Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Stochastic Processes and their Applications 116 (2006) 905–916
stochastic processes and their applications
www.elsevier.com/locate/spa

Iterated Brownian motion in bounded domains in $\mathbb{R}^n$

Erkan Nane*

Department of Mathematics, Purdue University, West Lafayette, IN 47906, United States

Received 2 May 2005; received in revised form 5 October 2005; accepted 5 October 2005
Available online 4 January 2006

Abstract

Let $\tau_D(Z)$ be the first exit time of iterated Brownian motion from a domain $D \subset \mathbb{R}^n$ started at $z \in D$ and let $P_z[\tau_D(Z) > t]$ be its distribution. In this paper we establish the exact asymptotics of $P_z[\tau_D(Z) > t]$ over bounded domains as an extension of the result in [R.D. DeBlassie, Iterated Brownian motion in an open set, Ann. Appl. Probab. 14 (3) (2004) 1529–1558], for $z \in D$:

$$P_z[\tau_D(Z) > t] \approx t^{1/2} \exp\left(-\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}\right), \quad \text{as } t \to \infty.$$

We also study asymptotics of the life time of Brownian-time Brownian motion (BTBM), $Z_t^1 = z + X(Y(t))$, where X_t and Y_t are independent one-dimensional Brownian motions.

MSC: 60J65; 60K99

Keywords: Iterated Brownian motion; Brownian-time Brownian motion; Exit time; Bounded domain

1. Introduction and statement of main results

Properties of iterated Brownian motion (IBM) analogous to the properties of Brownian motion have been studied extensively by several authors [1–3,6–8,11,13,16,18,22,24]. Several other iterated processes including Brownian-time Brownian motion (BTBM) have also been studied [1, 2,19]. One of the main differences between these iterated processes and Brownian motion is that they are not Markov processes. However, these processes have connections with the parabolic operator $\frac{1}{8}\Delta^2 - \frac{\partial}{\partial t}$, as described in [2,13].

* Tel.: +1 765 496 4320; fax: +1 765 494 0548.
E-mail address: enane@math.purdue.edu.

0304-4149/$ - see front matter
doi:10.1016/j.spa.2005.10.007

To define the iterated Brownian motion Z_t started at $z \in \mathbb{R}$, let X_t^+, X_t^- and Y_t be three independent one-dimensional Brownian motions, all started at 0. Two-sided Brownian motion is defined by

$$X_t = \begin{cases} X_t^+, & t \geq 0 \\ X_{(-t)}^-, & t < 0. \end{cases}$$

Then the iterated Brownian motion started at $z \in \mathbb{R}$ is

$$Z_t = z + X(Y_t), \quad t \geq 0.$$

In $\mathbb{R}^n$, one requires $X^\pm$ to be independent n-dimensional Brownian motions. This is the version of the iterated Brownian motion due to Burdzy; see [6].

In what follows, we will write $f \approx g$ and $f \lesssim g$ to mean that, for some positive C_1 and C_2, $C_1 \leq f/g \leq C_2$ and $f \leq C_1 g$, respectively. We will also write $f(t) \sim g(t)$, as $t \to \infty$, to mean that $f(t)/g(t) \to 1$, as $t \to \infty$.

Let τ_D be the first exit time of Brownian motion from a domain $D \subset \mathbb{R}^n$. The large time behavior of $P_z[\tau_D > t]$ has been studied for several types of domains, including general cones [5,12], parabola-shaped domains [4,21], twisted domains [14] and bounded domains [23]. Our aim in this article is to do the same for the exit time of IBM over bounded domains in $\mathbb{R}^n$ and for the exit times of BTBM over several domains in $\mathbb{R}^n$.

In particular, the large time asymptotics of the lifetime of Brownian motion in general cones have been studied by several people including Burkholder [9], DeBlassie [12] and Bañuelos and Smits [5]. Let D be an open cone with vertex 0 such that $S^{n-1} \cap D$ is regular for the Laplace-Beltrami operator $L_{S^{n-1}}$ on the sphere S^{n-1}. Then, for some $p(D) > 0$ (see [12] and [5]),

$$P_x[\tau_D > t] \sim C(x) t^{-p(D)}, \quad \text{as } t \to \infty.$$

Now let $D \subset \mathbb{R}^n$. Let $\tau_D(Z) = \inf\{t \geq 0 : Z_t \notin D\}$ be the first exit time of Z_t from D. When D is a generalized cone, using the results of Bañuelos and Smits, DeBlassie [13] obtained, for $z \in D$, as $t \to \infty$,

$$P_z[\tau_D(Z) > t] \approx \begin{cases} t^{-p(D)}, & p(D) < 1 \\ t^{-1} \ln t, & p(D) = 1 \\ t^{-(p(D)+1)/2}, & p(D) > 1. \end{cases}$$

For parabola-shaped domains, the study of exit time asymptotics for Brownian motion was initiated by Bañuelos, DeBlassie and Smits [4] to answer the question: Are there domains in $\mathbb{R}^n$ for which the distribution of the exit time is sub-exponential? They showed that, for the parabola $\mathcal{P} = \{(x, y) : x > 0, |y| < A\sqrt{x}\}$, $A > 0$, there exist positive constants A_1 and A_2 such that, for $z \in \mathcal{P}$,

$$-A_1 \leq \liminf_{t\to\infty} t^{-\frac{1}{3}} \log P_z[\tau_{\mathcal{P}} > t] \leq \limsup_{t\to\infty} t^{-\frac{1}{3}} \log P_z[\tau_{\mathcal{P}} > t] \leq -A_2.$$

Subsequently, Lifshits and Shi [21] found that the above limit exists for parabola-shaped domains $P_\alpha = \{(x, Y) \in \mathbb{R} \times \mathbb{R}^{n-1} : x > 0, |Y| < Ax^\alpha\}$, $0 < \alpha < 1$ and $A > 0$, in any dimension, for $z \in P_\alpha$,

$$\lim_{t\to\infty} t^{-\left(\frac{1-\alpha}{1+\alpha}\right)} \log P_z[\tau_\alpha > t] = -l, \tag{1.1}$$

where

$$l=\left(\frac{1+\alpha}{\alpha}\right)\left(\frac{\pi J_{(n-3)/2}^{2/\alpha}}{A^2 2^{(3\alpha+1)/\alpha}((1-\alpha)/\alpha)^{(1-\alpha)/\alpha}}\frac{\Gamma^2\left(\frac{1-\alpha}{2\alpha}\right)}{\Gamma^2\left(\frac{1}{2\alpha}\right)}\right)^{\frac{\alpha}{(\alpha+1)}}. \tag{1.2}$$

Here $J_{(n-3)/2}$ denotes the smallest positive zero of the Bessel function $J_{(n-3)/2}$ and Γ is the Gamma function.

Using the results for Brownian motion in parabola-shaped domains we established in [22] with l given by (1.2), for $z \in P_\alpha$,

$$\lim_{t\to\infty} t^{-\left(\frac{1-\alpha}{3+\alpha}\right)} \log P_z[\tau_\alpha(Z)>t] = -\left(\frac{3+\alpha}{2+2\alpha}\right)\left(\frac{1+\alpha}{1-\alpha}\right)^{\left(\frac{1-\alpha}{3+\alpha}\right)}\pi^{\left(\frac{2-2\alpha}{3+\alpha}\right)} l^{\left(\frac{2+2\alpha}{3+\alpha}\right)}.$$

For many bounded domains $D \subset \mathbb{R}^n$, the asymptotics of $P_z[\tau_D > t]$ are well-known (see [23] for a more precise statement of this). For $z \in D$,

$$\lim_{t\to\infty} e^{\lambda_D t} P_z[\tau_D > t] = \psi(z)\int_D \psi(y)\mathrm{d}y, \tag{1.3}$$

where λ_D is the first eigenvalue of $\frac{1}{2}\Delta$ with Dirichlet boundary conditions and ψ is its corresponding eigenfunction.

In [13], DeBlassie proved in the case of iterated Brownian motion in bounded domains for $z \in D$,

$$\lim_{t\to\infty} t^{-1/3}\log P_z[\tau_D(Z)>t] = -\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}. \tag{1.4}$$

The limits (1.3) and (1.4) are very different, in that the latter involves taking the logarithm which may kill many unwanted terms in the exponential. It is then natural to ask if it is possible to obtain an analogue of (1.3) for IBM. That is, to remove the log in (1.4). In this paper we prove the following theorem.

Theorem 1.1. *Let $D \subset \mathbb{R}^n$ be the bounded domain for which (1.3) holds pointwise and let λ_D and ψ be as above. Then, for $z \in D$,*

$$\begin{aligned}2C(z) &\le \liminf_{t\to\infty} t^{-1/2}\exp\left(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}\right)P_z[\tau_D(Z)>t]\\ &\le \limsup_{t\to\infty} t^{-1/2}\exp\left(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}\right)P_z[\tau_D(Z)>t] \le \pi C(z),\end{aligned}$$

where $C(z) = \lambda_D\sqrt{2\pi/3}(\psi(z)\int_D \psi(y)\mathrm{d}y)^2$.

We also obtain a version of the above Theorem 1.1 for another closely related process, the so called Brownian-time Brownian motion (BTBM). To define this, let X_t and Y_t be two independent one-dimensional Brownian motions, all started at 0. BTBM is defined to be $Z_t^1 = x + X(|Y_t|)$. Properties of this process and its connections to PDEs have been studied in [1,2] and [19]. Analogous to Theorem 1.1, we have the following result for this process.

Theorem 1.2. *Let $D \subset \mathbb{R}^n$, λ_D and ψ be as in the statement of Theorem 1.1. Let $\tau_D(Z^1)$ be the first exit time of BTBM from D. Then, for $z \in D$,*

$$\lim_{t\to\infty} t^{-1/6} \exp\left(\frac{3}{2}2^{-2/3}\pi^{2/3}\lambda_D^{2/3}t^{1/3}\right) P_z[\tau_D(Z^1) > t] = C(\lambda_D)\psi(z)\int_D \psi(y)\mathrm{d}y,$$

where $C(\lambda_D) = \pi^{-1/6}2^{13/6}3^{-1/2}\lambda_D^{1/3}$. This limit is uniform on compact subsets of D.

Notice that the limits in Theorems 1.1 and 1.2 are different, even at the exponential level. We obtain the following inequality between distributions of $\tau_D(Z)$ and $\tau_D(Z^1)$.

Theorem 1.3. *Let $D \subset \mathbb{R}^n$. Then, for all $z \in D$ and all $t > 0$,*

$$P_z[\tau_D(Z) > t] \leq 2P_0[\tau_D(Z^1) > t].$$

Remark 1.1. Notice that, from the theorems proved in this paper, the reverse inequality in Theorem 1.3 cannot hold for all large t, in the case of domains $D \subset \mathbb{R}^n$ considered (i.e. bounded domains with regular boundary, parabola-shaped domains, twisted domains).

The paper is organized as follows. In Section 2 we give some preliminary lemmas to be used in the proof of the main results. Theorem 1.1 is proved in Section 3. Section 4 is devoted to prove Theorem 1.2 and some other results on the exit time asymptotics of BTBM over several domains. In Section 5, we compare the exit time distributions of IBM and BTBM. In Section 6, we prove several asymptotic results to be used in the proof of the main results.

2. Preliminaries

In this section we state some preliminary facts that will be used in the proof of the main results.

The main fact is the following Tauberian theorem ([15, Laplace transform method,1958, Chapter 4]). Laporte [20] also studied this type of integral. Let h and f be continuous functions on $\mathbb{R}$. Suppose that f is non-positive and has a global max at x_0, $f'(x_0) = 0$, $f''(x_0) < 0$ and $h(x_0) \neq 0$ and $\int_{-\infty}^{\infty} h(x)\exp(\lambda f(x)) < \infty$ for all $\lambda > 0$. Then, as $\lambda \to \infty$,

$$\int_0^\infty h(x)\exp(\lambda f(x))\mathrm{d}x \sim h(x_0)\exp(\lambda f(x_0))\sqrt{\frac{2\pi}{\lambda|f''(x_0)|}}. \tag{2.1}$$

It can easily be seen from the Laplace transform method that, as $\lambda \to \infty$,

$$\int_0^\infty \exp(-\lambda(x + x^{-2}))\mathrm{d}x \sim \exp(-3\lambda 2^{-2/3})\sqrt{\frac{2^{4/3}\pi}{3\lambda}}. \tag{2.2}$$

Similarly, as $t \to \infty$,

$$\int_0^\infty \exp\left(-\frac{at}{u^2} - bu\right)\mathrm{d}u \sim \sqrt{\frac{\pi}{3}}2^{2/3}a^{1/6}b^{-2/3}t^{1/6}\exp(-3a^{1/3}b^{2/3}2^{-2/3}t^{1/3}). \tag{2.3}$$

This follows from Eq. (2.2) and after making the change of variables $u = (atb^{-1})^{1/3}x$.

Finally, we obtain, as $t \to \infty$,

$$\int_0^\infty u \exp\left(-\frac{at}{u^2} - bu\right) \mathrm{d}u \sim 2\sqrt{\frac{\pi}{3}} a^{1/2} b^{-1} t^{1/2} \exp(-3a^{1/3} b^{2/3} 2^{-2/3} t^{1/3}). \tag{2.4}$$

3. Iterated Brownian motion in bounded domains

If $D \subset \mathbb{R}^n$ is an open set, write

$$\tau_D^\pm(z) = \inf\{t \geq 0 : X_t^\pm + z \notin D\},$$

and if $I \subset \mathbb{R}$ is an open interval, write

$$\eta_I = \eta(I) = \inf\{t \geq 0 : Y_t \notin I\}.$$

Recall that $\tau_D(Z)$ stands for the first exit time of iterated Brownian motion from D. As in DeBlassie [13, Section 3], we have by the continuity of the paths for $Z_t = z + X(Y_t)$, if f is the probability density of $\tau_D^\pm(z)$,

$$P_z[\tau_D(Z) > t] = \int_0^\infty \int_0^\infty P_0[\eta_{(-u,v)} > t] f(u) f(v) \mathrm{d}v \mathrm{d}u. \tag{3.1}$$

The proof of Theorem 1.1. The following is well known:

$$P_0[\eta_{(-u,v)} > t] = \frac{4}{\pi} \sum_{n=0}^\infty \frac{1}{2n+1} \exp\left(-\frac{(2n+1)^2\pi^2}{2(u+v)^2} t\right) \sin\frac{(2n+1)\pi u}{u+v}, \tag{3.2}$$

(see [17, pp. 340–342]).
Let $\epsilon > 0$. From Lemma 6.1, choose $M > 0$ so large that

$$(1-\epsilon)\frac{4}{\pi} \mathrm{e}^{-\frac{\pi^2 t}{2}} \sin \pi x \leq P_x[\eta_{(0,1)} > t] \leq (1+\epsilon)\frac{4}{\pi} \mathrm{e}^{-\frac{\pi^2 t}{2}} \sin \pi x, \tag{3.3}$$

for $t \geq M$, uniformly $x \in (0, 1)$. Letting $0 < \delta < 1/2$, from the Jordan inequality for the sine function in the interval $(0, \pi/2]$,

$$2x \leq \sin \pi x \leq \pi x, \quad x \in (0, \delta]. \tag{3.4}$$

For a bounded domain with a regular boundary, it is well known (see [23, pp. 121–127]) that there exists an increasing sequence of eigenvalues, $\lambda_1 < \lambda_2 \leq \lambda_3 \cdots$, and eigenfunctions ψ_k corresponding to λ_k such that,

$$P_z[\tau_D \leq t] = \sum_{k=1}^\infty \exp(-\lambda_k t) \psi_k(z) \int_D \psi_k(y) \mathrm{d}y. \tag{3.5}$$

From the arguments in [13, Lemma A.4],

$$f(t) = \frac{\mathrm{d}}{\mathrm{d}t} P_z[\tau_D \leq t] = \sum_{k=1}^\infty \lambda_k \exp(-\lambda_k t) \psi_k(z) \int_D \psi_k(y) \mathrm{d}y. \tag{3.6}$$

Finally, choose $K > 0$ so large that

$$A(z)(1-\epsilon) \exp(-\lambda_D u) \leq f(u) \leq A(z)(1+\epsilon) \exp(-\lambda_D u)$$

for all $u \geq K$, where

$$A(z) = \lambda_1 \psi_1(z) \int_D \psi_1(y) \mathrm{d}y = \lambda_D \psi(z) \int_D \psi(y) \mathrm{d}y.$$

We further assume that t is so large that $K < \delta\sqrt{t/M}$. Define A for $\delta < 1/2$, $K > 0$ and $M > 0$ as

$$A = \left\{(u, v): \ K \leq u \leq \delta\sqrt{\frac{t}{M}}, \frac{1-\delta}{\delta} u \leq v \leq \sqrt{\frac{t}{M}} - u\right\}.$$

On the set A, since $\delta < 1/2$, we have $v \geq (\frac{1}{\delta} - 1)u > u > K$ and $u + v > \frac{u}{\delta}$; this gives $\frac{u}{u+v} \leq \delta$.

By Eqs. (3.3) and (3.4), $P_z[\tau_D(Z) > t] = P[\eta_{(-\tau_D^-(z), \tau_D^+(z))} > t]$ is

$$\geq C^1 \int_K^{\delta\sqrt{t/M}} \int_{(1-\delta)u/\delta}^{\sqrt{t/M}-u} \frac{u}{(u+v)} \exp\left(-\frac{\pi^2 t}{2(u+v)^2}\right) \exp(-\lambda_D(u+v)) \mathrm{d}v \mathrm{d}u,$$

where $C^1 = C^1(z) = 4(4/\pi)A(z)^2(1-\epsilon)^3$. Changing the variables $x = u + v, z = u$, the integral is

$$= C^1 \int_K^{\delta\sqrt{t/M}} \int_{z/\delta}^{\sqrt{t/M}} \frac{z}{x} \exp\left(-\frac{\pi^2 t}{2x^2}\right) \exp(-\lambda_D x) \mathrm{d}x \mathrm{d}z,$$

and, reversing the order of integration,

$$\begin{aligned}
&= C^1 \int_{K/\delta}^{\sqrt{t/M}} \int_K^{\delta x} \frac{z}{x} \exp\left(-\frac{\pi^2 t}{2x^2}\right) \exp(-\lambda_D x) \mathrm{d}z \mathrm{d}x \\
&= C^1/2 \int_{K/\delta}^{\sqrt{t/M}} \frac{1}{x} \exp\left(-\frac{\pi^2 t}{2x^2}\right) \exp(-\lambda_D x)(\delta^2 x^2 - K^2) \mathrm{d}x \\
&\geq \delta^2 C^1/2 \int_{K/\delta}^{\sqrt{t/M}} x \exp\left(-\frac{\pi^2 t}{2x^2}\right) \exp(-\lambda_D x) \mathrm{d}x - I,
\end{aligned}$$

where

$$I = (C^1/2)K^2 \int_0^\infty \frac{1}{x} \exp\left(-\frac{\pi^2 t}{2x^2}\right) \exp(-\lambda_D x) \mathrm{d}x.$$

From the Laplace transform method, Eq. (2.1), there exists $C_0 > 0$ such that, as $t \to \infty$,

$$I \sim C_0 t^{-1/6} \exp\left(-\frac{3}{2}\pi^{2/3}\lambda_D^{2/3} t^{1/3}\right). \tag{3.7}$$

From Eq. (2.4), as $t \to \infty$,

$$\int_0^\infty x \exp\left(-\frac{\pi^2 t}{2x^2}\right) \exp(-\lambda_D x) \mathrm{d}x \sim 2\sqrt{\frac{\pi}{3}} \left(\frac{\pi^2}{2}\right)^{1/2} \lambda_D^{-1} t^{1/2} \exp\left(-\frac{3}{2}\pi^{2/3}\lambda_D^{2/3} t^{1/3}\right). \tag{3.8}$$

Now, for some $c_1 > 0$,

$$\int_0^{K/\delta} x \exp\left(-\frac{\pi^2 t}{2x^2} - \lambda_D x\right) \mathrm{d}x$$

$$\leq \mathrm{e}^{-\pi^2\delta^2 t/2K^2} \int_0^{K/\delta} x \exp(-\lambda_D x)\mathrm{d}x \lesssim \mathrm{e}^{-c_1 t}, \tag{3.9}$$

and

$$\begin{aligned} &\int_{\sqrt{t/M}}^{\infty} x \exp\left(-\frac{\pi^2 t}{2x^2}\right) \exp(-\lambda_D x)\mathrm{d}x \leq \int_{\sqrt{t/M}}^{\infty} x \exp(-\lambda_D x)\mathrm{d}x \\ &\quad = (\sqrt{t/M}\lambda_D^{-1} + \lambda_D^{-2}) \exp(-\lambda_D \sqrt{t/M}). \end{aligned} \tag{3.10}$$

Now, from Eqs. (3.7)–(3.10), we get

$$\begin{aligned} &\liminf_{t\to\infty} t^{-1/2} \exp\left(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3} t^{1/3}\right) P_z[\tau_D(Z) > t] \\ &\quad \geq \delta^2 (C^1/2) 2\sqrt{\frac{\pi}{3}} \left(\frac{\pi^2}{2}\right)^{1/2} \lambda_D^{-1}. \end{aligned} \tag{3.11}$$

For the upper bound for $P[\tau_D(Z) > t]$ from Eq. (3.10) in [13],

$$P_z[\tau_D(Z) > t] = 2\int_0^{\infty}\int_u^{\infty} P_{\frac{u}{u+v}}\left[\eta_{(0,1)} > \frac{t}{(u+v^2)}\right] f(u) f(v)\mathrm{d}v\mathrm{d}u. \tag{3.12}$$

We define the following sets that make up the domain of integration:

$$\begin{aligned} A_1 &= \{(u,v) : v \geq u \geq 0, u+v \geq \sqrt{t/M}\}, \\ A_2 &= \{(u,v) : u \geq 0, v \geq K, u \leq v, u+v \leq \sqrt{t/M}\}, \\ A_3 &= \{(u,v) : 0 \leq u \leq v \leq K\}. \end{aligned}$$

Over the set A_1 we have, for some $c > 0$,

$$\begin{aligned} &\int\int_{A_1} P_{\frac{u}{u+v}}\left[\eta_{(0,1)} > \frac{t}{(u+v)^2}\right] f(u) f(v)\mathrm{d}v\mathrm{d}u \\ &\quad \leq \int\int_{A_1} f(u) f(v)\mathrm{d}v\mathrm{d}u \leq \exp(-c\sqrt{t/M}). \end{aligned} \tag{3.13}$$

The Eq. (3.13) follows from the distribution of τ_D from Lemma 2.1 in [22].

Since on A_3, $t/(u+v)^2 \geq M$,

$$\begin{aligned} &\int\int_{A_3} P_{\frac{u}{u+v}}\left[\eta_{(0,1)} > \frac{t}{(u+v)^2}\right] f(u) f(v)\mathrm{d}v\mathrm{d}u \\ &\quad \leq \int_0^K\int_0^K \exp\left(-\frac{\pi^2 t}{2(u+v)^2}\right) f(u) f(v)\mathrm{d}v\mathrm{d}u. \\ &\quad \leq \exp\left(-\frac{\pi^2 t}{8K^2}\right)\int_0^K\int_0^K f(u) f(v)\mathrm{d}v\mathrm{d}u \leq \exp\left(-\frac{\pi^2 t}{8K^2}\right). \end{aligned} \tag{3.14}$$

Let $C_1 = C_1(z) = 2\pi(4/\pi)A(z)^2(1+\epsilon)^3$. For the integral over A_2 we get,

$$\begin{aligned} &\int\int_{A_2} P_{\frac{u}{u+v}}\left[\eta_{(0,1)} > \frac{t}{(u+v)^2}\right] f(u) f(v)\mathrm{d}v\mathrm{d}u \\ &\quad \leq C_1 \int_0^K \int_K^{\sqrt{t/M}-u} f(u) \exp\left(-\frac{\pi^2 t}{2(u+v)^2} - \lambda_D v\right)\mathrm{d}v\mathrm{d}u \end{aligned}$$

$$
\begin{aligned}
&+C_1\int_K^{1/2\sqrt{t/M}}\int_u^{\sqrt{t/M}-u}\frac{u}{u+v}\exp\left(-\frac{\pi^2 t}{2(u+v)^2}-\lambda_D(u+v)\right)\mathrm{d}v\mathrm{d}u\\
&=I+II. \qquad (3.15)
\end{aligned}
$$

Changing variables $u+v=z$, $u=w$,

$$
\begin{aligned}
I &= \int_0^K\int_K^{\sqrt{t/M}-u}\exp\left(-\frac{\pi^2 t}{2(u+v)^2}\right)f(u)\exp(-\lambda_D v)\mathrm{d}v\mathrm{d}u\\
&\le \int_0^K\int_{w+K}^{\sqrt{t/M}}\exp\left(-\frac{\pi^2 t}{2z^2}\right)f(w)\exp(-\lambda_D z)\exp(\lambda_D w)\mathrm{d}z\mathrm{d}w\\
&\le \exp(\lambda_D K)\int_0^K f(w)\mathrm{d}w\int_0^\infty\exp\left(-\frac{\pi^2 t}{2z^2}\right)\exp(-\lambda_D z)\mathrm{d}z\\
&\lesssim t^{1/6}\exp\left(-\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}\right). \qquad (3.16)
\end{aligned}
$$

Eq. (3.16) follows from Eq. (2.3), with $a=\pi^2/2$, $b=\lambda_D$.

Changing variables $u+v=z$, $u=w$,

$$
\begin{aligned}
II &= C_1\int_K^{1/2\sqrt{t/M}}\int_u^{\sqrt{t/M}-u}\frac{u}{(u+v)}\exp\left(-\frac{\pi^2 t}{2(u+v)^2}-\lambda_D(u+v)\right)\mathrm{d}v\mathrm{d}u\\
&\le C_1\int_K^{1/2\sqrt{t/M}}\int_{2w}^{\sqrt{t/M}}\frac{w}{z}\exp\left(-\frac{\pi^2 t}{2z^2}-\lambda_D z\right)\mathrm{d}z\mathrm{d}w\\
&= C_1\int_{2K}^{\sqrt{t/M}}\int_K^{z/2}\frac{w}{z}\exp\left(-\frac{\pi^2 t}{2z^2}-\lambda_D z\right)\mathrm{d}w\mathrm{d}z \qquad (3.17)\\
&\le C_1/8\int_{2K}^{\sqrt{t/M}}z\exp\left(-\frac{\pi^2 t}{2z^2}-\lambda_D z\right)\mathrm{d}z\\
&\le (1+\epsilon)(C_1/8)2\sqrt{\frac{\pi}{3}}\left(\frac{\pi^2}{2}\right)^{1/2}\lambda_D^{-1}t^{1/2}\left(-\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}\right). \qquad (3.18)
\end{aligned}
$$

Eq. (3.17) follows by changing the order of the integration. Also, Eq. (3.18) follows from Eq. (2.4).

Now, from Eqs. (3.13), (3.14), (3.16) and (3.18) we obtain

$$
\begin{aligned}
&\limsup_{t\to\infty} t^{-1/2}\exp\left(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}\right)P_z[\tau_D(Z)>t]\\
&\quad\le (1+\epsilon)\left(\frac{C_1}{8}\right)2\sqrt{\frac{\pi}{3}}\left(\frac{\pi^2}{2}\right)^{1/2}\lambda_D^{-1}. \qquad (3.19)
\end{aligned}
$$

Finally, from Eqs. (3.11) and (3.19) and letting $\epsilon\to 0$, $\delta\to 1/2$,

$$
\begin{aligned}
2C(z) &\le \liminf_{t\to\infty} t^{-1/2}\exp\left(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}\right)P_z[\tau_D(Z)>t]\\
&\le \limsup_{t\to\infty} t^{-1/2}\exp\left(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}\right)P_z[\tau_D(Z)>t]\le \pi C(z),
\end{aligned}
$$

where $C(z)=\lambda_D\sqrt{2\pi/3}(\psi(z)\int_D\psi(y)\mathrm{d}y)^2$. □

4. The process Z_t^1; Brownian-time Brownian motion

In this section we study Brownian-time Brownian motion (BTBM), Z_t^1 started at $z \in \mathbb{R}$. Let X_t and Y_t be two independent one-dimensional Brownian motions, all started at 0. BTBM is defined to be $Z_t^1 = x + X(|Y_t|)$. In $\mathbb{R}^n$, we require X to be independent one-dimensional iterated Brownian motions. If $D \subset \mathbb{R}^n$ is an open set, write

$$\tau_D(z) = \inf\{t \geq 0 : X_t + z \notin D\},$$

and if $I \subset \mathbb{R}$ is an open interval, we write

$$\eta_I = \inf\{t \geq 0 : Y_t \notin I\}.$$

Let $\tau_D(Z^1)$ stand for the first exit time of BTBM from D. We have, by the continuity of paths,

$$P_z[\tau_D(Z^1) > t] = P[\eta(-\tau_D(z), \tau_D(z)) > t]. \tag{4.1}$$

Theorem 4.1. *Let $0 < \beta$. Let ξ be a positive random variable such that*

$$-\log P[\xi > t] \sim ct^\beta, \quad \text{as } t \to \infty.$$

If ξ is independent of the Brownian motion Y, then

$$-\log P[\eta_{(-\xi,\xi)} > t] \sim 2^{-\frac{2\beta}{2+\beta}} \left(\frac{2+\beta}{2}\right) c^{2/(2+\beta)} \beta^{-\beta/(2+\beta)} \pi^{2\beta/(2+\beta)} t^{\beta/(2+\beta)},$$

as $t \to \infty$.

Proof. The proof follows similar to the proof of Theorem 3.1 in [22], by integration by parts,

$$P[\eta_{(-\xi,\xi)} > t] = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}u} P_0(\eta_{(-u,u)} > t) P[\xi > u] \mathrm{d}u. \tag{4.2}$$

We use the distribution of $\eta_{(-u,u)}$ given in (3.2). We use the asymptotics from Eq. (6.1) on the set $A = \{u > 0 : \ K \leq u \leq \sqrt{t/M}\}$. For the lower bound we use Lemma 2.4 in [22], but for the upper bound we use the deBruijn Tauberian Theorem as in [22, Lemma 2.2]. □

From Theorem 4.1 we obtain similar results for the asymptotic distribution of the first exit time of Z^1 from the interior of several open sets $D \subset \mathbb{R}^n$.

Corollary 4.1. *Let $0 < \alpha < 1$. Let $P_\alpha = \{(x, Y) \in \mathbb{R} \times \mathbb{R}^{n-1} : x > 0, |Y| < Ax^\alpha\}$. Then, for $z \in P_\alpha$,*

$$\lim_{t\to\infty} t^{-\left(\frac{1-\alpha}{3+\alpha}\right)} \log P_z[\tau_\alpha(Z^1) > t] = -2^{\left(\frac{2\alpha-2}{3+\alpha}\right)} \left(\frac{3+\alpha}{2+2\alpha}\right) \left(\frac{1+\alpha}{1-\alpha}\right)^{\left(\frac{1-\alpha}{3+\alpha}\right)} \pi^{\left(\frac{2-2\alpha}{3+\alpha}\right)} l^{\left(\frac{2+2\alpha}{3+\alpha}\right)},$$

where l is the limit given by (1.2).

Corollary 4.2. *Let $D \subset \mathbb{R}^2$ be a twisted domain with growth radius γr^p, $\gamma > 0$, $0 < p < 1$. Then, for $z \in D$,*

$$\lim_{t\to\infty} t^{-\left(\frac{1-p}{p+3}\right)} \log P_z[\tau_D(Z^1) > t] = -2^{\left(\frac{2p-2}{3+p}\right)} \left(\frac{3+p}{2+2p}\right) \left(\frac{1+p}{1-p}\right)^{\left(\frac{1-p}{3+p}\right)} \pi^{\left(\frac{2-2p}{3+p}\right)} l_1^{\left(\frac{2+2p}{3+p}\right)},$$

where l_1 is the limit given by the limit in [14, Theorem 1.1].

Remark 4.1. Notice that there is only a constant difference in the limit of the asymptotic distribution of $\tau_D(Z)$ and in that of $\tau_D(Z^1)$ (compare with the results in Nane [22]).

Proof of Theorem 1.2. From Eqs. (3.2), (3.6) and (4.1),

$$P_z[\tau_D(Z^1) \le t] = \int_0^\infty P_0[\eta_{(-u,u)} > t] f(u)\mathrm{d}u$$
$$= \frac{4}{\pi}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\lambda_k\psi_k(z)\int_D \psi_k(y)\mathrm{d}y \int_0^\infty \exp\left(-\frac{(2n+1)^2\pi^2 t}{8u^2} - \lambda_k u\right)\mathrm{d}u. \quad (4.3)$$

From Eq. (2.3), for each n, k we have, with $a = \frac{(2n+1)^2\pi^2}{8}$ and $b = \lambda_k$,

$$\int_0^\infty \exp\left(-\frac{(2n+1)^2\pi^2 t}{8u^2} - \lambda_k u\right)\mathrm{d}u$$
$$\sim \pi^{5/6}2^{1/6}3^{-1/2}(2n+1)^{1/3}\lambda_k^{-2/3}t^{1/6}\exp\left(-\frac{3}{2}(2n+1)^{2/3}\pi^{2/3}\lambda_k^{2/3}2^{-2/3}t^{1/3}\right).$$

With this, Eq. (4.3) becomes

$$\int_0^\infty P_0[\eta_{(-v,v)} > t] f(v)\mathrm{d}v$$
$$\sim \frac{4}{\pi}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\lambda_k\psi_k(z)\int_D \psi_k(y)\mathrm{d}y$$
$$\times \pi^{5/6}2^{1/6}3^{-1/2}(2n+1)^{1/3}\lambda_k^{-2/3}t^{1/6}\exp\left(-\frac{3}{2}2^{-2/3}(2n+1)^{2/3}\pi^{2/3}\lambda_k^{2/3}t^{1/3}\right). \quad (4.4)$$

To get the desired result, we must prove that the following series converge absolutely, which implies that the first term in the series in (4.4) is the dominant term,

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(2n+1)^{-2/3}\lambda_k^{1/3}\exp\left(-\frac{3}{2}2^{-2/3}(2n+1)^{2/3}\pi^{2/3}\lambda_k^{2/3}\delta/2\right) < \infty.$$

The series in n for k fixed is

$$\sum_{n=0}^{\infty}(2n+1)^{-2/3}\exp\left(-\frac{3}{2}2^{-2/3}(2n+1)^{2/3}\pi^{2/3}\lambda_k^{2/3}\delta/2\right)$$
$$\le \frac{\exp\left(-\frac{3}{2}2^{-2/3}\pi^{2/3}\lambda_k^{2/3}\delta/2\right)}{1-\exp\left(-\frac{3}{2}\pi^{2/3}\lambda_1^{2/3}\delta/2\right)}.$$

Since, for $\delta > 0$,

$$\sum_{k=1}^{\infty}\exp\left(-\frac{3}{2}2^{-2/3}\pi^{2/3}\lambda_k^{2/3}\delta/3\right) \le \infty,$$

we are done. This follows from Weyl's asymptotic formula for the eigenvalues λ_k, $\lambda_k \ge C_{n,D}k^{n/2}$, see [10], where $C_{n,D}$ depends only on the dimension n and the domain D, independent of k. From the above Eq. (4.4), the constant $C(\lambda_D) = \pi^{-1/6}2^{13/6}3^{-1/2}\lambda_D^{1/3}$, where $\lambda_D = \lambda_1$ is the first eigenvalue of the Dirichlet Laplacian in D. □

5. Comparison of IBM and BTBM

Proof of Theorem 1.3. From Eq. (3.10) in [13], we get

$$\begin{aligned} P_z[\tau_D(Z) > t] &= 2\int_0^\infty \int_u^\infty P_0[\eta_{(-u,v)} > t] f(u) f(v) \mathrm{d}v \mathrm{d}u \\ &\le 2\int_0^\infty \int_u^\infty P_0[\eta_{(-v,v)} > t] f(u) f(v) \mathrm{d}v \mathrm{d}u \qquad (5.1) \\ &\le 2\int_0^\infty \int_0^\infty P_0[\eta_{(-v,v)} > t] f(u) f(v) \mathrm{d}v \mathrm{d}u \\ &= 2\int_0^\infty P_0[\eta_{(-v,v)} > t] f(v) \mathrm{d}v \\ &= 2 P_z[\tau_D(Z^1) > t]. \qquad (5.2) \end{aligned}$$

The inequality (5.1) follows from the fact that $(-u, v) \subset (-v, v)$. The equality (5.2) follows from Eq. (4.1). □

Let ϕ be an increasing function. If we multiply the inequality in the Theorem 1.3 by the derivative of ϕ and integrate in time, we get

$$E_z(\phi(\tau_D(Z))) \le 2E_z(\phi(\tau_D(Z^1))).$$

In particular, for $p \ge 1$,

$$E_z((\tau_D(Z))^p) \le 2E_z((\tau_D(Z^1))^p).$$

6. Asymptotics

In this section we will prove some lemmas that were used in Sections 3 and 4. The following lemma is proved in [13, Lemma A1] (it also follows from more general results on "intrinsic ultracontractivity"). We include it for completeness.

Lemma 6.1. *As $t \to \infty$,*

$$P_x[\eta_{(0,1)} > t] \sim \frac{4}{\pi} \mathrm{e}^{-\frac{\pi^2 t}{2}} \sin \pi x, \quad \textit{uniformly for } x \in (0, 1).$$

We will next prove similar results to Nane [22, Lemma 4.2] that will be used for the process Z^1.

Lemma 6.2. *Let $B = \{u > 0 : t/u^2 > M\}$ for M large. Then, on B,*

$$\frac{\mathrm{d}}{\mathrm{d}u} P_0[\eta_{(-u,u)} > t] \sim \exp\left(-\frac{\pi^2 t}{8u^2}\right) \frac{\pi t}{u^3}. \qquad (6.1)$$

Proof. If we differentiate $P_0[\eta_{(-u,u)} > t]$, which is given in (3.2), we get

$$\frac{\mathrm{d}}{\mathrm{d}u} P_0[\eta_{(-u,u)} > t] = \frac{\pi t}{u^3} \sum_{n=0}^\infty (2n+1)(-1)^n \exp\left(-\frac{(2n+1)^2\pi^2}{8u^2} t\right).$$

The result follows from this. □

Acknowledgments

I would like to thank Professor Rodrigo Bañuelos, my academic advisor, for suggesting this problem to me and for his guidance on this paper. I would also like to thank the anonymous referee for his help with the presentation of the paper.

The author was supported in part by NSF Grant # 9700585-DMS.

References

[1] H. Allouba, Brownian-time processes: The pde connection and the corresponding Feynman-Kac formula, Trans. Amer. Math. Soc. 354 (11) (2002) 4627–4637.

[2] H. Allouba, W. Zheng, Brownian-time processes: The pde connection and the half-derivative generator, Ann. Probab. 29 (2) (2001) 1780–1795.

[3] R. Bañuelos, R.D. DeBlassie, The exit distribution for iterated Brownian motion in cones, 2005 (preprint).

[4] R. Bañuelos, R.D. DeBlassie, R. Smits, The first exit time of planar Brownian motion from the interior of a parabola, Ann. Probab. 29 (2) (2001) 882–901.

[5] R. Bañuelos, R. Smits, Brownian motion in cones, Probab. Theory Related Fields 108 (1997) 299–319.

[6] K. Burdzy, Some path properties of iterated Brownian motion, in: E. Çinlar, K.L. Chung, M.J. Sharpe (Eds.), Seminar on Stochastic Processes, Birkhäuser, Boston, 1993, pp. 67–87.

[7] K. Burdzy, Variation of iterated Brownian motion, in: D.A. Dawson (Ed.), Workshops and Conference on Measure-valued Processes, Stochastic Partial Differential Equations and Interacting Particle Systems, Amer. Math. Soc., Providence, RI, 1994, pp. 35–53.

[8] K. Burdzy, D. Khoshnevisan, Brownian motion in a Brownian crack, Ann. Appl. Probab. 8 (3) (1998) 708–748.

[9] D.L. Burkholder, Exit times of Brownian motion, harmonic majorization and Hardy spaces, Adv. Math. 26 (1977) 182–205.

[10] I. Chavel, Eigenvalues in Riemannian Geometry, Academic Press, 1984.

[11] E. Csàki, M. Csörgő, A. Földes, P. Révész, The local time of iterated Brownian motion, J. Theoret. Probab. 9 (3) (1996) 717–743.

[12] R.D. DeBlassie, Exit times from cones in $\mathbb{R}^n$ of Brownian motion, Probab. Theory Related Fields 74 (1987) 1–29.

[13] R.D. DeBlassie, Iterated Brownian motion in an open set, Ann. Appl. Probab. 14 (3) (2004) 1529–1558.

[14] R.D. DeBlassie, R. Smits, Brownian motion in twisted domains, Trans. Amer. Math. Soc. 357 (3) (2005) 1245–1274.

[15] N.G. De Bruijn, Asymptotic Methods in Analysis, North-Holland Publishing Co., Amsterdam, 1957.

[16] N. Eisenbum, Z. Shi, Uniform oscillations of the local time of iterated Brownian motion, Bernoulli 5 (1) (1999) 49–65.

[17] W. Feller, An Introduction to Probability Theory and its Applications, Wiley, New York, 1971.

[18] D. Khoshnevisan, T.M. Lewis, Stochastic calculus for Brownian motion in a Brownian fracture, Ann. Appl. Probab. 9 (3) (1999) 629–667.

[19] D. Khoshnevisan, T.M. Lewis, Chung's law of the iterated logarithm for iterated Brownian motion, Ann. Inst. H. Poincaré Probab. Statist. 32 (3) (1996) 349–359.

[20] O. Laporte, Absorption coefficients for thermal neutrons, Phys. Rev. 52 (1937) 72–74.

[21] M. Lifshits, Z. Shi, The first exit time of Brownian motion from a parabolic domain, Bernoulli 8 (6) (2002) 745–765.

[22] E. Nane, Iterated Brownian motion in parabola-shaped domains, Potential Anal. (in press).

[23] S.C. Port, C.J. Stone, Brownian Motion and Classical Potential Theory, Academic, New York, 1978.

[24] Y. Xiao, Local times and related properties of multidimensional iterated Brownian motion, J. Theoret. Probab. 11 (2) (1998) 383–408.