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Superposition in Branching Allocation Problems*

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1. INTRODUCTION

A superposition principle is proven valid for linear allocation problems [1] occurring when several companies merge or when small firms "spin off" from a parent organization. This principle permits superposition of optimal policies for ordinary dynamic programming problems formed from the branches of the larger problem. Certain inhomogeneities and nonlinearities can be tolerated.

2. NOTATION AND SUPERPOSITION THEOREM

Consider the following linear converging branch [2] multistage decision problem, shown schematically in Fig. 1.

$$\max_{y_1,\cdots,y_{N+P}} \left\{ \sum_{n=1}^{N+P} [g_n y_n + h_n(x_n - y_n)] \right\}$$

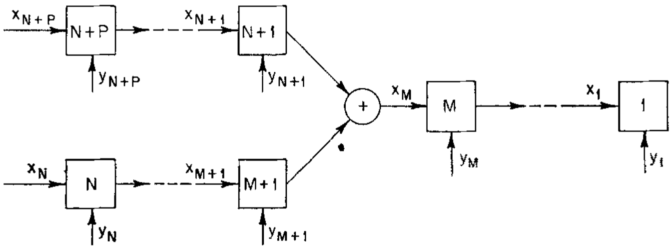

FIG. 1. Converging branches

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with transition functions

$$x_M = a_{N+1}y_{N+1} + b_{N+1}(x_{N+1} - y_{N+1}) + a_{M+1}y_{M+1} + b_{M+1}(x_{M+1} - y_{M+1}) \tag{1 : M}$$

$$x_n = a_{n+1}y_{n+1} + b_{n+1}(x_{n+1} - y_{n+1});$$

$$n = 1, \cdots, M-1, M+1, \cdots, N-1, N+1, \cdots, N+P-1 \tag{1 : n}$$

$$0 \leqslant y_n \leqslant x_n\,, \qquad n = 1, \cdots, N+P$$

and where a_n, b_n, g_n, and h_n are real constants. Let $y_n{}^*$ be the optimal policy, $n = 1, \cdots, N+P$, and $x_n{}^*$ be the resulting optimal states, $n = 1, \cdots, N-1, N+1, \cdots, N+P-1$.

We now consider two serial systems derived from the above branched system. Let serial problem I be:

$$\max_{y_1{}', \ldots, y_N{}'} \left\{ \sum_{n=1}^{N} [g_n y_n{}' + h_n(x_n{}' - y_n{}')] \right\}$$

with

$$x_n' = a_{n+1}y_{n+1}' + b_{n+1}(x_{n+1}' - y_{n+1}'); \qquad n = 1, \cdots, N-1 \tag{2}$$

and

$$0 \leqslant y_n' \leqslant x_n', \qquad n = 1, \cdots, N$$

and let $y_n'{}^*$, $n = 1, \cdots, N$, be the optimal policy for this problem, and $x_n'{}^*$, $n = 1, \cdots, N-1$, the resulting optimal states.

Let serial problem II be:

$$\max_{y_1'', \ldots, y_M'', y_{N+1}'', \ldots, y_{N+P}''} \left\{ \sum_{n=1}^{M} [g_n y_n'' + h_n(x_n'' - y_n'')] + \sum_{n=N+1}^{N+P} [g_n y_n'' + h_n(x_n'' - y_n'')] \right\}$$

with

$$x_n'' = a_{n+1}y_{n+1}'' + b_{n+1}(x_{n+1}'' - y_{n+1}''),$$

$$n = 1, \cdots, M-1, N+1, \cdots, N+P-1$$

$$x_M'' = a_{N+1}y_{N+1}'' + b_{N+1}(x_{N+1}'' - y_{N+1}'')$$

and

$$0 \leqslant y_n'' \leqslant x_n''\,, \qquad n = 1, \cdots, M, N+1, \cdots, N+P$$

and let $y_n''{}^*$, $n = 1, \cdots, M, N+1, \cdots, N+P$, be the optimal policy for this problem, and $x_n''{}^*$, $n = 1, \cdots, M, N+1, \cdots, N+P-1$, the resulting optimal states.

Superposition Theorem

(i) The qualitative policies for all problems are the same:

$$\frac{y_n}{x_n} = \begin{cases} y_n'/x_n' & \text{if} \quad x_n' \neq 0 \\ 0, & \text{if} \quad x_n' = 0 \end{cases} \qquad n = 1, \cdots, N \tag{3'}$$

and

$$\frac{y_n}{x_n} = \begin{cases} y_n''/x_n'' & \text{if} \quad x_n'' \neq 0 \\ 0 & \text{if} \quad x_n'' = 0 \end{cases} \qquad n = 1, \cdots, M, N+1, \cdots, N+P \tag{3''}$$

(ii) Superposition of the quantitative policies for problems I and II gives the quantitative policy for the branch problem:

$$\left.\begin{aligned} x_n &= x_n' + x_n'' \\ y_n &= y_n' + y_n'' \end{aligned}\right\} \qquad n = 1, \cdots, M$$

$$\left.\begin{aligned} x_n &= x_n' \\ y_n &= y_n' \end{aligned}\right\} \qquad n = M+1, \cdots, N$$

$$\left.\begin{aligned} x_n &= x_n'' \\ y_n &= y_n'' \end{aligned}\right\} \qquad n = N+1, \cdots, N+P \tag{4:n}$$

3. Proof

Let

$$f_n(x_n, x_{N+1}, y_{N+1}) \equiv \max_{y_1, \cdots, y_n} \left\{ \sum_{i=1}^{n} [g_i y_i + h_i(x_i - y_i)] \right\} \qquad n = 1, \cdots, N.$$

Then one can show by induction on n that for the branch problem,

$$\begin{aligned} & f_n(x_n, x_{N+1}, y_{N+1}) \\ &= \max_{0 \leqslant y_n \leqslant x_n} \{\lambda_n y_n + \mu_n x_n + \delta_n[a_{N+1} y_{N+1} + b_{N+1}(x_{N+1} - y_{N+1})]\} \\ &= k_n x_n + \delta_n[a_{N+1} y_{N+1} + b_{N+1}(x_{N+1} - y_{N+1})], \qquad n = 1, \cdots, N \end{aligned} \tag{5}$$

where

$$\left.\begin{aligned} k_0 &\equiv 0, \\ \lambda_n &\equiv g_n - h_n + k_{n-1}(a_n - b_n) \\ \mu_n &\equiv h_n + k_{n-1} b_n \\ k_n &\equiv \max\{\mu_n, \lambda_n + \mu_n\} \end{aligned}\right\} \qquad n = 1, \cdots, N$$

and

$$\delta_n = \begin{cases} 0, & \text{for} \quad n = 1, \cdots, M \\ k_M, & \text{for} \quad n = M + 1, \cdots, N. \end{cases}$$

Then the optimal decisions, $y_n{}^*$, are given by

$$\frac{y_n^*}{x_n^*} = \begin{cases} 0, & \text{if} \quad \lambda_n \leqslant 0 \\ 1, & \text{if} \quad \lambda_n \geqslant 0 \end{cases}; \qquad n = 1, \cdots, N$$

where

$$x_N^* \equiv x_N$$

and

$$x_n^* = a_{n+1} y_{n+1}^* + b_{n+1}(x_{n+1}^* - y_{n+1}^*),$$

for

$$n = 1, \cdots, M - 1, M + 1, \cdots, N - 1 \tag{6:n}$$

and

$$x_M^* = a_{M+1} y_{M+1}^* + b_{M+1}(x_{M+1}^* - y_{M+1}^*) + a_{N+1} y_{N+1}^* + b_{N+1}(x_{N+1}^* - y_{N+1}^*) \tag{6:M}$$

This holds for all values of x_{N+1} and y_{N+1}, and in particular when $x_{N+1} = y_{N+1} = 0$, which is the case for serial problem I, the optimal decisions and states of which are $y_n'^*$ and $x_n'^*$, respectively. Therefore, $y_n'^* / x_n'^* = y_n^* / x_n^*$, as asserted in Eq. (3′). A similar argument can be used to prove Eq. (3″).

Since $x_N \equiv x_N'$, Eq. (4 : n) for $n = M + 1, \cdots, N$, is proven inductively using Eqs. (3′) and (6 : n). The proof for $n = N + 1, \cdots, N + P$ is similar, based on the identity of x_{N+P} and x_{N+P}''.

In serial problem I, $x_{N+1}' = y_{N+1}' \equiv 0$ and Eq. (6 : n) becomes

$$x_n'^* = a_{n+1} y_{n+1}'^* + b_{n+1}(x_{n+1}'^* - y_{n+1}'^*); \qquad n = 1, \cdots, M. \tag{6′:n}$$

Similarly for serial problem II, $x_{M+1}'' = y_{M+1}'' \equiv 0$ so that

$$x_M''^* = a_{N+1} y_{N+1}''^* + b_{N+1}(x_{N+1}''^* - y_{N+1}''^*) \tag{6″:M}$$

and

$$x_n''^* = a_{n+1} y_{n+1}''^* + b_{n+1}(x_{n+1}''^* - y_{n+1}''^*); \qquad n = 1, \cdots, M - 1. \tag{6″:n}$$

Combination of Eqs. (6 : n), (6 : M), (6′ : n), (6″ : M), and (6″ : n) with Eqs. (3′) and (3″) gives by induction

$$x_n^* = x_n'^* + x_n''^*; \qquad n = 1, \cdots, M. \tag{4:n}$$

4. Discussion

The above results also hold for more general systems. First, the transition functions may be written as inhomogeneous linear expressions containing a constant, K_n:

$$x_n = a_{n+1}y_{n+1} + b_{n+1}(x_{n+1} - y_{n+1}) + K_n$$

since adding a constant to the homogeneous linear transitions will not affect the *qualitative* policy, i.e., the y_n/x_n.

Second, the theorem is also valid for those systems in which Eq. (1 : M), the transition function at the branching junction, has the more general form:

$$x_M = \gamma[a_{N+1}y_{N+1} + b_{N+1}(x_{N+1} - y_{N+1})] + \varphi[a_{M+1}y_{M+1} + b_{M+1}(x_{M+1} - y_{M+1})] \qquad (7 : M)$$

where γ and φ are any real constants.

Third, the above results generalize to large systems comprised of any number of linear branches, so that each branch may be analyzed independently of the others.

Generally, the method of superposition is applicable only to initial value [2], linear converging branch problems or to final value [2], linear diverging branch problems (which are mathematically equivalent). However, if a nonlinear branch is adjoined to a linear system, the optimal qualitative decisions in the linear portion are unaffected by the introduction of the branch. This is clear from Eq. (5), which could just as well have been written as

$$f_n(x_n, x_{N+1}, y_{N+1}) = k_n x_n + \delta_n[\Phi(x_{N+1}, y_{N+1})]; \qquad n = 1, \cdots, N \qquad (8 : n)$$

where $\Phi(x_{N+1}, y_{N+1})$ is any analytic function, without affecting the subsequent analysis and proof.

These results have an economic interpretation. Consider a firm which has worked out an optimal policy for a linear allocation problem. Even if an unknown number of mergers at arbitrary future times were to add allocation capital to the system, the original qualitative plan would still be optimal—even if the merging firms were nonlinear. Moreover, the original *quantitative* plan remains optimal until the first merger takes place. Therefore long range planners with linear allocation problems need never worry about their policies being upset by future mergers.

In [2] it is shown that for general return and transition functions, diverging branch problems can be solved with no more effort than that needed for the same size serial problem, whereas the treatment required for converging branch problems is more complicated. We have shown that for the linear case, converging branches may be readily solved by superposition. For linear diverging branch problems in which one is free to choose the branch inputs, the analysis is simpler yet. Consider, for example, the system shown schematically in Fig. 2. The total return for stage $M + 1$ plus the returns for all

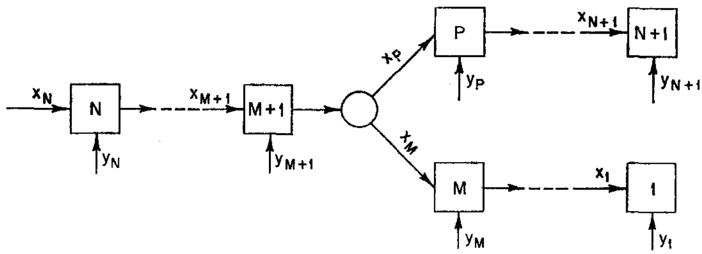

FIG. 2. Diverging branches

stages to the right is

$$f_{M+P-N+1}(x_{M+1}) = \max_{y_{M+1}} \{g_{M+1}y_{M+1} + h_{M+1}(x_{M+1} - y_{M+1}) + k_P x_P + k_M x_M\}.$$

We lose no generality in assuming that

$$x_M + x_P = a_{M+1}y_{M+1} + b_{M+1}(x_{M+1} - y_{M+1})$$

Since the branch inputs are decision variables in this problem, one simply chooses $x_M^* = 0$ when $k_P \geqslant k_M$, and $x_P^* = 0$ when $k_P < k_M$. Thus, in every case, one of the branches receives no input, and is effectively removed from the system.

REFFRENCES

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