimorphism $R \to S$. If $R$ is a right peak ring and $\mathrm{mod}_{sp}(S)$ is of finite type then there is an equivalence of categories

$$\mathrm{mod}_{sp}(R)/[\mathrm{mod}_{sp}(S)] \cong \mathrm{mod}_{sp}^{-}((\mathbf{R}_{\widetilde{\mathbb{K}}})_*)$$

where $\widetilde{\mathbb{K}}_G = \mathrm{Hom}_S({}_GM_S, \mathrm{mod}_{sp}(S))$ and $\mathrm{mod}_{sp}^{-}(\mathbf{R}_*)$ is the full subcategory of $\mathrm{mod}_{sp}(\mathbf{R}_*)$ consisting of modules having no injective direct summands. As a consequence we get a constructive method for solving schurian vector space PI-categories of finite type and useful information about the structure of their subspace categories (Theorem 4.4). If $\mathbb{K}_F$ is such a vector space category then the method allows us to reduce in finite number of steps the classification of indecomposable objects in $\mathscr{U}(\mathbb{K}_F)$ to the well-known classification of the indecomposable modules over hereditary PI-rings of the form

$$\begin{pmatrix} G & {}_GN_{F'} \\ 0 & F' \end{pmatrix}$$

where $G$ and $F'$ are division rings and $(\dim_G N)(\dim N_{F'}) \leqslant 3$ (see [7]).

In Section 5 we extend the differentiation algorithm [4] from $\ell$-hereditary 1-Gorenstein rings to right peak rings. It follows that the algorithm can be used in the investigation of subspace categories of vector space categories.

The results of this paper are accounced in [24]. They were presented at the Annual Algebra Conference in Chiba (Japan) in July 1982. A particular case when $\mathbb{K}_F$ is special schurian was studied in [22]. It was shown there that special schurian vector space categories correspond to $\ell$-hereditary right $QF$-2 artinian rings under the map $\mathbb{K}_F \mapsto \mathbf{R}_{\mathbb{K}}$. In this case $\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$ is the category of $\ell$-hereditary $\mathbf{R}_{\mathbb{K}}$-modules in the sense of [4, Definition 1.3]. It follows that the results of this paper extend some of the results in [3]. In our study of the socle projective modules we follow some concepts in [3, 4].

Throughout this paper $\mathrm{Mod}(R)$ denotes the category of all right $R$-modules and $\mathrm{mod}(R)$ is the full subcategory of $\mathrm{Mod}(R)$ consisting of finitely generated modules. Given a module $X$ we denote by $E(X)$, $P(X)$, $\mathrm{soc}(X)$ and $\mathrm{top}(X)$ the injective envelope, the projective cover, the socle and the top of $X$, respectively. The direct sum of $t$ copies of $X$ is denoted by $X^t$. The Jacobson radical of $R$ will be denoted by $J = J(R)$. Finally, we denote by $R^{\mathrm{op}}$ the ring opposite to $R$.

## 2. Right Peak Rings and Their Socle Projective Modules

Definition 2.1. A ring $R$ is called a right peak ring if $\mathrm{soc}(R_R)$ is essential in $R$ and has the form $P^t$ where $P$ is a simple projective module. In this case $P$ is called the right peak of $R$. $R$ is said to be a left peak ring if $R^{\mathrm{op}}$ is a right peak ring.

Proposition 2.2. *Let $R$ be a basic semiperfect ring. The following statements are equivalent*:

(a) *$R$ is a right peak ring.*

(b) *$R$ has a triangular form*

$$R \cong \begin{pmatrix} A & {}_AM_F \\ 0 & F \end{pmatrix}$$

*where $F$ is a division ring, $A$ is a semiperfect ring and ${}_AM_F$ is an $A-F$-bimodule which is $A$-faithful and finite dimensional over $F$.*

(c) *$R$ has the form*

$$R \cong \begin{bmatrix} F_1 & {}_1M_2 & \cdots & {}_1M_n & {}_1M_{n+1} \\ {}_2M_1 & F_2 & \cdots & {}_2M_n & {}_2M_{n+1} \\ \vdots & & \ddots & & \vdots \\ {}_nM_1 & {}_nM_2 & \cdots & F_n & {}_nM_{n+1} \\ 0 & 0 & \cdots & 0 & F_{n+1} \end{bmatrix}$$

*where $F_1, \ldots, F_n$ are local rings, $F_{n+1}$ is a division ring, ${}_iM_j$ are $F_i - F_j$-bimodules, ${}_jM_{n+1}$ are finite dimensional over $F_{n+1}$, the multiplication is given by $F_i - F_k$-bilinear maps*

$$c_{ijk} : M_j \otimes {}_jM_k \to {}_iM_k,$$

*with $\otimes = \otimes_{F_j}$, and the map*

$$\bar{c}_{ijn+1} : {}_iM_j \to \mathrm{Hom}_{F_{n+1}}({}_jM_{n+1}, {}_iM_{n+1})$$

*adjoint to $c_{ijn+1}$ is injective for all $i$ and $j$.*

*Proof.* (a) ⇒ (c) Let $R = P_1 \oplus \cdots \oplus P_{n+1}$ be a decomposition of $R$ into a direct sum of indecomposable right ideals and suppose that $P_{n+1}$ is simple. If we put

$$F_i = \operatorname{End}(P_i), \qquad {}_iM_j = \operatorname{Hom}_R(P_j, P_i)$$

and if we take for $c_{ijk}$ the morphism composition map then obviously $R$ is isomorphic to the matrix ring required in (c). Since $\operatorname{soc}(R_R) \cong P^t_{n+1}$ then $\bar{c}_{ijn+1}$ are injective and (c) follows.

(c) ⇒ (b) Obvious.

(b) ⇒ (a) If $S \subset \operatorname{soc}(R_R)$ is a simple nonprojective module then the kernel of the projective cover $f\colon P(S) \to S$ map is nonzero. Hence, via the ring isomorphism in (b), $f$ corresponds to an element $f' \in A$ such that $f'M = 0$; a contradiction. Then the proof is complete.

Throughout this section we suppose that $R$ is a semiperfect right peak ring and we fix a decomposition

$$R = P_1 \oplus \cdots \oplus P_n \oplus P_{n+1}$$

where $P_i$ are indecomposable right modules. We identify $R$ with its matrix form in Proposition 2.2 and we identify $P_i$ with the indecomposable $i$th row ideal.

The ring $R$ is called *schurian* if $F_1,..., F_{n+1}$ are division rings.

Suppose that $R$ is a right peak schurian ring. If $R$ is artinian we associate to $R$ a *value scheme* $(\mathbf{I}_R, \mathbf{d})$ where $\mathbf{I}_R = \{1,..., n+1\}$ and $\mathbf{d}$ is a pair of $n+1 \times n+1$ matrices $(d_{ij})$, $(d'_{ij})$ with

$$d_{ij} = \dim({}_iM_j)_{F_j} \qquad \text{and} \qquad d'_{ij} = \dim_{F_i}({}_iM_j) \qquad \text{for} \quad i = j.$$

We put $d'_{ii} = d_{ii} = 0$ for all $i$. Usually we will consider $(\mathbf{I}_R, \mathbf{d})$ as a set of point $\mathbf{I}_R$ together with a set of dashed arrows

$$i \overset{(d_{ij}, d'_{ij})}{- - - - \rightarrow} j$$

with $d_{ij}$ and $d'_{ij}$ nonzero. We will write $i - - \rightarrow j$ if $d_{ij} = d'_{ij} = 1$; we write

$$i \xrightarrow{(d_{ij}, d'_{ij})} j$$

if ${}_iM_j \neq 0$ and there is no $s \neq i,j$ with ${}_iM_s \neq 0$ and ${}_sM_j \neq 0$.

Note that if $P_{n+1}$ is a right peak of $R$ then there is a valued arrow in $(\mathbf{I}_R, \mathbf{d})$ from arbitrary $j$ into $n+1$. This fact motivates the name "peak."

PROPOSITION 2.3. *Suppose $R$ is a schurian right peak ring. Then*

(a) $c_{iji}=0$ *for all* $i \neq j$.

(b) *If* ${}_iM_{n+1}$ *is a simple* $F_i-F_{n+1}$*-bimodule for any* $i$, *then for any* $i \neq j$ *either* ${}_iM_j=0$ *or* ${}_jM_j=0$. *In this case* $(\mathbf{I}_R, \mathbf{d})$ *is a valued poset with a unique maximal element and* $R$ *has an upper triangular matrix form.*

*Proof.* (a) If $c_{iji} \neq 0$ then there are nonzero maps $g: P_j \to P_i$ and $f: P_i \to P_j$ such that $c_{iji}(f \otimes g)=fg=1$. Hence $i=j$ and (a) follows.

(b) Suppose that ${}_iM_j \neq 0$ and ${}_jM_i \neq 0$. If $P_{n+1}$ is a right peak of $R$ then by our assumption and Proposition 2.2 the composed map

$$ {}_iM_j \otimes {}_jM_i \otimes {}_iM_{n+1} \xrightarrow{1 \otimes c_{jin+1}} {}_iM_j \otimes {}_jM_{n+1} \xrightarrow{c_{ijn+1}} {}_iM_{n+1} $$

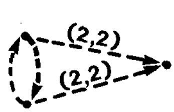
is a nonzero surjection. On the other hand the map is zero because we know from (a) that $c_{iji}=0$. We get a contradiction, which finishes the proof.

Now we give a simple example of a right peak ring which is not an upper triangular matrix ring.

EXAMPLE. Let $K$ be a field and

$$ R=\begin{bmatrix} K & K & K^2 \\ K & K & K^2 \\ 0 & 0 & K \end{bmatrix}. $$

We take for $c_{123}: K \otimes K^2 \to K^2$ the projection on the first coordinate, for $c_{213}: K \otimes K^2 \to K^2$ the projection on the second coordinate and $c_{121}= c_{212}=0$. It is clear that $R$ is a right peak ring and $(\mathbf{I}_R, \mathbf{d})$ has the form




A module $X_R$ will be called *socle projective* if $\operatorname{soc}(X_R)$ is a projective and an essential submodule of $X_R$.

Given a ring $R$ we denote by $\operatorname{Mod}_{sp}(R)$ the category of all socle projective right $R$-modules. We will denote by $\operatorname{mod}_{sp}(R)$ the full subcategory of $\operatorname{Mod}_{sp}(R)$ consisting of finitely generated modules. Finally, we denote by $\operatorname{mod}_{ti}(R)$ the category of finitely generated right $R$-modules with injective top.

If $R$ is a right peak ring of the form 2.2(c) then we will identify in this paper any right $R$-module $X$ with a system $(X_i, {}_j\varphi_i)_{i,j \leqslant n+1}$ where $X_i$ is the right $F_i$-module $\operatorname{Hom}_R(P_i, X)$ and ${}_j\varphi_i: X_i \otimes {}_iM_j \to X_j$ is the corresponding $F_j$-linear map induced by the multiplication in $R$ (see [4, 20]).

Throughout this paper we will frequently use the following characterization of modules in $\mathrm{mod}_{sp}(R)$ and in $\mathrm{mod}_{ti}(R)$.

PROPOSITION 2.4. *Let $R$ be a semiperfect ring of the form* 2.2(c) *and let $X = (X_i, {}_j\varphi_i)$ be a finitely generated right $R$-module.*

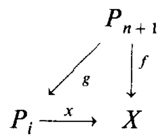
(a) *If $P_{n+1}$ is a right peak of $R$ then* $\mathrm{soc}(X)$ *is projective and essential if and only if the map*

$$_{n+1}\bar{\varphi}_i : X_i \to \mathrm{Hom}_{F_{n+1}}({}_iM_{n+1}, X_{n+1})$$

*adjoint to ${}_{n+1}\varphi_i$ is injective for any $i \leqslant n$.*

(b) *If $P_1^R$ is a left peak of $R$ then* $\mathrm{top}(P_1)$ *is a unique injective simple right $R$-module and* $\mathrm{top}(X)$ *is injective if and only if the map ${}_i\varphi_1 : X_1 \otimes {}_1M_i \to X_i$ is surjective for any $i$. Here $P_1^R = \mathrm{Hom}_R(P_1, R)$.*

*Proof.* First we note that ${}_{n+1}\varphi_i(x \otimes g) = xg$ for any $x \in X_i = \mathrm{Hom}_R(P_i, X)$ and $g \in {}_iM_{n+1} = \mathrm{Hom}_R(P_{n+1}, P_i)$. In order to prove (a) suppose that $\mathrm{soc}(X)$ is projective and take a nonzero element $x$ in $X_i$. Since there is a commutative diagram




with $f \neq 0$, then ${}_{n+1}\varphi_i(x \otimes g) = f \neq 0$ and therefore ${}_{n+1}\bar{\varphi}_i$ is injective.

Conversely suppose that ${}_{n+1}\bar{\varphi}_i$ is injective for all $i$ and let $S$ be a simple submodule of $X$. If $t: P_i \to S$ is the projective cover of $S$ then there is $g \in {}_iM_{n+1}$ such that ${}_{n+1}\varphi_i(t \otimes g) = tg \neq 0$. Hence $S$ is projective and (a) is proved. The proof of the statement (b) is similar and we leave it to the reader.

Suppose $R$ is a right peak schurian ring of the form 2.2(c). Following Ringel [15] we say that the bimodule ${}_iM_j$ has the *constant dimension property* if the dimensions of the iterated dual bimodules ${}_iM_j^{ij\cdots ij}$ and ${}_iM_j^{ji\cdots ji}$ over $F_i$ and $F_j$ are finite and equal to $d'_{ij}$ and $d_{ij}$, respectively. Here we put

$$_iN_j^k = \mathrm{Hom}_{F_k}({}_iN_j, F_k)$$

for $k = i, j$ and any $F_i - F_j$-bimodule ${}_iN_j$.

We say that $R$ has the *constant dimension property* if $R$ is a schurian artinian and the bimodules ${}_iM_j$ have the constant dimension property for all $i \neq j$.

We recall from [7, Proposition 1.3] that any schurian artinian PI-ring has the constant dimension property.

In the study of schurian vector space categories we will need the following simple result.

PROPOSITION 2.5. *Suppose that $\Lambda = P_1 \oplus \cdots \oplus P_n$ is a basic schurian right artinian ring with the constant dimension property, $P_1, \ldots, P_n$ are indecomposable right ideals in R and let*

$$\tilde{\Lambda} = \mathrm{End}(Q_1 \oplus \cdots \oplus Q_n)$$

*where $Q_j$ is the injective envelope of* $\mathrm{top}(P_j)$.

(a) *If $G_i = \mathrm{End}(P_i)$ and ${}_iN_j = \mathrm{Hom}(P_j, P_i)$ then for every j the module $Q_j$ is finitely generated and has the form*

$$Q_j = ({}_1N_j^j, \ldots, {}_{j-1}N_j^j, G_j, {}_{j+1}N_j^j, \ldots, {}_nN_j^j, {}_k\psi_i^j)$$

*where ${}_i\psi_k^j : {}_kN_j^j \otimes {}_kN_i \to {}_iN_j^j$ is such that its $G_j$-dual corresponds via the isomorphism $\mathrm{Hom}_{G_j}({}_kN_j^j \otimes {}_kN_i, G_j) \cong \mathrm{Hom}_{G_k}({}_kN_i, {}_kN_j)$ to the map $\bar{c}_{kij} : {}_iN_j \to \mathrm{Hom}_{G_k}({}_kN_i, {}_kN_j)$ adjoint to $c_{kij}$. Moreover $\tilde{\Lambda}$ has the form*

$$\tilde{\Lambda} \cong \begin{bmatrix} G_1 & {}_1N_2^{21} & \cdots & {}_1N_{n-1}^{n-11} & {}_1N_n^{n1} \\ {}_2N_1^{12} & G_2 & \cdots & {}_2N_{n-1}^{n-12} & {}_2N_n^{n2} \\ \vdots & & \ddots & \vdots & \vdots \\ {}_nN_1^{1n} & {}_nN_2^{2n} & \cdots & {}_nN_{n-1}^{n-1n} & G_n \end{bmatrix}$$

*and there is a Morita duality $D: \mathrm{mod}(\Lambda) \to (\mathrm{mod}(\tilde{\Lambda}^{op}))^{op}$.*

(b) *If $\Lambda$ is a right peak ring then $\tilde{\Lambda}$ is also a right peak ring and $(\mathbf{I}_\Lambda, \mathbf{d})$ coincides with $(\mathbf{I}_{\tilde{\Lambda}}, \tilde{\mathbf{d}})$.*

(c) *If $\Lambda$ is a left peak ring then $Q_1, \ldots, Q_n$ have injective tops, $\tilde{\Lambda}$ is a left peak ring and $(\mathbf{I}_\Lambda, \mathbf{d})$ coincides with $(\mathbf{I}_{\tilde{\Lambda}}, \tilde{\mathbf{d}})$.*

*Proof.* We recall that given a finitely generated projective right $\Lambda$-module $P$ we have a pair of adjoint functors

$$\mathrm{Mod}(\mathrm{End}(P)) \underset{r}{\overset{L}{\rightleftarrows}} \mathrm{Mod}(\Lambda)$$

defined by formulas $r(Y_\Lambda) = \mathrm{Hom}_\Lambda(P, Y)$, $L(X) = \mathrm{Hom}_{\mathrm{End}(P)}(\mathrm{Hom}_\Lambda(P, \Lambda), X)$. The functor $L$ is full, faithful and carries over injectives into injectives. Moreover $rL \cong \mathrm{id}$.

Now taking for $P$ the module $P_j$ we get the indecomposable injective $\Lambda$-module $L(G_j)$ which is obviously the injective envelope of $\mathrm{top}(P_j)$ and has the form required in the statement (a). Note also that we have a right $\tilde{\Lambda}$-module decomposition $\tilde{\Lambda} = P'_1 \oplus \cdots \oplus P'_n$ where

$$P'_j = \mathrm{Hom}(Q_1 \oplus \cdots \oplus Q_n, Q_j)$$

is indecomposable. Since we have $G_i - G_j$-bimodule isomorphisms

$$\mathrm{Hom}_{\tilde{\Lambda}}(P'_j, P'_i) \cong \mathrm{Hom}_{\Lambda}(Q_j, Q_i) \cong {}_iN_j^{ji}$$

then (a) follows.

In order to prove (b) suppose that $P_n$ is a right peak of $\Lambda$. We will prove that $P'_n$ is a right peak of $\tilde{\Lambda}$. For this purpose take a nonzero map $f: Q_j \to Q_i$, $i \neq j$. By our assumption there is a commutative diagram

$$\begin{array}{ccc} P(Q_j) & \xrightarrow{\;u\;} & Q' \\ \big\downarrow{\scriptstyle t} & \swarrow{\scriptstyle w} & \\ Q_j & & \end{array}$$

where $t$ is a projective cover of $Q_j$, $u$ is a monomorphism and $Q'$ is a direct sum of copies of $Q_n$. The restriction $g$ of the map $w$ to a suitable summand $Q_n$ of $Q'$ has the property $gf \neq 0$. It follows from Proposition 2.2 that $P'_n$ is a right peak of $\tilde{\Lambda}$. The remaining part of (b) is a consequence of (a).

Now suppose that $P_1^{\Lambda}$ is a left peak of $\Lambda$. Then by Proposition 2.2 the map $\bar{c}_{1ij}$ is injective for all $i$ and $j$. Hence the map ${}_i\psi_1^j$ is surjective for all $i$ and $j$ and by Proposition 2.4 $\mathrm{top}(Q_j)$ is injective. Then in view of the duality $D$ the module $(P'_1)^{\tilde{\Lambda}}$ is a left peak of $\tilde{\Lambda}$. Since the remaining part of (c) follows from (a) the proof is complete.

A module $N$ in $\mathrm{mod}_{sp}(R)$ is said to be *sp-injective* if $N$ is injective with respect to those monomorphisms in $\mathrm{mod}_{sp}(R)$ whose cokernels have projective socles.

PROPOSITION 2.6. *If*

$$R = \begin{pmatrix} A & {}_AM_F \\ 0 & F \end{pmatrix}$$

*is a right peak ring and* ${}_AM_F^F = \mathrm{Hom}_F({}_AM_F, F)$ *is A-noetherian then*

$$R^{\nabla} = \begin{pmatrix} F & {}_AM_F^F \\ 0 & A \end{pmatrix}$$

*is a left peak ring and there is an equivalence of categories*

$$\nabla: \mathrm{mod}_{sp}(R) \to \mathrm{mod}_{ti}(R^{\nabla})$$

*with the following properties*:

(a) *Let* $K, L, N$ *be modules in* $\mathrm{mod}_{sp}(R)$. *A sequence*

$$0 \to K \to L \to N \to 0$$

*is exact in* $\mathrm{mod}(R)$ *if and only if the induced sequence*

$$0 \to \nabla(K) \to \nabla(L) \to \nabla(N) \to 0$$

*is exact in* $\mathrm{mod}(R^{\nabla})$.

(b) *A module* $X$ *in* $\mathrm{mod}_{sp}(R)$ *is sp-injective if and only if* $X \cong \nabla^{-1}(Q)$ *where* $Q$ *is an injective module in* $\mathrm{mod}(R^{\nabla})$ *and* $E(\nabla(X)) \in \mathrm{mod}_{ti}(R^{\nabla})$.

*Proof.* Since ${}_A M_F^F$ is a faithful $A$-module then by Proposition 2.2 $R^{\nabla}$ is a left peak ring. Let $X$ be a module in $\mathrm{mod}_{sp}(R)$. By Proposition 2.4 $X$ can be identified with a triple $(X'_A, X''_F, t)$ where $t: X'_A \otimes {}_A M_F \to X''_F$ is an $F$-homomorphism such that its adjoint map $\bar{t}: X'_A \to \mathrm{Hom}_F({}_A M_F, X''_F)$ is injective. We put $\nabla(X) = (Y'_F, Y''_A, s)$ where $Y'_F = X''_F$, $Y''_A = \mathrm{Coker}\,\bar{t}$ and $s: Y'_F \otimes_F ({}_A M_F^F) \to Y''_A$ is the composition of the cokernel map with the natural isomorphism $Y'_F \otimes_F ({}_A M_F^F) \cong \mathrm{Hom}_F({}_A M_F, Y'_F)$. We define $\nabla$ on maps in a natural way and we get a covariant additive functor. By Proposition 2.4 $\nabla(X)$ is a module in $\mathrm{mod}_{ti}(R^{\nabla})$ and $\nabla$ is dense. Now it is easy to check that $\nabla$ is an equivalence satisfying (a) and (b). We leave it to the reader (compare [4, Propositions 1.6 and 1.7]).

Our previous results together with [18] yield

COROLLARY 2.7. *Let* $R$ *be a right peak ring. If* $R$ *is either an artinian PI-ring or has the constant dimension property then every module in* $\mathrm{mod}_{sp}(R)$ *has an sp-injective envelope in* $\mathrm{mod}_{sp}(R)$. *Moreover there is a duality* $D\nabla: \mathrm{mod}_{sp}(R) \to \mathrm{mod}_{sp}(\widetilde{(R^{\nabla})^{op}})$.

We finish this Section by a useful result on rings having both the left and the right peaks.

PROPOSITION 2.8. *Let* $R$ *be a schurian artinian ring with a right peak* $P_{n+1}$ *and suppose that* $R$ *has the constant dimension property. The following three conditions are equivalent*:

(a) $E(R_R)$ *is projective.*

(b) $R$ *is a right* $QF-3$ *ring.*

(c) $R$ *has a left peak* $P_1^R = \mathrm{Hom}_R(P_1, R)$ *and* $d_{1n+1} = d'_{1n+1} = 1$. *Furthermore, if* $E(R_R)$ *is projective then* $E(P_{n+1})$ *is a unique indecomposable projective-injective right* $R$*-module, any indecomposable module* $X$ *in*

$\mathrm{mod}_{sp}(R)$ *with* $X_1 \neq 0$ *is isomorphic to* $E(P_{n+1})$ *and any indecomposable module* $Y$ *in* $\mathrm{mod}_{ti}(R)$ *with* $Y_{n+1} \neq 0$ *is isomorphic to* $E(P_{n+1})$.

*Proof.* (a) $\Rightarrow$ (b) $E(P_{n+1})$ is a faithful projective module.

(b) $\Rightarrow$ (c) If $R$ is a right $QF-3$ ring then $E(P_{n+1})$ is projective and by Proposition 2.2 the module $\mathrm{Hom}_R(E(P_{n+1}), R)$ is a left peak of $R$ satisfying the condition (c).

(c) $\Rightarrow$ (a) It is enough to prove that $P_1 \cong E(P_{n+1})$. For this purpose we note that by our assumption and Proposition 2.2 the $F_i$-homomorphism

$$\bar{c}_{1in+1}: {}_1M_i \to \mathrm{Hom}_{F_{n+1}}({}_iM_{n+1}, {}_1M_{n+1}) \cong {}_iM^{n+1}_{n+1}$$

adjoint to $c_{1in+1}$ is injective for every $i$. Then we have defined an $R$-monomorphism $\bar{c}: P_1 \to E(P_{n+1})$. Since $P_1^R$ is a left peak of $R$ then by Proposition 2.2 the $F_i$-homomorphism

$$\bar{\bar{c}}_{1in+1}: {}_iM_{n+1} \to \mathrm{Hom}_{F_1}({}_1M_i, {}_1M_{n+1}) \cong {}_1M^1_i$$

adjoint to $c_{1in+1}$ is injective. Now in view of the constant dimension property $\bar{c}_{1in+1}$ is bijective for every $i$. Hence $\bar{c}$ is an isomorphism and (a) follows.

Now suppose that $E(R_R)$ is projective. If $X$ is an indecomposable module in $\mathrm{mod}_{sp}(R)$ with $X_1 \neq 0$ then there is a nonzero map $P_1 \to X$ which is an isomorphism because $P_1 \cong E(P_{n+1})$. If $Y$ is an indecomposable module in $\mathrm{mod}_{ti}(R)$ with $Y_{n+1} \neq 0$ then there is a commutative diagram

$$\begin{array}{ccc} P_{n+1} & \hookrightarrow & Y \\ & \searrow & \downarrow u \\ & & P_1 \end{array}$$

with $u \neq 0$ and the projective cover of $Y$ has the form $P_1^s \xrightarrow{(t_k)} Y$. Hence $ut_k \neq 0$ for some $k$. Since $\mathrm{End}(P_1)$ is a division ring then $u$ is an isomorphism and the proof of the Proposition is complete.

*Remark* 2.9. (a) Suppose $I$ is a finite partially ordered set and $F$ is a division ring. Denote by $I^*$ the enlargement of $I$ by a unique maximal element and by $I_\phi$ the enlargement of $I$ by a unique minimal element. Then the incidence ring $FI^*$ of $I^*$ with coefficients in $F$ is a right peak ring, $FI_\phi$ is a left peak ring, $(FI^*)^\nabla \cong FI_\phi$, $\mathrm{mod}_{sp}(FI^*)$ is the category *I-sp* of $I$-spaces in the sense of Gabriel (see [17]) and $\mathrm{mod}_{ti}(FI_\phi)$ is the category *I-fsp* of $I$-factor spaces, i.e., $FI_\phi$-modules $(X_i, {}_i\varphi_j)_{i,j \in I_\phi}$ such that ${}_i\varphi_j: X_j \to X_i$ is an epimorphism for $j \leqslant i$ in $I_\phi$. In this case the functor $\nabla$: *I-sp* $\to$ *I-fsp* carries over an $I$-space $(U, U_i)_{i \in I}$ with $U_i \subseteq U$ into $(X_i, {}_j\varphi_i)$ where $X_\phi = U$, $X_i = U/U_i$ and ${}_j\varphi_i: X_i \to X_j$ are the natural epimorphisms for all $i \leqslant j$ in $I_\phi$. Note also that $(\mathbf{I}_{FI^*}, \mathbf{d})$ is the poset $I^*$.

(b) Suppose that $\Omega = \{X_1, \ldots, X_n\}$ is a set of finitely generated indecomposable socle projective modules over a right peak artinian ring $R$. If the right peak $P_{n+1}$ belongs to $\Omega$ then the ring

$$E = \mathrm{End}(X_1 \oplus \cdots \oplus X_n)$$

is a right peak ring. If, in addition, $E(P_{n+1})$ belongs to $\Omega$ then $R$ is both a left and a right peak ring. This observation plays a key role in the definition of a differentiation of a right peak ring with respect to a smooth indecomposable projective right module (see Section 5).

## 3. A Coxeter Scheme of a Vector Space Category

A vector space category $\mathbb{K}_F$ is called *semiperfect* if $\mathbb{K}$ is a Krull–Schmidt category and every indecomposable object in $\mathbb{K}$ has local endomorphism ring.

Throughout this paper $\mathbb{K}_F$ is a vector space catagory defined by the faithful additive functor

$$|-|: \mathbb{K} \to \mathrm{mod}(F)$$

where $F$ is a division ring. We will suppose (for simplicity) that the number of isomorphism classes of indecomposable objects in $\mathbb{K}$ is finite and we fix their representatives $K_1, \ldots, K_n$.

We call $\mathbb{K}_F$ *schurian* if the semiperfect ring

$$E = \mathrm{End}(K_1 \oplus \cdots \oplus K_n)$$

is schurian. We put

$$F_{n+1} = F \qquad \text{and} \qquad F_j = \mathrm{End}(K_j) \qquad \text{for} \quad j = 1, \ldots, n,$$

and we define $F_i - F_k$-bimodules ${}_iK_k$ by the formulas

$$\begin{aligned} {}_iK_{n+1} &= {}_{F_i}|K_i|_F && \text{for} \quad i \leqslant n, \\ {}_iK_j &= \mathbb{K}(K_j, K_i) && \text{for} \quad i,j \leqslant n, \quad i \neq j, \end{aligned}$$

where $\mathbb{K}(K_j, K_i)$ denotes the group of all maps from $K_j$ into $K_i$ in $\mathbb{K}$.

Since the $E - F$-bimodule

$${}_EK_F = {}_E|K_1 \oplus \cdots \oplus K_n|_F$$

is $E$-faithful then by Proposition 3.1 the ring

$$\mathbf{R}_{\mathbb{K}} = \begin{pmatrix} E & {}_EK_F \\ 0 & F \end{pmatrix}$$

is a right peak ring and there is a ring isomorphism

$$\mathbf{R}_{\mathbb{K}} \cong \begin{bmatrix} F_1 & {}_1K_2 & \cdots & {}_1K_n & {}_1K_{n+1} \\ {}_2K_1 & F_2 & \cdots & {}_2K_n & {}_2K_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ {}_nK_1 & {}_nK_2 & \cdots & F_n & {}_nK_{n+1} \\ 0 & 0 & \cdots & 0 & F_{n+1} \end{bmatrix}$$

where the multiplication in the matrix ring is given by the $F_i - F_s$-linear maps $c_{ijs}: {}_iK_j \otimes {}_jK_s \to {}_iK_s$ defined by the formula

$$\begin{aligned} c_{ijs}(f \otimes g) &= fg && \text{for} \quad i, j, s \leqslant n, \\ &= |f|(g) && \text{for} \quad s = n+1, \quad i, j \leqslant n. \end{aligned}$$

The ring $\mathbf{R}_{\mathbb{K}}$ is called the *right peak ring associated to* $\mathbb{K}_F$. The vector space catagory $\mathbb{K}_F$ is said to be *artinian* if $\mathbf{R}_{\mathbb{K}}$ is both left and right artinian.

We are going to define for any schurian vector space category $\mathbb{K}_F$ with the constant dimension property for $\mathbf{R}_{\mathbb{K}}$ a Coxeter scheme which is a sequence of functors having properties analogous to the Coxeter functors [5, 21]. For this purpose we need the following definitions and notations.

We denote by $\mathscr{V}(\mathbb{K}_F)$ the *factor space category* defined as follows. The objects of $\mathscr{V}(\mathbb{K}_F)$ are triples $(V, X, t)$ where $X$ is an object in $\mathbb{K}$, $V$ is a finite-dimensional right $F$-module and $t: |X| \to V$ is an $F$-linear map. A map $(V, X, t) \to (V', X', t')$ in $\mathscr{V}(\mathbb{K}_F)$ is a pair of maps $(f, g)$, $f \in \operatorname{Hom}_F(V, V')$, $g \in \mathbb{K}(X, X')$ such that $t' |g| = ft$.

The category $\mathscr{V}(\mathbb{K}_F)$ has a useful matrix interpretation similar to that one given in [22].

If $(V, X, t)$ is either an object of $\mathscr{V}(\mathbb{K}_F)$ or an object of $\mathscr{U}(\mathbb{K}_F)$ we define its *coordinate vector* $\mathbf{cdn}(V, X, t) \in \mathbb{Z}^{n+1}$ by the formula

$$\mathbf{cdn}(V, X, t) = (s_1, ..., s_n, s_{n+1})$$

where $s_{n+1} = \dim V_F$ and $X \cong K_1^{s_1} \oplus \cdots \oplus K_n^{s_n}$. If all $s_1, ..., s_n$ are nonzero then $(V, X, t)$ is called *exact*.

Now suppose that $R$ is a right peak schurian ring. For any module $X = (X_i, {}_i\varphi_j)$ in $\operatorname{mod}_{sp}(R)$ we put

$$\mathbf{dim}\, X = (x_1, ..., x_n, x_{n+1})$$

where $x_i = \dim(X_i)_{F_i}$. If $P_{n+1}$ is not a direct summand of $X$ and the projective cover of $X$ has the form $P(X) = P_1^{s_1} \oplus \cdots \oplus P_n^{s_n}$ then we put

$$\mathbf{cdn}(X) = (s_1, ..., s_n, s_{n+1})$$

where $s_{n+1} = \dim(X_{n+1})_F$. If all $s_1, ..., s_n, s_{n+1}$ are nonzero we call $X$ *exact* (compare [3, 8]).

Now we define a pair of additive functors

$$\mathscr{U}(\mathbb{K}_F) \underset{S^+}{\overset{S^-}{\rightleftarrows}} \mathscr{V}(\mathbb{K}_F)$$

by formulas $S^-(U, X, t) = (\operatorname{Coker} t, X, t')$, $S^+(V, X, t) = (\operatorname{Ker} t, X, t'')$ where $t'$ and $t''$ is the cokernel and the kernel map, respectively. We define $S^-$ and $S^+$ on morphisms in a natural way.

The proof of the following simple lemma is left to the reader.

LEMMA 3.1. *The functors $S^-$ and $S^+$ have the following properties*:

(a) *Let $A$ be an indecomposable object in $\mathscr{V}(\mathbb{K}_F)$. Then $S^+A = 0$ if and only if $A \cong (F, 0, 0)$. If $S^+A \neq 0$ then there is an isomorphism $S^-S^+A \cong A$.*

(b) *If $A$ and $B$ are indecomposable objects in $\mathscr{V}(\mathbb{K}_F)$ such that $S^+A \neq 0$ and $S^+B \neq 0$ then $S^+$ induces an isomorphism* $\operatorname{Hom}(A, B) \cong \operatorname{Hom}(S^+A, S^+B)$.

(c) *The properties* (a) *and* (b) *with $S^+$ and $S^-$ interchanged.*

The following simple result plays an important role in our further consideration.

LEMMA 3.2. *Let $\mathbb{K}_F$ be an arbitrary vector space category and let* $\operatorname{pr}(E)$ *be the category of finitely generated projective right modules over the ring* $E = \operatorname{End}(K_1 \oplus \cdots \oplus K_n)$. *Then*

(a) *There exists an equivalence of categories $\omega: \mathbb{K} \to \operatorname{pr}(E)$ such that the diagram*

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{\;|-|\;} & \operatorname{mod}(F) \\ \downarrow{\scriptstyle \omega} & \nearrow{\scriptstyle -\otimes_E K_F} & \\ \operatorname{pr}(E) & & \end{array}$$

*is commutative up to a natural equivalence $\omega(-) \otimes_E K_S \to |-|$.*

(b) *$\omega$ induces a full and faithful embedding*

$$\omega^{+}: \mathscr{V}(\mathbb{K}_F) \to \mathrm{mod}(\mathbf{R}_{\mathbb{K}}).$$

*The image of $\omega^{+}$ consists of all $\mathbf{R}_{\mathbb{K}}$-modules $(X'_E, X''_F, \varphi: X' \otimes {}_E K_F \to X''_F)$ with $X'_E$ in* pr$(E)$ *and* $\dim X''_F$ *finite.*

*Proof.* We put $\omega(-) = \mathbb{K}(K_1 \oplus \cdots \oplus K_n, -)$. It is well known that $\omega$ is an equivalence of categories.

We recall from [12] that there are equivalences of categories

$$\begin{aligned} E - F\text{-bimodules} &\cong \mathrm{Add}(\mathrm{pr}(E) \otimes \mathrm{pr}(F)^{\mathrm{op}}, \mathscr{A}b) \\ &\cong \mathrm{Add}(\mathrm{pr}(E), \mathrm{Add}(\mathrm{pr}(F)^{\mathrm{op}}, \mathscr{A}b)) \\ &\cong \mathrm{Add}(\mathbb{K}, \mathrm{mod}(F)) \end{aligned}$$

and the bimodule ${}_E K_F$ corresponds to the functor $|-|$ via the composed equivalence. If $\bar{\mathbf{K}}: \mathrm{pr}(E) \otimes \mathrm{pr}(F)^{\mathrm{op}} \to \mathscr{A}b$ is the functor corresponding to ${}_E K_F$ then the Yoneda Lemma and the adjoint formula yield

$$\omega(X) \otimes {}_E K_F \cong \bar{\mathbf{K}}(\omega(X), F) \cong |X|_F$$

and (a) follows. Since (b) is an immediate consequence of (a) then the proof is complete.

*Remarks.* (1) Lemma 3.2 remains valid if we replace the division ring $F$ by an arbitrary artinian ring. This generalization is useful in solving matrix problems which are more general then the classification of indecomposables in $\mathscr{U}(\mathbb{K}_F)$. An interesting example of this kind is the category of representations of a pair of partially ordered sets.

(2) It follows from Lemma 3.2 that the category $\mathscr{U}(\mathbb{K}_F)$ is equivalent to the category of ${}_E K_F$-matrices in the sense of Drozd [9] (published in 1972!).

Following an idea of Drozd [8] we define a functor

$$H: \mathscr{V}(\mathbb{K}_F) \to \mathrm{mod}_{sp}((\mathbf{R}_{\mathbb{K}})$$

as the composition of two functors

$$\mathscr{V}(\mathbb{K}_F) \xrightarrow{\omega^{+}} \mathrm{mod}(\mathbf{R}_{\mathbb{K}}) \xrightarrow{\Theta} \mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$$

where $\omega^{+}$ is the full and faithful embedding in Lemma 3.2 and $\Theta$ is the left adjoint functor to the natural embedding $\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}}) \hookrightarrow \mathrm{mod}(\mathbf{R}_{\mathbb{K}})$.

Since

$$\mathbf{R}_{\mathbb{K}} = \begin{pmatrix} E & {}_E K_F \\ 0 & F \end{pmatrix}$$

then any right $\mathbf{R}_{\mathbb{K}}$-module $X$ is a triple $(X'_E, V_F, t)$ where $t: X' \otimes_E K_F \to V_F$ is an $F$-linear map. $X$ is in the image of $\omega^+$ if and only if $X'_E$ is a projective $E$-module. It is easy to see that

$$\Theta(X) = (X''_E, V_F, t')$$

where $X''_E$ is the image of the map $\bar{t}: X'_E \to \operatorname{Hom}_F({}_E K_F, V_F)$ adjoint to $t$ and $t'$ is the map adjoint to the inclusion $X''_E \hookrightarrow \operatorname{Hom}_F({}_E K_F, V_F)$.

We denote by $\mathscr{V}_0(\mathbb{K}_F)$ the full subcategory of $\mathscr{V}(\mathbb{K}_F)$ consisting of objects without direct summands of the form $(0, X, 0)$ where $X$ is an object in $\mathbb{K}$.

For any objects $A$ and $B$ in an additive category $\mathscr{C}$ we put

$$J(A, B) = \{f \in \mathscr{C}(A, B),\ 1_A - gf \text{ is invertible for all } g \in \mathscr{C}(B, A)\},$$
$$J^2(A, B) = \{t \in J(A, B),\ t = gf \text{ with } f \in J(A, X), g \in J(X, B)\}$$

(see [12]).

Finally, we say that a module $X$ over a right peak ring $R$ has a *perfect projective cover* if the kernel of the projective cover $P(X) \to X$ has the form $P^t_{n+1}$ for some $t$(compare [3]).

One of the main results of this paper is the following theorem.

THEOREM 3.3. *If $\mathbb{K}_F$ is a semiperfect vector space category then the functor $H: \mathscr{V}(\mathbb{K}_F) \to \operatorname{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$ has the following properties*:

(1) *$H$ is full and dense.*

(2) *A morphism $h: A \to B$ in $\mathscr{V}(\mathbb{K}_F)$ belongs to the kernel of the natural epimorphism*

$$\alpha: (A, B) \to \operatorname{Hom}_{R_{\mathbb{K}}}(H(A), H(B))$$

*induced by $H$ if and only if $h$ can be factored through an object $(0, K, 0)$ where $K$ is an object in $\mathbb{K}$. If $A$ and $B$ are indecomposable objects in $\mathscr{V}_0(\mathbb{K}_F)$, $H(A)$ is not simple and $H(B)$ has a perfect projective cover then $\alpha$ is an isomorphism.*

(3) *$H(A) = 0$ if and only if $A \cong (0, K, 0)$ for some object $K$ in $\mathbb{K}$.*

(4) *If $A$ and $B$ are indecomposable objects in $\mathscr{V}_0(\mathbb{K}_F)$ then a map $h: A \to B$ is irreducible in $\mathscr{V}(\mathbb{K}_F)$ if and only if $H(h)$ is irreducible.*

(5) *$H$ induces a representation equivalence of categories*

$$H: \mathscr{V}_0(\mathbb{K}_F) \to \operatorname{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$$

*such that* $\mathbf{cdn}(A) = \mathbf{cdn}(H(A))$ *for any object $A$ in $\mathscr{V}_0(\mathbb{K}_F)$. Moreover, if*

$$\mathbf{cdn}(A) = (s_1, \dots, s_n, s_{n+1})$$

*and* $H(A) = (X_i, {}_j\varphi_i)_{i,j \leqslant n+1} = (X'_E, X_{n+1}, \varphi\colon X' \otimes_E K_F \to X_{n+1})$ *then*

$$s_i = \dim(X_i/\bar{X}_i)_{F_i/J(F_i)}$$

*where*

$$\bar{X}_i = \operatorname{Im}\left(X_i \otimes J(F_i) \oplus \bigoplus_{j \neq i} X_j \otimes {}_jK_i \xrightarrow{({}_j\varphi_i)} X_i\right)$$

*and the E-projective cover of* $X'_E$ *has the form*

$$\bar{P}_1^{s_1} \oplus \cdots \oplus \bar{P}_n^{s_n} \to X'_E \to 0$$

*where* $\bar{P}_j$ *is the E-projective cover of the simple E-module* $\operatorname{top}(P_j)$.

*Proof.* Suppose that $A$ is an indecomposable object in $\mathscr{V}(\mathbb{K}_F)$ and let $\omega^+(A) = X = (X'_E, V_F, t)$, $\Theta(X) = (X''_E, V_F, t')$. Then either $t = 0$ or $t$ is surjective. In the first case either $X'_E = 0$ and $V_F = F$ or $A$ has the form $(0, K_i, 0)$. In the second case $X'_E$ is the $E$-projective cover of $X''_E$. Hence the property (1) easily follows. In order to prove (2) suppose that $\omega^+(B) = (Y'_E, W_F, s)$ and $H(B) = (Y''_E, W_F, s')$. We note that $X'_E$ and $Y'_E$ are projective. If $\omega^+(h) = (f, g)$ and $H(h) = 0$ then $g = 0$ and we have a commutative diagram

$$\begin{array}{ccccc}
X'_E & \xrightarrow{p} & X''_E & \longrightarrow & 0 \\
\downarrow{\scriptstyle u} & & \downarrow & & \\
\bar{P}_E & \longrightarrow & 0 & & \\
\downarrow{\scriptstyle u'} & & \downarrow & & \\
Y'_E & \xrightarrow{p'} & Y''_E & \longrightarrow & 0
\end{array}$$

where $\bar{P}_E$ is the projective cover of $\operatorname{Ker} p'$ and $f = u'u$. Hence (2) and (3) follow.

(4) It follows from (2) that $H$ induces an isomorphism

$$(A, B)/J^2(A, B) \cong \operatorname{Hom}(H(A), H(B))/J^2(H(A), H(B)).$$

Since $A$ and $B$ are indecomposable then $H(A)$ and $H(B)$ are also indecomposable. Then $f\colon A \to B$ (resp. $H(f)$) is irreducible if and only if $f$ is not an isomorphism and $f \notin J^2(A, B)$ (resp. $H(f) \notin J^2(H(A), H(B))$). Hence (4) follows.

Since the statement (5) follows immediately from (1) by applying standard projective cover arguments the proof is complete.

*Remark* (3). In the case when $\dim |K_j|_F = 1$ for $j = 1,..., n$ the functor $H$ was defined in [22] by a slightly different formula not involving projective covers.

We can use the functor $\Theta$ to prove that there are almost split sequences in $\mathrm{mod}_{sp}(R)$ provided $R$ is an artin algebra with a right peak (compare [2]). We have the following result.

PROPOSITION 3.4. *Let R be an artinian right peak ring.*

(a) *If* $0 \to X \to^{u} Y$ *is a left minimal almost split monomorphism* [1] *in* $\mathrm{mod}(R)$ *and* $X$ *is a module in* $\mathrm{mod}_{sp}(R)$ *then* $0 \to X \to^{u'} \Theta(Y)$ *is a minimal left almost split monomorphism in* $\mathrm{mod}_{sp}(R)$ *where* $u'$ *is the composition of* $u$ *and the natural epimorphism* $Y \to \Theta(Y)$.

(b) *If R is an artin algebra then for any module X in* $\mathrm{mod}_{sp}(R)$ *there exist a left and a right minimal almost split maps in* $\mathrm{mod}_{sp}(R)$.

*Proof.* (a) follows immediately from the fact that any map $Y \to Z$ with $Z$ in $\mathrm{mod}_{sp}(R)$ has a factorization through $Y \to \Theta(Y)$. Now in view of the duality $D\nabla\colon \mathrm{mod}_{sp}(R) \to \mathrm{mod}_{sp}(\widetilde{R^{\nabla}})^{\mathrm{op}}$ the statement (b) follows from (a) because we know from [1] that there are almost split sequences in $\mathrm{mod}(R)$.

Given a vector space category $\mathbb{K}_F$ we define a new vector space category $\mathbb{K}_F^*$ which is the category $\mathbb{K}^{\mathrm{op}}$ together with the composed functor

$$\mathbb{K}^{\mathrm{op}} \xrightarrow{|-|^{\mathrm{op}}} \mathrm{mod}(F)^{\mathrm{op}} \xrightarrow{(-)^*} \mathrm{mod}(F^{\mathrm{op}})$$

where $(-)^*$ is the $F$-duality. Note that there is a ring isomorphism $\mathbf{R}_{\mathbb{K}^*} \cong (\mathbf{R}_{\mathbb{K}}^{\nabla})^{\mathrm{op}}$ which implies

$$\mathbf{R}_{\mathbb{K}^*}^{\nabla} \cong \mathbf{R}_{\mathbb{K}}^{\mathrm{op}}$$

(we use the notation in Proposition 2.6).

Now suppose that $\mathbf{R}_{\mathbb{K}}$ is either schurian with the constant dimension property or is an artinian PI-ring. We define a functor

$$G\colon \mathscr{U}(\mathbb{K}_F) \to \mathrm{mod}_{sp}((\mathbf{R}_{\mathbb{K}})_*)$$

as the composition of four functors

$$\begin{array}{ccc} \mathscr{U}(\mathbb{K}_F) \xrightarrow{(-)^*} \mathscr{V}(\mathbb{K}_F^*) \xrightarrow{H^{\mathrm{op}}} & (\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}^*}))^{\mathrm{op}} & \\ & \big\downarrow \nabla^{\mathrm{op}} & \\ & (\mathrm{mod}_{ti}(\mathbf{R}_{\mathbb{K}}^{\mathrm{op}}))^{\mathrm{op}} & \xrightarrow{D} \mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}})_* \end{array}$$

where $(U, X, t)^* = (U^*, X, t^*)$, $\nabla$ is the equivalence in Proposition 2.6,

$$(\mathbf{R}_{\mathbb{K}})_* = (\widetilde{\mathbf{R}_{\mathbb{K}}^{\mathrm{op}}})^{\mathrm{op}}$$

and $D$ is a Morita duality (see Proposition 2.5 and [18]). It follows from Proposition 2.5 that if $\mathbf{R}_{\mathbb{K}}$ has the constant dimension property then

$$(\mathbf{R}_{\mathbb{K}})_* = \begin{bmatrix} F_1 & {}_1K_2^{12} & \cdots & {}_1K_n^{1n} & {}_1K_{n+1}^{1n+1} \\ {}_2K_1^{21} & F_2 & \cdots & {}_2K_n^{2n} & {}_2K_{n+1}^{2n+1} \\ \vdots & & \ddots & \vdots & \vdots \\ {}_nK_1^{n1} & {}_nK_2^{n2} & \cdots & F_n & {}_nK_{n+1}^{nn+1} \\ 0 & 0 & \cdots & 0 & F_{n+1} \end{bmatrix}.$$

In order to formulate main properties of the functor $G$ we need some notation.

Let $R$ be a right peak ring with the constant dimension property and suppose that its value scheme $(\mathbf{I}_R, \mathbf{d})$ has no oriented cycles. We define reflections

$$\boldsymbol{\delta}_i : \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}, \qquad i = 1,..., n+1,$$

by the formula $\boldsymbol{\delta}_i(x_1,..., x_{n+1}) = (y_1,..., y_{n+1})$ where $y_j = x_j$ for $j \neq i$,

$$y_i = d'_{in+1}x_{n+1} - (d'_{1i}x_1 + \cdots + d'_{i-1i}x_{i-1} + x_i + d_{ii+1}x_{i+1} + \cdots + d_{in}x_n)$$

for $i \leqslant n$ and

$$y_{n+1} = -x_{n+1} + d_{1n+1}x_1 + \cdots + d_{nn+1}x_n.$$

The composed map

$$\boldsymbol{\delta} = \boldsymbol{\delta}_1 \cdots \boldsymbol{\delta}_n \boldsymbol{\delta}_{n+1}$$

will be called the *Coxeter transformation of the scheme* $(\mathbf{I}_R, \mathbf{d})$.

Suppose that $(\mathbf{I}_R, \mathbf{d})$ is *symmetrizable* in the sense that there are natural numbers $f_1,..., f_{n+1}$ such that $d_{ij}f_j = f_i d'_{ij}$ for all $i$ and $j$. We associate to $(\mathbf{I}_R, \mathbf{d})$ the rational Tits quadratic form

$$q(x_1,..., x_{n+1}) = \sum_{i=1}^{n+1} x_i^2 f_i + \sum_{\substack{i,j=1\\ i\neq j}}^{n} x_i x_j f_i d'_{ij} - \left(\sum_{i=1}^{n} x_i f_i d'_{in+1}\right) x_{n+1}.$$

If $\mathbf{e}_1,..., \mathbf{e}_{n+1}$ is the standard basis of $\mathbb{Q}^{n+1}$ and $B$ is the symmetric bilinear form associated to $q$ then $B(\mathbf{e}_i, \mathbf{e}_i) = f_i$,

$$\boldsymbol{\delta}_i(\mathbf{x}) = \mathbf{x} - \frac{2B(\mathbf{x}, \mathbf{e}_i)}{B(\mathbf{e}_i, \mathbf{e}_i)} \mathbf{e}_i$$

and $q(\mathbf{x}) = q(\boldsymbol{\delta}_i(\mathbf{x}))$ for $\mathbf{x} \in \mathbb{Q}^{n+1}$ and $i = 1,..., n+1$.

By an easy induction we can prove the following useful result.

LEMMA 3.5. *If* $(\mathbf{I}_R, \mathbf{d})$ *has no oriented cycles then*

$$\boldsymbol{\delta}_1 \cdots \boldsymbol{\delta}_n(x_1,..., x_{n+1}) = (y_1,..., y_{n+1})$$

*if and only if* $y_{n+1} = x_{n+1}$ *and*

$$\sum_{j=1}^{i-1} d_{ji} x_j + x_i + y_i + \sum_{j=i+1}^{n} d'_{ij} y_j = d'_{in+1} x_{n+1}$$

*for* $i = 1,..., n$.

The proof is left to the reader.

Let us denote by $\mathscr{U}_0(\mathbb{K}_F)$ the full subcategory of $\mathscr{U}(\mathbb{K}_F)$ consisting of objects having no direct summands of the form $(0, K, 0)$ with $K \neq 0$.

THEOREM 3.6. *Let* $\mathbb{K}_F$ *be a semiperfect vector space category and suppose that the ring* $\mathbf{R}_{\mathbb{K}}$ *has a Morita duality. Then*

(a) *The functor G has the properties* (1)–(4) *and the first part of* (5) *in Theorem* 3.4 *with H,* $\mathscr{V}(\mathbb{K}_F)$ *and G,* $\mathscr{U}(\mathbb{K}_F)$ *interchanged.*

(b) *If* $\mathbf{R}_{\mathbb{K}}$ *has the constant dimension property and* $(\mathbf{I}_{\mathbf{R}_{\mathbb{K}}}, \mathbf{d})$ *has no oriented cycles then*

$$\mathbf{cdn}(G(A)) = \boldsymbol{\delta}_1 \cdots \boldsymbol{\delta}_n(\mathbf{cdn}(A))$$

*for any indecomposable object A in* $\mathscr{U}_0(\mathbb{K}_F)$ *such that the modules* $H(A^*)$ *and* $G(A)$ *have perfect projective covers.*

*Proof.* (a) follows immediately from Theorem 3.4 and the definition of $G$. In order to prove (b) suppose that $H(A^*)$ and $G(A)$ have perfect projective covers and let

$$\mathbf{cdn}(A) = (s_1,..., s_{n+1}), \qquad \mathbf{cdn}(G(A)) = (s'_1,..., s'_{n+1}),$$
$$\mathbf{dim}\, H(A^*) = (x_1,..., x_{n+1}), \qquad \mathbf{dim}\, G(A) = (x'_1,..., x'_{n+1}).$$

Consider the module $\omega^+(A^*) = ({}_E\bar{P}, V, h)$ over the ring $\mathbf{R}_{\mathbb{K}^*} \cong (\mathbf{R}_{\mathbb{K}}^{\nabla})^{\mathrm{op}}$ where $h: ({}_E K_F^F) \otimes {}_E\bar{P} \to {}_F V$ is an $F$-linear map. By our assumption ${}_E\bar{P}$ is projective of the form ${}_E\bar{P} \cong (\bar{P}_1^E)^{s_1} \oplus \cdots \oplus (\bar{P}_n^E)^{s_n}$ where $\bar{P}_j^E = \mathrm{Hom}_E(\bar{P}_j, E)$ (see Theorem 3.4). Moreover $\bar{h}: {}_E\bar{P} \to \mathrm{Hom}_F({}_E K_F^F, V) \cong {}_E K_F \otimes {}_F V$ is injective, $\dim {}_E\bar{P} = (x_1,..., x_n)$ and $\nabla H(A^*) = (V, {}_E Y, t)$ where ${}_E Y = \mathrm{Coker}\, \bar{h}$ and $t: {}_E K_F \otimes {}_F V \to {}_E Y$ is the natural epimorphism. Hence $x_i = \sum_{j=1}^{i-1} d_{ji} s_j + s_i$ for $i \neq n+1$. Since obviously $\dim \nabla H(A^*) = \dim G(A)$ then using the same type of arguments as above we conclude that $x'_i = \sum_{j=i+1}^{n} s'_j d'_{ij} + s'_i$ for $i \neq n+1$.

Consequently the equality $Y = \operatorname{Coker} \bar{h}$ yields $s_{n+1} = s'_{n+1}$ and $x_i + x'_i = d'_{in+1} s_{n+1}$ for $i = 1, \ldots, n$. Then the required equality follows from Lemma 3.5 and the proof is complete.

Now suppose that $\mathbf{R}_{\mathbb{K}}$ is either an artinian PI-ring or an artinian schurian ring with the constant dimension property. We define two maps

$$|\operatorname{mod}_{sp}(\mathbf{R}_{\mathbb{K}})| \underset{\Delta^-}{\overset{\Delta^+}{\rightleftarrows}} |\operatorname{mod}_{sp}(\mathbf{R}_*)|, \ \mathbf{R}_* = (\mathbf{R}_{\mathbb{K}})_*,$$

where $|\operatorname{mod}_{sp}(T)|$ denotes the set of isomorphism classes of modules in $\operatorname{mod}_{sp}(T)$. We will call $\Delta^+$ and $\Delta^-$ *Coxeter maps* of $\mathbf{R}_{\mathbb{K}}$.

Let $N = (X_E, V_F, t\colon X_E \otimes {}_E K_F \to V_F)$ be a module in $\operatorname{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$. Consider the sequence

$$0 \longrightarrow W_F \xrightarrow{w} P_E \otimes {}_E K_F \xrightarrow{t(v \otimes 1)} V_F$$

where $v\colon P_E \to X_E$ is the projective cover of $X_E$ and $W_F = \operatorname{Ker} t(v \otimes 1)$. Let

$$\tilde{w}\colon P_E^E \to \operatorname{Hom}_F({}_E K_F^F, W_F^F) \cong {}_E K_F \otimes {}_F(W_F^F)$$

be the image of $w$ under the composed isomorphism

$$\begin{aligned} \operatorname{Hom}_F(W_F, P_E \otimes {}_E K_F) &\xrightarrow[\approx]{*} \operatorname{Hom}_F((P_E \otimes {}_E K_F)^F, W_F^F) \\ &\cong \operatorname{Hom}_F(\operatorname{Hom}_E(P_E, {}_E K_F^F), W_F^F) \\ &\cong \operatorname{Hom}_F({}_E K_F^F \otimes {}_E(P_E^E), W_F^F) \\ &\cong \operatorname{Hom}_E(P_E^E, \operatorname{Hom}_F({}_E K_F^F, W_F^F)) \end{aligned}$$

where $Y_E^E = \operatorname{Hom}_E(Y_E, E)$. Note that if $f \in P_E^E$ and $g \in {}_E K_F^F$ then

$$\tilde{w}(f)(g) = [g \otimes f]\, w$$

where $[g \otimes f]\colon P_E \otimes {}_E K_F \to F$ is defined by the formula

$$[g \otimes f](p \otimes k) = g(f(p) \cdot k).$$

Now consider the module $({}_E Y, W_F^F, u)$ where $u\colon {}_E K_F \otimes {}_F(W_F^F) \to {}_E Y$ is the cokernel of $\tilde{w}$. We put

$$\Delta^+ N = D({}_E Y, W_F^F, u)$$

where $D$ is the duality in the definition of $G$.

The map $\Delta^-$ is defined analogously.

Now suppose that $N$ is not projective. If $p: P(N) \to N$ is the projective cover of $N$ then the module

$$\tilde{N} = P(N)/\mathrm{soc}(\mathrm{Ker}\, p)$$

is isomorphic to $N' = (P_E, V_F, t(v \otimes 1))$. Indeed, by the projectivity of $P(N)$ there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}\, p & \longrightarrow & P(N) & \xrightarrow{p} & N & \longrightarrow & 0 \\ & & \uparrow \cup & & \uparrow \mathrm{id} & & \uparrow (t(v \otimes 1), \mathrm{id}) & & \\ 0 & \longrightarrow & \mathrm{Ker}\, p' & \xrightarrow{j} & P(N) & \xrightarrow{p'} & N' & \longrightarrow & 0 \end{array}$$

and it is ealsy to see that $\mathrm{soc}(K\mathrm{er}\, p) = \mathrm{Ker}\, p'$, as required. Since obviously $\mathrm{Ker}\, p' = (U_i, {}_j\psi_i)$ with $U_{n+1} = W_F$ and $U_i = 0$ for $i \neq n+1$ then the cokernel of the map $j_*: \mathrm{Hom}_{\mathbf{R}_{\mathbb{K}}}(P(N), R_{\mathbb{K}}) \to \mathrm{Hom}_{\mathbf{R}_{\mathbb{K}}}(\mathrm{Ker}\, p', \mathbf{R}_{\mathbb{K}})$ is isomorphic to the $\mathbf{R}_{\mathbb{K}}^{\mathrm{op}}$-module $({}_E Y, W_F^F, u)$. On the other hand $\mathrm{Coker}\, j_*$ is the transpose module $\mathrm{tr}\, \tilde{N}$.

Since we know from Proposition 2.2 and Lemma 3.2 that any right peak ring is of the form $R = \mathbf{R}_{\mathbb{K}}$ with $\mathbb{K} = \mathrm{pr}(A)$ then the above remarks together with Proposition 3.4 yield

COROLLARY 3.7. *Let $R$ be an artinian right peak ring which is either a PI-ring or has the constant dimension property. If $N$ is an indecomposable nonprojective module in* $\mathrm{mod}_{sp}(R)$ *and $L$ is an indecomposable non-sp-injective module in* $\mathrm{mod}_{sp}(\mathbf{R}_*)$ *then*

$$\Delta^+ N \cong D\, \mathrm{tr}\, \tilde{N}, \qquad \Delta^- L \cong \Theta\, \mathrm{tr}\, D(L)$$

*where* tr *is the Auslander's transpose. If, in addition, $R$ is an artin algebra then $\mathbf{R}_* \cong R$ and there are almost split sequences in* $\mathrm{mod}_{sp}(R)$ *of the forms*

$$0 \to \Delta^+ N \to X \to N \to 0,$$

$$0 \to L \to Y \to \Delta^- L \to 0.$$

*Remark* (4). The Corollary 3.7 gives a useful method for calculating almost split sequences in $\mathrm{mod}_{sp}(R)$. It was already applied by Bünermann [6] in a particular case when $R$ is an incidence ring $FI^*$ of a partially ordered set (see Remark 2.9). In this case the maps $\Delta^+$ and $\Delta^-$ coincide with the corresponding functions $F$ and $\bar{F}$ of Drozd [8].

DEFINITION 3.8. Let $\mathbb{K}_F$ be a schurian artinian vector space category such that the ring $\mathbf{R}_{\mathbb{K}}$ has the constant dimension property. The Coxeter scheme of $\mathbb{K}_F$ is the following infinite diagram $\mathbf{Cox}(\mathbb{K}_F)$

$$
\begin{array}{ccccccccccc}
\cdots \underset{S^-}{\overset{S^+}{\rightleftarrows}} & \mathscr{U}(\mathbb{K}_{-2}) & & \mathscr{V}(\mathbb{K}_{-1}) \underset{S^-}{\overset{S^+}{\rightleftarrows}} & \mathscr{U}(\mathbb{K}_{-1}) & & \mathscr{V}(\mathbb{K}_0) \underset{S^-}{\overset{S^+}{\rightleftarrows}} & \mathscr{U}(\mathbb{K}_0) & & \mathscr{V}(\mathbb{K}_1) \underset{S^-}{\overset{S^+}{\rightleftarrows}} \cdots \\
& {\scriptstyle G_{-2}}\searrow & & \swarrow{\scriptstyle H_{-1}} \quad {\scriptstyle G_{-1}}\searrow & & \swarrow{\scriptstyle H_0} \quad {\scriptstyle G_0}\searrow & & \swarrow{\scriptstyle H_1} & & \\
\cdots & & \mathrm{mod}_{sp}(R_{-1}) & & & \mathrm{mod}_{sp}(R_0) & & & \mathrm{mod}_{sp}(R_1) & & \cdots \\
& & \downarrow & & & \downarrow & & & \downarrow & & \\
\cdots \underset{\Delta^-_{-2}}{\overset{\Delta^+_{-2}}{\rightleftarrows}} & & |\mathrm{mod}_{sp}(R_{-1})| & \underset{\Delta^-_{-1}}{\overset{\Delta^+_{-1}}{\rightleftarrows}} & & |\mathrm{mod}_{sp}(R_0)| & \underset{\Delta^-_0}{\overset{\Delta^+_1}{\rightleftarrows}} & & |\mathrm{mod}_{sp}(R_1)| & \underset{\Delta^-_1}{\overset{\Delta^+_1}{\rightleftarrows}} \cdots & \\
& & \downarrow{\scriptstyle \mathbf{cdn}} & & & \downarrow{\scriptstyle \mathbf{cdn}} & & & \downarrow{\scriptstyle \mathbf{cdn}} & & \\
\cdots \underset{\delta^{-1}}{\overset{\delta}{\rightleftarrows}} & & \mathbb{Z}^{n+1} & \underset{\delta^{-1}}{\overset{\delta}{\rightleftarrows}} & & \mathbb{Z}^{n+1} & \underset{\delta^{-1}}{\overset{\delta}{\rightleftarrows}} & & \mathbb{Z}^{n+1} & \underset{\delta^{-1}}{\overset{\delta}{\rightleftarrows}} \cdots &
\end{array}
$$

where $\mathbb{K}_0 = \mathbb{K}_F$, $R_0 = \mathbf{R}_{\mathbb{K}}$, $\mathbb{K}_i$ are vector space categories for $i = \pm 1, \pm 2, \ldots$, $R_i = \mathbf{R}_{\mathbb{K}_i}$, $R_{i+1} = (\mathbf{R}_{\mathbb{K}_i})_*$, $H_i$ and $G_i$ are the functors $H$ and $G$ taken for the category $\mathbb{K}_i$, $\Delta_i^+$ and $\Delta_i^-$ are appropriate maps $\Delta^+$ and $\Delta^-$, $\boldsymbol{\delta}$ is the Coxeter transformation of the value scheme $(\mathbf{I}_{R_{\mathbb{K}}}, \mathbf{d})$ and the middle vertical arrows denote functions which assign to each module its isomorphism class.

It follows from Lemma 3.2 and the definitions of $H$ and $G$ that $\mathbf{Cox}(\mathbb{K}_F)$ exists and is uniquely determined by $\mathbb{K}_F$ up to a natural equivalence. Moreover, since any semiperfect right peak ring $R$ has the form $R \cong \mathbf{R}_{\mathbb{K}}$ then any schurian right peak ring with the constant dimension property admits a Coxeter scheme.

*Remarks.* (5) If $\mathbb{K}_F$ has infinitely many pairwise nonisomorphic indecomposable objects the Coxeter scheme of $\mathbb{K}_F$ can be defined analogously. In this case we replace the ring $R_i$ by an appropriate factor of the tensor category of the species $(F_j, {}_jK_s^{(i)})$ with an obvious commutativity condition (see [20]).

(6) The notion of the vector space category and its Coxeter scheme admit useful generalizations. The obvious one we get by taking instead of the division ring $F$ a product of division rings. In this case the corresponding ring $\mathbf{R}_{\mathbb{K}}$ has a projective and essential right socle. A particular case of it was considered in [14]. A more interesting generalization we get by taking for $F$ a hereditary artin algebra (compare [25]).

We denote by $\underline{\mathrm{mod}}_{sp}(R)$ (resp. by $\overline{\mathrm{mod}}_{sp}(R)$) the factor category of $\mathrm{mod}_{sp}(R)$ modulo the ideal consisting of all maps which admit a factorization through a projective (resp. $sp$-injective) module. The corresponding Hom functor is denoted by $\underline{\mathrm{Hom}}$ and $\overline{\mathrm{Hom}}$, respectively.

THEOREM 3.9. *Let* $\mathbf{Cox}(\mathbb{K}_F)$ *be the Coxeter scheme of a schurian artinian vector space category* $\mathbb{K}_F$ *for which* $\mathbf{R}_{\mathbb{K}}$ *has the constant dimension property. Then*

(1) *The ring* $R_i$ *has the constant dimension property and* $(\mathbf{I}_{R_i}, \mathbf{d}^{(i)})$ *is isomorphic to* $(\mathbf{I}_{\mathbf{R}_{\mathbb{K}}}, \mathbf{d})$ *for every i.*

(2) *If* $A$ *is an indecomposable object in* $\mathscr{V}(\mathbb{K}_i)$ *then* $\Delta_i^+ H_i(A) \cong G_i S^+(A)$ *and* $H_i(A) \cong \Delta_i^- G_i S^+(A)$ *provided the terms are nonzero.*

(3) *The functors* $S^+G_i$, $H_i$ *and* $H_i S^-$, $G_i$ *induce two equivalences of categories*

$$\underline{\mathrm{mod}}_{sp}(R_i) \underset{\Delta_i^-}{\overset{\bar{\Delta}_i^+}{\rightleftarrows}} \overline{\mathrm{mod}}_{sp}(R_{i+1})$$

*each inverse to the other. Moreover* $\overline{\Delta_i^+ N} \cong \bar{\Delta}_i^+ N$ *and* $\underline{\Delta_i^- M} \cong \underline{\Delta}_i^- M$ *for every module N in* $\mathrm{mod}_{sp}(R_i)$ *and every M in* $\mathrm{mod}_{sp}(R_{i+1})$.

(4) *All categories* $\mathrm{mod}_{sp}(R_i)$ *have the same number of indecomposables.*

(5) *Let N and M be indecomposable modules in* $\mathrm{mod}_{sp}(R_i)$. *Then*

(i) $\Delta_i^+ N = 0$ *if and only if N is projective. If* $\Delta_i^+ N \neq 0$ *then*

$$\Delta_i^- \Delta_i^+ N \cong N \qquad \textit{and} \qquad \mathrm{End}(N)/J^2 \cong \mathrm{End}(\Delta_i^+ N)/J^2.$$

(ii) $\Delta_{i-1}^- N = 0$ *if and only if N is sp-injective. If* $\Delta_{i-1}^- N \neq 0$ *then*

$$\Delta_{i-1}^+ \Delta_{i-1}^- N \cong N \qquad \textit{and} \qquad \mathrm{End}(N)/J^2 \cong \mathrm{End}(\Delta_{i-1}^- N)/J^2.$$

(iii) *If* $(\mathbf{I}_{\mathbf{R}_{\mathbb{K}}}, \mathbf{d})$ *has no oriented cycles,* $\Delta_i^+ N \neq 0$ (resp. $\Delta_{i-1}^- N \neq 0$) *and the modules* $\nabla^{-1} D^{-1} \Delta_i^+ N$, $\Delta_i^+ N$ (resp. $\nabla^{-1} D^{-1} N$, $\Delta_{i-1}^- N$) *have perfect projective covers then*

$$\mathbf{cdn}(\Delta_i^+ N) = \boldsymbol{\delta}(\mathbf{cdn}(N)) \qquad \textit{and} \qquad \mathrm{Hom}_{R_i}(M, N) \cong \mathrm{Hom}_{R_{i+1}}(\Delta_i^+ M, \Delta_i^+ N),$$

(resp. $\mathbf{cdn}(\Delta_{i-1}^- N) = \boldsymbol{\delta}(\mathbf{cdn}(N))$ *and* $\mathrm{Hom}_{R_i}(M, N) \cong \mathrm{Hom}_{R_{i-1}}(\Delta_{i-1}^- M, \Delta_{i-1}^- N)$) *provided* $\Delta_i^+ M \neq 0$ (resp. $\Delta_{i-1}^- M \neq 0$).

*Proof.* Since $R_{i+1} = (R_i)_*$ then (1) follows form the remark after the definition of $G$ because $\mathbf{R}_{\mathbb{K}}$ has the constant dimension property.

(2) follows immediately from the definition of $\Delta^+$ and $\Delta^-$.

(3) Denote by $\mathfrak{A}_i$ (resp. by $\mathfrak{B}_i$) the two-sided ideal in the category $\mathscr{V}(\mathbb{K}_i)$ (resp. in $\mathscr{U}(\mathbb{K}_i)$) consisting of maps having a factorization through a direct sum of objects of the forms $(F, 0, 0)$, $(0, K_j, 0)$ and $(|X|, X, \mathrm{id})$. It is

easy to see that an indecomposable object $A$ in $\mathscr{V}(\mathbb{K}_i)$ is of one of the forms $(|K_j|, K_j, \mathrm{id})$, $(F, 0, 0)$ if and only if $H_i(A)$ is nonzero projective. Then by Theorem 3.3 $H_i$ induces an equivalence

$$\mathscr{V}(\mathbb{K}_i)/\mathfrak{A}_i \to \underline{\mathrm{mod}}_{sp}(R_i).$$

Next we note that an indecomposable object $B$ in $\mathscr{U}(\mathbb{K}_i)$ is of one of the forms $(F, 0, 0)$, $(|K_j|, K_j, \mathrm{id})$ if and only if $H(A^*)$ is nonzero projective in $\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}_i^*})$. Since there is a commutative diagram

$$\begin{array}{ccc} (\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}_i^*}))^{\mathrm{op}} & \xrightarrow{\nabla^{\mathrm{op}}} & (\mathrm{mod}_{ti}(R_i^{\mathrm{op}}))^{\mathrm{op}} \\ \downarrow D & & \downarrow D \\ \mathrm{mod}_{ti}(R_{i+1}^{\nabla}) & \xrightarrow{\nabla^{-1}} & \mathrm{mod}_{sp}(R_{i+1}) \end{array}$$

then by Proposition 2.6 $H(A^*)$ is projective if and only if $G_i(A)$ is $sp$-injective. Consequently $G_i$ induces an equivalence

$$\mathscr{U}(\mathbb{K}_i)/\mathfrak{B}_i \to \overline{\mathrm{mod}}_{sp}(R_{i+1})$$

which together with the equivalence $\mathscr{V}(\mathbb{K}_i)/\mathfrak{A}_i \cong \mathscr{U}(\mathbb{K}_i)/\mathfrak{B}_i$ induced by $S^+$ (see Lemma 3.1) proves the statements (3), (4) and (i), (ii) in (5). The isomorphisms of appropriate endomorphism rings modulo $J^2$ in (i) and (ii) follow from Theorems 3.3 and 3.6, and an obvious observation that if $X$ is an indecomposable nonprojective (resp. non-$sp$-injective) module then any endomorphism of $X$ which can be factored through a projective (resp. $sp$-injective) module belongs to $J^2\,\mathrm{End}(X)$.

In order to prove (iii) suppose $\Delta_i^+ N \neq 0$. Then $S^+\tilde{N} \neq 0$ and from the definition of $S^+$ it follows that

$$\boldsymbol{\delta}_{n+1}(\mathbf{cdn}(N)) = \boldsymbol{\delta}_{n+1}(\mathbf{cdn}(\tilde{N})) = \mathbf{cdn}(S^+\tilde{N}) = \mathbf{cdn}\ \Theta(S^+\tilde{N})^*.$$

Thus (iii) follows from Theorem 3.6 and the theorem is proved.

From the proof of Theorem 3.9 immediately follows

COROLLARY 3.10. *Let $\mathbb{K}_F$ be a vector space category. If $\mathbf{R}_{\mathbb{K}}$ is an artinian PI-ring then the statements* (2), (3), (4) *and* (5)(i)–(ii) *are true.*

We finish this section by a useful characterization of schurian vector space $K$-categories of finite representation type which extends the result of Drozd [8].

THEOREM 3.11. *Let $\mathbb{K}_F$ be a schurian vector space category such that the ring $\mathbf{R}_{\mathbb{K}}$ is a finite-dimensional algebra over a field $K$. Then the value scheme $(\mathbf{I}_{\mathbf{R}_{\mathbb{K}}}, \mathbf{d})$ is symmetrizable and the quadratic form $q$ associated to $(\mathbf{I}_{\mathbf{R}_{\mathbb{K}}}, \mathbf{d})$ is weakly positive (i.e., $q(\mathbf{x}) > 0$ for any nonzero $\mathbf{x} \in \mathbb{Z}^{n+1}$ with* *nonnegative coordinates) if and only if* $\mathscr{U}(\mathbb{K}_F)$ *is of finite representation type. Moreover, if q is weakly positive then:*

(a) *For every indecomposable module X in* $\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$ *there is an indecomposable projective* $\mathbf{R}_{\mathbb{K}}$*-module* $P_j$ *such that* $\mathrm{End}(X) \cong \mathrm{End}(P_j) \cong F_j$. *If in addition X and* $\Delta^+X$ *are exact then*

$$\mathbf{cdn}(\Delta^+X) = \delta(\mathbf{cdn}(X)) \qquad \text{and} \qquad \mathrm{Hom}(N, X) \cong \mathrm{Hom}(\Delta^+N, \Delta^+X)$$

*for every indecomposable module N in* $\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$ *with* $\Delta^+N \neq 0$.

(b) *If X is an indecomposable module in* $\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$ *with* $\mathrm{End}(X) \cong F_j$ *then* $q(\mathbf{cdn}(X)) = f_j$.

(c) *For every indecomposable module X in* $\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$ *there is an integer j such that* $\Delta^{+j}X$ *is either projective or is not exact (note that in* $\mathbf{Cox}(\mathbb{K}_F)$ $\Delta_i^+ = \Delta^+$ *for all i).*

(d) *Every indecomposable module in* $\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$ *is uniquely determined by its composition factors.*

*Proof.* Suppose that $\mathscr{U}(\mathbb{K}_F)$ is of finite representation type. Given $\mathbf{s} = (s_1, \ldots, s_n, s_{n+1}) \in \mathbb{N}^{n+1}$ we consider the algebraic variety

$$\mathfrak{X}_s = \mathrm{Hom}_{\mathbf{R}_{\mathbb{K}}}(P_1^{s_1} \oplus \cdots \oplus P_n^{s_n}, Q^{s_{n+1}})$$

where $Q$ is the injective envelope of the simple projective module $P_{n+1}$. There is an obvious action of the algebraic group

$$\mathfrak{G}_{\mathbf{s}} = Gl(Q^{s_{n+1}}) \times Gl(P_1^{s_1} \oplus \cdots \oplus P_n^{s_n})$$

on $\mathfrak{X}_{\mathbf{s}}$. Note that $f, g \in \mathfrak{X}_{\mathbf{s}}$ belong to the same $\mathfrak{G}_{\mathbf{s}}$-orbit if and only if the $\mathbf{R}_{\mathbb{K}}$-modules $\mathrm{Im} f$ and $\mathrm{Im}\, g$ are isomorphic. Since by Theorem 3.3 $\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$ is of finite representation type then there is only finitely many $\mathfrak{G}_{\mathbf{s}}$-orbits in $\mathfrak{X}_{\mathbf{s}}$. Hence we conclude that $\dim \mathfrak{G}_{\mathbf{s}} > \dim \mathfrak{X}_{\mathbf{s}}$ and therefore $q(\mathbf{s}) > 0$ (we take for $f_i$ in the form $q$ the dimension of the division ring $F_i = \mathrm{End}(P_i)$ over the field $K$).

Suppose conversely that $q$ is weakly positive. In view of Theorems 3.3, 3.6 and 3.9 the statements (a)–(c) can be proved by applying arguments of Drozd [8]. In particular one can show that every exact module in $\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$ has a perfect projective cover (compare [8, Lemma 2]). Next we conclude from (b) that $\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$ is of finite representation type because the set of indecomposable modules $N$ in $\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$ with $q(\mathbf{cdn}(N)) = f_j$ is finite (see [8, Appendix]). Furthermore, $N$ is determined up to isomorphism by $\mathbf{cdn}(N)$. Then (d) follows from the equality

$$x_i = s_i + \sum_{j=1}^{i-1} d'_{ij} s_j, \qquad i \leqslant n,$$

established in the proof of Theorem 3.6, where $\mathbf{dim}\, N = (x_1,..., x_{n+1})$, because without loss of generality we can suppose that $N$ is exact and therefore $N$ has a perfect projective cover by a remark above. The theorem is proved.

*Remark* (7). In the case $R$ is a schurian right peak PI-ring and ${}_jM_{n+1}$ is a simple bimodule for any $j = 1,..., n$ the previous results allow us to introduce the notion of preprojectivity and of preinjectivity in $\mathrm{mod}_{sp}(R)$ in a way similar to that in [3, 21]. Criteria for $\mathrm{mod}_{sp}(R)$ to be of finite type similar to those in [3, 21] can be given. In particular one can prove that $\mathrm{mod}_{sp}(R)$ is of finite type if and only if the preprojective component in $\mathrm{mod}_{sp}(R)$ is finite. In this case there is no oriented cycle of irreducible maps in $\mathrm{mod}_{sp}(R)$.

## 4. A Triangular Reduction

Our main purpose in this section is to describe an algorithm for the classification of the indecomposable subspaces of schurian vector space categories of finite type. The algorithm is obtained by combining the results in Section 3 together with the method applied by Ringel [17, 2.5, 2.6] (compare [13]).

Let

$$R = \begin{pmatrix} F & {}_FM_S \\ 0 & S \end{pmatrix}$$

be an artinian ring with a division ring $F$ and an $F-S$-bimodule ${}_FM_S$. Then $\mathrm{mod}(R)$ can be identified with the category $\imath({}_FM_S)$ of all triples $X = (X'_F, X''_S, t)$ where $X'_F$ is a finite-dimensional vector space over $F$, $X''_S$ is a module in $\mathrm{mod}(S)$ and $t: X' \otimes {}_FM_S \to X''_S$ is a homomorphism of $S$-modules. The map adjoint to $t$ is denoted by

$$\bar{t}: X'_F \to \mathrm{Hom}_S({}_FM_S, X''_S).$$

The category $\mathbb{K}^R = \mathrm{Hom}_S({}_FM_S, \mathrm{mod}(S))$ together with the embedding functor $|-|: \mathbb{K}^R \to \mathrm{Mod}(F)$ will be denoted by $\mathbb{K}^R_F$ (see [17]).

Suppose that $\mathbb{K} = \mathbb{K}^R$ has finitely many pairwise nonisomorphic indecomposable objects and that $\mathrm{Im}\, |-| \subseteq \mathrm{mod}(F)$. If $\mathbf{R}_{\mathbb{K}}$ is either an artinian *PI*-ring or is schurian with the constant dimension property we define the functor

$$G_+ : \mathrm{mod}(R) \to \mathrm{mod}_{sp}((\mathbf{R}_{\mathbb{K}})_*)$$

as the composition of two functors

$$\mathrm{mod}(R) \xrightarrow{\Phi} \mathscr{U}(\mathbb{K}_F) \xrightarrow{G} \mathrm{mod}_{sp}((\mathbf{R}_{\mathbb{K}})_*)$$

where $\mathbb{K}_F = \mathbb{K}^R_F$ and $\Phi(X) = (X'_F, \mathrm{Hom}_S({}_FM_S, X''_S), \bar{t})$ see [17, 2.5]).

If $\mathbb{K}^R$ has infinitely many nonisomorphic indecomposable objects the functor $G_+$ can be defined analogously. In this case we replace $\mathrm{mod}_{sp}((\mathbf{R}_{\mathbb{K}})_*)$ by the category of socle projective representations of an appropriate species with a commutativity condition (see [20] and Section 6A).

Note that there is an obvious embedding of $\mathrm{mod}(S)$ into $\mathrm{mod}(R)$. We denote by $[\mathrm{mod}(S)]$ the two-sided ideal in $\mathrm{mod}(R)$ consisting of those $R$-homomorphisms that admit a factorization through an $S$-module. The factor category of $\mathrm{mod}(R)$ modulo the ideal $[\mathrm{mod}(S)]$ is denoted by $\mathrm{mod}(R)/[\mathrm{mod}(S)]$. Finally, we denote by $\mathfrak{M}_R^S$ the full subcategory of $\mathrm{mod}(R)$ consisting of modules having no direct summands in $\mathrm{mod}(S)$.

We have the following reduction theorem which generalizes [17, 2.5, 2.6] and [23, Proposition 2.2].

THEOREM 4.1. (1) *The functor $G_+$ is full, dense and induces a representation equivalence*

$$G'_+ : \mathfrak{M}_R^S \to \mathrm{mod}_{sp}((\mathbf{R}_{\mathbb{K}})_*)$$

*as well as an equivalence of categories*

$$\mathrm{mod}(R)/[\mathrm{mod}(S)] \cong \mathrm{mod}_{sp}((\mathbf{R}_{\mathbb{K}})_*).$$

(2) $\# \mathrm{mod}(R) = \# \mathrm{mod}(S) + \# \mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$ *where $\#$ means the number of indecomposable modules.*

(3) *If $X$ and $Y$ are indecomposable modules in $\mathfrak{M}_R^S$ then the map $f: X \to Y$ is irreducible in* $\mathrm{mod}(R)$ *if and only if $G_+(f)$ is irreducible.*

(4) *Suppose $f'': X''_S \to Y''_S$ is irreducible in* $\mathrm{mod}(S)$. *If* $\mathrm{Hom}_S(M_S, X''_S) = 0$ *then* $(0, f''): (0, X''_S, 0) \to (0, Y''_S, 0)$ *is irreducible in* $\mathrm{mod}(R)$. *If* $\mathrm{Hom}_S(M_S, X''_S) \neq 0$ *and* $\mathrm{Hom}_S(M_S, Y''_S) = 0$ *then* $(0, f'')$: $(\mathrm{Hom}_S({}_FM_S, X''_S), X''_S, \overline{\mathrm{id}}) \to (0, Y''_S, 0)$ *is irreducible in* $\mathrm{mod}(R)$.

*Proof.* (1) It follows from Theorems 3.3 and 3.6 that $G_+(f) = 0$ if $f$ belongs to $[\mathrm{mod}(S)]$. Conversely, suppose $f = (f', f''): (X'_F, X''_S, t) \to (Y'_F, Y''_S, u)$ is a map in $\mathrm{mod}(R)$ such that $G_+(f) = 0$. Without loss of generality we can suppose that $X$ and $Y$ are indecomposable and $t \neq 0$, $u \neq 0$. It follows from Theorems 3.3 and 3.6 that $f' = 0$. Since the diagram

$$\begin{array}{ccc} X'_F & \xrightarrow{\bar{t}} & \mathrm{Hom}_S({}_FM_S, X''_S) \\ \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle f''_*} \\ Y'_F & \xrightarrow{\bar{u}} & \mathrm{Hom}_S({}_FM_S, Y''_S) \end{array}$$

is commutative then $f$ has a facturization $X \to (0, Y''_S, 0) \to Y$, as we required. Since obviously $G_+$ is full and dense then it induces the equivalence required in (1). Now in order to prove that $G'_+$ is a representation equivalence it is enough to show that the residue functor $\mathfrak{M}_R^S \to \mathrm{mod}(R)/[\mathrm{mod}(S)]$ reflects isomorphisms. For this purpose suppose $f: X \to Y$ is an isomorphism modulo $[\mathrm{mod}(S)]$ where $X$ and $Y$ are in $\mathfrak{M}_R^S$. Then there is an $R$-homomorphism $g: Y \to X$ such that $1 - gf$ and $1 - fg$ have a factorization through $S$-modules. It follows that $1 - gf \in J(\mathrm{End}(X))$, $1 - fg \in J(\mathrm{End}(Y))$ [19, Lemma 1.1] and hence $gf$ and $fg$ are invertible. Consequently $f$ is an isomorphism, as required.

(3) It follows from (1) that the kernel of the surjection

$$\mathrm{Hom}_R(X, Y) \to \mathrm{Hom}(G_+(X), G_+(Y))$$

induced by $G_+$ is contained in $J^2(X, Y)$. Hence there is an isomorphism

$$\mathrm{Hom}_R(X, Y)/J^2(X, Y) \cong \mathrm{Hom}(G_+(X), G_+(Y))/J^2(G_+(X), G_+(Y))$$

and (3) follows.

Since (2) is a consequence of (1), the first part of (4) is proved in [17, 2.6] and the second one can be easily verified using the definition of the irreducible map, then the theorem is proved.

Now suppose that $R$ is an artinian right peak ring. It follows that $S$ is a right peak ring and $\mathrm{soc}(M_S)$ is projective. Note also that an $R$-module $X = (X'_F, X''_S, t)$ has a projective socle if and only if $X''_S$ has a projective socle and $t$ is injective.

Let $\tilde{\mathbb{K}}_F^R$ be the category $\tilde{\mathbb{K}}^R = \mathrm{Hom}_S(M_S, \mathrm{mod}_{sp}(S))$ together with the embedding functor $|-|: \tilde{\mathbb{K}}^R \to \mathrm{Mod}(F)$. Suppose that $\tilde{\mathbb{K}} = \tilde{\mathbb{K}}^R$ has finitely many pairwise nonisomorphic indecomposable objects and that $\mathrm{Im}\,|-| \subseteq \mathrm{mod}(F)$. If $\mathbf{R}_{\tilde{\mathbb{K}}}$ is either an artinian PI-ring or is schurian with the constant dimension property we define the functor

$$\tilde{G}_+ : \mathrm{mod}_{sp}(R) \to \mathrm{mod}_{sp}((\mathbf{R}_{\tilde{\mathbb{K}}})_*)$$

as the composition of two functors

$$\mathrm{mod}_{sp}(R) \xrightarrow{\Phi'} \mathscr{U}(\tilde{\mathbb{K}}_F) \xrightarrow{G} \mathrm{mod}_{sp}((\mathbf{R}_{\tilde{\mathbb{K}}})_*)$$

where $\tilde{\mathbb{K}}_F = \tilde{\mathbb{K}}_F^R$ and $\Phi'(X) = \Phi(X)$. Finally, we denote by $\tilde{\mathfrak{M}}_R^S$ the full subcategory of $\mathfrak{M}_R^S$ consisting of socle projective modules.

Using the same type of arguments as in the proof of Theorem 4.1 we can prove the following result.

THEOREM 4.2. (1) *The functor $\tilde{G}_+$ is full and induces a representation equivalence*

$$\tilde{G}'_+ : \tilde{\mathfrak{M}}_R^S \to \mathrm{mod}_{sp}^-((\mathbf{R}_{\tilde{\mathbb{K}}})_*)$$

*as well as an equivalence of categories*

$$\operatorname{mod}_{sp}(R)/[\operatorname{mod}_{sp}(S)] \cong \operatorname{mod}_{sp}^{-}((\mathbf{R}_{\widetilde{\mathbb{K}}})_*)$$

*where* $\widetilde{\mathbb{K}}_F = \widetilde{\mathbb{K}}_F^R$ *and* $\operatorname{mod}_{sp}^{-}((\mathbf{R}_{\widetilde{\mathbb{K}}})_*)$ *denotes the full subcategory of* $\operatorname{mod}_{sp}((\mathbf{R}_{\widetilde{\mathbb{K}}})_*)$ *consisting of modules having no injective summand.*

(2) $\# \operatorname{mod}_{sp}(R) = \# \operatorname{mod}_{sp}(S) + \# \operatorname{mod}_{sp}(\mathbf{R}_{\widetilde{\mathbb{K}}}) - 1.$

(3) *If $X$ and $Y$ are indecomposable modules in $\widetilde{\mathfrak{M}}_R^S$ then the map $f\colon X \to Y$ is irreducible in* $\operatorname{mod}_{sp}(R)$ *if and only if $\widetilde{G}_+(f)$ is irreducible.*

(4) *Let $f''\colon X_S'' \to Y_S''$ be an irreducible map in* $\operatorname{mod}_{sp}(S)$. *If* $\operatorname{Hom}_S(M_S, X_S'') = 0$ *then* $(0, f'')\colon (0, X_S'', 0) \to (0, Y_S'', 0)$ *is irreducible in* $\operatorname{mod}_{sp}(R)$. *If* $\operatorname{Hom}_S(M_S, X_S'') \neq 0$ *and* $\operatorname{Hom}_S(M_S, Y_S'') = 0$ *then* $(0, f'')\colon (\operatorname{Hom}_S({}_FM_S, X_S''), X_S'', \overline{\mathrm{id}}) \to (0, Y_S'', 0)$ *is irreducible in* $\operatorname{mod}_{sp}(R)$.

As an immediate consequence of Theorem 4.1 and Remark 2.9 we get

COROLLARY 4.3. *If the ring $R$ in Theorem* 4.1 *is such that the right peak ring $\mathbf{R}_{\mathbb{K}}$ with $\mathbb{K}_F = \mathbb{K}_F^R$ is isomorphic to the incidence ring $FI^*$ of some finite partially ordered set $I^*$ then $G_+$ induces a commutative diagram*

$$\begin{array}{ccc}
\operatorname{mod}(R) & \xrightarrow{\;G_+^I\;} & I - sp \\
 & \searrow^{\pi} & \uparrow{\scriptstyle \cong\, \overline{G}_+} \\
 & & \operatorname{mod}(R)/[\operatorname{mod}(S)]
\end{array}$$

*where $\pi$ is the residue class functor and $\overline{G}_+$ is an equivalence of categories.*

Now we consider an artinian ring $\Lambda$ of the form

$$\Lambda = \begin{pmatrix} S & {}_SN_F \\ 0 & F \end{pmatrix}$$

where $F$ is a division ring and ${}_SN_F$ is an $S-F$-bimodule such that $\dim N_F$ is finite. Then we have a vector space category

$$\mathbb{T}_F^\Lambda = \operatorname{mod}(S) \otimes {}_SN_F$$

and we define a functor

$$H_+ : \operatorname{mod}(\Lambda) \to \operatorname{mod}_{sp}(\mathbf{R}_{\mathbb{T}}), \qquad \mathbb{T}_F = \mathbb{T}_F^\Lambda,$$

as the composition of two functors

$$\operatorname{mod}(\Lambda) \xrightarrow{\Psi} \mathscr{V}(\mathbb{T}_F^\Lambda) \xrightarrow{H} \operatorname{mod}_{sp}(\mathbf{R}_{\mathbb{T}})$$

where $\Psi(X_S', X_F'', t\colon X' \otimes {}_SN_F \to X_F'') = (X_F'', X' \otimes {}_SN_F, t)$.

Using the same type of arguments as in the proof of Theorem 4.1 one can prove the following

THEOREM 4.1′. *The functor $H_+$ is full, dense and there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{mod}(\Lambda) & \xrightarrow{H_+} & \mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{T}}) \\ & \searrow^{\pi} & \uparrow{\scriptstyle \cong}\, \bar{H}_+ \\ & & \mathrm{mod}(\Lambda)/[\mathrm{mod}(S)] \end{array}$$

*where $\pi$ is the residue class functor and $\tilde{H}_+$ is an equivalence of categories. The statement* (2)–(4) *in Theorem* 4.1 *with $R$, $\mathbb{K}$, $G$ and $\Lambda$, $\mathbb{T}$, $H$ interchanged are also true.*

The reduction procedure given by one of the functors $G_+$, $\tilde{G}_+$ and $H_+$ will be called a *triangular reduction.*

DEFINITION. A right peak ring $R$ is called *sp*-representation finite if $\mathrm{mod}_{sp}(R)$ is of finite representation type.

THEOREM 4.4. *Let $R$ be a basic nonsemisimple schurian right peak PI-ring. The following conditions are equivalent:*

(a) *$R$ is sp-representation finite.*

(b) *$R$ is artinian and there is a ring isomorphism*

$$R \cong \begin{pmatrix} G & {}_G M_S \\ 0 & S \end{pmatrix}$$

*where $S$ is sp-representation finite right peak PI-ring, $G$ is a division ring and $\mathbf{R}_{\tilde{\mathbb{K}}}$ with $\tilde{\mathbb{K}}_G = \tilde{\mathbb{K}}_G^R$ is sp-representation finite schurian PI-ring.*

(c) *There are sp-representation finite schurian right peak PI-rings $S$, $R'$ and a proper ring epimorphism $R \to S$ such that*

$$\mathrm{mod}_{sp}(R)/[\mathrm{mod}_{sp}(S)] \cong \mathrm{mod}_{sp}^{-}(R').$$

(d) *There are a sequence $0 = \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}_m = \mathrm{mod}_{sp}(R)$ of full subcategories of $\mathrm{mod}_{sp}(R)$, division PI-rings $G_i$, $G_i'$ and $G_i - G_i'$-bimodules $N^{(i)}$ with $(\dim_{G_i} N^{(i)})(\dim N^{(i)}_{G_i'}) \leqslant 3$ such that*

$$\mathfrak{A}_i/[\mathfrak{A}_{i-1}] \cong \mathrm{mod}_{sp}^{-} \begin{pmatrix} G_i & N^{(i)} \\ 0 & G_i' \end{pmatrix}, \qquad i = 1, \ldots, m.$$

*Proof.* First we will prove by induction on $r = \# \operatorname{mod}_{sp}(R)$ that (a) implies (b) as well as the following condition:

($\mathbf{e}(R)$) End($X$) *is a division* PI-*ring for every indecomposable module X in* $\operatorname{mod}_{sp}(R)$.

Since $r$ is finite then any bimodule ${}_jM_{n+1}$ in the matrix presentation of $R$ in Proposition 2.2 is simple (see [7]). Moreover, since $\bar{c}_{ijn+1}$ is injective for all $i$ and $j$ then $\dim_{F_i}({}_iM_j)$ and $\dim({}_iM_j)_{F_j}$ are finite and therefore $R$ is artinian. Now we conclude from Proposition 2.3 that $R$ has the form required in (b) and $\# \operatorname{mod}_{sp}(S) < r$. By the inductive assumption ($\mathbf{e}(S)$) holds and therefore the category $\widetilde{\mathbb{K}}_G^R$ is schurian. Note also that the dimension of $\operatorname{Hom}_S({}_GM_S, Y_S)$ over $G$ is finite for any indecomposable module $Y_S$ in $\operatorname{mod}_{sp}(S)$ because otherwise one can construct infinitely many pairwise nonisomorphic indecomposable modules in $\operatorname{mod}_{sp}(R)$ of the form $(G^t, \operatorname{Hom}_S({}_GM_S, Y_S), u_t)$, $t = 1, 2, \ldots$. Then we are in the position of Theorem 4.2 and therefore $\# \operatorname{mod}_{sp}(\mathbf{R}_{\widetilde{\mathbb{K}}}) < r$ provided $S$ is not a division ring. Thus ($\mathbf{e}(\mathbf{R}_{\widetilde{\mathbb{K}}})$) holds and hence (b) and ($\mathbf{e}(R)$) follow from Theorem 4.2. If $S$ is a division ring then (b) and $\mathbf{e}(R)$) follow from [7].

The implication (a) ⇒ (d) can be proved similarly. Since (b) ⇒ (c) follows from Theorem 4.2 and each of the conditions (b)–(d) implies (a) the theorem is proved.

COROLLARY 4.5. *Let R be a schurian artinian PI-ring with* $\operatorname{soc}(R_R)$ *projective. Then R is of finite representation type if and only if*

$$R \cong \begin{pmatrix} G & {}_GM_S \\ 0 & S \end{pmatrix}$$

*where G is a division ring, S is of finite representation type,* $\operatorname{soc}(S_S)$ *is projective and the ring* $\mathbf{R}_{\mathbb{K}}$ *with* $\mathbb{K}_G = \mathbb{K}_G^R$ *is schurian and sp-representation finite PI-ring. In particular,* $\operatorname{mod}(R)$ *is schurian when R is of finite representation type.*

*Proof.* Use Theorem 4.1 and apply arguments in the proof of Theorem 4.4.

Note that Theorems 3.11 and 4.4 describe two different algorithms for solving schurian vector space PI-categories of finite representation type and for calculating their indecomposable subspaces. It follows from Theorem 4.4 that if $\mathscr{U}(\mathbb{K}_F)$ is of finite type then functors $H$ and $\tilde{G}_+$ allow us to reduce in a finite number of steps the classification of indecomposables in $\mathscr{U}(\mathbb{K}_F)$ to the well-known classification of indecomposable modules over hereditary PI-rings of the form

$$\begin{pmatrix} G & {}_GN_{F'} \\ 0 & F' \end{pmatrix}$$

where $G$ and $F$ are division PI-rings and $(\dim_G N)(\dim N_{F'}) \leqslant 3$. Note also that Theorems 4.1 and 4.4 describe a constructive method for the classification of indecomposable modules over a large class of triangulated PI-rings of finite representation type including schurian factors of hereditary PI-rings. The method is illustrated by Corollary 4.5.

We finish this section by giving useful criteria for $\mathbb{K}_F^R$ to be a vector space category.

LEMMA 4.6. *Let $R$ be an artinian ring of the form*

$$R = \begin{pmatrix} F & {}_F M_S \\ 0 & S \end{pmatrix}$$

*where $S$ is a ring of finite representation type and let $\mathbb{K}_F^R = \operatorname{Hom}_S({}_F M_S, \operatorname{mod}(S))$. Then the ring $\mathbf{R}_{\mathbb{K}}$ with $\mathbb{K} = \mathbb{K}_F^R$ is left artinian. $\mathbf{R}_{\mathbb{K}}$ is an artinian right peak ring if and only if the dimension of the right vector space $\operatorname{Hom}_S({}_F M_S, X_S)$ over $F$ is finite for any indecomposable module $X_S$ in $\operatorname{mod}(S)$.*

*Proof.* Since $S$ is of finite representation type then End $(X)$ is an artinian ring for any module $X$ in $\operatorname{mod}(S)$ and the left $\operatorname{End}(X)$-module $\operatorname{Hom}_S(Y, X)$ is artinian for any $X$ and $Y$ in $\operatorname{mod}(S)$ (see [21, Sect. 2]). It follows that the ring $E$ is artinian and the left $E$-module ${}_E K$ (in the notation of Section 3) is artinian. Hence $\mathbf{R}_{\mathbb{K}}$ is left artinian and the required equivalence easily follows.

LEMMA 4.7. *Suppose $F$ is a division PI-ring, $S$ is an artinian PI-ring and ${}_F M_S$ is an $F-S$-bimodule such that ${}_F M$ and $M_S$ are both finitely generated. Then $\dim \operatorname{Hom}_S({}_F M_S, X_S)_F$ is finite for any $X_S$ in $\operatorname{mod}(S)$ and $\mathbb{K}_F = \operatorname{Hom}_S({}_F M_S, \operatorname{mod}(S))$ is a vector space category. If, in addition, $S$ is of finite representation type then the right peak ring $\mathbf{R}_{\mathbb{K}}$ is artinian.*

*Proof.* The second part follows from the first one and Lemma 4.6. The first part will be proved by induction on $m$ where $J(S)^m = 0$ and $J(S)^{m-1} \neq 0$.

If $m = 1$ then $S$ is a semisimple PI-ring and the lemma follows from [7, Proposition 1.3]. If $m \geqslant 2$ then we consider the exact sequence of $F-S$-bimodules

$$0 \to MJ(S)^{m-1} \to M \to M/MJ(S)^{m-1} \to 0.$$

Since $S$ is a PI-ring then the minimal injective cogenerator in $\operatorname{Mod}(S)$ is finitely generated [18] and therefore it is enough to prove the lemma for $X_S$

finitely generated and injective. Given such module $X_S$ we consider the induced exact sequence of right $F$-vector spaces

$$0 \to \mathrm{Hom}_S({}_FM/MJ(S)^{m-1}, X) \to \mathrm{Hom}_S({}_FM, X) \to \mathrm{Hom}_S({}_FMJ(S)^{m-1}, X) \to 0.$$

The dimensions of the left- and of the right-hand terms in the sequence are finite by the inductive assumption. Hence also $\dim \mathrm{Hom}_S({}_FM, X)_F$ is finite and the proof is complete.

## 5. A Differentiation

We show in this section how the differentiation algorithm defined for $\ell$-hereditary 1-Gorenstein rings in [4] can be extended to right peak rings. The extended algorithm can be successfully applied in the investigation of arbitrary vector space categories.

Let $R$ be a basic right peak ring. We keep the notation of Section 2.

Definition 5.1 (compare [4]). An indecomposable projective right $R$-module $P_s$ is called smooth if $F_s$ is a division ring, $\dim_{F_s}({}_sM_{n+1}) = \dim({}_sM_{n+1})_{F_{n+1}} = 1$, there is no $j \neq s$, $n+1$ in $\mathbf{I}_R$ with ${}_sM_j \neq 0$, ${}_jM_{n+1} \neq 0$, $c_{sjn+1} \neq 0$, every nonzero map $c_{tsn+1}$ is surjective, the right peak ring

$$T = \mathrm{End}\left(\bigoplus_{j \in \mathbf{I}_R \setminus s^\nabla} P_j\right) \qquad \text{with} \quad s^\nabla = \{j \in \mathbf{I}_R, {}_jM_s \neq 0\}$$

has up to isomorphisms only finitely many indecomposable modules $L_1, \ldots, L_t$ in $\mathrm{mod}_{sp}(T)$ and

$$\Gamma = \mathrm{End}(L_1 \oplus \cdots \oplus L_t)$$

is an artinian ring with the left Morita duality.

Note that $\Gamma$ is a right peak ring. If $T$ has a Morita duality then $\Gamma$ is also a left peak ring.

If $P_s$ is smooth then $R$ has the form

$$R \cong \begin{pmatrix} S & {}_SM_T \\ 0 & T \end{pmatrix}$$

where $S = \mathrm{End}(\bigoplus_{j \in s^\nabla} P_j)$ is a right peak ring and

$${}_SM_T = \mathrm{Hom}_R\left(\bigoplus_{i \in \mathbf{I}_R \setminus s^\nabla} P_i, \bigoplus_{j \in s^\nabla} P_j\right).$$

Following [4] we consider the commutative diagram

$$\begin{array}{ccccc}
(\mathrm{mod}_{sp}(\Gamma^{\mathrm{op}}))^{\mathrm{op}} & \xrightarrow{D} & \mathrm{mod}_{ti}(\Gamma') & \xrightarrow{\nabla^{-1}} & \mathrm{mod}_{sp}(T_s) \\
\uparrow{\scriptstyle \gamma} & & & & \cup \\
\mathrm{mod}_{sp}(T) & & \xrightarrow{\quad\quad\tau\quad\quad} & & \mathscr{I}_{sp}(T_s)
\end{array}$$

where $\Gamma'$ is the ring Morita dual to $\Gamma$, $T_s$ is such that $T_s^{\nabla}=\Gamma'$, $\gamma$ is the Yoneda embedding given by $\gamma(L)=\mathrm{Hom}_T(L, L_1 \oplus \cdots \oplus L_t)$ and $\mathscr{I}_{sp}(T_s)$ is the category of $sp$-injective modules in $\mathrm{mod}_{sp}(T_s)$. It follows from Proposition 2.6 that $\tau$ is an equivalence of categories. The right peak ring

$$R_s' = \begin{pmatrix} \bar{S} & {}_{\bar{S}}N_{T_s} \\ 0 & T_s \end{pmatrix}$$

with $\bar{S}=\mathrm{End}(\oplus_{i\in s^{\nabla}\setminus\{s\}} P_i)$ and ${}_{\bar{S}}N_{T_s}=\tau(\mathrm{Hom}_R(\oplus_{j\in \mathrm{I}_R\setminus s^{\nabla}} P_j,\ \oplus_{i\in s^{\nabla}\setminus\{s\}} P_i))$ is called the *differential* of $R$ with respect to $P_s$.

Following [4] we define the functor

$$\Phi_s : \mathrm{mod}_{sp}(R) \to \mathrm{mod}_{sp}(R_s')$$

as follows. Let ${}_S\tilde{M}_{T_s}=\tau({}_SM_T)$. According to Proposition 2.4 any module $X$ in $\mathrm{mod}_{sp}(R)$ can be identified with a triple $(X_S', X_T'', t)$ where $X_S'$ is and $S$-module, $X_T''$ is a module in $\mathrm{mod}_{sp}(T)$ and

$$t: X_S' \to \mathrm{Hom}_T({}_SM_T, X_T'') \cong \mathrm{Hom}_{T_s}({}_S\tilde{M}_{T_s}, \tau(X_T''))$$

is an $S$-monomorphism. Then the $F_s$-linear map

$$t_s : X_s' \to \mathrm{Hom}_{T_s}({}_s\tilde{M}_{T_s}, \tau(X_T'')) \cong \mathrm{soc}(\tau(X_T''))$$

is injective and similarly as in [4] one can find a submodule $Y_{T_s}$ of $\tau(X_T'')$ such that $\mathrm{soc}(Y_{T_s}) = \mathrm{Im}\, t_s$. A simple analysis shows that there is a factorization

$$\begin{array}{ccc}
X_{\bar{S}}' & \xrightarrow{\bar{t}} & \mathrm{Hom}_{T_s}({}_{\bar{S}}N_{T_s}, \tau(X_T'')) \\
& \searrow{\scriptstyle \tilde{t}} & \cup \\
& & \mathrm{Hom}_{T_s}({}_{\bar{S}}N_{T_s}, Y_{T_s})
\end{array}$$

where $X_{\bar{S}}'$ is the image of $X_S'$ under the restriction functor $\mathrm{mod}_{sp}(S) \to \mathrm{mod}_{sp}(\bar{S})$. We put $\Phi_s(X) = (X_{\bar{S}}', Y_{T_s}, \tilde{t})$. We define $\Phi_s$ on maps in a natural way. If we denote by $\mathrm{mod}_{sp}^s(R)$ the full subcategory of $\mathrm{mod}_{sp}(R)$ consisting of modules having no direct summands $Y$ with $Y_s = 0$ then using the same type of arguments as in [4] one can prove

THEOREM 5.2. *If $R$ is a right peak ring and $P_s$ is a smooth indecomposable projective right $R$-module then $R_s'$ is a right peak ring and $\Phi_s$ induces an equivalence of categories*

$$\mathrm{mod}_{sp}(R)/[\mathrm{mod}_{sp}(T)] \cong \mathrm{mod}_{sp}(R_s')$$

*as well as a representation equivalence*

$$\Phi_s' : \mathrm{mod}_{sp}^s(R) \to \mathrm{mod}_{sp}(R_s').$$

We note that in contrast to the Nazarova–Rojter differentiation and to the differentiation of $\ell$-hereditary 1-Gorenstein rings in [4] our differentiation of right peak rings requires no restriction for the width of $\mathbf{I}_R \setminus s^\nabla$ and for the ring $R$. The only condition we need is that $P_s$ is smooth. Unfortunately, we do not know how to define a differentiation with respect to a non-smooth projective in $R$.

It would be useful to have a differentiation with respect to a pair of points for right peak rings analogous to that one of Zavadskij [26]. It could be successfully used in the study of schurian vector space categories of tame type.

*Remark* 5.3. The assumption that the ring $T$ in the differentiation procedure is *sp*-representation finite is not essential. If $T$ is not *sp*-representation finite then we define $\Gamma$ as the ring (without unity) of the category $\mathrm{mod}_{sp}(T)$ and we easily modify the definition of $R_s'$.

## 6. FINAL REMARKS

**A**. The results of this paper can be successfully applied in the investigation of tame algebras in a way similar to that in [17]. Since in this case vector space categories with infinitely many indecomposable objects appear rather frequently we show for the convenience of the reader how the results of this paper can be extended to this more general case.

Let $\mathbb{K}_F$ be an arbitrary vector space category. Suppose $K_i$, $i \in I$, is a set of representatives of indecomposable objects in $\mathbb{K}$, $F_i = \mathrm{End}(K_i)$ and ${}_iK_j = \mathbb{K}(K_j, K_i)$ for $i, j \in I$. Suppose also that $F_i$ are local rings for all $i$. Denote by $I^*$ the set $I$ enlarged by an element $m$ and put $F_m = F$, ${}_iK_m = {}_{F_i}|K_i|_F$ for $i \in I$. Consider the species $\mathscr{M}_{\mathbb{K}} = (F_i, {}_iK_j)_{i,j \in I^*}$ with the commutativity condition $\mathbf{c} = (c_{ijk})$ defined by the formula in the definition of $\mathbf{R}_{\mathbb{K}}$ in Section 3 (see [20]). Following [20] we denote by $\imath(\mathscr{M}_{\mathbb{K}}, \mathbf{c})$ the category of finitely generated representations of $\mathscr{M}_{\mathbb{K}}$ satisfying the commutativity condition $\mathbf{c}$ and by $\imath_{sp}(\mathscr{M}_{\mathbb{K}}, \mathbf{c})$ the full subcategory of $\imath(\mathscr{M}_{\mathbb{K}}, \mathbf{c})$ consisting of representations with finitely generated projective and essential socle. It is easy that $(X_i, {}_i\varphi_j)_{i,j\in I^*}$ is an object in $\imath_{sp}(\mathscr{M}_{\mathbb{K}}, c)$ if and only if $\dim(X_m)_F$ is finite and the $F_i$-linear map ${}_m\bar{\varphi}_i: X_i \to \mathrm{Hom}_{F_m}({}_iK_m, X_m)$ is injective for all $i \in I$. If $I$ is finite then there is an equivalence of categories $\imath_{sp}(\mathscr{M}_{\mathbb{K}}, \mathbf{c}) \cong \mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$. Similarly as in Section 3 one can define functors

$$H: \mathscr{T}(\mathbb{K}_F) \to \imath_{sp}(\mathscr{M}_{\mathbb{K}}, \mathbf{c}), \qquad G: \mathscr{U}(\mathbb{K}_F) \to \imath_{sp}((\mathscr{M}_{\mathbb{K}})_*, \mathbf{c}_*)$$

and one can extend Theorems 3.3, 3.6 and 3.9 to this more general situation. A counterpart of Theorems 4.2 and 5.2 is also true. The details are left to the reader.

**B.** By a *generalized vector space category* (or a vector space category over several division rings) we mean an additive category $\mathbb{K}$ together with a faithful additive functor $|-|: \mathbb{K} \to \mathrm{mod}(F)$ where $F$ is a finite product of division rings. The categories $\mathscr{U}(\mathbb{K}_F)$ and $\mathscr{T}(\mathbb{K}_F)$ are defined in a natural way. It is an easy observation that most of the results of this paper also remain true for generalized vector space categories if we replace right peak rings by semiperfect rings with essential and projective right socle. Note that a corresponding generalized triangular reduction derived from Theorem 4.2 (with a finite product of division rings instead of $G$) allows us to shorten the reduction procedure. For example, we consider the ring

$$R = \begin{pmatrix} F & 0 & F & F \\ & F & F & F \\ & & G & F \\ & \bigcirc & & F \end{pmatrix},$$

where $G \subset F$ are division rings with $\dim F_G = \dim F = 2$. The generalized triangular reduction reduces in one step the classification of indecomposable modules in $\mathrm{mod}_{sp}(R)$ to the one in

$$\mathrm{mod}_{sp}\begin{pmatrix} G & F \\ 0 & F \end{pmatrix} \qquad \text{and in} \qquad \mathrm{mod}_{sp}\begin{pmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix}$$

over hereditary rings, whereas the usual reduction is longer.

It is easy to see that a differentiation algorithm analogous to that in Section 5 can also be defined for semiperfect rings with essential and projective right socle and can be applied in the investigation of generalized vector space categories. The algorithm was already applied in [14] in the study of socle projective modules over hereditary algebras.

It would be interesting to have a diagrammatic characterization of representation finite generalized vector space categories similar to that announced in [11].

**C.** Our interpretation of $\mathscr{U}(\mathbb{K}_F)$ in terms of $\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}})$ allows us to apply the covering technique [10] to the non-schurian vector space categories of finite type and connect them with the schurian ones. We will discuss the problem elsewhere.

**D.** By a slight generalization of the results in [22, Sect. 2] we get a useful Kleisli category interpretation of the factor space category $\mathscr{F}(\mathbb{K}_F)$.

*Note added in proof.* In this note we want to formulate some useful consequences of the results of the paper which are often used in applications of vector space categories.

**7.1.** It is useful to consider the functor $H^*\colon \mathscr{U}(\mathbb{K}_F) \to (\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}^*}))^{\mathrm{op}}$ which is the composed functor $\mathscr{U}(\mathbb{K}_F) \xrightarrow{(-)^*} \mathscr{F}(\mathbb{K}_F^*)^{\mathrm{op}} \xrightarrow{H^{\mathrm{op}}} (\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}^*}))^{\mathrm{op}}$. The functor $H^*$ is full, dense and $\mathrm{Ker}\, H^* = [(0, K_1, 0),..., (0, K_n, 0)]$.

**7.2.** Suppose that $\mathbb{K}_F$ is of the poset type, i.e., $\mathbb{K}_F$ is schurian and $\dim |K_i|_F = 1$, $\dim_{F_i} |K_i| = 1$ for $i = 1,..., n$. Then $I = \mathbf{I}_{R_{\mathbb{K}}} - \{n+1\}$ is a poset, $\mathbf{R}_{\mathbb{K}} \cong FI^*$, $\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}}) \cong \mathrm{mod}_{sp}((\mathbf{R}_{\mathbb{K}})_*) \cong I\text{-}sp$, and $\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}^*}) \cong I^{\mathrm{op}}\text{-}sp$ (see Remark 2.9). If

$$\acute{H}\colon \mathscr{F}(\mathbb{K}_F) \to I\text{-}sp, \qquad \acute{G}\colon \mathscr{U}(\mathbb{K}_F) \to I\text{-}sp, \qquad \acute{H}^*\colon \mathscr{U}(\mathbb{K}_F) \to I^{\mathrm{op}}\text{-}sp$$

are the compositions of $H$, $G$, $H^*$ with the corresponding equivalences above then [22] yields the following result. Given an indecomposable object $\mathbf{C} = (U_F, X, \varphi)$, $X \cong K_1^{s_1} \oplus \cdots \oplus K_n^{s_n}$, such that $\acute{H}(\mathbf{C}) = (M, M_i)$, $\acute{G}(\mathbf{C}) = (L, L_i)$, $\acute{H}^*(\mathbf{C}) = (N, N_i)$ are nonzero then $M = L = U_F$, $N = U_F^*$, and

$$M_j = \mathrm{Im}\Big( \bigoplus_{i \leqslant j} |K_i^{s_i}|_F \xrightarrow{u_j} |X|_F \xrightarrow{\varphi} U_F \Big), \quad L_j = \mathrm{Ker}\Big( U_F \xrightarrow{\varphi} |X|_F \xrightarrow{\pi_j} \bigoplus_{i \geqslant j} |K_i^{s_i}|_F \Big),$$

$$N_j = \mathrm{Im}\Big( \bigoplus_{i \geqslant j} |K_i^{s_i}|_F^* \xrightarrow{u_j} |X|_F^* \xrightarrow{\varphi^*} U_F^* \Big),$$

$$s_j = \dim\Big( M_j \Big/ \sum_{i<j} M_i \Big)_F = \dim\Big( \bigcap_{i>j} L_i / L_j \Big)_F = \dim\Big( N_j \Big/ \sum_{i>j} N_i \Big)_{F^{\mathrm{op}}},$$

where $u_j$ and $\pi_j$ are the natural embeddings and projections, respectively.

**7.3.** In the situation of Section 4 the functor

$$H_+^* = H^*\Phi\colon \mathrm{mod}(R) \to (\mathrm{mod}_{sp}(\mathbf{R}_{\mathbb{K}^*}))^{\mathrm{op}}$$

is full, dense and $\mathrm{Ker}\, H_+^* = [\mathrm{mod}(S)]$. $H_+^*$ is very useful in studying modules over arbitrary artinian rings $R$.

**7.4.** In view of 7.2 Corollary 4.3 can be completed as follows. Let $\hat{H}_+^*\colon \mathrm{mod}(R) \to I^{\mathrm{op}}\text{-}sp$ be the composition of $H_+^*$ with the corresponding equivalence in 7.2 and let $Z = (U_F, Y_S, \varphi)$ be an indecomposable right module over $R = \left(\begin{smallmatrix} F & M \\ 0 & S \end{smallmatrix}\right)$ with $U_F \neq 0$ and $Y_S \cong L_1^{s_1} \oplus \cdots \oplus L_n^{s_n}$ where $L_1,..., L_n$ are pairwise nonisomorphic indecomposable modules in $\mathrm{mod}(S)$. We put $K_i = \mathrm{Hom}_S({}_FM_S, L_i)$ and $I = \{1,..., n\}$. If $G'_+(Z) = (M', M'_i)$, $\hat{H}_+^*(Z) = (N, N_i)$ then $M' = U_F$, $N = U_F^*$ and

$$s_j = \dim\Big( \bigcap_{i>j} M'_i / M'_j \Big)_F = \dim\Big( N_j \Big/ \sum_{i>j} N_i \Big)_{F^{\mathrm{op}}}.$$

For a discussion of these results and covering type results for vector space categories we refer to the author's note, A module-theoretical approach to vector space categories ("Proceedings, Conference on Abelian Groups and Modules, Udine, 1984," Springer-Verlag, Vienna).

Let us also mention that in the author's note Representations of partially ordered sets, vector space categories and socle projective modules (Paderborn, 1983, pp. 1–141), $F$-moduled categories $\mathbb{K}_F$ over an arbitrary ring $F$ are studied as well as corresponding categories $\mathscr{U}(\mathbb{K}_F)$ and $\mathscr{V}(\mathbb{K}_F)$ are developed. Most of the results of this paper can be generalized to the case of moduled categories. This allows us to get a generalization of Zavadskij's [26] differentiation with respect to a pair of points for right peak rings and to obtain its functorial interpretation (see the author's note, On vector space categories and differentiations of right peak rings, *in* "Proceedings, International Conference on Representations of Algebras IV, Ottawa, Carleton University, August 1984").

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