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The average CRI-length of a controlled ALOHA collision resolution algorithm*

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Abstract

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We investigate the expected CRI-length (collision resolution interval) of a hybrid collision resolution algorithm based on slotted ALOHA with controlled retransmission probability, e.g., we study the average number of slots necessary for the resolution of an initial collision of multiplicity n. An algorithm similar to binary exponential backoff for adjusting the retransmission probability w.r.t. the channel load is used prior to the application of the ALOHA resolution algorithm, thus operating it in the region of nonexponential behaviour. Mellin-transform techniques are used for the derivation of an asymptotic expression for the desired quantity, which turns out to be $O(n \log n)$.

1. Introduction

This paper deals with the analysis of a *collision resolution algorithm* (CRA) for networks based on random multiple access channels. With this type of networks, all stations (i.e., transmitting/receiving units) share a single communication channel. Data are sent in *packets*, without any centralized channel arbitration mechanism. Thus, a distributed algorithm for resolving conflicts arising from simultaneous transmission attempts of multiple stations is needed.

The whole subject came up with the development of the ALOHA system at the University of Hawaii in the late 1970s. Since this time, a number of varieties of the original ALOHA algorithm and, most important, a family of *tree algorithms* have been proposed, which offer better characteristics, e.g., average packet throughput; cf. [4] for an overview.

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An example is the well-known *slotted* ALOHA algorithm, which works as follows: If a station has been involved in a collision, it transmits its packet in each subsequent slot with a fixed probability p until a successful transmission of the packet occurs. Packets are assumed to have fixed size and fit into exactly one slot. Note that as a collision causes the destruction of all packets sent, it may be detected by all stations via certain checksumming methods.

It is well-known that ALOHA-based algorithms lack a very important feature: *stability*. A network built on ALOHA possibly reaches a state where the retransmission activity drops the useful throughput to zero. In order to circumvent such instable behaviour, a method for controlling the retransmission activity seems to be reasonable. A suitable idea for slotted ALOHA is to adjust the retransmission probability according to the number of conflicting stations.

The collision resolution algorithm proposed in this paper is based on this idea. It falls in the category of *hybrid algorithms*. The resolution of an initial collision of n stations is performed in two phases:

(1) *Estimation.* First, an adaptive *estimation strategy* is employed in order to determine an estimated number of colliders n', which is close to n *almost surely*.

(2) *Collision resolution.* If n is estimated, the ordinary slotted ALOHA algorithm with retransmission probability $p=1/n'$ is used to resolve the collision.

We shall investigate the (average) CRI-length L_n of this collision resolution algorithm. The *CRI-length* (*collision resolution interval*) equals the number of slots necessary for resolving an initial collision of n transmitters. Note that new packets, i.e., those generated during a resolution process, are assumed to get delayed until its completion. In this case, the CRI-length is not influenced by the underlying packet-generating process. It permits a model-independent estimation of the performance of a CRA. We should mention that this parameter is well known from the throughput analysis of tree algorithms; see [6] for a survey. The analysis of the CRI-length of the ordinary slotted ALOHA algorithm, which provides some necessary results for this paper, is contained in [8].

The paper is organized as follows: The introduction in Section 1 is followed by a detailed description of our controlled ALOHA collision resolution algorithm and a discussion of our final results in Section 2. Sections 3 and 4 are devoted to some preliminaries and methodological notes, especially concerning the Mellin-transform techniques. The analysis of the estimation algorithm is contained in Section 5, and the treatment of the resolution algorithm may be found in Section 6. Several technical lemmas are given in Section 7, and some conclusions contained in Section 8 complete the paper.

2. The controlled ALOHA algorithm and final results

Our hybrid controlled ALOHA collision resolution algorithm uses an estimation algorithm already introduced in [5] for determining the multiplicity of the initial collision n. It may be viewed as a slight modification of the *binary exponential backoff* policy suggested by Metcalfe and Boggs for the *Ethernet*; see [7] for a survey. It works as follows: Starting from an idle channel, each station may send a newly generated packet immediately in the next slot, with a fixed initial (re)transmission probability. At the end of each subsequent collision slot, the retransmission probability is decremented by multiplying it with a fixed *backoff*, until the first noncollision slot occurs.

To be more specific, any transmitter involved in the initial collision performs the following steps in order to execute the (distributed) algorithm:

$k := 0$;
repeat
  $k := k + 1$;
  Transmit with probability a^{-k}
until "no collision"

Note that the (truly distributed!) algorithm performs a repeated test of the hypothesis $n > a^k$ since the expected number of active transmitters in the kth iteration is na^{-k}, and any collision supports the hypothesis. That is, the algorithm supplies a power of a fixed backoff $1 < a < \infty$, which is close to n with high probability, i.e., almost surely.

Transmission attempts for new packets, i.e., packets generated during the estimation and resolution of a (previous) initial collision, are considered to get delayed until the whole process is complete. In fact, they probably will cause the next initial collision. Due to this assumption, all stations performing the estimation algorithm coincide in obtaining the same retransmission probability $p = ca^{-k}$ for the subsequent slotted ALOHA algorithm, where c is an arbitrary positive constant and a^{-k} is the inverse of the estimation of n supplied by the estimation algorithm. This adjusted probability is used for the retransmission of the packet in each subsequent slot until a successful transmission occurs.

The following theorems state our final results. The first one deals with the estimation algorithm only. It turns out that the expected number of iterations of the **repeat**-loop E_n, which equals the number of slots wasted for the estimation of n, is $\mathrm{O}(\log n)$.

Theorem 2.1 (Estimation algorithm). *The average number of slots E_n which are wasted for the estimation of the multiplicity of an initial collision of n stations by our estimation algorithm with backoff $a > 1$, is given by*

$$E_n = \frac{\lambda^*(0)}{\log a} \log_a n + \frac{\Lambda(\log_a n)}{\log a} \log_a n + \mathrm{O}(1),$$

where

$$\Lambda(u) = \sum_{k \neq 0} \lambda^*(\chi_k) \mathrm{e}^{-2k\pi \mathrm{i} u} \quad \textit{with } \chi_k = \frac{2k\pi \mathrm{i}}{\log a},$$

$$\lambda^*(s) = \int_0^\infty (1+x)\mathrm{e}^{-x} \prod_{j \geq 1} (1 - \mathrm{e}^{-xa^j}(1 + xa^j)) x^{s-1} \, \mathrm{d}x.$$

On the other hand, the expected number L_n of slots occupied for the subsequent ALOHA resolution is shown to be $O(n \log n)$. Thus, the overhead resulting from the estimation algorithm is negligible; the asymptotic order of the CRI-length W_n of the whole controlled ALOHA algorithm is $O(n \log n)$, too.

Theorem 2.2 (Controlled ALOHA). *The average CRI-length W_n for the resolution of an initial collision of n stations by our controlled* ALOHA *algorithm with backoff $a>1$ and initial transmission probability $c \cdot a$, $0<c<1$, is given by*

$$W_n = \frac{\psi^*(0) + a\psi^*(-1)}{c} n \log_a n + \frac{\Psi_0(\log_a n) + a\Psi_{-1}(\log_a n)}{c} n \log_a n$$

$$+ \frac{1}{c \log a}(\omega^*(0) + a\omega^*(-1) + (\gamma - 1)(\psi^*(0) + a\psi^*(-1)))n$$

$$+ \frac{1}{c \log a}(\Omega_0(\log_a n) + a\Omega_{-1}(\log_a n) + (\gamma - 1)(\Psi_0(\log_a n)$$

$$+ a\Psi_{-1}(\log_a n)))n + O(n^\varepsilon)$$

with γ denoting Euler's constant, $\varepsilon > 0$ arbitrary, and the abbreviations

$$\Psi_t(u) = \sum_{k \neq 0} \psi^*(t + \chi_k) e^{-2k\pi i u} \quad \textit{with } \chi_k = \frac{2k\pi i}{\log a},$$

$$\psi^*(s) = \int_0^\infty e^{-x/a} \prod_{j \geq 0} (1 - e^{-xa^j}(1 + xa^j)) x^{s-1} \, dx,$$

$$\Omega_t(u) = \sum_{k \neq 0} \omega^*(t + \chi_k) e^{-2k\pi i u},$$

$$\omega^*(s) = \int_0^\infty e^{-x/a} h(cx/a) \prod_{j \geq 0} (1 - e^{-xa^j}(1 + xa^j)) x^{s-1} \, dx,$$

$$h(x) = \sum_{j \geq 1} \frac{x^j}{j j!}.$$

This result, whose type is well known from the analysis of various computer science problems (tree-based data structures, for example), requires additional remarks.

(1) The CRI-length of any of the various "tree" algorithms is shown to be $O(n)$ (see [6] for details), which establishes the worse performance of ALOHA-type algorithms once more: Our result shows that, despite the improvement w.r.t. the ordinary slotted ALOHA algorithm, our controlled algorithm is unstable, too.

(2) The restriction $c<1$ is necessary for keeping the whole formula valid. The minimal asymptotic CRI-length is obtained by the choice $c=1$, but the precision of the expansion is reduced to

$$W_n = \frac{\psi^*(0) + a\psi^*(-1)}{c} n \log_a n + \frac{\Psi_0(\log_a n) + a\Psi_{-1}(\log_a n)}{c} n \log_a n + O(n)$$

in this case. For $c>1$, one obtains exponentially increasing W_n as $n \to \infty$; we shall not treat this case in detail.

These results are a consequence of operating the ALOHA resolution process at a limit. A little increase of the parameter c causes the whole system coming into a region of exponential behaviour, hence we may conclude, that the system is very sensitive to statistical fluctuations of the estimation process, too. Details about the behaviour may be found by the analysis of higher moments of the CRI-length.

(3) The functions $\Lambda(u)$, $\Psi_t(u)$, and $\Omega_t(u)$ are periodic, with period 1, mean 0 and small absolute values.

(4) The function $h(x)$ is related to an exponential integral by $\mathrm{Ei}(x)=\gamma+\log x+h(x)$; see equation (2) and [1] for additional informations.

(5) The order of the function represented by the infinite product in the expressions of $\psi^*(s)$ and $\omega^*(s)$ is

$$\prod_{j\geqslant 0}(1-\mathrm{e}^{-xa^j}(1+xa^j))=\begin{cases}\mathrm{O}(x^d) & \text{for all } d\geqslant 0, \text{ as } x\to 0,\\ \mathrm{O}(1), & \text{as } x\to\infty.\end{cases}$$

(6) Discarding the low-valued functions $\Psi_t(\log_a n)$ and $\Omega_t(\log_a n)$ in the expression for L_n, we obtain $L_n \approx C_1 n \log n + C_2 n$. Computing this constant using the double-precision integration routine D01AMF from the Fortran NAG library on a CDC CYBER 860 computer yields the tables 1 and 2, which agree well with the results of a computer simulation, too.

Table 1
Constant C_1

	c									
a	1.00	0.95	0.90	0.85	0.80	0.75	0.70	0.65	0.60	0.55
3	1.35	1.42	1.50	1.59	1.68	1.80	1.92	2.07	2.25	2.45
2	0.91	0.96	1.02	1.08	1.14	1.22	1.31	1.41	1.52	1.66
1.5	0.65	0.68	0.72	0.76	0.81	0.87	0.93	1.00	1.08	1.18
1.1	0.35	0.37	0.39	0.41	0.44	0.46	0.50	0.54	0.58	0.63
1.01	0.20	0.21	0.22	0.23	0.25	0.26	0.28	0.30	0.33	0.36

Table 2
Constant C_2

	c									
a	1.00	0.95	0.90	0.85	0.80	0.75	0.70	0.65	0.60	0.55
3	—	1.62	1.54	1.48	1.42	1.38	1.33	1.30	1.27	1.24
2	—	1.86	1.74	1.65	1.57	1.50	1.45	1.40	1.35	1.31
1.5	—	2.24	2.05	1.91	1.79	1.69	1.61	1.54	1.47	1.41
1.1	—	4.16	3.54	3.09	2.78	2.49	2.27	2.08	1.93	1.80
1.01	—	15.3	11.2	8.6	6.8	5.6	4.7	4.0	3.3	2.9

3. Preliminaries and methodological notes

Since we are interested in the number of slots required for the resolution of an initial collision with multiplicity n, which is the sum of the contributions from the estimation algorithm and the ALOHA algorithm, we use further conditioning to obtain

$$\begin{aligned} W_n &= E[\text{CRI-length} \mid n] \\ &= E[E[k+\text{CRI-length ALOHA} \mid \text{estimation with } k \text{ iterations}, n] \mid n] \\ &= E[\text{iterations of estimation} \mid n] + E[E[\text{CRI-length ALOHA} \mid \text{estimation with } k \text{ iterations}, n] \mid n] \\ &= E_n + L_n. \end{aligned}$$

The conditional probability $p_{n,k}$, that the estimation algorithm terminates after the kth iteration (if at all it reached it) is the probability of having one or zero active transmitters in the kth step, which yields

$$p_{n,k} = (1-a^{-k})^{n-1} na^{-k} + (1-a^{-k})^n.$$

Thus, the probability $q_{n,k}$ of reaching at least the iteration k is

$$q_{n,k} = \begin{cases} (1-p_{n,1})(1-p_{n,2})\cdots(1-p_{n,k-1}) & \text{for } k>1, \\ 1 & \text{for } k=1. \end{cases}$$

Obviously, the unconditional probability of termination after iteration k is computed by $q_{n,k}p_{n,k} = q_{n,k} - q_{n,k+1}$, and the expected number of iterations of the estimation algorithm is

$$E_n = \sum_{k \geqslant 1} k q_{n,k} p_{n,k} = \sum_{k \geqslant 1} q_{n,k}.$$

Denoting by $L_{n,p}$ the expected CRI-length of the ordinary slotted ALOHA algorithm with retransmission probability p, we obtain for the expected number of slots used for the ALOHA resolution process

$$L_n = n \sum_{k \geqslant 1} q_{n,k} a^{-k} (1-a^{-k})^{n-1} L_{n-1, ca^{-k}} + \sum_{k \geqslant 1} q_{n,k} (1-a^{-k})^n L_{n, ca^{-k}}.$$

This becomes clear by mentioning the fact that termination after the kth iteration implies a retransmission probability $p = ca^{-k}$, by convention. Section 5 deals with the computation of E_n, whereas Section 6 contains the derivation of L_n. Both problems are solved with a similar approach introduced in [5], based on replacing $q_{n,k+1}$ by a function $\phi(na^{-k})$ defined by

$$\phi(x) = \prod_{j \geqslant 0} (1 - \mathrm{e}^{-xa^j}(1+xa^j)).$$

This follows from using the exponential approximation $(1-p)^n \approx \mathrm{e}^{-np}$ for large n and small p in the expressions of $q_{n,k}$ and $p_{n,k}$, respectively. The extension to an infinite product is only for simplification and has little influence, since the supplementary factors are very close to 1.

Rewriting our expressions, we obtain the so-called harmonic sums, e.g., sums looking like $h(n)=\sum_k a_k f(nb_k)$, which are treatable by Mellin-transform techniques for obtaining asymptotic expansions. This integral transformation allows a "separation" of a harmonic sum and yields its asymptotic expansion by means of singularity analysis of the transform; see Section 4 for a summary.

The remaining problem is the estimation of the error terms resulting from the various replacement operations. These computations are tedious but straightforward, and are given here for the sake of completeness.

This is the right place for collecting the results from the analysis of the ordinary slotted ALOHA algorithm; see [8] for details. The expected CRI-length of the slotted ALOHA algorithm with arbitrary retransmission probability p has the asymptotic expression:

$$L_{n,p}=\begin{cases} \dfrac{H_n-1}{p}-H_n+\dfrac{h(np)}{p}+\mathrm{O}(\mathrm{e}^{np}) & \text{for } p\leqslant n^{-0.51}/2\mathrm{e}, \\ \dfrac{1}{np^2(1-p)^{n-1}}\left(1+\mathrm{O}\left(\dfrac{\log^4 n}{np}\right)\right) & \text{for } p\geqslant n^{-0.99}, \end{cases} \tag{1}$$

where H_n denotes the harmonic numbers and

$$h(x)=\sum_{j\geqslant 1}\frac{x^j}{jj!}=\frac{\mathrm{e}^x-x-1}{x}\,\theta(x) \quad \text{and } 1\leqslant\theta(x)\leqslant 2. \tag{2}$$

Moreover, for all values of p, we have the exact value

$$L_{n,p}=1+\sum_{j=2}^{n}\frac{1}{pj(1-p)^{j-1}}. \tag{3}$$

Equation (1) is the result of the application of simple asymptotic methods to a suitable sum involving binomial coefficients, which is related to (3). On the other hand, (2) is a consequence of the application of a generating-function method based on contour integration; the interested reader is referred to [8].

4. Mellin-transform techniques

This section contains a summary of theorems concerning the Mellin integral transformation, which is some kind of Laplace transformtion; see [2] for a very complete theory. Furthermore, it is a powerful tool in the asymptotic analysis, and applicable to a wide variety of problems.

We restrict ourselves to a short summary of theorems (without any proof), and refer to [3] for application-oriented details.

Definition 4.1 (*Mellin Transform*). The Mellin transform of a continuous, real-valued function $f(x)$ is the complex-valued function

$$f^*(s) = \mathcal{M}[f(x); s] = \int_0^\infty f(x) x^{s-1} \, dx,$$

if the integral is absolutely convergent in the region $a < \Re(s) < b$. This region is called the fundamental strip of $f^*(s)$.

Theorem 4.2 (Existence and analyticity). *If there are two complex numbers a, b, with $\Re(-a) < \Re(-b)$ and the property*

$$f(x) = \begin{cases} O(x^a) & \text{for } x \to 0, \\ O(x^b) & \text{for } x \to \infty, \end{cases}$$

e.g., when the order of $f(x)$ near zero is larger than its order near infinity, then the transform $f^*(s)$ exists in the fundamental strip $\langle -a, -b \rangle$, and is analytic within the whole region.

Lemma 4.3 (Transform). *If the transform $f^*(s)$ of the function $f(x)$ exists in the fundamental strip $\langle a, b \rangle$, we obtain, for a real $c > 0$,*

$$\mathcal{M}[f(cx); s] = c^{-s} f^*(s) \quad \text{within the fundamental strip } \langle a, b \rangle.$$

Theorem 4.4 (Asymptotic expansion). *Given $f^*(s)$ in the fundamental strip $\langle a, b \rangle$, satisfying certain smallness conditions towards $i\infty$ for $b \leqslant \Re(s) \leqslant M$ (to the right of the fundamental strip), a pole of the transform*

$$f^*(s) \sim \sum_{n=0}^{N} \frac{d_{n,k}}{(s - b_k)^{n+1}} \quad \text{for } b \leqslant \Re(b_k) < M \text{ and } s \to b_k$$

translates into a term of the asymptotic expansion for $x \to \infty$ by

$$\sum_{n=0}^{N} \frac{-d_{n,k}}{n!} (-\log x)^n x^{-b_k} \quad \text{for } x \to \infty.$$

Moreover, the complete expansion of $f(x)$ yields

$$f(x) = \sum_k \text{term resulting from } b_k + O(x^{-M}) \quad \text{for } x \to \infty.$$

We should mention that the requested smallness conditions are proved to be satisfied within the fundamental strip for each transform. The last theorem is a consequence of the classical inversion theorem

$$f(x) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} f^*(s) x^{-s} \, ds,$$

where c lies in the fundamental strip of $f^*(s)$. The idea behind Theorem 4.4 is to extend the contour by a large rectangular one in the right halfplane, and to take the residues of the newly enclosed singularities into account. The smallness conditions ensure vanishing contributions resulting from the horizontal segments when expanding the contour to $\pm i\infty$.

5 Estimation algorithm

With the definitions and principles defined in Section 3, we investigate the expected number of iterations of the estimation algorithm E_n by treating the harmonic sum

$$e_n = \sum_{k \geq 1} \phi(na^{1-k})$$

instead, and estimating the error term afterwards. Using Lemma 4.3, the Mellin transform of e_n yields

$$e^*(s) = \sum_{k \geq 1} a^{(k-1)s} \phi^*(s) = \frac{\phi^*(s)}{1-a^s}.$$

Due to the easily established order of the function

$$\phi(x) = \begin{cases} O(x^d) & \text{for all } d \geq 0, \text{ as } x \to 0 \\ O(1) & \text{as } x \to \infty, \end{cases}$$

Theorem 4.2 yields the fundamental strip $\langle -\infty, 0 \rangle$. To find the major term of the asymptotic expansion, we need a meromorphic continuation of $\phi^*(s)$ to the right of the fundamental strip. Since we are interested in terms of order larger than O(1), we have to take into account singularities of the function with $\Re(s)=0$ only. The continuation is easily provided by introducing the function.

$$\lambda(x) = \phi(ax) - \phi(x) = (1+x)e^{-x} \prod_{j \geq 1} (1 - e^{-xa^j}(1 + xa^j)),$$

which is of order

$$\lambda(x) = \begin{cases} O(x^d) & \text{for all } d \geq 0, \text{ as } x \to 0 \\ O(x^{-d}) & \text{for all } d \geq 0, \text{ as } x \to \infty. \end{cases}$$

Thus, $\lambda^*(s)$ has the fundamental strip $\langle -\infty, +\infty \rangle$, e.g., the transform is an entire function. Applying Lemma 4.3 to the definition of $\lambda(x)$, we obtain

$$\lambda^*(s) = a^{-s}\phi^*(s) - \phi^*(s),$$

which yields the Laurent expansion

$$\phi^*(s) = -\frac{\lambda^*(s)}{1-a^{-s}} = -\frac{\lambda^*(\chi_k)}{\log a} \frac{1}{s-\chi_k} + O(1) \quad \text{with } \chi_k = \frac{2k\pi i}{\log a}.$$

The singularities are poles of first order, resulting from zeros of the denominator. Computing the residues is straightforward and, hence, suppressed. We finally obtain

$$e^*(s)=\frac{\phi^*(s)}{1-a^s}=\frac{\lambda^*(\chi_k)}{\log^2 a}\frac{1}{(s-\chi_k)^2}+\mathrm{O}\left(\frac{1}{s-\chi_k}\right).$$

Because of the smallness of $\lambda^*(s)$ and the discreteness of the poles resulting from the denominator, there are no problems in establishing the smallness conditions necessary for Theorem 4.4.

The main term comes from the double pole, which contributes a term of order $\log n$ to the expansion; the remainder is O(1), e.g.,

$$e_n=\sum_k \frac{\lambda^*(\chi_k)}{\log^2 a}\log n n^{-\chi_k}=\frac{\lambda^*(0)}{\log a}\log_a n+\frac{\Lambda(\log_a n)}{\log a}\log_a n+\mathrm{O}(1),$$

with the abbreviation

$$\Lambda(u)=\sum_{k\neq 0}\lambda^*(\chi_k)\mathrm{e}^{-2k\pi iu}.$$

The last problem is the estimation of the error term resulting from investigating e_n instead of E_n. Using the bounds concerning $q_{n,k}$ and $\phi(na^{1-k})$ from Lemmas 7.1 and 7.2 in the Appendix, we obtain for $v=\lceil \log_a n\rceil$

$$\begin{aligned}E_n-e_n&=\sum_{k\geq 1}(q_{n,k}-\phi(na^{1-k}))\\&=\sum_{k\leq v+\sqrt{v}}\mathrm{O}\left(\frac{\log n}{n}\right)+\sum_{k>v+\sqrt{v}}(q_{n,k+1}-\phi(na^{-k}))\\&<\mathrm{O}\left(\frac{\log^2 n}{n}\right)+2\sum_{k>\sqrt{v}}a^{-k(k+1)}<\mathrm{O}\left(\frac{\log^2 n}{n}\right)+2\sum_{k>v}a^{-k}=\mathrm{O}\left(\frac{\log^2 n}{n}\right),\end{aligned}$$

which finally proves Theorem 2.1.

6. Resolution algorithm

This section deals with an asymptotic expansion of the expected number L_n of slots necessary for resolving an initial collision of multiplicity n. From Section 3, we have the expression

$$L_n=n\sum_{k\geq 1}q_{n,k}a^{-k}(1-a^{-k})^{n-1}L_{n-1,ca^{-k}}+\sum_{k\geq 1}q_{n,k}(1-a^{-k})^n L_{n,ca^{-k}}.$$

This quantity depends on the parameter c, and easy considerations show that the restriction $c\leq 1$ is necessary for obtaining a nonexponential behaviour of L_n. Using the asymptotic expansion of $L_{n,p}$ for large p from (1) yields

$$G_{n,c,a^{-k}}=(1-a^{-k})^{n-1}L_{n,ca^{-k}}\sim\frac{a^{2k}}{nc^2}\left(\frac{1-a^{-k}}{1-ca^{-k}}\right)^{n-1}$$

for small values of k. Because

$$\frac{1-a^{-k}}{1-ca^{-k}}=1+(c-1)\frac{a^{-k}}{1-ca^{-k}}=1+d_k \quad \text{with } d_k>0 \text{ for } c>1, \tag{4}$$

we obtain exponentially increasing contributions to the sum for L_n. Thus, we have to restrict the choices of the free parameter c to the range $c \leqslant 1$.

Replacing $L_{n-1,p}$ by $L_{n,p}$ according to (3), we obtain

$$\begin{aligned} L_n &= n \sum_{k\geqslant 1} q_{n,k} a^{-k}(1-a^{-k})^{n-1} L_{n,ca^{-k}} + \sum_{k\geqslant 1} q_{n,k}(1-a^{-k})^n L_{n,ca^{-k}} \\ &\quad - \sum_{k\geqslant 1} q_{n,k}\left(\frac{1-a^{-k}}{1-ca^{-k}}\right)^{n-1} \\ &= (n-1) \sum_{k\geqslant 1} q_{n,k} a^{-k}(1-a^{-k})^{n-1} L_{n,ca^{-k}} \\ &\quad + \sum_{k\geqslant 1} q_{n,k}(1-a^{-k})^{n-1} L_{n,ca^{-k}} + \mathrm{O}(\log n) \end{aligned} \tag{5}$$

with $\mathrm{O}(\log n)$ coming from the result $\sum_{k\geqslant 1} q_{n,k} = \mathrm{O}(\log n)$ in Section 5. Similar to the previous derivations, we replace $q_{n,k}$ with $\phi(na^{1-k})$ and, by using the exponential approximation and the expansion of $L_{n,p}$ for small p from (1), $G_{n,c,a^{-k}}$ with

$$g_{n,c,a^{-k}} = \mathrm{e}^{-na^{-k}}((H_n-1)c^{-1}a^k - H_n + c^{-1}a^k h(nca^{-k})), \tag{6}$$

and consider the resulting expression l_n instead. The error term is computed later, a significant contribution results (in the case $c=1$) from discarding the remainder of $L_{n,ca^{-k}}$ only. We obtain

$$\begin{aligned} l_n &= (n-1)(H_n-1)c^{-1}a_0(n) - (n-1)H_n a_1(n) + (n-1)c^{-1}b_0(n) \\ &\quad + (H_n-1)c^{-1}a_{-1}(n) - H_n a_0(n) + c^{-1}b_{-1}(n), \end{aligned}$$

with the abbreviations

$$\begin{aligned} a_t(n) &= \sum_{k\geqslant 1} \phi(na^{1-k})\mathrm{e}^{-na^{-k}} a^{-tk}, \\ b_t(n) &= \sum_{k\geqslant 1} \phi(na^{1-k})\mathrm{e}^{-na^{-k}} h(nca^{-k}) a^{-tk}. \end{aligned}$$

Both sums are harmonic; hence, the Mellin-transform techniques are applicable for obtaining asymptotic expansions. We start with rewriting

$$a_t(n) = \sum_{k\geqslant 1} \psi(na^{1-k}) a^{-tk} \quad \text{with } \psi(x) = \phi(x)\mathrm{e}^{-x/a}.$$

Due to the easily established bounds

$$\psi(x) = \begin{cases} \mathrm{O}(x^d) & \text{for all } d\geqslant 0, \text{ as } x\to 0, \\ \mathrm{O}(x^{-d}) & \text{for all } d\geqslant 0, \text{ as } x\to\infty, \end{cases}$$

and by Theorem 4.2 the transform $\psi^*(s)$ is an entire function. We obtain

$$a_t^*(s) = \psi^*(s) \sum_{k \geq 1} a^{(k-1)s-tk} = a^{-t} \frac{\psi^*(s)}{1-a^{s-t}}$$

$$= -\frac{a^{-t}\psi^*(t+\chi_k)}{\log a} \frac{1}{s-(t+\chi_k)} + \mathrm{O}(1) \quad \text{with } \chi_k = \frac{2k\pi \mathrm{i}}{\log a}.$$

We suppress the derivation of the Laurent expansion around the simple-pole singularities $s = t + \chi_k$, which is straightforward. Applying Theorem 4.4 yields, for an arbitrary but fixed M,

$$a_t(n) = \sum_k \frac{a^{-t}\psi^*(t+\chi_k)}{\log a} n^{-(t+\chi_k)} + \mathrm{O}(n^{-M})$$

$$= \frac{a^{-t}\psi^*(t)}{\log a} n^{-t} + \frac{a^{-t}}{\log a} \Psi_t(\log_a n) n^{-t} + \mathrm{O}(n^{-M}),$$

with the abbreviation

$$\Psi_t(u) = \sum_{k \neq 0} \psi^*(t+\chi_k) \mathrm{e}^{-2k\pi \mathrm{i} u}.$$

The same procedure is used for treating the harmonic sum for $b_t(n)$, e.g.,

$$b_t(n) = \sum_{k \geq 1} \omega(na^{1-k}) a^{-tk} \quad \text{with } \omega(x) = \phi(x) \mathrm{e}^{-x/a} h(xc/a).$$

Using (2), we obtain the bounds

$$\omega(x) = \begin{cases} \mathrm{O}(x^d) & \text{for all } d \geq 0, \text{ as } x \to 0, \\ \mathrm{O}(x^{-1}) & \text{for } c = 1, \text{ as } x \to \infty, \\ \mathrm{O}(x^{-d}) & \text{for all } d \geq 0 \text{ and } c < 1, \text{ as } x \to \infty. \end{cases}$$

Thus, the transform $\psi^*(s)$ is analytic for $c \leq 1$ within the plane to the left of $\Re(s) = 1$, and even entire for $c < 1$. In addition to the remarks at the beginning of the chapter we should note the case $c > 1$, which is not treatable by this technique, because the fundamental strip of the transform becomes empty. Similar to the previous derivations, we obtain

$$b_t^*(s) = \omega^*(s) \sum_{k \geq 1} a^{(k-1)s-tk} = a^{-t} \frac{\omega^*(s)}{1-a^{s-t}}$$

$$= -\frac{a^{-t}\omega^*(t+\chi_k)}{\log a} \frac{1}{s-(t+\chi_k)} + \mathrm{O}(1) \quad \text{with } \chi_k = \frac{2k\pi \mathrm{i}}{\log a}.$$

Applying Theorem 4.4 yields for a fixed $\varepsilon>0$

$$b_t(n)=\sum_k \frac{a^{-t}\omega^*(t+\chi_k)}{\log a} n^{-(t+\chi_k)}+\mathrm{O}(n^{\varepsilon-1})$$

$$=\frac{a^{-t}\omega^*(t)}{\log a} n^{-t}+\frac{a^{-t}}{\log a}\Omega_t(\log_a n)n^{-t}+\mathrm{O}(n^{\varepsilon-1})$$

with the abbreviation

$$\Omega_t(u)=\sum_{k\neq 0}\omega^*(t+\chi_k)\mathrm{e}^{-2k\pi i u}.$$

The order of the remainder is determined by the restriction for analyticity of the transform $\omega^*(s)$ in the case $c=1$. If $c<1$, we obtain a remainder $\mathrm{O}(n^{-M})$ for an arbitrary but fixed M.

Substituting our results in the expression for l_n and discarding the terms of smaller order yields

$$l_n=n\frac{(H_n-1)}{c\log a}(\psi^*(0)+a\psi^*(-1))+n\frac{(H_n-1)}{c\log a}(\Psi_0(\log_a n)+a\Psi_{-1}(\log_a n))$$

$$+\frac{n}{c\log a}(\omega^*(0)+a\omega^*(-1))+\frac{n}{c\log a}(\Omega_0(\log_a n)+a\Omega_{-1}(\log_a n))+\mathrm{O}(n^{\varepsilon})$$

$$=\frac{\psi^*(0)+a\psi^*(-1)}{c} n\log_a n+\frac{\Psi_0(\log_a n)+a\Psi_{-1}(\log_a n)}{c} n\log_a n$$

$$+\frac{1}{c\log a}(\omega^*(0)+a\omega^*(-1)+(\gamma-1)(\psi^*(0)+a\psi^*(-1)))n$$

$$+\frac{1}{c\log a}(\Omega_0(\log_a n)+a\Omega_{-1}(\log_a n)+(\gamma-1)(\Psi_0(\log_a n)$$

$$+a\Psi_{-1}(\log_a n)))n+\mathrm{O}(n^{\varepsilon}),$$

where we used the well-known expansion $H_n=\log n+\gamma+\mathrm{O}(1/n)$ with γ denoting Euler's constant.

Finally, we estimate the errors resulting from the replacement operations. We shall show, by some tedious computations, that $L_n-l_n=\mathrm{O}(n^{\varepsilon})$, with a fixed $\varepsilon>0$, thus covered by the remainder already established (if $c<1$). If $c=1$, we find an $\mathrm{O}(n)$, error term mainly by neglecting the $\mathrm{O}(\mathrm{e}^{np})$ term in the expansion of $L_{n,p}$.

For the sake of simplification, we use the abbreviations q, ϕ, G, g, and e for $q_{n,k}$, $\phi(na^{1-k})$, $G_{n,c,a^{-k}}$, $g_{n,c,a^{-k}}$, and $(n-1)a^{-k}+1$, respectively. Recalling (5), we obtain

$$|L_n-l_n|\leqslant\sum_{k\geqslant 1}|eqG-e\phi g|+\mathrm{O}(\log n).$$

With $v = \lceil \log_a n \rceil$, we find

$$\sum_{k \geq 1} |eqG - e\phi g| = \sum_{k \geq 1} e|qG - \phi G + \phi G - \phi g|$$

$$\leq \sum_{k \geq 1} e|q - \phi|G + \sum_{k < v/1.9} e\phi G + \sum_{k < v/1.9} e\phi g$$

$$+ \sum_{k \geq v/1.9} e\phi |G - g|$$

$$= F_1 + F_2 + F_3 + F_4.$$

Splitting the range of summation at $k = v/1.9$ is justified later in this section; note that 1.9 is arbitrarily chosen; any number <2 would do it as well. We start with using (1) for large p to establish the bound

$$G_{n,c,a^{-k}} = (1 - a^{-k})^{n-1} L_{n,ca^{-k}} = \mathrm{O}\left(\frac{a^{2k}}{n}\left(\frac{1 - a^{-k}}{1 - ca^{-k}}\right)^{n-1}\right) = \mathrm{O}\left(\frac{a^{2k}}{n}\right) \tag{7}$$

for $c \leq 1$. Thus, it suffices to estimate

$$S_t(n) = \sum_{k \geq 0} (q_{n,k+1} - \phi(na^{-k}))a^{tk},$$

because the desired quantity F_1 is computed to be

$$F_1 = \mathrm{O}(S_1(n)) + \mathrm{O}\left(\frac{S_2(n)}{n}\right).$$

Splitting the range of summation into two parts and using the bounds from Section 7, we obtain

$$\sum_{k < v + \sqrt{v}} (q_{n,k+1} - \phi(na^{-k}))a^{tk} = \mathrm{O}\left(\frac{\log n}{n}\right) \sum_{k < v + \sqrt{v}} a^{tk}$$

$$= \mathrm{O}\left(\frac{\log n}{n} n^t a^{t\sqrt{\log_a n}}\right) = \mathrm{O}(n^{t-1+\varepsilon}) \quad \text{with } \varepsilon > 0.$$

The last step is established by noting that

$$a^{t\sqrt{\log_a n}} = (a^{t \log_a n})^{1/\sqrt{\log_a n}} = n^{t/\sqrt{\log_a n}} = \mathrm{O}(n^{\varepsilon}).$$

The second part yields

$$\sum_{k \geq v + \sqrt{v}} (q_{n,k+1} - \phi(na^{-k}))a^{tk} = \sum_{k \geq v + \sqrt{v}} \mathrm{O}\left(\frac{1}{a^{(k-v)(k-v+1)}}\right) a^{tk}$$

$$= a^{tv} \sum_{k \geq \sqrt{v}} \mathrm{O}\left(\frac{1}{a^{k(k+1)}}\right) a^{tk} = \mathrm{O}\left(n^t \sum_{k \geq \sqrt{v}} a^{-k^2 + tk}\right)$$

$$= \mathrm{O}(n^t n^{-1} a^{t\sqrt{\log_a n}}) = \mathrm{O}(n^{t-1+\varepsilon}).$$

Hence, we find $S_t(n)=O(n^{t-1+\varepsilon})$ and, therefore, $F_1=O(n^{\varepsilon})$. In order to compute F_2, we recall (4) and use the exponential approximation to estimate (7) by

$$G_{n,c,a^{-k}}=O\left(\frac{a^{2k}}{n}\,e^{(c-1)na^{-k}}\right)=O\left(\frac{a^{2k}}{n}\,e^{(c-1)\sqrt{n}}\right)\quad\text{for } k\leqslant v/1.9.$$

Thus, it suffices to consider

$$T_t(n)=\sum_{k<v/1.9}\phi(na^{1-k})a^{tk}$$

because

$$F_2=O(T_1(n)e^{(c-1)\sqrt{n}})+O\left(\frac{T_2(n)}{n}\,e^{(c-1)\sqrt{n}}\right).$$

But, treating $T_t(n)$ is simple by noting that $\phi(na^{1-k})=O(1)$, which follows from $q_{n,k}\leqslant 1$ and Lemma 7.1. Therefore, we may estimate the sum by the finite geometric series

$$T_t(n)=O\left(\sum_{k<v/1.9}a^{tk}\right)=O(n^{t/1.9}),$$

which finally yields

$$F_2=\begin{cases}O(n^{\varepsilon}) & \text{for } c<1,\\ O(n^{0.51}) & \text{for } c=1.\end{cases}$$

F_3 is treated in a similar manner, and we obtain

$$g_{n,c,a^{-k}}=\left(\frac{a^{2k}}{n}\,e^{(c-1)na^{-k}}\right)=O\left(\frac{a^{2k}}{n}\,e^{(c-1)\sqrt{n}}\right)\quad\text{for } k\leqslant v/1.9$$

by a crude estimation of (6) using (2). Therefore, we have

$$F_3=O(T_1(n)e^{(c-1)\sqrt{n}})+O\left(\frac{T_2(n)}{n}\,e^{(c-1)\sqrt{n}}\right)=\begin{cases}O(n^{\varepsilon}) & \text{for } c<1,\\ O(n^{0.51}) & \text{for } c=1.\end{cases}$$

The last problem in this section is the estimation of F_4, resulting from the neglection of the remainder of $L_{n,p}$. Let

$$l_{n,p}=\frac{H_n-1}{p}-H_n+\frac{h(np)}{p}$$

denote the substitution for $L_{n,p}$; see (6). Recalling (1) and using a more precise form of the exponential approximation, we obtain for $k\geqslant v/1.9$

$$\begin{aligned}G_{n,c,a^{-k}}-g_{n,c,a^{-k}}&=(1-a^{-k})^{n-1}(l_{n,ca^{-k}}+O(e^{nca^{-k}}))-e^{-na^{-k}}l_{n,ca^{-k}}\\&=e^{-na^{-k}}(1+O(na^{-2k}))(l_{n,ca^{-k}}+O(e^{nca^{-k}}))-e^{-na^{-k}}l_{n,ca^{-k}}\\&=O(na^{-2k}e^{-na^{-k}}l_{n,ca^{-k}})+O(e^{(c-1)na^{-k}})\\&=F_{4,1}+F_{4,2}.\end{aligned}$$

The first term is evaluated with different approaches for $c<1$ and $c=1$, respectively. For $c<1$, we make use of the computations at the beginning of the section to obtain an asymptotic expression for

$$F_{4,1}=O(n(n-1)(H_n-1)c^{-1}a_2(n)-n(n-1)H_na_3(n)+n(n-1)c^{-1}b_2(n)$$
$$+n(H_n-1)c^{-1}a_1(n)-nH_na_2(n)+c^{-1}nb_1(n)),$$

which is similar to the expression for l_n. The restriction $c<1$ ensures that the transform $b^*(s)$ is an entire function and implies that $b_t(n)$ has the same remainder as $a_t(n)$. We obtain $F_{4,1}=O(n^\varepsilon)$ in this case. The choice $c=1$ is handled by using $na^{-2k}=O(n^{-\varepsilon})$ for $k\geqslant v/1.9$, which yields $F_{4,1}=O(n^{-\varepsilon}l_n)=O(n)$. Estimation of the second term requires the use of the Mellin transform once more. Denoting $c_t(n)$ by

$$c_t(n)=\sum_{k\geqslant 1}\phi(na^{1-k})e^{(c-1)na^{-k}}a^{-tk},$$

we obtain

$$F_{4,2}=O(nc_1(n))+O(c_0(n)).$$

Rewriting the harmonic sum yields

$$c_t(n)=\sum_{k\geqslant 1}\sigma(na^{1-k})a^{-tk}\quad\text{with } \sigma(x)=\phi(x)e^{(c-1)x/a}.$$

The following bounds are straightforward:

$$\sigma(x)=\begin{cases} O(x^d) & \text{for all } d\geqslant 0, \text{ as } x\to 0,\\ O(1) & \text{for } c=1, \text{ as } x\to\infty,\\ O(x^{-d}) & \text{for all } d\geqslant 0 \text{ and } c<1, \text{ as } x\to\infty.\end{cases}$$

Thus, the transform $\sigma^*(s)$ is entire for $c<1$ only. In the case $c=1$, we have $\sigma(x)=\phi(x)$, and the results from Section 5 are valid. However, we obtain

$$c_t^*(s)=\sigma^*(s)\sum_{k\geqslant 1}a^{(k-1)s-tk}=a^{-t}\frac{\sigma^*(s)}{1-a^{s-t}}.$$

Applying Theorem 4.4 yields

$$c_t(n)=\begin{cases} O(n^{-t}) & \text{for } c<1, \text{ or } c=1 \text{ and } t<0,\\ O(\log n) & \text{for } c=1 \text{ and } t=0,\\ O(1) & \text{for } c=1 \text{ and } t>0\end{cases}$$

and, therefore, $F_{4,2}=O(1)$, and $F_{4,2}=O(n)$ for $c<1$ and $c=1$, respectively. We finally obtain

$$F_4=\begin{cases} O(n^\varepsilon) & \text{for } c<1,\\ O(n) & \text{for } c=1.\end{cases}$$

This completes our investigations of the quantity L_n; Theorem 2.2 is a simple consequence.

7. Technical lemmas

This section contains technical lemmas concerning the probabilities $q_{n,k}$ and the function $\phi(na^{1-k})$. The function provides a very good approximation for $q_{n,k}$: the second lemma establishes a uniform bound $q_{n,k+1}-\phi(na^{-k})=O(\log n/n)$.

In addition, for $k>v$ with $v=\lceil \log_a n\rceil$, both $q_{n,k}$ and $\phi(na^{-k})$ decrease very fast with increasing k, as shown by Lemma 7.1.

Lemma 7.1 (Smallness). *For $n\geqslant 2$ and $v=\lceil \log_a n\rceil$, both $q_{n,k+1}$ and $\phi(na^{-k})$ are less than $a^{-(k-v)(k-v+1)}$.*

Proof. Recalling the definition of $q_{n,k}$ from Section 3, we estimate the jth term $1-p_{n,j}$ of the product by

$$\begin{aligned}1-p_{n,j}&=1-(1-a^{-j})^{n-1}na^{-j}-(1-a^{-j})^n\\&=1-(1-a^{-j})^n\left(1+\frac{na^{-j}}{1-a^{-j}}\right)\\&<1-(1-a^{-j})^n(1+na^{-j})\\&<1-(1-na^{-j})(1+na^{-j})\\&=n^2a^{-2j}\leqslant a^{-2(j-v)}.\end{aligned}$$

In order to establish this bound for $\phi(na^{-k})$, we use

$$\begin{aligned}\phi(na^{-k})&=\prod_{j\geqslant 0}(1-e^{-na^{j-k}}(1+na^{j-k}))\\&=\prod_{j=1}^{k}(1-e^{-na^{-j}}(1+na^{-j}))\prod_{j\geqslant 0}(1-e^{-na^{j}}(1+na^{j})).\end{aligned}\tag{8}$$

Obviously, the second product is less than 1, and the jth term in the first product is bounded by

$$1-e^{-na^{-j}}(1+na^{-j})\leqslant 1-(1-a^{-j})^n(1+na^{-j})<a^{-2(j-v)},$$

where we used the well-known relation $e^{-x}\geqslant(1-x/n)^n$ and the previous observations, too. Hence, we obtain

$$q_{n,k+1}=\prod_{j=1}^{k}(1-p_{n,j})<\prod_{j=v}^{k}(1-p_{n,j})<\prod_{j=v}^{k}a^{-2(j-v)}=a^{-(k-v)(k-v+1)},$$

and, with the same approach, $\phi(na^{-k})<a^{-(k-v)(k-v+1)}$. □

Lemma 7.2 (Uniform bound). *For $n \geq 2$, the difference $q_{n,k+1} - \phi(na^{-k})$ is uniformly bounded by* $O(\log n/n)$.

Proof. For $k \geq 2v$, this is a trivial corollary of Lemma 7.1. Considering the other case, we introduce the abbreviation

$$t_{n,k+1} = \prod_{j=1}^{k} (1 - e^{-na^{-j}}(1 + na^{-j})),$$

which corresponds to $q_{n,k+1}$. Because of

$$\begin{aligned}\log \prod_{j \geq 0} (1 - e^{-na^j}(1 + na^j)) &= \sum_{j \geq 0} O(e^{-na^j}(1 + na^j)) \\ &= O\left(e^{-n}(n+1) \sum_{j \geq 0} \frac{1 + na^j}{n+1} e^{(1-a^j)n}\right) \\ &= O((n+1)e^{-n}) = O(e^{-n/2}),\end{aligned}$$

we obtain the estimation $1 + O(e^{-n/2})$ for the second product in expression (8). Therefore, it suffices to establish the bound

$$t_{n,k+1} - q_{n,k+1} = O\left(\frac{\log n}{n}\right)$$

by providing both an upper and a lower bound for $t_{n,k+1}$ in terms of $q_{n,k+1}$. Using the relation $0 \leq e^{-x} - (1 - x/n)^n \leq n^{-1}x^2 e^{-x}$ from [9, p. 242], we obtain

$$e^{-na^{-j}} - (1 - a^{-j})^n \leq e^{-na^{-j}} na^{-2j} < \frac{1}{n}$$

by noting that $e^{-x} < 1/x^2$ for $x > 0$. Now we are ready for treating the upper bound

$$\begin{aligned}t_{n,k+1} &\leq \prod_{j=1}^{k} (1 - (1 - a^{-j})^n (1 + na^{-j})) \\ &= \prod_{j=1}^{k} (1 - (1 - a^{-j})^n - (1 - a^{-j})^{n-1} na^{-j} + (1 - a^{-j})^{n-1} na^{-2j}) \\ &= q_{n,k+1} \prod_{j=1}^{k} \left(1 + \frac{(1 - a^{-j})^{n-1} na^{-2j}}{1 - (1 - a^{-j})^n - (1 - a^{-j})^{n-1} na^{-j}}\right) \\ &\leq q_{n,k+1} \prod_{j=1}^{k} \left(1 + \frac{e^{-(n-1)a^{-j}} na^{-2j}}{1 - (1 - a^{-j})^n - (1 - a^{-j})^{n-1} na^{-j}}\right) \\ &\leq q_{n,k+1} \prod_{j=1}^{k} \left(1 + \frac{C}{n(1 - (1 - a^{-j})^n - (1 - a^{-j})^{n-1} na^{-j})}\right) \\ &\leq q_{n,k+1} \left(1 + \frac{C}{n(1 - (1 - a^{-j})^n - (1 - a^{-j})^{n-1} na^{-j})}\right)^k,\end{aligned}$$

where we used the previous estimates, too. C denotes a sufficiently large constant. Similarly, the lower bound yields

$$t_{n,k+1} \geqslant \prod_{j=1}^{k} (1-(1-a^{-j})^n(1+na^{-j})-e^{-na^{-j}}na^{-2j}(1+na^{-j}))$$

$$\geqslant q_{n,k+1}\left(1-\frac{C}{n(1-(1-a^{-j})^n-(1-a^{-j})^{n-1}na^{-j})}\right)^k,$$

by using the easily established result $e^{-x}<C/x^3$ for a sufficiently large C and substituting $x=na^{-j}$. Remembering the restriction $k<2v$ and using the exponential approximation yields

$$\left(1\pm\frac{C}{n(1-(1-a^{-j})^n-(1-a^{-j})^{n-1}na^{-j})}\right)^k=1+O\left(\frac{\log n}{n}\right),$$

which finally establishes the result. □

8. Conclusions

We derived an asymptotic expression for the average length of a collision resolution interval when resolving a collision of multiplicity n for a hybrid collision resolution algorithm similar to the exponential backoff policy used for the well-known Ethernet. It consists of an estimation phase for determining the multiplicity of the initial collision, and the ordinary slotted ALOHA with adjusted retransmission probability for actual collision resolution. Unlike the usual analysis of ALOHA-type algorithms, which relies on queuing theory, we used certain asymptotic methods based on Mellin transforms to obtain our result.

Our investigations allow an estimation of the performance of our hybrid algorithm w.r.t. the well-known tree algorithms: Since the latter provide a CRI-length of order $O(n)$ as $n\to\infty$, they are superior to our algorithm, which yields an $O(n\log n)$ result. In addition, this fact gives some insight into the (already known) instability of such controlled ALOHA algorithms: In [6] it was shown that stability would require a CRI-length of order $O(n)$ as $n\to\infty$. Thus, even controlling the retransmission probability is not sufficient to guarantee stability.

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