Optimally pebbling hypercubes and powers

David Moews*

Center for Communications Research, 4320 Westerra Court, San Diego, CA 92121, USA

Received 13 August 1997; revised 2 March 1998; accepted 9 March 1998

Abstract

We point out that the optimal pebbling number of the n-cube is \((\frac{1}{2})^{2^n}+O(\log n)\), and explain how to approximate the optimal pebbling number of the nth cartesian power of any graph in a similar way. © 1998 Elsevier Science B.V. All rights reserved

Let G be a graph. By a distribution of pebbles on G we mean a function \(a: V(G) \rightarrow \mathbb{Z}_{\geq 0}\); we usually write \(a(v)\) as \(a_v\), and call \(a_v\) the number of pebbles on \(v\). A pebbling move on a distribution changes the distribution by removing 2 pebbles from some vertex with at least 2 pebbles and placing 1 additional pebble on some adjacent vertex. Call a distribution \(a\) good if, for all vertices \(v\), there is some sequence of pebbling moves starting from \(a\) and ending with at least one pebble on \(v\). The pebbling number \(f(G)\) of a graph \(G\) was introduced by Chung [1]; it is the smallest \(n\) such that, if a distribution \(a\) of pebbles on \(G\) uses a total of \(n\) pebbles, i.e., \(\sum_v a_v = n\), then \(a\) is good. Chung answered a question of Lagarias and Saks by showing that the pebbling number \(f(Q_n)\) of the n-cube equals \(2^n\), and used her methods to prove a number-theoretic result of Lemke and Kleitman [1,4] (also, see [2] for a correction.) Pachtor et al. [5] introduced the dual concept of the optimal pebbling number, \(\text{opt}(G)\), of a graph \(G\); this is the smallest \(n\) such that there exists some good distribution \(a\) of pebbles on \(G\) with a total of \(n\) pebbles used. Such a distribution is called an optimal pebbling. Pachtor et al. also asked what the optimal pebbling number of \(Q_n\) is.

To help compute \(\text{opt}(Q_n)\), and later the optimal pebbling number of the cartesian power of a graph, we define continuous analogs of these concepts. Define a continuous distribution of pebbles on \(G\) to be a function \(a: V(G) \rightarrow \mathbb{R}_{\geq 0}\), and a continuous pebbling move on a distribution \(a\) to be a move that changes the distribution by, for some \(\delta \geq 0\) and adjacent vertices \(v\) and \(w\), decreasing \(a_v \geq \delta\) by \(\delta\) and adding \(\delta/2\)

* E-mail: dmoews@xraysgi.ims.uconn.edu.

0012-365X/98/$19.00 Copyright © 1998 Elsevier Science B.V. All rights reserved

PII S0012-365X(98)00154-X
to \( a_w \). We define good continuous distributions just as we defined good distributions; a continuous distribution \( a \) on \( G \) will evidently be good just when
\[
\sum_v a_v 2^{-d(v, w)} \geq 1
\]
for all vertices \( w \) of \( G \), where \( d(v, w) \) is the distance between vertices \( v \) and \( w \) of \( G \).

(We set \( d(v, w) = \infty \) if \( v \) and \( w \) are not connected in \( G \), and we set \( 2^{-\infty} = 0 \).)

We can now define the continuous optimal pebbling number, \( \text{ofc}(G) \), and continuous optimal pebblings in a way analogous to \( \text{of}(G) \) and optimal pebblings. For graphs \( G \) and \( H \), let the cartesian product, \( G \times H \), of \( G \) and \( H \) have \( V(G \times H) = V(G) \times V(H) \) and
\[
E(G \times H) = \{ \{(v, x), (v, y)\} | v \in V(G), \{x, y\} \in E(H)\}
\]
\[
\cup \{\{(v, x), (w, x)\} | \{v, w\} \in E(G), x \in V(H)\}.
\]

Let the \( n \)th cartesian power of \( G \), \( G^n \), be the graph obtained by taking the cartesian product of \( n \) copies of \( G \).

**Theorem 1.** For all \( G \) and \( H \), \( \text{ofc}(G \times H) = \text{ofc}(G) \text{ofc}(H) \).

**Proof.** (\( \leq \)): If \( a \) is a continuous optimal pebbling of \( G \), and \( b \) of \( H \), and if we define \( c \) by \( c_{(v, x)} = a_v b_x \), then \( c \) is a good continuous distribution on \( G \times H \) with a total of \( \text{ofc}(G) \text{ofc}(H) \) pebbles.

(\( \geq \)): Let \( c \) be a good continuous distribution on \( G \times H \). Then for all \( v \),
\[
1 \leq \sum_{w, y} c_{(w, y)} 2^{-d(v, w) - d(x, y)}
\]
\[
= \sum_w \left( \sum_y c_{(w, y)} 2^{-d(x, y)} \right) 2^{-d(v, w)}
\]
so for all \( x \), putting \( \sum_y c_{(w, y)} 2^{-d(x, y)} \) pebbles on \( w \) is a good continuous distribution on \( G \), and therefore, for all \( x \),
\[
\text{ofc}(G) \leq \sum_w \sum_y c_{(w, y)} 2^{-d(x, y)}
\]
\[
= \sum_y \left( \sum_w c_{(w, y)} \right) 2^{-d(x, y)}
\]
which implies that putting \( \sum_w c_{(w, y)}/\text{ofc}(G) \) pebbles on \( y \) is a good continuous distribution on \( H \); therefore, \( \sum_y \sum_w c_{(w, y)}/\text{ofc}(G) \geq \text{ofc}(H) \), so \( \sum_{w, y} c_{(w, y)} \) is at least \( \text{ofc}(G) \text{ofc}(H) \), as desired. \( \Box \)

Since a good distribution is also a good continuous distribution, \( \text{of}(G) \geq \text{ofc}(G) \) for all \( G \). Let \( P_2 \) be the path with two vertices; then the \( n \)-cube, \( Q_n \), is \( P_2^n \). It is easy to
see that \( \text{ofc}(P_2) = \frac{4}{3} \) (a continuous optimal pebbling has \( \frac{1}{3} \) of a pebble on each vertex), and consequently, \( \text{ofc}(Q_n) \geq \text{ofc}(Q_n) = \left(\frac{4}{3}\right)^n \). What is interesting is that this is also an approximate upper bound.

Let the covering radius of a subset \( W \) of \( V(G) \) be the smallest \( d \) such that all vertices \( v \) of \( G \) are at distance no more than \( d \) from some member of \( W \). In [3] we find the following theorem.

**Theorem 2.** For all \( n \) and \( 0 < \rho < n/2 \), there exists a subset \( W \) of \( V(Q_n) \) with covering radius \( \rho \) and \( |W| = 2^k \), where \( k \leq n(1 - H(\rho/n)) + 2 \log_2 n \). Here, \( H(x) = -x \log_2 x - (1 - x) \log_2(1 - x) \).

We can use this to prove our upper bound.

**Corollary 3.** \( \text{ofc}(Q_n) = \left(\frac{4}{3}\right)^n + O(\log n) \).

**Proof.** Let \( W \) be as in Theorem 2. If we put \( 2^\rho \) pebbles on each vertex of \( W \), this will be a good distribution on \( Q_n \), and it will use \( 2^\rho + k \) pebbles. If \( \rho \) is approximately \( \gamma n \), we can approximate \( \rho + k \) by \( n(1 - x - H(x)) \). The minimum of \( x - H(x) \) is at \( x = \frac{1}{2} \), so let \( \rho = \lceil n/3 \rceil \). Since \( H \) is increasing on \( [0, \frac{1}{2}] \), for \( n \geq 2 \), \( H(\rho/n) \geq H(\frac{1}{2}) = -\frac{3}{2} + \log 2 3 \). Then

\[
\rho + k \leq \lceil n/3 \rceil + n(1 - H(\rho/n)) + 2 \log_2 n \\
\leq n/3 + n(1 - (-2/3 + \log_2 3)) + 2 \log_2 n + 1 \\
= (2 - \log_2 3)n + O(\log n).
\]

This completes the proof. \( \square \)

In [3], Theorem 2 is proved probabilistically: \( W \) is chosen randomly from a set of cardinality \( 2^k \) subsets of \( V(Q_n) \), and it is shown that there is a positive probability that \( W \) has small enough covering radius. This suggests the possibility that we can find an upper bound on \( \text{ofc}(G^n) \) in the same manner, and indeed this is the case.

In the remainder of the paper, we will let \( 0 \cdot \infty = 0 \).

**Lemma 4.** Let \( G \) be a graph, let \( \text{ofc}(G) = b \), and let \( a \) be a continuous optimal pebbling of \( G \). Then for all vertices \( w \) of \( G \),

\[
\sum_y a_y d(w, y) 2^{-d(w, y)} \leq \log_2 b.
\]

**Proof.** If we set \( 0 \log_2 0 = 0 \), then \( x \log_2 x \) is convex for nonnegative \( x \), so for all nonnegative \( x_y \) and \( c_y \) with \( \sum_y c_y = 1 \),

\[
\sum_y c_y x_y \log_2 x_y \geq \left( \sum_y c_y x_y \right) \log_2 \left( \sum_y c_y x_y \right).
\]
Setting $c_y = a_y/b$ and $x_y = 2^{-d(w,y)}$ and rearranging then gives
\[
\sum_y a_y d(w, y) 2^{-d(w,y)} \leq \log_2 \sum_y a_y 2^{-d(w,y)}.
\]
Since $a$ is good, we have $\sum_y a_y 2^{-d(w,y)} \geq 1$, so we have the desired result. □

**Theorem 5.** For all graphs $G$, $ofc(G^n) = ofc(G)^n + O(\log n)$.

**Proof.** Let $n \geq 2$, let $V(G) = \{x_1, \ldots, x_m\}$, let $D$ be the maximum diameter of any connected component of $G$, let $ofc(G) = b$, let $a$ be a continuous optimal pebbling of $G$, and let $x_i = a_i/b$ for all $v \in V(G)$. Let $A_0 = [n \log_2 b]$, and fix $\theta \in \mathbb{R}_{>0}$. For $A = 0, \ldots, A_0$, let $A_A = b^n 2^{-d} n^\theta$. Define a probability distribution on $V(G^n)$ by giving vertex $(v_1, \ldots, v_n)$ probability $\prod_i x_{v_i}$. For each $A = 0, \ldots, A_0$, independently select, with replacement, $|A_A|$ vertices in $V(G^n)$ according to this probability distribution; call the set of selected vertices $S_A$. For each $A$, place $2^{A + \theta D}$ pebbles on each vertex in $S_A$.

This gives us our distribution of pebbles; we use no more than
\[
\sum_{A=0}^{A_0} |A_A| 2^{D + \theta D} \leq ((A_0 + 1)b^n n^\theta + 2^{A_0 + 1} - 1)2^{\theta D}
\]
pebbles in all.

The resultant distribution will be good if, for each $v$, there is some $A$ such that $v$ is within distance $A + \theta D$ of one of the vertices in $S_A$, and this will happen with positive probability if, for each vertex $v$, the probability of such a $A$ failing to exist is less than $\frac{\theta}{n}$. Fix a typical vertex $v = (v_1, \ldots, v_n)$, and let $i_v$ be the number of indices $i$ with $v_i = w$. For some other vertex $v' = (v'_1, \ldots, v'_n)$, let $j_{wy}$ be the number of indices $i$ with $v_i = w$ and $v'_i = y$. Consider the set $T$ of all vertices $v'$ such that, for some fixed $l_{wy}$'s, $j_{wy} = l_{wy}$ for all $w$ and $y$. Each member of this set has distance $\sum_w l_{wy} d(w, y)$ from $v$, and is selected with probability $\prod_y x_{v'}^{\sum_y l_{wy}}$. The probability that no vertex in $T$ is in $S_A$ is thus
\[
p = \left(1 - |T| \prod_y x_{v'}^{\sum_y l_{wy}} \right)^{|A_A|},
\]
and
\[
|T| = \prod_w \left(\sum_{l_{wx}} l_{wx} \cdots l_{wx_n} \right).
\]

Fix some nonnegative real $\lambda_{wy}$'s; let $\lambda_{wy} = 0$ if $x_y = 0$ or $d(w,y) = \infty$, and let $\sum_y \lambda_{wy} = 1$ for all $w$. For all $w$ and $y$, let $l_{wy}$ be $\lambda_{wy} l_w$, rounded either to the next larger or next smaller integer in such a way that the condition $\sum_y l_{wy} = i_w$ holds for all $w$. We wish to find a bound for $p$ in terms of the $\lambda_{wy}$'s.
Since $l_{wy}$ and $\lambda_{wy}i_w$ are both in some interval $[r, r + 1]$, $r \in \mathbb{Z} \geq 0$, it follows that

$$l_{wy} \leq \Gamma(\lambda_{wy}i_w + 1) \max(l_{wy}, 2/\sqrt{\pi}),$$

and it follows from Stirling’s approximation that

$$\text{for } z > 0, \quad e < \Gamma(z + 1) \leq \left(\frac{z}{e}\right)^z \left(\sqrt{2\pi z} + 1\right);$$

hence,

$$|T| = \prod_w \frac{i_w!}{l_{wx_1}! \cdots l_{wx_m}!} \geq \frac{1}{n^m} \prod_w \frac{\Gamma(t_w + 1)}{\Gamma(\lambda_{wx_i}i_w + 1)} \prod_{\lambda_{wx_i}i_w + 1}$$

$$\geq \frac{1}{n^m(\sqrt{2\pi n + 1})} \prod_w \left(\frac{i_w!}{(\lambda_{wx_i})!} \cdots \left(\frac{i_{wx_m}}{(\lambda_{wx_m})!}\right)\right);$$

Also, we will have $\sum_w l_{wy} = \sum_w \lambda_{wy}i_w = 0$ if $\alpha_y = 0$; for other $y$, $l_{wy} \leq \lceil \lambda_{wy}i_w \rceil \leq \lambda_{wy}i_w + 1$, so $\sum_w l_{wy} \leq m + \sum_w \lambda_{wy}i_w$; therefore,

$$\prod_y \sum_w l_{wy} \geq \prod_{\alpha_y \neq 0} \beta^m \prod_{\alpha_y \neq 0} \sum_{\lambda_{wy} \neq 0} \lambda_{wy}i_w;$$

and then

$$p \leq \left(1 - \frac{1}{n^m(\sqrt{2\pi n + 1})} \prod_{\alpha_y \neq 0} \beta^m \prod_{\lambda_{wy} \neq 0} \left(\frac{\alpha_y}{\lambda_{wy}}\right)^{\lambda_{wy}i_w}\right) \cdot \Delta. \quad (1)$$

To satisfy our distance constraint, we wish to have

$$\sum_w l_{wy}d(w, y) \leq \Delta + m^2D. \quad (2)$$

If $d(w, y) = \infty$, $l_{wy} = 0$. Otherwise, $l_{wy} \leq \lambda_{wy}i_w + 1$, and $d(w, y) \leq D$, so

$$\sum_w l_{wy}d(w, y) \leq m^2D + \sum_w \sum_y \lambda_{wy}d(w, y),$$

and to satisfy (2) it will do to have

$$\sum_w \sum_y \lambda_{wy}d(w, y) \leq \Delta. \quad (3)$$

Now, set

$$\lambda_{wy} = \frac{\alpha_y - d(w, y)}{\sum_z \alpha_z^{2 - d(w, z)}}. \quad (4)$$

Since $a$ is a continuous optimal pebbling on $G$, for all $w$ there must exist some $y$ in the same connected component as $w$ with $a_y \neq 0$. Hence the denominator in (4)
is always nonzero. It is clear that \( \sum_y \lambda_{wy} = 1 \) for all \( w \), and that \( \lambda_{wy} = 0 \) if \( x_y = 0 \) or \( d(w,y) = \infty \). It follows from Lemma 4 that with our choice of \( \lambda_{wy} \)'s, the left-hand side of (3) is no bigger than \( n \log_2 b \leq \Delta_0 \), so we can let \( \Delta \) be the ceiling of the left-hand side of (3). Then

\[
\prod_{\lambda_{wy} \neq 0} \left( \frac{x_y}{\lambda_{wy}} \right)^{i_w} = \prod_{\lambda_{wy} \neq 0} \left( \frac{\sum_z x_z 2^{-d(w,z)}}{\lambda_{wy}^{i_w}} \right) \\
\geq 2^{d-1} \prod_w \left( \sum_z x_z 2^{-d(w,z)} \right)^{i_w} \\
\geq 2^{d-1} b^{-n},
\]

since \( a \) is a continuous optimal pebbling of \( G \). Substituting this into (1), we then find that

\[
p \leq \left( \frac{1 - \frac{1}{n^{m^2}(\sqrt{2\pi n} + 1)^m} 2^{d-1} b^{-n} \prod_{x \neq 0} x^m_{\lambda_x}^{A_\Delta}}{n^{m^2}(\sqrt{2\pi n} + 1)^m} \right)^{A_{d}},
\]

or, using \( 1 - x \leq e^{-x} \),

\[
p \leq \exp \left( -\frac{1}{n^{m^2}(\sqrt{2\pi n} + 1)^m} 2^{d-1} b^{-n} A_{\Delta} \prod_{x \neq 0} x^m_{\lambda_x} \right).
\]

We want to have \( \log p < -n \log m \) for large \( n \). Recalling that \( A_{d} 2^{d} = b^n n^\theta \), we see that this will be true if \( \theta > \frac{1}{2} m^2 + 1 \), so we are done. \( \Box \)

References