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The provably terminating operations of the subsystem PETJ of explicit mathematics

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ABSTRACT

In Spescha and Strahm (2009) [15], a system PET of explicit mathematics in the style of Feferman (1975, 1978) [6,7] is introduced, and in Spescha and Strahm (in press) [16] the addition of the join principle to PET is studied. Changing to intuitionistic logic, it could be shown that the provably terminating operations of $\mathsf{PETJ}^{\mathsf{i}}$ are the polytime functions on binary words. However, although strongly conjectured, it remained open whether the same holds true for the corresponding theory PETJ with classical logic. This note supplements a proof of this conjecture.

1. Introduction

In Spescha and Strahm [15], a system PET of explicit mathematics in the style of Feferman [6,7] is presented. PET, a natural extension of the first order applicative theory PT introduced and analyzed by Strahm [17,18], formalizes a weak explicit type system with restricted elementary comprehension so that its provably terminating operations coincide with the functions on binary words that are computable in polynomial time. In Spescha [14] and Spescha and Strahm [16] the addition of the join principle to PET is studied. Changing to intuitionistic logic, it could be shown that the provably terminating operations of $\mathsf{PETJ}^{\mathsf{i}}$ are the polytime functions on binary words. However, although strongly conjectured, it remained open whether the same holds true for the corresponding theory PETJ with classical logic. This note supplements a proof of this conjecture. More precisely, we show that for each term t,

$$\mathsf{PT} \vdash t \in (\mathsf{W}^n \to \mathsf{W}) \iff \mathsf{PETJ} \vdash t \in (\mathsf{W}^n \to \mathsf{W}),$$

where $t \in (\mathsf{W}^n \to \mathsf{W})$ is short for $(\forall x_1, \ldots, x_n \in \mathsf{W})(tx_1 \cdots x_n \in \mathsf{W})$. By [17], we then know that $\mathsf{PETJ} \vdash t \in (\mathsf{W}^n \to \mathsf{W})$ iff there exists a polytime function $\mathcal{F} : \mathbb{W}^n \to \mathbb{W}$ on the standard words so that for all $w_1, \ldots, w_n \in \mathbb{W}$, $\mathsf{PT} \vdash \overline{\mathcal{F}(w_1, \ldots, w_n)} = t\overline{w}_1 \cdots \overline{w}_n$.

PT is an applicative theory that formalizes a combinatory algebra featuring the combinators k and s, and additionally describes the structure of binary words $(\mathbb{W}, \epsilon, 0, 1, S_0, S_1, +, \times, \sqsubseteq)$ equipped with the two successor functions $S_i(w) = wi$ ($i \in \{0, 1\}$), word concatenation, word multiplication and an initial subword relation. This is implemented by adding a unary relation symbol W to recognize words, and terms to perform the indicated operation and to decide the subword relation. Further, $s \leq t := 0 \times s \sqsubseteq 0 \times t$ and $s \leq_{\mathsf{W}} t := \mathsf{W}(s) \wedge s \leq t$.

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Further, the theory PT is equipped with the following induction principle: if $g \in (\mathsf{W} \to \mathsf{W})$, B is positive and W-free and $X := \{x : (\exists y \leq_{\mathsf{W}} gx)B(g, x, y)\}$, then

$$\mathsf{Prog}_{\sqsubset}(X) \ \to \ \mathsf{W} \subseteq X, \qquad (\Sigma^b_{\mathsf{W}}\text{-}\mathsf{I})$$

where $\mathsf{Prog}_{\sqsubset}(X) := \epsilon \in X \wedge (\forall x \in \mathsf{W})(x \in X \to \mathsf{s}_0 x \in X \wedge \mathsf{s}_1 x \in X)$. This partly technically motivated induction principle allows us to prove that each polytime function can be represented by a term $t_F \in (\mathsf{W}^n \to \mathsf{W})$. We consider it more convenient to work with the induction principle ($\mathfrak{S}$-I) instead, which states induction for so-called *simple* formulas: if A is a positive formula that does not contain the variable v, then A^v is simple, where A^v is obtained from A by replacing each occurrence of $\mathsf{W}(s)$ by $s \leq_{\mathsf{W}} v$. Accordingly, for each word b, the class $X := \{x : A^b(x)\}$ is called *simple*. The word b is also referred to as a *bound for* X. The theory $\mathsf{PT}_{\mathfrak{S}}$ is obtained from PT by replacing its induction principle by induction on notation for simple classes. We will see that $\mathsf{PT}_{\mathfrak{S}}$ proves each instance of the induction principle of PT and still has the same provably terminating operations.

However, more important to the determination of the provably total operations of PETJ is the observation that the provably terminating operations of $\mathsf{PT}_{\mathfrak{S}}$ are not affected by a further strengthening with the bounding principle (BP) which asserts that each simple class $X \subseteq \mathsf{W}$ is bounded by some word w: for each simple formula $A^v(u)$ of the language L of PT,

$$(\forall b \in \mathsf{W})(\exists c \in \mathsf{W})[\forall x(A^b(x) \to x \in \mathsf{W}) \to \forall x(A^b(x) \to x \leq c)]. \qquad (\mathsf{BP})$$

To formulate the theory PET, the language of PT is enriched by second order variables intended to range over types, which are tied to the first order part by a naming relation $R(U, s)$, stating that s is a name of the type U. Further, there are additional constants to generate names of types. The type existence principles of PET are such that each simple class is a type. Therefore, it is straightforward to extend a model of $\mathsf{PT}_{\mathfrak{S}}$ to a model of PET where each type is simple. However, if a is a name and $(\forall x \,\dot{\in}\, a)\exists X \mathsf{R}(X, gx)$, then the additional join principle of PETJ claims that $X := \{(x, y) : x \,\dot{\in}\, a \wedge y \,\dot{\in}\, gx\}$ is a type. Now each sequence $(w_x : x \,\dot{\in}\, a)$ of bounds for the types $(gx : x \,\dot{\in}\, a)$ may be unbounded in W and X may not be simple. Yet, as we shall show, for each model $\mathbb{E}'$ of PT+ (BP) there is an elementary equivalent model $\mathbb{E}$ that extends to a model of PETJ where each type is simple. This validates the aforementioned conjecture.

The provably terminating operations of PT+ (BP) are determined considering the auxiliary theory $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I}$, which proves each instance of (BP) and whose provably total functions coincide with those of PT. Thereby, $\mathsf{PT}^{\dagger}_{\mathfrak{S}}$ extends $\mathsf{PT}_{\mathfrak{S}}$ by asserting that exponentiation is not a total operation on words, i.e. $\exp \notin (\mathsf{W} \to \mathsf{W})$, where exp is a fixed term so that $\exp \epsilon = 0$ and $\exp(wi) = \exp w + \exp w$ for each word w and $i \in \{0, 1\}$. And the induction principle (I) applies also to classes that are defined by searching for a word in the downset $X{\downarrow}$ of some simple class $X \subseteq \mathsf{W}$, where $X{\downarrow} := \{x : (\exists y \in X)(x \leq_{\mathsf{W}} y)\}$. $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I}$ then entails (BP): the strengthening of $\mathsf{PT}_{\mathfrak{S}}$ by the assertion $\exp \notin (\mathsf{W} \to \mathsf{W})$ ensures that W is not simple, for otherwise, also $Y := \{x : (\exists y \in \mathsf{W})(\exp x = y)\}$ would be simple, and $\exp \in (\mathsf{W} \to \mathsf{W})$ would follow by ($\mathfrak{S}$-I). And $\exp \notin (\mathsf{W} \to \mathsf{W})$ in conjunction with (I) ensures that no simple class $X \subseteq \mathsf{W}$ is unbounded, for otherwise, $X{\downarrow} = W$ and the progressivity of Y together with (I) would imply $W \subseteq Y$.

We would like to point out that a similar strengthening for applicative theories is used in Probst [13] to prove the existence of pseudo-hierarchies in subsystems of explicit mathematics. There, the subsystem EMA of explicit mathematics which formalizes a Mahlo universe is strengthened to $\mathsf{EMA}^{\dagger}$ by the assertion $\neg\mathsf{TI}_{\lhd}(|\mathsf{T}|)$, stating that transfinite induction up to the proof-theoretic ordinal of EMA fails (which by the way is $\varphi\omega 00$ as shown in Jäger and Strahm [12]). That the provably total operations of PT and $\mathsf{PT}^{\dagger}$ coincide can be seen as an analog to the observation that EMA and $\mathsf{EMA}^{\dagger}$ have still the same proof-theoretic ordinal (cf. Jäger and Probst [11]).

The paper is organized as follows: Section 2 recalls the theory PT, Section 3 takes a first glance at the theory PETJ, and in Section 4 we present the boundedness principle (BP) and the auxiliary theory $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I}$ which proves each instance of (BP). The interlude in Section 5 elaborates on the observation that PT_0 and PT prove the same positive $\exists$-sequents (sequents without $\forall$-quantifiers), where PT_0 is PT with induction restricted to formulas without $\forall$-quantifiers. In Section 6, we show that $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I}$ and PT + (UP) prove the same positive sequents, where (UP) denotes Cantini's uniformity principle (cf. [4]) that claims for each positive formula $A(u, v)$,

$$\forall x(\exists y \in \mathsf{W})A(x, y) \to (\exists y \in \mathsf{W})\forall x A(x, y). \qquad (\mathsf{UP})$$

Further, we prove that $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I}$ and PT_0 derive the same positive $\exists$-sequents; in particular they still have the same provably terminating operations. In Section 7 we then conclude with a transformation of a model of PT into a model of PETJ which preserves the validity of first order formulas.

2. The theory PT

The theory PT is introduced and analyzed in Strahm [17]. Among other things, it is shown there that each polytime function can be represented by a term of PT, and further, by providing a realization interpretation of PT in the open term model $\mathcal{M}(\lambda\eta)$, that the provably total operations of PT are the operation on words computable in polynomial time. As introduced in [17], the underlying logic of PT is the logic of partial terms due to Beeson [1,2]. From a conceptual point of view, it is natural to regard terms as possibly non-terminating operations, yet from a technical viewpoint it is simpler to deal with the extension PT + (tot) which claims that application is total (i.e. $\forall x, y(x \cdot y){\downarrow}$) so that each term has a value. This allows us to formulate a corresponding theory that we also denote by PT, but whose underlying logic is classical predicate logic. The presentation of this theory is the purpose of this section.

The theory PT is an applicative theory that formalizes a combinatory algebra featuring the combinators k and s, and in addition, specifies the structure $(\mathbb{W}, \epsilon, 0, 1, S_0, S_1, +, \times, \sqsubseteq)$, where $\mathbb{W} = \{0,1\}^*$ is the set of finite binary words, ϵ is the empty word, $S_0(w) = w0$ and $S_1(w) = w1$ are the two successor functions, $v+w = vw$ is word concatenation and $v\times w$ is word multiplication which concatenates v the length $|w|$ of w times with itself. Further, $v \sqsubseteq w$ states that v is an initial subword of w, that is, w is of the form vv' for some word v'. PT is formulated in the first order language L that contains the constants k, s (combinators), p, p_0, p_1 (pairing and projections) $\mathsf{d_W}$ (definition by cases on binary words), ϵ (empty word), s_0, s_1 (binary successors), $\mathsf{p_W}$ (binary predecessor), $+$, $\times$ (word concatenation and multiplication), and $\mathsf{c}_\sqsubseteq$ (initial subword relation). Further, L is equipped with a unary relation symbol W (binary words), a binary relation symbol $=$ (equality) and a binary function symbol $\cdot$ (application). The lower case letters $a, b, c, x, y, z, u, v, w, g, h, \ldots$ (possibly with subscripts) are used to denote variables. The terms of L, usually denoted by $r, s, t, \ldots$, are inductively generated from the variables and the constants by means of application $\cdot$. We keep writing st or $s(t)$ for $\cdot(s,t)$, and similarly $s+t$ and $s\times t$ for $+st$ and $\times st$. The formulas $A, B, C, \ldots$ are built from the atoms $s = t$, $\mathsf{W}(s)$ and the negated atoms $\sim(s = t)$, $\sim\mathsf{W}(s)$ by closing under conjunction, disjunction and quantification. *Positive* formulas are build from (positive) atoms only. $\sim(s = t)$ is usually written as $s \neq t$ and negation is defined using de Morgan's law and the law of double negation. The connectives $\rightarrow$ and $\leftrightarrow$ are defined in the usual way. Formulas that do not contain the relation symbol W are called W-free. As usual, we write $t[\vec{s}/\vec{u}]$ and $A[\vec{s}/\vec{u}]$ for the term and formula obtained from t and A by substituting all occurrences of the terms $\vec{s}$ for the variables $\vec{u}$. If a formula was introduced as $A(\vec{u})$, then $A(\vec{s})$ is short for $A[\vec{s}/\vec{u}]$. Further, $0 := \mathsf{s}_0\epsilon$, $1 := \mathsf{s}_1\epsilon$, $(s,t) := \mathsf{p}st$, $(s_1, \ldots, s_n, s_{n+1}) := (s_1, (s_2, \ldots, s_{n+1}))$, $(s)_0 := \mathsf{p}_0 s$, $(s)_1 := \mathsf{p}_1 s$, $s \sqsubseteq t := \mathsf{c}_\sqsubseteq st = 0$, $s \le t := 0\times s \sqsubseteq 0\times t$, and $\mathsf{W}(t_1, \ldots, t_n)$ and $t_1, \ldots, t_n \in \mathsf{W}$ are shorthand notations for $\mathsf{W}(t_1), \ldots, \mathsf{W}(t_n)$. Finally, $s \le_{\mathsf{W}} t := \mathsf{W}(s) \wedge s \le t$.

The underlying logic of PT is classical logic. Following Strahm [17], we formulate PT in Gentzen's classical sequent calculus LK. We assume that the reader is familiar with LK as it is presented, for example, in Girard [10]. Sequents are formal expressions of the form $\Gamma \Rightarrow \Delta$, where $\Gamma, \Delta, \ldots$ range over finite sequences of formulas of L. As usual, the intended interpretation of a sequent $A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n$ is that $\bigwedge_{1\le i\le n} A_i$ entails $\bigvee_{1\le i\le m} B_i$. Further, if $\Gamma = A_1, \ldots, A_n$, then $\mathrm{SET}(\Gamma) = \{A_1, \ldots, A_n\}$. Instead of $\emptyset \Rightarrow \Gamma$ and $\Gamma \Rightarrow \emptyset$, we just write Γ and $\Gamma \Rightarrow$, respectively. Also, we just mention the main formulas of the axioms, that is, if $\Gamma \Rightarrow \Delta$ is a displayed sequent in a subsequent list of axioms, then also $\Gamma' \Rightarrow \Delta'$ is an axiom if $\mathrm{SET}(\Gamma) \subseteq \mathrm{SET}(\Gamma')$ and $\mathrm{SET}(\Delta) \subseteq \mathrm{SET}(\Delta')$. Similarly for rules: the inference

$$\frac{\Gamma_i \Rightarrow \Delta_i \ (i \in I)}{\Gamma \Rightarrow \Delta} \ !u!$$

indicates that

$$\frac{\Gamma_i' \Rightarrow \Delta_i' \ (i \in I)}{\Gamma' \Rightarrow \Delta'}$$

is a rule provided

(i) $u \notin \mathsf{FV}(\Gamma', \Delta')$,
(ii) $\mathrm{SET}(\Gamma_i) \subseteq \mathrm{SET}(\Gamma_i')$ and $\mathrm{SET}(\Delta_i) \subseteq \mathrm{SET}(\Delta_i')$ $(i \in I)$,
(iii) $\mathrm{SET}(\Gamma) \cup \bigcup_{i\in I}(\mathrm{SET}(\Gamma_i') - \mathrm{SET}(\Gamma_i)) \subseteq \mathrm{SET}(\Gamma')$,
(iv) $\mathrm{SET}(\Delta) \cup \bigcup_{i\in I}(\mathrm{SET}(\Delta_i') - \mathrm{SET}(\Delta_i)) \subseteq \mathrm{SET}(\Delta')$.

The logical rules of LK are listed below. t ranges over terms, A over atoms and B, C range over formulas of L:

$$\frac{\Rightarrow}{A \Rightarrow A} \qquad \frac{\Rightarrow B}{\neg B \Rightarrow}\ (\neg\mathrm{L}) \qquad \frac{B \Rightarrow}{\Rightarrow \neg B}\ (\neg\mathrm{R}) \qquad \frac{\Rightarrow B \quad B \Rightarrow}{\Rightarrow}$$

$$\frac{\Rightarrow B, C}{\Rightarrow B \vee C} \qquad \frac{B, C \Rightarrow}{B \wedge C \Rightarrow} \qquad \frac{\Rightarrow B \quad \Rightarrow C}{\Rightarrow B \wedge C} \qquad \frac{B \Rightarrow \quad C \Rightarrow}{B \vee C \Rightarrow}$$

$$\frac{\Rightarrow B(t)}{\Rightarrow \exists x B(x)} \qquad \frac{\Rightarrow B(u)}{\Rightarrow \forall x B(x)}\ !u! \qquad \frac{B(u) \Rightarrow}{\exists x B(x) \Rightarrow}\ !u! \qquad \frac{B(t) \Rightarrow}{\forall x B(x) \Rightarrow}$$

$$\frac{\Rightarrow B, C}{\Rightarrow C, B} \qquad \frac{B, C \Rightarrow}{C, B \Rightarrow} \qquad \frac{\Rightarrow B, B}{\Rightarrow B} \qquad \frac{B, B \Rightarrow}{B \Rightarrow}.$$

Next, we present the non-logical axioms of PT^-, i.e. PT without induction. Sequents are separated by ";" and r, s, t range over terms of L. The theory PT^- comprises the axioms listed below.

(i) $s = s$; $s = t \Rightarrow t = s$; $s = t, t = r \Rightarrow s = r$; $s = t, \mathsf{W}(s) \Rightarrow \mathsf{W}(t)$.
(ii) $\mathsf{k}st = s$; $\mathsf{s}rst = (rt)(st)$; $\mathsf{p}_0(s,t) = s$; $\mathsf{p}_1(s,t) = t$.
(iii) $\mathsf{W}(r), \mathsf{W}(s), r = s \Rightarrow \mathsf{d_W}t_1t_2rs = t_1$; $\mathsf{W}(r), \mathsf{W}(s) \Rightarrow r = s, \mathsf{d_W}t_1t_2rs = t_2$.

(iv) $\mathsf{W}(\epsilon)$; $\mathsf{p_W}\epsilon = \epsilon$; $\mathsf{W}(s) \Rightarrow \mathsf{W}(\mathsf{s}_i s)$ $(i \in \{0, 1\})$; $\mathsf{W}(s) \Rightarrow \mathsf{W}(\mathsf{p_W}(s))$.
$\mathsf{W}(s) \Rightarrow \mathsf{p_W}(\mathsf{s}_i s) = s$ $(i \in \{0, 1\})$; $\mathsf{W}(s) \Rightarrow \epsilon = s, \mathsf{s}_0(\mathsf{p_W} s) = s, \mathsf{s}_1(\mathsf{p_W} s) = s$.
$\mathsf{W}(s), \mathsf{s}_0 s = \mathsf{s}_1 s \Rightarrow$; $\mathsf{W}(s), \mathsf{s}_0 s = \epsilon \Rightarrow$; $\mathsf{W}(s), \mathsf{s}_1 s = \epsilon \Rightarrow$.

(v) $\mathsf{W}(r), \mathsf{W}(s) \Rightarrow \mathsf{c}_\sqsubseteq rs = 0, \mathsf{c}_\sqsubseteq rs = 1$; $\mathsf{W}(s), \mathsf{c}_\sqsubseteq s\epsilon = 0 \Rightarrow s = \epsilon$ and $\mathsf{c}_\sqsubseteq \epsilon\epsilon = 0$.
$\mathsf{W}(r), \mathsf{W}(s), \mathsf{c}_\sqsubseteq rs = 0 \Rightarrow \mathsf{c}_\sqsubseteq r(\mathsf{p_W} s) = 0, r = s$.
$\mathsf{W}(r), \mathsf{W}(s), \mathsf{c}_\sqsubseteq r(\mathsf{p_W} s) = 0 \Rightarrow \mathsf{c}_\sqsubseteq rs = 0$.

(vi) $\mathsf{W}(r), \mathsf{W}(s) \Rightarrow \mathsf{W}(r{+}s)$,
$\mathsf{W}(s) \Rightarrow s{+}\epsilon = s$; $\mathsf{W}(r), \mathsf{W}(s) \Rightarrow r{+}(\mathsf{s}_i s) = \mathsf{s}_i(r{+}s)$ $(i \in \{0, 1\})$.

(vii) $\mathsf{W}(r), \mathsf{W}(s) \Rightarrow \mathsf{W}(r{\times}s)$,
$\mathsf{W}(s) \Rightarrow s{\times}\epsilon = \epsilon$ and $\mathsf{W}(r), \mathsf{W}(s) \Rightarrow r{\times}(\mathsf{s}_i s) = (r{\times}s){+}s$ $(i \in \{0, 1\})$.

We point out that the main formulas of all non-logical axioms and rules (including equality) are positive, so that a standard partial cut-elimination argument allows us to restrict to derivations where all cut-formulas are positive. If we are only interested in positive sequents, then also both $\neg$-rules are admissible. We write $\mathrel{|\!\!\frac{\;\;}{*}}\ \Gamma \Rightarrow \Delta$ to indicate that $\Gamma \Rightarrow \Delta$ is the end-sequent of a derivation where all cut-formulas are positive and no $\neg$-rule is used.

Lemma 1 (*Partial Cut-Elimination*). *If $\Gamma \Rightarrow \Delta$ is a positive sequent of* L *formulas, then* $\mathsf{PT} \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathsf{PT} \mathrel{|\!\!\frac{\;\;}{*}} \Gamma \Rightarrow \Delta$.

The following two lemmas are folklore, too.

Lemma 2 (*λ-Abstraction*). *For each* L *term t and all variables u there is an* L *term $\lambda x.t[x/u]$ with $\mathsf{FV}(\lambda x.t[x/u]) = \mathsf{FV}(t){-}\{u\}$, so that PT^- proves $(\lambda x.t[x/u])u = t$.*

Lemma 3 (*Recursion*). *There exists a closed term* rec *of* L *so that PT^- proves $\mathsf{rec} fx = f(\mathsf{rec} f)x$.*

One of the design goals of PT has been that the provably total functions on words are exactly the polytime functions on words as characterized by Cobham [5]. Depending on constants $\vec{c} = c_1, \ldots, c_m$, meant to denote words, we inductively define a set $\mathcal{PT}_{\vec{c}}$ of function symbols. In a second step, we assign to each function symbol $F^n \in \mathcal{PT}_{\vec{c}}$ (where the superscript ^{n} indicates that F is an n-ary function symbol) a function $\mathcal{F} : \mathbb{W}^n \to \mathbb{W}$ on the standard words and an L term $f \in (\mathsf{W}^n \to \mathsf{W})$ that represents $\mathcal{F}$, i.e. for all $\vec{w} \in \mathbb{W}$,

$$\mathsf{PT}_{\vec{c}} \vdash \overline{\mathcal{F}(\vec{w})} = f(\overline{w}_1, \ldots, \overline{w}_n),$$

where $\mathsf{T}_{\vec{c}}$ denotes the theory $\mathsf{T} + \mathsf{W}(\vec{c})$, $\overline{\epsilon} := \epsilon$, and $\overline{wi} := \mathsf{s}_i \overline{w}$ $(i \in \{0, 1\})$. The set $\mathcal{PT}_{\vec{c}}$ contains the 0-ary function symbols c_i $(1 \le i \le m)$, function symbols P^n_i $(n > i)$ for projections, and binary function symbols for $+$, $\times$ and the characteristic function $\mathsf{CHR}^2_\sqsubseteq$ of $\sqsubseteq$ restricted to words. Further, if $G^m, F^n_1, \ldots, F^n_m \in \mathcal{PT}_{\vec{c}}$, then $\mathsf{CMP}(G, F_1, \ldots, F_m)$ is an n-ary function symbol which is in $\mathcal{PT}_{\vec{c}}$, and if $F^{n+2}_0, F^{n+2}_1, G^n, B^{n+1} \in \mathcal{PT}_{\vec{c}}$, then $\mathsf{BRC}(F, G, B)$ is an $n{+}1$-ary function symbol which is in $\mathcal{PT}_{\vec{c}}$. If $\vec{c} = \epsilon, 0, 1$, then we just write $\mathcal{PT}$ for $\mathcal{PT}_{\vec{c}}$, and if $\vec{c} = \epsilon, 0, 1, \alpha$, then we write $\mathcal{PT}_\alpha$ instead. In the following, α will denote a non-standard word so that $\exp \alpha \notin \mathsf{W}$.

With each symbol $F^n \in \mathcal{PT}$ we associate a function $\mathsf{fun}(F) : \mathbb{W}^n \to \mathbb{W}$ in the expected way. If F, G, H are function symbols, then we simply write $\mathcal{F}, \mathcal{G}, \mathcal{H}$ instead of $\mathsf{fun}(F), \mathsf{fun}(G), \mathsf{fun}(H)$ for the corresponding functions. To the constants $\epsilon, 0, 1$ we assign $\epsilon, 0, 1 \in \mathbb{W}$, $\mathcal{P}_i(w_1, \ldots, w_n) := w_i$, $\mathsf{CHR}_\sqsubseteq(v, w) \in \{0, 1\}$ and $\mathsf{CHR}_\sqsubseteq(v, w) = 0$ iff $v \sqsubseteq w$, and $+$ and $\times$ are word concatenation and word multiplication. If $F^n = \mathsf{CMP}(G^m, F^n_1, \ldots, F^n_m)$, then $\mathcal{F} := \mathsf{CMP}(\mathcal{G}, \mathcal{F}_1, \ldots, \mathcal{F}_m)$ and if $H^{n+1} = \mathsf{BRC}(F^{n+2}_0, F^{n+2}_1, G^n, B^{n+1})$, then $\mathcal{H} := \mathsf{BRC}(\mathcal{F}_0, \mathcal{F}_1, \mathcal{G}, \mathcal{B})$. Thereby,

$$\mathcal{F}(\vec{x}) := \mathcal{G}(\mathcal{F}_1(\vec{x}), \ldots, \mathcal{F}_m(\vec{x})),$$
$$\mathcal{H}(\vec{x}, \epsilon) := \mathcal{G}(\vec{x}), \text{ and}$$
$$\mathcal{H}(\vec{x}, yi) := \mathcal{F}_i(\vec{x}, y, \mathcal{H}(\vec{x}, y))|\mathcal{B}(\vec{x}, y) \text{ for } i \in \{0, 1\},$$

where $x|z$ denotes the truncation of x to the length $|z|$ of z. The set of polytime functions on $\mathbb{W}$ is given by $\{\mathcal{F} : F \in \mathcal{PT}\}$.

To represent the polytime functions in $\mathsf{PT}_{\vec{c}}$, we assign to each function symbol $F^n \in \mathcal{PT}_{\vec{c}}$ a closed term $f := \mathsf{Term}(F)$ of $\mathsf{L}(\vec{c})$. Again, we write f, g, h instead of $\mathsf{Term}(F), \mathsf{Term}(G), \mathsf{Term}(H)$. For a constant c_i, $\mathsf{Term}(c_i) := c_i$, $\mathsf{Term}(S_i) := \mathsf{s}_i$ $(i \in \{0, 1\})$ and $\mathsf{chr}_\sqsubseteq := \lambda x.\mathsf{c}_\sqsubseteq(x)_0(x)_1$. Further,

$$\mathsf{Term}(\mathsf{CMP}(G, F_1, \ldots, F_n)) := \mathsf{cmp}(g, f_1, \ldots, f_n),$$
$$\mathsf{Term}(\mathsf{BRC}(F_0, F_1, G, B)) := \mathsf{brc}(f_0, f_1, g, b),$$

where cmp, brc and $\mathsf{Term}(\mathcal{P}_i)$ satisfy the expected equations. PT^- proves that such terms exist. Strahm [17] gives an explicit construction of these terms. Induction on notations is required to show that for $F^{n+2}_0, F^{n+2}_1, G^n, B^{n+1} \in \mathcal{PT}_{\vec{c}}$,

$$f_0 \in (\mathsf{W}^{n+2} \to \mathsf{W}) \wedge f_1 \in (\mathsf{W}^{n+2} \to \mathsf{W}) \wedge g \in (\mathsf{W}^n \to \mathsf{W}) \wedge b \in (\mathsf{W}^{n+1} \to \mathsf{W})$$
$$\to \ \mathsf{brc}(f_0, f_1, g, b) \in (\mathsf{W}^{n+1} \to \mathsf{W}).$$

In particular, one has to prove that

$$(\forall y \in \mathsf{W})(\exists z \le_{\mathsf{W}} b(\vec{x}, y))(\mathsf{brc}(f_0, f_1, g, b)(\vec{x}, y) = z).$$

Therefore, PT is equipped with the induction rule (Σ^b_{W}-I), which claims for each formula $A(u) := (\exists y \leq_{\mathsf{W}} tu)B(t, u, y)$, $y \notin \mathsf{FV}(t)$, B positive and W-free,

$$\frac{\mathsf{W}(u) \Rightarrow \mathsf{W}(tu) \quad \Rightarrow A(\epsilon) \quad \mathsf{W}(u), A(u) \Rightarrow A(\mathsf{s}_i u)}{\mathsf{W}(s) \Rightarrow A(s)}\ !u!. \tag{Σ^b_{W}-I}$$

It is detailed in [17] how this induction principle entails that $f \in (\mathsf{W}^n \to \mathsf{W})$ for each $F^n \in \mathcal{PT}$. Analogously, $\mathsf{PT}_{\vec{c}}$ proves $f \in (\mathsf{W}^n \to \mathsf{W})$ for each $F^n \in \mathcal{PT}_{\vec{c}}$. Hence, we can assign to each such function symbol F^n and each model $\mathbb{E} = (\mathcal{V}, \mathsf{W}, \ldots)$ of $\mathcal{PT}_{\vec{c}}$ a function $\mathcal{F} := \mathsf{fun}(F, \mathbb{E}) : \mathsf{W}^n \to \mathsf{W}$, $\mathcal{F}(w_1, \ldots, w_n) := f^{\mathbb{E}}(w_1, \ldots, w_n)$. Whether $\mathcal{F}$ is a function on the standard words $\mathbb{W}$ or the words W of some non-standard model will be clear from the context or of no importance.

3. The theories PETJ

As in Spescha and Strahm [15], and Spescha [14], the first order part of PETJ is the applicative theory PT. Note that the present version of PET also contains a constant all that serves as dual to the type-forming operation dom.

The theory PETJ is formulated in the language L^2 that extends L by second order variables $U, V, W \ldots X, Y, Z$, a binary relation $s \in U$ (elementhood), a binary relation $\mathsf{R}(U, s)$ (s is a name of the type U), and fresh constants id, $\mathsf{i_w}$, dom, all, inv, un, int and j to generate names of types. The additional atoms of L^2 are the expressions of the form $s \in U$ and $\mathsf{R}(U, s)$, where s is a term of L. The formulas of L^2 are then built from the atoms as before, but additionally we close under second order quantification. The formula $U = V$ is an abbreviation for $\forall x[x \in U \leftrightarrow x \in V]$. Further, we write $\mathsf{R}(s)$ for $\exists X \mathsf{R}(X, s)$, $s \doteq Z$ for $\exists X[\mathsf{R}(X, s) \wedge X = Z]$, $s \doteq t$ for $\exists X[\mathsf{R}(X, s) \wedge \mathsf{R}(X, t)]$, and $s \dot{\in} t$ for $\exists X[\mathsf{R}(X, t) \wedge s \in X]$. The logical axioms and rules of PETJ are those for classical second oder logic, and the non-logical axioms and rules of PETJ are those of PT without induction, plus the following three groups of axioms concerning types. PET is PETJ without the axioms for the constant j, and PET^- and PETJ^- are PET and PETJ without type induction, respectively.

The ontological axioms of PETJ:

$$\exists x \mathsf{R}(U, x) \text{ and } \mathsf{R}(U, u) \wedge U = V \to R(V, u) \text{ and } \mathsf{R}(U, u) \wedge R(V, u) \to U = V.$$

The type forming axioms of PETJ:

(i) $\mathsf{id} \doteq \{(x, x) : x = x\}$.
(ii) $u \in \mathsf{W} \to \mathsf{i_w} u \doteq \{x : x \leq_{\mathsf{W}} u\}$.
(iii) $\mathsf{R}(u) \to \mathsf{all}\, u \doteq \{x : \forall y[(x, y) \dot{\in} u]\}$.
(vi) $\mathsf{R}(u) \to \mathsf{dom}\, u \doteq \{x : \exists y[(x, y) \dot{\in} u]\}$.
(v) $\mathsf{R}(u) \wedge \mathsf{R}(v) \to \mathsf{int}(u, v) \doteq \{x : x \dot{\in} u \wedge x \dot{\in} v\}$.
(vi) $\mathsf{R}(u) \wedge \mathsf{R}(v) \to \mathsf{un}(u, v) \doteq \{x : x \dot{\in} u \vee x \dot{\in} v\}$.
(vii) $\mathsf{R}(u) \to \mathsf{inv}(u, g)) \doteq \{x : gx \dot{\in} u\}$.
(viii) $\mathsf{R}(u) \wedge g \in (u \to \mathsf{R}) \to \mathsf{j}(u, g) \doteq \{(x, y) : x \dot{\in} u \wedge y \dot{\in} gx\}$, where $g \in (u \to \mathsf{R})$ is short for $(\forall x \dot{\in} u)\mathsf{R}(gx)$.

The induction axiom of PETJ:

$$\mathsf{Prog}_{\sqsubset}(U) \to \mathsf{W} \subseteq U, \tag{T-I}$$

where $\mathsf{Prog}_{\sqsubset}(U) := \epsilon \in U \wedge (\forall x \in \mathsf{W})(x \in U \to \mathsf{s}_0 x \in U \wedge \mathsf{s}_1 x \in U)$ expresses that U is progressive.

A structure $\mathcal{M} = (\mathbb{E}, \mathcal{R}, \ldots)$ for L^2 consists of a structure $\mathbb{E} = (\mathsf{V}_{\mathbb{E}}, \mathsf{W}_{\mathbb{E}}, \ldots)$ for L, an interpretation $\mathcal{R} \subseteq \mathsf{Pow}(\mathsf{V}_{\mathbb{E}}) \times \mathsf{V}_{\mathbb{E}}$ of the naming relation R and interpretations $\underline{\mathsf{id}}, \underline{\mathsf{i_w}}, \underline{\mathsf{all}}, \underline{\mathsf{dom}}, \underline{\mathsf{inv}}, \underline{\mathsf{int}}, \underline{\mathsf{un}}, \underline{\mathsf{j}}$ of the constants id, $\mathsf{i_w}$, all, dom, inv, int, un, j. Note that $\mathcal{R}$ also specifies the domain $\mathcal{T} = \{X : \exists x \mathcal{R}(X, x)\}$ of the type variables. Then, $\in$ is interpreted as the standard elementhood relation restricted to $\mathsf{V}_{\mathbb{E}} \times \mathcal{T}$. Occasionally, we identify $\mathcal{R}$ with its range, that is, $x \in \mathcal{R}$ is then read as $\exists X \mathcal{R}(X, x)$. $\mathcal{M}$ is a model of PETJ if it satisfies all axioms and rules of PETJ.

4. Extensions of PT and the bounding principle (BP)

The theory $\mathsf{PT}_{\mathfrak{S}}$ is obtained from PT by replacing the induction rule (Σ^b_{W}-I) by an induction rule ($\mathfrak{S}$-I) that claims induction on notation for so-called *simple* formulas. If A is a positive formula of L and u does not occur in A, then A^u is obtained from A by replacing each occurrence of an expression $\mathsf{W}(s)$ by $s \leq_{\mathsf{W}} u$. The formula $B(u) := A^u$ is then called *simple* w.r.t. u, and we write $B(u) \in \mathfrak{S}(u)$. The class $\mathfrak{S}(u)$ of simple formulas w.r.t. u is closed under conjunction, disjunction and quantification of variables different from u.[1] An alternative definition of the class $\mathfrak{S}(u)$ is given below.

Lemma 4. *For each variable u, the class of simple formulas w.r.t. u, denoted by $\mathfrak{S}(u)$, is inductively defined as follows:*

(i) *If $u \notin \mathsf{FV}(s = t)$ and $u \notin \mathsf{FV}(r)$, then $s = t$ and $r \leq_{\mathsf{W}} u$ are in $\mathfrak{S}(u)$.*
(ii) *If A and B are in $\mathfrak{S}(u)$, then $A \wedge B$ and $A \vee B$ are in $\mathfrak{S}(u)$.*
(iii) *If A is in $\mathfrak{S}(u)$, then $\forall x A[x/v]$ and $\exists x A[x/v]$ are in $\mathfrak{S}(u)$.*

---

[1] Simple formulas are essentially Σ^b_{T} formulas (cf. Spescha and Strahm [15]) with $\forall$-quantifiers.

The induction rule of $\mathsf{PT}_{\mathfrak{S}}$ states that for each L formula $A(u, w) \in \mathfrak{S}(w)$,

$$\frac{\Rightarrow A(\epsilon, w) \qquad A(u, w) \Rightarrow A(\mathsf{s}_i u, w) \quad (i \in \{0, 1\})}{\mathsf{W}(w), \mathsf{W}(s) \Rightarrow A(s, w)}\ !u!. \tag{$\mathfrak{S}$-I}$$

Further, we say that a class X is *simple*, if there is a formula $A(u, v) \in \mathfrak{S}(v)$ and a word a so that $X = \{x : A(x, a)\}$.

Lemma 5. $\mathsf{PT}_{\mathfrak{S}}$ *proves each instance of* (Σ^b_{W}-I).

Proof. As shown in Spescha [14], there is a closed term max of L (which is a feasible functional in the sense of Cook and Kapron) so that $\mathsf{PT}_{\mathfrak{S}}$ proves

$$t \in (\mathsf{W} \to \mathsf{W}) \to (\forall x, y \in \mathsf{W})[x \sqsubseteq y \to tx \leq_{\mathsf{W}} \mathsf{max}tx \leq_{\mathsf{W}} \mathsf{max}ty].$$

Let $X := \{x : (\exists y \leq_{\mathsf{W}} tx)B(y, t, x)\}$ and assume that $y \notin \mathsf{FV}(t)$, B positive and W-free, and $\mathsf{Prog}_{\sqsubset}(X)$. If s is a word, then

$$Y := \{x : \mathsf{c}_{\sqsubseteq}xs = 1 \vee (x \sqsubseteq s \wedge (\exists y \leq_{\mathsf{W}} \mathsf{max}ts)(y \leq tx)B(y, t, x))\}$$

is also progressive and simple, thus $s \in Y$. But this implies $s \in X$. □

Without affecting the provably total operations of $\mathsf{PT}_{\mathfrak{S}}$, we can extend $\mathsf{PT}_{\mathfrak{S}}$ by the bounding principle (BP) which asserts that each simple class $X \subseteq \mathsf{W}$ is bounded by some word w: for each formula $A(u, v) \in \mathfrak{S}(v)$, referred to below as $A^v(u)$,

$$(\forall b \in \mathsf{W})(\exists c \in \mathsf{W})[\forall x(A^b(x) \to x \in \mathsf{W}) \to \forall x(A^b(x) \to x \leq c)]. \tag{BP}$$

We shall see in Section 7 that for each model $\mathbb{E}$ of PT+ (BP), there is an elementary equivalent model $\mathbb{E}' \equiv \mathbb{E}$ so that the simple classes of $\mathbb{E}'$ are the types of a model $(\mathbb{E}', \mathcal{R})$ of PETJ.

To handle the theory PT+ (BP) we have a look at the extension $\mathsf{PT}^{\dagger}_{\mathfrak{S}}+\mathsf{I}$ of $\mathsf{PT}_{\mathfrak{S}}$ which proves that each simple subclass of W is bounded in W, but cannot prove more operations to be total than $\mathsf{PT}_{\mathfrak{S}}$. To introduce $\mathsf{PT}^{\dagger}_{\mathfrak{S}}+\mathsf{I}$, we fix an L term exp so that, provably in PT^-, $\exp \epsilon = 0$ and $\exp(wi) = \exp w + \exp w$ for each word w and $i \in \{0, 1\}$. Further, if T is an L theory, then $\mathsf{T}^{\dagger}$ is formulated in the language $\mathsf{L}(\alpha)$, comprises the axioms of T adjusted to the new language $\mathsf{L}(\alpha)$, and, in addition,

$$\Rightarrow \mathsf{W}(\alpha) \qquad \text{and} \qquad \mathsf{W}(\exp \alpha) \Rightarrow . \tag{$\neg$(exp)}$$

$\mathsf{PT}^{\dagger}_{\mathfrak{S}}+\mathsf{I}$ extends $\mathsf{PT}^{\dagger}_{\mathfrak{S}}$ by the induction rule

$$\frac{A(u, w) \Rightarrow \mathsf{W}(u) \quad \Rightarrow C(\epsilon, w) \quad C(u, w) \Rightarrow C(\mathsf{s}_i u, w)}{\mathsf{W}(w), \mathsf{W}(s) \Rightarrow C(s, w)}\ !u! \tag{I}$$

where a $A(u, w) \in \mathfrak{S}(w)$ and $C(u, w) := \exists y[A(y, w) \wedge \exists z(z \leq_{\mathsf{W}} y \wedge B(u, w, y, z))]$ with B positive and W-free.

Lemma 6. $\mathsf{PT}^{\dagger}_{\mathfrak{S}}+\mathsf{I}$ *proves each instance of* (BP).

Proof. Suppose that $X \subseteq \mathsf{W}$ is simple but unbounded. Then

$$Y := \{x : \exists y(y \in X \wedge \exists z(z \leq_{\mathsf{W}} y \wedge z = \exp x))\}$$

is progressive. Applying (I) yields $\exp \in (\mathsf{W} \to \mathsf{W})$. A contradiction! □

5. Interlude: positive theories and ∀-quantifiers

This section elaborates on the following observation made in Strahm [17]. For the theory PT, universal quantifiers do not play a role in the sense that the only non-logical rule that may contain universal quantifiers, the induction rule (Σ^b_{W}-I), can be restricted to formulas that do not contain universal quantifiers, and the resulting theory still has the same provably total operations as PT. The question suggests itself, whether a tailored fragment without universal quantifiers of a theory whose non-logical axioms can be presented by positive sequents still proves the same positive sequents without universal quantifiers as the full theory.

Let L_1 be a first order language, that besides other function and relation symbols, contains constants ($\mathsf{e}_i : i \in \mathbb{N}$). The idea is that in all theories under consideration below, the constants e_i are *free*: that is, if $\Gamma \Rightarrow \Delta$ is an axiom, then so is $\Gamma[s/\mathsf{e}_i] \Rightarrow \Delta[s/\mathsf{e}_i]$, for each term s, and accordingly for rules. In other words, when considering models, we are free to assign any value to a constant e_i. Positive formulas are built from (positive) atoms only, by closing under ∧, ∨ and quantifications. A sequent $\Gamma \Rightarrow \Delta$ is positive if it contains only positive formulas. Further, a ∃-sequent does not contain universal quantifiers, and a sequent $\Gamma \Rightarrow \Delta$ is *good* if it is positive and, in addition, Γ does not contain universal quantifiers. A theory T is good, if all its axioms and rules can be presented by good sequents. Note that a good theory can comprise Cantini's uniformity principle (cf. [4]) if it is presented as a rule: for each positive formula $A(u, v)$,

$$\frac{\Gamma \Rightarrow \forall x(\exists y \in W)A(x, y), \Delta}{\Gamma \Rightarrow (\exists y \in W)\forall x A(x, y), \Delta}.$$

A restriction $\restriction$ is a substitution that replaces each universally bound variable in a formula by a constant e_i so that variables bound by the same $\forall$-quantifier are replaced by the same constant. LK^{e} is LK without both $\neg$-rules, and with all logical axioms and rules restricted to positive $\exists$-sequents. As a substitute for the left $\forall$-rule, we take all its restrictions,

$$\frac{\Gamma, A(s) \Rightarrow \Delta}{\Gamma, A(\mathsf{e}_i) \Rightarrow \Delta}, \quad \text{for each constant } \mathsf{e}_i. \tag{e_i-left}$$

The right $\forall$-rule is dropped as all its restrictions are admissible. $\mathsf{LK}^{\exists}$ is LK^{e} without the e_i-rules.

To a positive theory T we assign a theory T^{e} that derives positive $\exists$-sequents as follows: the underlying logic of T^{e} is LK^{e}, and its non-logical axioms and rules are all the restrictions of the non-logical axioms and rules of T. Further, $\mathsf{T}^{\exists}$ is T^{e} with the underlying logic $\mathsf{LK}^{\exists}$. In this section, we will prove the following theorem.

Theorem 7. *Let T be a good L_1 theory. If $\Gamma \Rightarrow \Delta$ is a positive $\exists$-sequent, then*

$$\mathsf{T} \vdash \Gamma \Rightarrow \Delta \implies \mathsf{T}^{\exists} \vdash \Gamma \Rightarrow \Delta.$$

We start with the following observation.

Lemma 8. *Let T be a positive theory and $\Gamma \Rightarrow \Delta$ a positive sequent of L_1 formulas. If $\mathsf{T} \vdash \Gamma \Rightarrow \Delta$, then $\mathsf{T}^{\mathsf{e}} \vdash \Gamma\restriction \Rightarrow \Delta\restriction$ for all restrictions $\restriction$.*

Proof. By induction on the depth of the proof. In the case of a left $\forall$-rule, the I.H. and an application of a left e-rule yield the claim. In the case of a right $\forall$-rule, we have $\mathsf{T} \overset{n}{\vdash} \Gamma \Rightarrow A(u), \Delta$. Then, also $\mathsf{T} \overset{n}{\vdash} \Gamma \Rightarrow A(\mathsf{e}_i), \Delta$ for each i. By I.H. $\mathsf{T}^{\mathsf{e}} \overset{n}{\vdash} \Gamma\restriction \Rightarrow A(\mathsf{e}_{i_0})\restriction, \Delta\restriction$ for an i_0 so that $\forall x A(x)\restriction = A(\mathsf{e}_{i_0})\restriction$. In the case of the other rules, the I.H. applies directly. □

If T is a good theory, then the system $\mathsf{T}^{\mathsf{e}}_{\sigma}$ derives so-called annotated sequents. An annotated sequent $\Sigma \Rightarrow \Delta$ is a sequent, where a natural number is assigned to each occurrence of a constant e_i in Σ. For instance, we write $A(\mathsf{e}_i^n) \Rightarrow$ to indicate that the number n is assigned to the displayed occurrence of e_i. Further, we just write e_i instead of e_i^0. A sequent $\Sigma \Rightarrow \Delta$ is an axiom of $\mathsf{T}^{\mathsf{e}}_{\sigma}$ iff deleting all annotations yields an axiom of T^{e} and each constant e_i in Σ is annotated by 0. Further, a rule is a rule of $\mathsf{T}^{\mathsf{e}}_{\sigma}$ iff the corresponding rule with the annotations deleted is a rule of T^{e} and the annotations remain unchanged except for the following cases. The annotation of the constant e_i introduced by an e_i-rule has to be bigger than all annotations in the term t that is substituted by e_i, i.e.

$$\frac{A(t(\mathsf{e}_{i_1}^{n_1}, \ldots, \mathsf{e}_{i_k}^{n_k})) \Rightarrow}{A(\mathsf{e}_i^n) \Rightarrow} \quad \text{where } n > n_i \text{ for } 1 \le i \le k.$$

In the case of contraction and a context sharing rule, the bigger annotation is kept,

$$\frac{A(\mathsf{e}_{i_1}^{n_1}, \ldots, \mathsf{e}_{i_k}^{n_k}), A(\mathsf{e}_{i_1}^{m_1}, \ldots, \mathsf{e}_{i_k}^{m_k}) \Rightarrow}{A(\mathsf{e}_{i_1}^{l_1}, \ldots, \mathsf{e}_{i_k}^{l_k}) \Rightarrow} \quad \text{and} \quad \frac{A(\mathsf{e}^n), B \Rightarrow \quad A(\mathsf{e}^m), C \Rightarrow}{A(\mathsf{e}^l), B \vee C \Rightarrow},$$

where for $1 \le i \le k$, $l_i = \max(n_i, m_i)$ and $l = \max(n, m)$. The weight of an annotated formula is the sum of all its annotations. $\mathsf{T} \overset{n}{\underset{k}{\vdash}} \Gamma \Rightarrow \Delta$ states that the depth of the derivation is n and that the cut-rule is only applied if the weight of the cut-formula is less than k or if no constant e_i occurs in the cut-formula.

Lemma 9 (e-*Substitution*)**.** *If $\Gamma \Rightarrow \Delta$ is a positive $\exists$-sequent and t an e-free term, then*

$$\mathsf{T}^{\mathsf{e}}_{\sigma} \overset{n}{\underset{k}{\vdash}} \Gamma \Rightarrow \Delta \implies \mathsf{T}^{\mathsf{e}}_{\sigma} \underset{k}{\vdash} \Gamma[t/\mathsf{e}_i^0] \Rightarrow \Delta[t/\mathsf{e}_i].$$

Proof. By induction on n. If for instance $\Sigma, A(\mathsf{e}_i^n) \Rightarrow \Delta$ is obtained from $\Sigma, A(s) \Rightarrow \Delta$ by a left e_i-rule, then the I.H. yields $\Sigma[t/\mathsf{e}_i^0], A(s[t/\mathsf{e}_i^0])[t/\mathsf{e}_i^0] \Rightarrow \Delta[t/\mathsf{e}_i]$, and now the left e_i-rule yields $\Sigma[t/\mathsf{e}_i^0], A(\mathsf{e}_i^n)[t/\mathsf{e}_i^0] \Rightarrow \Delta[t/\mathsf{e}_i]$. If $\Sigma \Rightarrow B \wedge C, \Delta$ is obtained from $\Sigma_l \Rightarrow B, \Delta$ and $\Sigma_r \Rightarrow C, \Delta$, where Σ_l and Σ_r differ only w.r.t. the annotation, then the I.H. yields

$$\Sigma_l[t/\mathsf{e}_i^0] \Rightarrow B[t/\mathsf{e}_i], \Delta[t/\mathsf{e}_i] \quad \text{and} \quad \Sigma_r[t/\mathsf{e}_i^0] \Rightarrow C[t/\mathsf{e}_i], \Delta[t/\mathsf{e}_i].$$

Observe that a formula $A(\mathsf{e}_i^0)$ may be in Σ_l and $A(\mathsf{e}_i^1)$ may be the corresponding formula in Σ_r. In this case, after applying the $\wedge$-rule, $A(t)[t/\mathsf{e}_i^0], A(\mathsf{e}_i^1)[t/\mathsf{e}_i^0]$ are side formulas in the conclusion of this rule. An application of the e_i-rule and contraction help to obtain the claim. □

Lemma 10. *Suppose that $\mathsf{T}^{\mathsf{e}}_{\sigma} \overset{m}{\underset{k}{\vdash}} \Sigma, A(\mathsf{e}_i^n) \Rightarrow \Delta$. Then there are e-free terms $t_1, \ldots, t_l$ and $n' < n$ so that*

$$\mathsf{T}^{\mathsf{e}}_{\sigma} \underset{k}{\vdash} \Sigma, A(t_1), \ldots, A(t_l), A(\mathsf{e}_i^{n'}) \Rightarrow \Delta.$$

Proof. A simple induction on the depth of the derivation. □

Lemma 11 (e-*Cut Elimination*)**.** *Suppose that $\Sigma \Rightarrow \Delta$ is an $\exists$-sequent which is e-free:*

$$\mathsf{T}^{\mathsf{e}}_{\sigma} \overset{n}{\underset{k}{\vdash}} \Sigma \Rightarrow \Delta \implies \mathsf{T}^{\mathsf{e}}_{\sigma} \underset{0}{\vdash} \Sigma \Rightarrow \Delta.$$

Proof. Main induction on the weight k, side induction on the depth n. The only case where the I.H. does not apply directly is if

$$\frac{\mathsf{T}^{\mathsf{e}}_{\sigma} \mid\!\frac{n}{k'}\; \Sigma, A(\mathsf{e}^m) \Rightarrow \Delta \quad \mathsf{T}^{\mathsf{e}}_{\sigma} \mid\!\frac{n}{k'}\; \Sigma \Rightarrow A(\mathsf{e}), \Delta}{\mathsf{T}^{\mathsf{e}}_{\sigma} \mid\!\frac{n+1}{k}\; \Sigma \Rightarrow \Delta},$$

where we have picked an annotated constant e^m in A with $m > 0$. Lemma 10 yields $\mathsf{T}^{\mathsf{e}}_{\sigma} \mid\!\frac{}{k}\; \Sigma, A(t_1), \ldots, A(t_l), A(\mathsf{e}^{m'}) \Rightarrow \Delta$ and the I.H. and the e-substitution lemma imply that $\mathsf{T}^{\mathsf{e}}_{\sigma} \mid\!\frac{}{0}\; \Sigma \Rightarrow A(s), \Delta$ for each e-free term s. A couple of cuts yield $\mathsf{T}^{\mathsf{e}}_{\sigma} \mid\!\frac{}{m}\; \Sigma \Rightarrow \Delta$. Since $m < k$, the claims follow by the I.H. □

The theorem follows: if T is good and $\Gamma \Rightarrow \Delta$ is a positive $\exists$-sequent so that $\mathsf{T} \vdash \Gamma \Rightarrow \Delta$, then $\mathsf{T}^{\mathsf{e}}_{\sigma} \mid\!\frac{}{0}\; \Gamma \Rightarrow \Delta$. As $\Gamma \Rightarrow \Delta$ is e-free, no use of a left e-rule was made. Therefore, $\mathsf{T}^{\exists} \vdash \Gamma \Rightarrow \Delta$.

Let us apply the theorem. PT_0 is the theory PT with induction restricted to formulas without universal quantifiers. Then, $\mathsf{PT}^{\exists}$ is contained in PT_0.

Corollary 12. *If $\Gamma \Rightarrow \Delta$ is a positive $\exists$-sequent so that $\mathsf{PT} \vdash \Gamma \Rightarrow \Delta$, then also $\mathsf{PT}_0 \vdash \Gamma \Rightarrow \Delta$.*

6. The provably terminating operations of $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I}$

In this section, we show that the provably terminating operations of $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I}$ are still the polytime functions. Actually, we prove a bit more, namely that PT_0 and $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I}$ prove the same positive $\exists$-sequents, and that $\mathsf{PT} + (\mathsf{UP})$ and $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I} + (\mathsf{UP})$ prove the same positive sequents, where (UP) is Cantini's uniformity principle (cf. [4]) that claims for each positive formula $A(u, v)$,

$$\forall x(\exists y \in \mathsf{W})A(x, y) \Rightarrow (\exists y \in \mathsf{W})\forall x A(x, y). \tag{UP}$$

To obtain this result, we translate a sequent $\Gamma \Rightarrow \Delta$ which is provable in $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I}$ into a sequent which is provable in PT. This translation is inspired by a realizability interpretation in the style of Cantini [3] and Strahm [17]. However, in contrast to the realizability interpretation applied in [3,17] our translation does not depend on an open term model $\mathcal{M}(\lambda\eta)$ of PT, and allows us to handle extensions of PT that are consistent but no longer true in $\mathcal{M}(\lambda\eta)$, such as $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I}$.

To define the realizability interpretation we fix closed L terms pair, wl, wr so that (provably in PT) $\mathsf{pair} \in (\mathsf{W}^2 \to \mathsf{W})$, and for $s, t \in \mathsf{W}$, $\mathsf{wl}(\mathsf{pair}st)) = s$ and $\mathsf{wr}(\mathsf{pair}st) = t$. To be specific, we let pair be such that for all words w_0, w_1, $\mathsf{pair}w_0w_1 = w_0^*00w_1^*$, where $\epsilon^* := \epsilon$ and $wi^* := w^*1i$ for $i \in \{0, 1\}$.

Definition 13. To each positive $\mathsf{L}(\alpha)$ formula A, we assign a positive and W-free formula $u\ \mathbf{r}\ A$ with an additional fresh free variable u. To increase readability, we write $\langle s, t\rangle$ for $\mathsf{pair}st$, $u = \langle w_0, w_1\rangle \wedge A(w_0, w_1)$ for $u = \langle \mathsf{wl}u, \mathsf{wr}u\rangle \wedge A(\mathsf{wl}u, \mathsf{wr}u)$, and, similarly, $u = \langle i, w\rangle \wedge A(i, w)$ is to abbreviate $u = \langle \mathsf{wl}u, \mathsf{wr}u\rangle \wedge \mathsf{wl}u = \bar{i} \wedge A(\mathsf{wl}u, \mathsf{wr}u)$.

(i) $u\ \mathbf{r}\ \mathsf{W}(t) := u = t$,
(ii) $u\ \mathbf{r}\ s = t := u = \epsilon \wedge s = t$,
(iii) $u\ \mathbf{r}\ A \wedge B := u = \langle w_0, w_1\rangle \wedge w_0\ \mathbf{r}\ A \wedge w_1\ \mathbf{r}\ B$,
(iv) $u\ \mathbf{r}\ A \vee B := u = \langle i, w_1\rangle \wedge [(i = 0 \wedge w_1\ \mathbf{r}\ A) \vee (i = 1 \wedge w_1\ \mathbf{r}\ B)]$,
(v) $u\ \mathbf{r}\ \forall x A(x) := \forall x(u\ \mathbf{r}\ A(x))$,
(vi) $u\ \mathbf{r}\ \exists x A(x) := \exists x(u\ \mathbf{r}\ A(x))$.

To a positive sequent $\Gamma \Rightarrow \Delta$ with $\Gamma = A_0, \ldots, A_{m-1}$ and $\Delta = B_0, \ldots, B_{n-1}$, we then assign the sequent

$$u_0\ \mathbf{r}\ A_0, \ldots, u_{m-1}\ \mathbf{r}\ A_{m-1} \Rightarrow \bigvee_{0 \le i < n} [u = \langle \mathsf{base}_2(i), w_1\rangle \wedge w_1\ \mathbf{r}\ B_i],$$

where $\mathsf{base}_2(i)$ is $\overline{w}$ for w the binary representation of the number i. Moreover, the variables $u, u_0, \ldots, u_{m-1}$ are pairwise different and do not occur free in Γ, Δ. The left hand part of the sequent displayed above is often abbreviated as $\vec{u}\ \mathbf{r}\ \Gamma$, and the formula on the right as $u\ \mathbf{r}\ \Delta$.

By a simple induction on the build-up of positive formulas, the following observation is made.

Lemma 14. *If A is a positive L formula, then $\mathsf{PT}^- \vdash (\exists x \in \mathsf{W})(x\ \mathbf{r}\ A) \to A$.*

The converse direction fails. When trying to prove the claim by induction on the build-up of A, in the case that $A = \forall y B(y)$, the I.H. yields that $\forall y(\exists x \in \mathsf{W})(x\ \mathbf{r}\ B(y))$, yet we cannot infer $(\exists x \in \mathsf{W})\forall y(x\ \mathbf{r}\ B(y))$, that is $(\exists x \in \mathsf{W})(x\ \mathbf{r}\ \forall y B(y))$, unless we employ (UP).

Lemma 15. $\mathsf{PT}^- + (\mathsf{UP}) \vdash A \leftrightarrow (\exists x \in \mathsf{W})(x\ \mathbf{r}\ A)$, *for each positive L formula A.*

Recall that T_α extends a theory T by the assertion $\mathsf{W}(\alpha)$. For instance, if T is $\mathsf{PT}_{\mathfrak{S}}$, then T_α is $\mathsf{PT}^{\dagger}_{\mathfrak{S}}$ without the axiom $\mathsf{W}(\exp\alpha) \Rightarrow$. Because T_α only states that α is a word, $\mathsf{T}_\alpha \vdash \Gamma \Rightarrow \Delta$ implies $\mathsf{T} \vdash \Gamma[0/\alpha] \Rightarrow \Delta[0/\alpha]$. An induction on the depth of the derivation also yields that $\mathsf{T}^{\dagger} \vdash \Gamma \Rightarrow \Delta$ implies $\mathsf{T}_\alpha \vdash \Gamma \Rightarrow \Delta, \mathsf{W}(\exp\alpha)$.

Theorem 16. *Suppose that $A_0 \dots, A_{m-1} \Rightarrow B_0, \dots, B_{n-1}$ is a positive sequent of* L*. If* $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I} + (\mathsf{UP}) \vdash_{*} A_0 \dots, A_{m-1} \Rightarrow B_0, \dots, B_{n-1}$ *and* $\vec{u} = u_0, \dots, u_{m-1}$*, then there is an* $F^m \in \mathcal{PT}_\alpha$ *so that*

$$\mathsf{PT}^{\dagger} \vdash \mathsf{W}(u_0), \dots, \mathsf{W}(u_{m-1}), \vec{u}\ \mathbf{r}\ A_0, \dots, A_{m-1} \Longrightarrow f(\vec{u})\ \mathbf{r}\ B_0, \dots, B_{n-1}.$$

In the above theorem, we can replace $\mathsf{PT}^{\dagger}_{\mathfrak{S}}$ and $\mathsf{PT}^{\dagger}$ by $\mathsf{PT}_{\mathfrak{S},\alpha}$ and PT_α, since the axiom $\mathsf{W}(\exp\alpha) \Rightarrow$ is only needed to derive the sequent $\mathsf{W}(u), u\ \mathbf{r}\ \mathsf{W}(\exp\alpha) \Rightarrow$.

Corollary 17. *Let $\Gamma \Rightarrow \Delta$ be a positive and $\Gamma' \Rightarrow \Delta'$ a positive $\exists$-sequent of* L*. Then*

(i) $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I} + (\mathsf{UP}) \vdash \Gamma \Rightarrow \Delta$ *implies* $\mathsf{PT} + (\mathsf{UP}) \vdash \Gamma \Rightarrow \Delta$,
(ii) $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I} + (\mathsf{UP}) \vdash \Gamma' \Rightarrow \Delta'$ *implies* $\mathsf{PT}_0 \vdash \Gamma' \Rightarrow \Delta'$.

Proof. If $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I} + (\mathsf{UP}) \vdash \Gamma \Rightarrow \Delta$, then by the above Theorem there is an $F \in \mathcal{PT}_\alpha$ so that $\mathsf{PT}_\alpha + (\mathsf{UP}) \vdash \mathsf{W}(\vec{u}), \vec{u}\ \mathbf{r}\ \Gamma \Rightarrow f(\vec{u})\ \mathbf{r}\ \Delta$. As $\Gamma \Rightarrow \Delta$ is an L sequent, there is also an $F \in \mathcal{PT}$ so that $\mathsf{PT} + (\mathsf{UP}) \vdash \mathsf{W}(\vec{u}), \vec{u}\ \mathbf{r}\ \Gamma \Rightarrow f(\vec{u})\ \mathbf{r}\ \Delta$. Since PT proves that $f(\vec{u})$ is a word, the claim follows by Lemma 15. For the second claim, we use the result from the interlude that $\mathsf{PT}_0 \vdash \Gamma' \Rightarrow \Delta'$ iff $\mathsf{PT} \vdash \Gamma' \Rightarrow \Delta'$. □

Corollary 18. *If t is a closed term of* L *(so it does not contain the constant α) and* $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I} \vdash \mathsf{W}(\vec{v}) \Rightarrow \mathsf{W}(t(\vec{v}))$*, then there is a polytime function symbol $F \in \mathcal{PT}$ so that for all $\vec{w} \in \mathbb{W}$,*

$$\mathsf{PT} \vdash \overline{\mathcal{F}(w_1, \dots, w_n)} = t(\overline{w}_1, \dots, \overline{w}_n) = f(\overline{w}_1, \dots, \overline{w}_n).$$

Proof. Assume that t is a closed L term and $\mathsf{PT}^{\dagger}_{\mathfrak{S}} + \mathsf{I} \vdash \mathsf{W}(\vec{v}) \Rightarrow \mathsf{W}(t(\vec{v}))$. Then, $\mathsf{PT}_{\mathfrak{S},\alpha} + \mathsf{I} \vdash \mathsf{W}(\vec{v}) \Rightarrow \mathsf{W}(t(\vec{v})), \mathsf{W}(\exp\alpha)$. By Theorem 16, there are $F, G, H \in \mathcal{PT}_\alpha$ so that $\mathcal{F}(\vec{w}) = \langle \mathcal{G}(\vec{w}), \mathcal{H}(\vec{w}) \rangle$, and

$$\mathsf{PT}_\alpha \vdash \mathsf{W}(\vec{u}) \Rightarrow g(\vec{u}) = 0 \wedge t(\vec{u}) = h(\vec{u}), \qquad g(\vec{u}) = 1 \wedge h(\vec{u}) = \exp\alpha.$$

Suppose that for $\vec{w} \in \mathbb{W}$, $\mathcal{G}(\vec{w}) = 1$. Then, $\mathsf{PT}_\alpha \vdash \mathsf{W}(\exp\alpha)$ which is impossible. Hence, $\mathcal{G}(\vec{w}) = 0$ for all $\vec{w} \in \mathbb{W}$, and thus for all $\vec{w} \in \mathbb{W}$, $\mathsf{PT}_\alpha \vdash C(t, \vec{w})$, where $C(t, \vec{w}) := h(\overline{w}_1, \dots, \overline{w}_n) = t(\overline{w}_1, \dots, \overline{w}_n)$. Since t is an L term, $\mathsf{PT} \vdash C(t, \vec{w})[0/\alpha]$, and there is a function symbol $H' \in \mathcal{PT}$, so that $h[0/\alpha] = h'$. □

Definition 19. We assign to each formula A of $\mathsf{L}(\alpha)$ a rank $\mathsf{rk}(A) \in 0 \times \mathbb{W}$. The rank of a literal is 0, the rank of $A j B$ is $00 \times (\mathsf{rk}(A) + \mathsf{rk}(B) + 0)$ and the rank of $\mathcal{Q}xA$ is $\mathsf{rk}(A)$.

Lemma 20. *Let $A(v) \in \mathfrak{S}(v)$ and B positive and* W*-free.*

(i) $\mathsf{PT}^- \vdash w \in \mathsf{W}, u\ \mathbf{r}\ A(w) \Rightarrow u \le \overline{\mathsf{rk}(A)} \times \max(0, w)$.
(ii) $\mathsf{PT}^- \vdash u\ \mathbf{r}\ B \Rightarrow u \le \overline{\mathsf{rk}(B)}$.

Proof. Both claims are shown by induction on the build up of the formula. □

Proof (*Theorem* 16)**.** The proof is by induction on the depth of the derivation. Note that the handling of the $\forall$-right rule and the $\exists$-left rule are directly by the definition of the translation and do not require a substitution lemma: for if v is different from $\vec{u}$ and does not occur free in $A_0, \dots, A_{m-1}, \forall x B_0(x), \dots, B_{n-1}$, then

$$\mathsf{PT}^{\dagger} \vdash \mathsf{W}(\vec{u}), \vec{u}\ \mathbf{r}\ A_0, \dots, A_{m-1} \Longrightarrow f(u_0 \cdots u_{m-1})\ \mathbf{r}\ B_0(v), \dots, B_{n-1}$$

entails

$$\mathsf{PT}^{\dagger} \vdash \mathsf{W}(\vec{u}), \vec{u}\ \mathbf{r}\ A_0, \dots, A_{m-1} \Longrightarrow f(u_0 \cdots u_{m-1})\ \mathbf{r}\ \forall x B_0(x), \dots, B_{n-1}.$$

Clearly, $\mathsf{PT}^{\dagger}$ proves the translation of the sequents $\Rightarrow \mathsf{W}(\alpha)$ and $\mathsf{W}(\exp\alpha) \Rightarrow$. And the translation of (UP) trivially holds as $u\ \mathbf{r}\ \forall x(\exists y \in \mathsf{W})A(x, y)$ is logically equivalent to $u\ \mathbf{r}\ (\exists y \in \mathsf{W})\forall x A(x, y)$. The logical rules and ($\mathfrak{S}$-I) are essentially handled as in the proof of the realization theorem in [17]. Therefore, we just consider the case when the last rule applied is an instance of (I).

We work informally in $\mathsf{PT}^{\dagger}$ and tacitly use that $f \in (\mathsf{W}^n \to \mathsf{W})$ for each $F^n \in \mathcal{PT}_\alpha$ and that induction for W-free and positive formulas is available. In addition, we commit a slight abuse of notation in that we write $(f(\vec{s}))_0$ and $(f(\vec{s}))_1$ for $\mathsf{wl}(f(\vec{s}))$ and $\mathsf{wr}(f(\vec{s}))$ if $F \in \mathcal{PT}_\alpha$.

Assume that last rule applied is an instance of (I). Then, the I.H. provides polytime function symbols $F_\iota, F_\epsilon, F_0, F_1$ so that $\mathsf{PT}^{\dagger}$ proves the sequents

$$\mathsf{W}(\vec{a}, b), \vec{a}, b\ \mathbf{r}\ \Gamma, A(u, w) \Rightarrow f_\iota(\vec{a}, b)\ \mathbf{r}\ \mathsf{W}(u), \Delta,$$
$$\mathsf{W}(\vec{a}), \vec{a}\ \mathbf{r}\ \Gamma \Rightarrow f_\epsilon(\vec{a})\ \mathbf{r}\ C(\epsilon), \Delta,$$
$$\mathsf{W}(\vec{a}, b), \vec{a}, b\ \mathbf{r}\ \Gamma, C(u, w) \Rightarrow f_i(\vec{a}, u)\ \mathbf{r}\ C(\mathsf{s}_i u, w), \Delta \qquad i \in \{0, 1\}.$$

It is assumed that $C(u, w)$ is of the form $\exists y[A(y, w) \wedge \exists z(z \le_{\mathsf{W}} y \wedge B(u, y, z))]$ with $A(u, w) \in \mathfrak{S}(w)$ and B positive and W-free. Further, we suppose that the atom $w \in \mathsf{W}$ is an element of the sequence Γ. Our task is to find a function symbol F so that

$$\mathsf{PT}^{\dagger} \vdash \mathsf{W}(\vec{a}, b), \vec{a}\ \mathbf{r}\ \Gamma \Rightarrow f(\vec{a}, b)\ \mathbf{r}\ C(b), \Delta. \qquad (*)$$

Recall that $f(\vec{a}, b)$ is of the form $\langle i, w\rangle$, where i tells us which formula in the sequence on the right is realized. If $i = 0$, then w **r** $C(b)$, hence $w = \langle w_0, \langle\langle w_1, \epsilon\rangle, w_2\rangle\rangle$, and there is a y so that w_0 **r** $A(y)$, and further $w_1 \leq_{\mathsf{W}} y$ and w_2 **r** $B(b, y, w_1)$. By Lemma 20 we know that $w_0 \leq \mathsf{rk}(A) \times \max(0, \vec{a})$. Further, as shown e.g. in Ferreira [9], there is for each $F \in \mathcal{PT}$ a function symbol $F^+ \in \mathcal{PT}$ with the same ariety so that for all words $\vec{w}, \vec{v}, F(\vec{w}) \leq F^+(\vec{v})$ provided that $v_i \leq w_i$. It is completely straightforward to generalize this result to function symbols $F \in \mathcal{PT}_\alpha$. Now, let $H \in \mathcal{PT}_\alpha$ be such that for $t(\vec{a}) := \overline{\mathsf{rk}(A)} \times \max(0, \vec{a})$, $\mathsf{PT}^\dagger$ proves

$$h(\vec{a}, b) = f_\epsilon(\vec{a}) + f_\iota^+(\vec{a}, t(\vec{a})) + \langle 0, f_i^+(\vec{a}, \langle t(\vec{a}), \langle\langle f_\iota^+(\vec{a}, t(\vec{a})), \epsilon\rangle, \overline{\mathsf{rk}(B)}\rangle\rangle\rangle\rangle.$$

Next, let $F \in \mathcal{PT}_\alpha$ be such that $\mathsf{PT}^\dagger$ proves $F(\vec{a}, \epsilon) = F_\epsilon(\vec{a})$ and

$$f(\vec{a}, bi) = \begin{cases} f(\vec{a}, b)|h(\vec{a}, b) & : (f(\vec{a}, b))_0 \neq 0, \\ f_\iota(\vec{a}, (f(\vec{a}, b))_{1,0})|h(\vec{a}, b) & : (f(\vec{a}, b))_0 = 0, (f_\iota(\vec{a}, (f(\vec{a}, b))_{1,0}))_0 \neq 0, \\ f_i(\vec{a}, (f(\vec{a}, b))_1)|h(\vec{a}, b) & : (f(\vec{a}, b))_0 = 0, (f_\iota(\vec{a}, (f(\vec{a}, b))_{1,0}))_0 = 0. \end{cases}$$

Assume that $\mathsf{W}(\vec{a})$ and $\vec{a}$ **r** Γ. We show that for all words $b, f(\vec{a}, b)$ **r** $C(b), \Delta$ and $f(\vec{a}, b) < h(\vec{a}, b)$ by induction on b. For $b = \epsilon$ there is nothing to show. That the claim for bi ($i \in \{0, 1\}$) follows provided the claim holds for b is shown below, distinguishing the following two cases.

(i) $(f(\vec{a}, b))_0 \neq 0$. Then $f(\vec{a}, b)$ is of the form $\langle i, c\rangle$, $i > 0$ and c **r** B for some B in Δ. As $f(\vec{a}, bi) = f(\vec{a}, b)$, also $f(\vec{a}, bi)$ **r** $C(bi), \Delta$ and $f(\vec{a}, bi) < h(\vec{a}, b)$ by I.H.

(ii) $(f(\vec{a}, b))_0 = 0$. By I.H. $(f(\vec{a}, b))_1$ **r** $C(b)$. So $(f(\vec{a}, b))_1$ is of the form $w = \langle w_0, \langle\langle w_1, \epsilon\rangle, w_2\rangle\rangle$, and there is a y so that w_0 **r** $A(y)$ and $w_1 \leq_{\mathsf{W}} y$ and w_2 **r** $B(b, y, w_1)$. As $w_0 < \mathsf{rk}(A) \times \max(0, \vec{a}) = t(\vec{a}), f_\iota(\vec{a}, (f(\vec{a}, b))_{1,0}) \leq f_\iota^+(\vec{a}, t(\vec{a})) < h(\vec{a}, b)$. In the case $(f_\iota(\vec{a}, (f(\vec{a}, b))_{1,0}))_0 \neq 0, f(\vec{a}, bi) < h(\vec{a}, b)$, and $(f(\vec{a}, bi))_1$ **r** B for some B in Δ.
If $(f_\iota(\vec{a}, (f(\vec{a}, b))_{1,0}))_0 = 0$, then $w_1 \leq y = f_\iota(\vec{a}, w_0)$, and

$$f_i(\vec{a}, (f(\vec{a}, b))_1) = f_i(\vec{a}, \langle w_0, \langle\langle w_1, \epsilon\rangle, w_2\rangle\rangle) \leq f_i^+(\vec{a}, \langle t(\vec{a}), \langle\langle f_\iota^+(\vec{a}, t(\vec{a})), \epsilon\rangle, \mathsf{rk}(B)\rangle\rangle) < h(\vec{a}, b).$$

Thus, $f(\vec{a}, bi) < h(\vec{a}, b)$, and $(f(\vec{a}, bi))_1$ **r** $C(bi)$. □

7. From a model of PT + (BP) to a model of PETJ

In this section, we show that each model $\mathbb{E}' = (\mathsf{V}_{\mathbb{E}'}, \mathsf{W}_{\mathbb{E}'}, \ldots)$ of PT+ (BP) has an elementary extension $\mathbb{E} \succ \mathbb{E}'$ that expands to a model $\mathcal{M} = (\mathbb{E}, \mathcal{R}, \ldots)$ of PETJ where each type is simple. Hence, PETJ does not prove more L formulas than PT+ (BP), and the provably terminating operations of PETJ are still the polytime functions. Recall that $\mathbb{E} \succ \mathbb{E}'$ if $\mathsf{V}_{\mathbb{E}'} \subseteq \mathsf{V}_{\mathbb{E}}$, $\mathsf{W}_{\mathbb{E}'} \subseteq \mathsf{W}_{\mathbb{E}}$ and $R^{\mathbb{E}}\restriction\mathbb{E}' = R^{\mathbb{E}'}$ for each relation and function symbol R of L. In addition, each sentence A of the language $\mathsf{L}(c_v : v \in \mathsf{V}_{\mathbb{E}'})$ which extends L by constants for each element of $\mathsf{V}_{\mathbb{E}'}$ holds in $\mathbb{E}'$ iff it holds in $\mathbb{E}$.

There is a standard method introduced by Feferman in [8] to construct an interpretation for the naming relation and the types over an applicative structure such as PT+ (BP). Using this method, each model $\mathbb{E}' = (\mathsf{W}_{\mathbb{E}'}, \mathsf{V}_{\mathbb{E}'})$ of PT+ (BP) is easily expanded to a model $\mathcal{M} = (\mathbb{E}', \mathcal{R}, \ldots)$ of PETJ^-. As interpretations for the additional constants id, $\mathsf{i_W}$, all, dom, inv, int, un and j, the values of the following closed terms in the model $\mathbb{E}'$ are chosen: $\underline{\mathsf{id}} := \epsilon$, $\underline{\mathsf{i_W}} := \lambda x.(\epsilon, x)$, $\underline{\mathsf{all}} := \lambda x.(0, x)$, $\underline{\mathsf{dom}} := \lambda x.(1, x)$, $\underline{\mathsf{inv}} := \lambda xy.(00, (x, y))$, $\underline{\mathsf{int}} := \lambda xy.(01, (x, y))$, $\underline{\mathsf{un}} := \lambda xy.(10, (x, y))$ and $\underline{\mathsf{j}} := \lambda xy.(11, (x, y))$. Then, the naming relation $\mathcal{R} \subseteq \mathsf{Pow}(\mathsf{V}_{\mathbb{E}'}) \times \mathsf{V}_{\mathbb{E}'}$ is built by a non-monotone inductive definition over the structure $\mathbb{E}'$, specified by the L(P) formula $R(\mathsf{P}, u, v, w) := R_0(\mathsf{P}, u, v, w) \vee R_1(\mathsf{P}, u, v, w)$ displayed below. The language L(P) extends L by a set constant P that serves as a placeholder and the elementhood relation $s \in \mathsf{P}$. For an ordinal $\alpha \in \mathsf{ON}$, the αth stage $I_\alpha^{\mathbb{E}',A}$ of the inductive definition over $\mathbb{E}'$ specified by the L(P) formula $A(\mathsf{P}, u_1, \ldots, u_n)$ is defined as follows:

$$I_\alpha^{\mathbb{E}',A} := I_{<\alpha}^{\mathbb{E}',A} \cup \{(u_1, \ldots, u_n) : A(I_{<\alpha}^{\mathbb{E}',A}, u_1, \ldots, u_n)\}, \quad \text{where } I_{<\alpha}^{\mathbb{E}',A} := \bigcup_{\beta<\alpha} I_\beta^{\mathbb{E}',A},$$

and the formula A is evaluated in the structure $\mathbb{E}'$. More precisely, $I_\alpha^{\mathbb{E}',A}$ is the union of $I_{<\alpha}^{\mathbb{E}',A}$ and $\{w \in \mathsf{V}_{\mathbb{E}'} : \mathbb{E}' \models \exists\vec{x}[w = (x_1, \ldots, x_n) \wedge A(I_{<\alpha}^{\mathbb{E}',A}, x_1, \ldots, x_n)]\}$. Further, $I^{\mathbb{E}',A} := \bigcup_{\alpha\in\mathsf{ON}} I_\alpha^{\mathbb{E}',A}$. A simple cardinality argument yields that there is a $\gamma \in \mathsf{ON}$ so that $I_\gamma^{\mathbb{E}',A} = I_{<\gamma}^{\mathbb{E}',A}$. It follows that $I^{\mathbb{E}',A} = I_{<\gamma}^{\mathbb{E}',A}$. Also, if $(x_1, \ldots, x_n) \in I^{\mathbb{E}',A}$, then there is a β so that $(x_1, \ldots, x_n) \notin I_{<\beta}^{\mathbb{E}',A}$ and $A(I_{<\beta}^{\mathbb{E}',A}, \vec{x})$.

The two disjuncts of R have the forms $R_0(\mathsf{P}, u, v, w) := v = w = 0 \wedge R_0'(\mathsf{P}, u)$ and $R_1(\mathsf{P}, u, v, w) := w = 1 \wedge R_1'(\mathsf{P}, u, v)$. To keep the definitions readable, we regard $s = (x, y) \wedge A(x, y)$ as a shorthand notation for $s = ((s)_0, (s)_1) \wedge A((s)_0, (s)_1)$. The idea is that a is a name iff $(a, 0, 0) \in I^{\mathbb{E}',R}$, and that $x \mathrel{\dot{\in}} a$ iff $(a, 0, 0), (a, x, 1) \in I^{\mathbb{E}',R}$. $R_0'(\mathsf{P}, u)$ is the disjunction of the clauses

(i) $u = \epsilon$,
(ii) $u = (\epsilon, x) \wedge \mathsf{W}(x)$,
(iii) $u = (0, a) \wedge (a, 0, 0) \in \mathsf{P}$,
(iv) $u = (1, a) \wedge (a, 0, 0) \in \mathsf{P}$,

(v) $u = (00, (a, g)) \wedge (a, 0, 0) \in \mathsf{P}$,
(vi) $u = (01, (a, b)) \wedge (a, 0, 0) \in \mathsf{P} \wedge (b, 0, 0) \in \mathsf{P}$,
(vii) $u = (10, (a, b)) \wedge (a, 0, 0) \in \mathsf{P} \wedge (a, 0, 0) \in \mathsf{P}$,
(viii) $u = (11, (a, g)) \wedge (a, 0, 0) \in \mathsf{P} \wedge \forall x[(a, x, 1) \in \mathsf{P} \to (gx, 0, 0) \in \mathsf{P}]$,

and $R_1'(\mathsf{P}, u, v)$, which contains P only positively, is the disjunction of the clauses

(i) $u = \epsilon \wedge v = (x, x)$,
(ii) $u = (\epsilon, x) \wedge v \leq_{\mathsf{W}} x$,
(iii) $u = (0, a) \wedge \forall y[(a, (v, y), 1) \in \mathsf{P}]$,
(iv) $u = (1, a) \wedge \exists y[(a, (v, y), 1) \in \mathsf{P}]$,
(v) $u = (00, (a, g)) \wedge (a, gv, 1) \in \mathsf{P}$,
(vi) $u = (01, (a, b)) \wedge (a, v, 1) \in \mathsf{P} \wedge (b, v, 1) \in \mathsf{P}$,
(vii) $u = (10, (a, b)) \wedge [(a, v, 1) \in \mathsf{P} \vee (b, v, 1) \in \mathsf{P}]$,
(viii) $u = (11, (a, g)) \wedge v = (x, y) \wedge (a, x, 1) \in \mathsf{P} \wedge (gx, y, 1) \in \mathsf{P}$.

The next lemma is immediate by induction on α and is used tacitly in the following.

Lemma 21. *For all ordinals α, we have $(a, x, 1) \in I_\alpha^{\mathbb{E}',R}$ iff $(a, x, 1) \in I_\alpha^{\mathbb{E}',R_1}$.*

Next, let $\mathcal{R}^{\mathbb{E}'} := \{(X, a) : (a, 0, 0) \in I^{\mathbb{E}',R} \wedge X = \{x : (a, x, 1) \in I^{\mathbb{E}',R}\}\}$ and $\mathcal{R}' := \mathcal{R}^{\mathbb{E}'}$. Again, we write $\mathcal{R}'(a)$ for $\exists X \mathcal{R}'(X, a)$ which is equivalent to $(a, 0, 0) \in I^{\mathbb{E}',R}$, and thus let $\mathcal{R}'_\alpha(a) := (a, 0, 0) \in I_\alpha^{\mathbb{E}',R}$ and $\mathcal{R}'_{<\alpha}(a) := (a, 0, 0) \in I_{<\alpha}^{\mathbb{E}',R}$. Further, $\mathsf{ext}'(a)$ refers to the collection $\{x : (a, x, 1) \in I^{\mathbb{E}',R}\}$; $\mathsf{ext}'_\alpha(a)$ and $\mathsf{ext}'_{<\alpha}(a)$ are defined accordingly. Finally, $x \mathbin{\dot\in}' a$ is short for $a \in \mathcal{R}' \wedge x \in \mathsf{ext}'(a)$, and $a \mathbin{\dot\subseteq}' X$ stands for $a \in \mathcal{R}' \wedge \mathsf{ext}'(a) \subseteq X$.

Lemma 22. *If $a \in \mathcal{R}'_\alpha$, then $x \in \mathsf{ext}'(a)$ implies $x \in \mathsf{ext}'_\alpha(a)$.*

Proof. By induction on α. Assume that the claim holds for all ordinals below α and let $a \in \mathcal{R}'_\alpha$. If $x \in \mathsf{ext}'(a)$, then $R_1(I_{<\beta}^{\mathbb{E}',R_1}, a, x, 1)$ for some β, e.g. there are $b, g \in \mathsf{W}_{\mathbb{E}'}$ so that $B(\beta, a, b, g, x)$ holds, where $B(\beta, a, b, g, x))$ is

$$a = (11, (b, g)) \wedge x = ((x)_0, (x)_1) \wedge (b, (x)_0, 1) \in I_{<\beta}^{\mathbb{E}',R} \wedge (g(x)_0, (x)_1, 1) \in I_{<\beta}^{\mathbb{E}',R}.$$

$a \in \mathcal{R}'_\alpha$ implies $b \in \mathcal{R}'_{<\alpha}$ and $(\forall z \in \mathsf{ext}'_{<\alpha}(b))(gz \in \mathcal{R}'_{<\alpha})$. By I.H. $(x)_0 \in \mathsf{ext}'_{<\beta}(b)$ entails $(x)_0 \in \mathsf{ext}'_{<\alpha}(b)$, which then implies that $g(x)_0 \in \mathcal{R}'_{<\alpha}$. Now $(x)_1 \in \mathsf{ext}'_{<\beta}(g(x)_0)$ and the I.H. yield $(x)_1 \in \mathsf{ext}'_{<\alpha}(g(x)_0)$. Therefore $B(\alpha)$. Thus $x \in \mathsf{ext}'_\alpha(a)$. □

Lemma 23. *If $\mathbb{E}'$ is a model of $\mathsf{PT}_{\mathfrak{S}}$, then $(\mathbb{E}', \mathcal{R}', \underline{\mathsf{id}}, \underline{\mathsf{i}}_{\mathsf{w}}, \ldots)$ is a model of PETJ^-.*

Proof. We just check the axiom for join. If $b \in \mathcal{R}'$ and $(\forall z \in \mathsf{ext}'(b))(gz \in \mathcal{R}')$, then $R(I^{\mathbb{E}',R}, (\underline{\mathsf{j}}(b, g), 0, 0))$, i.e. $\underline{\mathsf{j}}(b, g) \in \mathcal{R}'_\alpha$ for some α. Further, if $(x, y) \in \mathsf{ext}'(\underline{\mathsf{j}}(b, g))$, then by Lemma 22, $(x, y) \in \mathsf{ext}'_\alpha(\underline{\mathsf{j}}(b, g))$. By the definition of R_1 and Lemma 22, this holds iff $x \in \mathsf{ext}'(b)$ and $y \in \mathsf{ext}'(gx)$. □

Lemma 24. *Each model $\mathbb{E}'$ of $\mathsf{PT}_{\mathfrak{S}}$ has an elementary extension $\mathbb{E} \succ \mathbb{E}'$ so that $I_{<\omega}^{R,\mathbb{E}} = I^{R,\mathbb{E}}$.*

Proof. Let $\mathbb{E}'$ be a model of $\mathsf{PT}_{\mathfrak{S}}$. Below, we give a set T of formulas that are finitely realizable, i.e. for each finite subset $G \subseteq T$, there is a model $\mathbb{E}_0$ of $\mathsf{PT}_{\mathfrak{S}}$ and $\mathsf{w} \in \mathsf{W}_{\mathbb{E}_0}$, $\mathcal{F} \subseteq \mathsf{V}_{\mathbb{E}_0}$, so that for each formula $C(\mathsf{P}, \mathsf{p}) \in G$, $\mathbb{E}_0 \models C(\mathcal{F}, \mathsf{w})$. For each $n \in \mathbb{N}$, $(A_i(u_1, \ldots, u_n) : i \in \mathbb{N})$ is an enumeration of the $\mathsf{L}(\mathsf{P}, \mathsf{p})$ formulas with free variables $u_1, \ldots, u_n$. Further, $s \in (\mathsf{P})_t$ abbreviates $(s, t) \in \mathsf{P}$ and $s \in (\mathsf{P})_{<_{\mathsf{W}} t} := (\exists x <_{\mathsf{W}} t)(s \in (\mathsf{P})_x)$. The set T comprises (i) and (ii), for each $i \in \mathbb{N}$ the formula (iii) and the formula (iv):

(i) $\{\overline{w} \leq_{\mathsf{W}} \mathsf{p} : w \in \mathbb{W}\} \cup \{\mathsf{W}(\mathsf{p})\}$,
(ii) $\{A : \mathsf{V}_{\mathbb{E}'} \models A,\ A \text{ an } \mathsf{L}(c_v : v \in \mathsf{V}_{\mathbb{E}'}) \text{ sentence.}\}$,
(iii) $\forall x[(\exists z \leq_{\mathsf{W}} \mathsf{p}) A_i(x, z) \to (\exists z \leq_{\mathsf{W}} \mathsf{p})(A_i(x, z) \wedge (\forall y <_{\mathsf{W}} z) \neg A_i(x, y))]$,
(iv) $(\forall z \leq_{\mathsf{W}} \mathsf{p})[(\mathsf{P})_{0 \times z} = \{(a, b, c) : (a, b, c) \in (\mathsf{P})_{<_{\mathsf{W}} 0 \times z} \vee R((\mathsf{P})_{<_{\mathsf{W}} 0 \times z}, a, b, c)\}]$.

Since T is finitely realizable, compactness provides a model $\mathbb{E} = (\mathsf{V}_{\mathbb{E}}, \mathsf{W}_{\mathbb{E}})$ of $\mathsf{PT}_{\mathfrak{S}}$ and $\mathsf{w} \in \mathsf{W}_{\mathbb{E}}$, $\mathcal{F} \subseteq \mathsf{V}_{\mathbb{E}}$ so that $\mathbb{E} \models C(\mathcal{F}, \mathsf{w})$ for each $C(\mathsf{P}, \mathsf{p}) \in T$. By (i) we have that w is non-standard, (ii) forces that $\mathbb{E}$ is an elementary extension of $\mathbb{E}'$, (iii) tells us that each non-empty subclass of $\{w : w \leq \mathsf{w}\}$ which is L-definable with parameters from $\mathsf{V}_{\mathbb{E}} \cup \{\mathcal{F}\}$ has a $<$-minimal element, and (iv) entails that for each word $b \in \mathsf{W}_{\mathbb{E}}$ of length $n \in \mathbb{N}$, $I_n^{\mathbb{E},R} = (\mathcal{F})_{0 \times b}$, and so $I_n^{\mathbb{E},R} \subseteq (\mathcal{F})_{0 \times c}$ for each non-standard word $c \leq_{\mathsf{W}} \mathsf{w}$.

It remains to show that $I_{<\omega}^{\mathbb{E},R} = I^{\mathbb{E},R}$. First, assume that $R_1(I_{<\omega}^{\mathbb{E},R}, a, x, 1)$. Since $R_1(\mathsf{P}, u, v, w)$ contains P only positively, $R_1((\mathcal{F})_{<_{\mathsf{W}} 0 \times b}, a, x, 1)$ holds in particular for each non-standard word b. There is no shortest non-standard word. By (ii), there is a shortest word b_0 so that $R_1((\mathcal{F})_{<_{\mathsf{W}} 0 \times b_0}, a, x, 1)$. So $b_0 \in \mathbb{W}$ and $R_1((\mathcal{F})_{<_{\mathsf{W}} 0 \times b_0}, a, x, 1)$, thus $(a, x, 1) \in (\mathcal{F})_{0 \times b_0} \subseteq I_{<\omega}^{\mathbb{E},R}$.

Secondly, if $R_0(I_{<\omega}^{\mathbb{E},R}, a, 0, 0)$, then we do a case distinction on the clauses of R_0'. We just consider the case where $a = \underline{\mathsf{j}}(a', g), 0, 0)$ for some $a', g \in \mathsf{W}_{\mathbb{E}}$. Then X with

$$\emptyset \neq X := \{b \in \mathsf{W}_{\mathbb{E}} : (\forall x \mathbin{\dot\in} a')(gx, 0, 0) \in (\mathcal{F})_{<_{\mathsf{W}} 0 \times b}\} \subseteq \{b \leq \mathsf{w} : b \notin \mathbb{W}\}$$

has a $\leq$-minimal element which is in $\mathbb{W}$. So $(a, 0, 0) \in I_{<\omega}^{\mathbb{E},R}$. □

For the rest of this section, $\mathbb{E}$ denotes a model of PT+ (BP) with $I^{R,\mathbb{E}}_{<\omega} = I^{R,\mathbb{E}}$. We let $\mathcal{R} := \mathcal{R}^{\mathbb{E}}$ and $\mathcal{T} := \{X : \exists x \mathcal{R}(X, x)\}$. It remains to show that each type in $\mathcal{T}$ is simple.

Since $\{(a, x, 1) : (a, x, 1) \in I^{\mathbb{E},R}\} = I^{\mathbb{E},R_1} = I^{\mathbb{E},R_1}_{<\omega}$, $(a, x, 1) \in I^{\mathbb{E},R}_n$ can be expressed by a positive formula $E_n(a, x)$, where $E_0(u, v) := R'_1(\emptyset, u, v)$ and $E_{n+1}(u, v) := R'_1(\{(b, y) : E_n(b, y)\}, u, v)$. As the extension of each type $X \in \mathcal{T}$ is now of the form $X = \{x : E_n(a, x)\}$, it suffices to show that for each name a, there is a word w so that $X = \{x : E_n(a, x)\} = \{x : E^w_n(a, x)\}$, which entails that X is simple. Since E_n is positive, we then also have $X = \{x : E_n(a, x)\} = \{x : E^v_n(a, x)\}$ for each word $v > w$. To flatten the notation, we set $E_n(u, v, w) := E^w_n(u, v)$.

If a name a is obtained using at most n-times join, then $a \in \mathcal{J}_n$. More precisely, the sets $\mathcal{J}_n$ $(n \in \mathbb{N})$ are inductively defined below. Note that $\bigcup_{n\in\mathbb{N}} \mathcal{J}_n = \mathcal{R}$.

(i) $\underline{\mathsf{id}} \in \mathcal{J}_0$ and $\underline{\mathsf{i_w}} a \in \mathcal{J}_0$ for each $a \in \mathsf{W}_{\mathbb{E}}$.
(ii) If $a, b \in \mathcal{J}_n$ and $x \in \mathsf{V}_{\mathbb{E}}$, then $\underline{\mathsf{all}}\, a$, $\underline{\mathsf{dom}}\, a$, $\underline{\mathsf{inv}}(a, x)$, $\underline{\mathsf{int}}(a, b)$, $\underline{\mathsf{un}}(a, b) \in \mathcal{J}_n$.
(iii) If $a \in \mathcal{J}_n$, then $a \in \mathcal{J}_{n+1}$. And if $b \in \mathcal{J}_n$ and $\underline{\mathsf{j}}(b, g) \in \mathcal{R}$, then $\underline{\mathsf{j}}(b, g) \in \mathcal{J}_{n+1}$.

There is an element $\underline{\mathsf{bdo}} \in \mathsf{V}_{\mathbb{E}}$, so that for each $a \in \mathcal{J}_0$, $\underline{\mathsf{bdo}}\, a$ is a bound for the type named a. Thereto, we choose $\underline{\mathsf{bdo}}$ such that the following equations hold in $\mathbb{E}$.

(i) $\underline{\mathsf{bdo}}\, \underline{\mathsf{id}} = \epsilon$ and $\underline{\mathsf{bdo}}\, (\underline{\mathsf{i_w}} a) = 0 \times a$.
(ii) $\underline{\mathsf{bdo}}\, (\underline{\mathsf{all}}\, a) = \underline{\mathsf{bdo}}\, (\underline{\mathsf{dom}}\, a) = \underline{\mathsf{bdo}}\, (\underline{\mathsf{inv}}(a, x)) = \underline{\mathsf{bdo}}\, a$.
(iii) $\underline{\mathsf{bdo}}\, (\underline{\mathsf{int}}(a, b)) = \underline{\mathsf{bdo}}\, (\underline{\mathsf{un}}(a, b)) = \underline{\mathsf{max}}(\underline{\mathsf{bdo}}\, a, \underline{\mathsf{bdo}}\, b)$, where $\underline{\mathsf{max}} \in \mathsf{V}_{\mathbb{E}}$ is such that for all $x, y \in 0 \times \mathsf{W}_{\mathbb{E}}$, $\underline{\mathsf{max}}(x, y)$ returns y if $x < y$, and x otherwise

Lemma 25. *If $a \in \mathcal{J}_0 \cap \mathcal{R}_n$, then $a \doteq \{x : E_n(a, x)\} = \{x : E_n(a, x, \underline{\mathsf{bdo}}\, a)\}$. In particular, the extension of each type in $\mathcal{J}_0$ is simple, which implies that $(\mathbb{E}, \mathcal{R})$ is a model of* PET.

Proof. By induction on n. If e.g. $\underline{\mathsf{un}}(a, b) \in \mathcal{R}_{n+1}$ and $c := \underline{\mathsf{bdo}}\, (\underline{\mathsf{un}}(a, b)))$, then by definition of E_{n+1}, $E_{n+1}(\underline{\mathsf{un}}(a, b), x, c)$ iff $E_n(a, x, c) \vee E_n(b, x, c)$. The I.H. yields that $E_n(a, x)$ iff $E_n(a, x, \underline{\mathsf{bdo}}\, a)$ iff $E_n(a, x, c)$. The same holds for b. The claim follows as E_{n+1} is positive. □

If $a \in \mathcal{R}_n$ is simple and $g \in (a \to \mathcal{J}_0)$, then $\{\underline{\mathsf{bdo}}\, (gx) : x \,\dot{\in}\, a\} \subseteq \mathsf{W}_{\mathbb{E}}$ is simple, and, due to Lemma 6, bounded by a word b. Hence, $\underline{\mathsf{j}}(a, g) \doteq \{(x, y) : x \,\dot{\in}\, a \wedge E_n(gx, y, b)\}$ is simple as well. However, as the bound of the type $\underline{\mathsf{j}}(a, g)$ cannot be computed, the above trick only works if $g \in (a \to \mathcal{J}_0)$. Therefore, we assign to each name a a name $\underline{\mathsf{bd}}\, a \in \mathcal{J}_0$ of a *set of names of bounds*. That is, $\underline{\mathsf{bd}}\, a \,\dot{\subseteq}\, \mathcal{J}_0$, and for each $b \,\dot{\in}\, \underline{\mathsf{bd}}\, a$, either $b = \underline{\mathsf{i_w}} w$ or b names a *set of names of bounds*. The support of a name $\underline{\mathsf{i_w}} w$ is given by $\mathsf{supp}(\underline{\mathsf{i_w}} w) := \{w\}$, and if z is a name of Z and Z is a *set of names of bounds*, then $\mathsf{supp}(z) := \mathsf{supp}(Z) := \bigcup\{\mathsf{supp}(x) : x \in Z\}$. The name $\underline{\mathsf{bd}}\, a$ will be such that $\mathsf{supp}(\underline{\mathsf{bd}}\, a) \subseteq \mathsf{W}_{\mathbb{E}}$ and, in addition, $\mathsf{supp}(a) \leq v \in \mathsf{W}$ implies $a \doteq \{x : E_n(a, x, v)\}$. This procedure is detailed below. Thereby, we let $\underline{\mathsf{bd}} \in \mathsf{W}_{\mathbb{E}}$ be such that the following equations hold.

(i) $\underline{\mathsf{bd}}\, \underline{\mathsf{id}} = \underline{\mathsf{inv}}(\underline{\mathsf{id}}, \lambda x.(x, \underline{\mathsf{i_w}}\epsilon))$, i.e. a name of the type $\{\underline{\mathsf{i_w}}\epsilon\}$.
(ii) $\underline{\mathsf{bd}}\, (\underline{\mathsf{i_w}} a) = \underline{\mathsf{inv}}(\underline{\mathsf{id}}, \lambda x.(x, \underline{\mathsf{i_w}} a))$, i.e. a name of the type $\{\underline{\mathsf{i_w}} a\}$.
(iii) $\underline{\mathsf{bd}}\, (\underline{\mathsf{all}}\, a) = \underline{\mathsf{bd}}\, (\underline{\mathsf{dom}}\, a) = \underline{\mathsf{bd}}\, (\underline{\mathsf{inv}}(a, x)) = \underline{\mathsf{bd}}\, a$.
(iv) $\underline{\mathsf{bd}}\, (\underline{\mathsf{int}}(a, b)) = \underline{\mathsf{bd}}\, (\underline{\mathsf{un}}(a, b)) = \underline{\mathsf{un}}(\underline{\mathsf{bd}}\, a, \underline{\mathsf{bd}}\, b)$.
(v) $\underline{\mathsf{bd}}\, (\underline{\mathsf{j}}(a, g))$ is the name of the type $\{\underline{\mathsf{bd}}\, (gx) : x \,\dot{\in}\, a\} \cup \{\underline{\mathsf{bd}}\, a\}$ given by
$\underline{\mathsf{un}}(\underline{\mathsf{dom}}\, (\underline{\mathsf{int}}(\underline{\mathsf{inv}}(\underline{\mathsf{id}}(\lambda x.(\underline{\mathsf{bd}}\, (g(x)_1), (x)_0)), \underline{\mathsf{inv}}(a, \lambda x.(x)_0)))), \underline{\mathsf{inv}}(\underline{\mathsf{id}}, \lambda x.\underline{\mathsf{bd}}\, a))$.

We say that $\mathsf{lv}(X) := 0$ iff $X \subseteq \mathcal{R} \wedge (\forall x \in X)((x)_0 = \epsilon)$ (i.e. X contains only names of the form $\underline{\mathsf{i_w}} w$). $\mathsf{lv}(a) = n$ is short for $\exists X[\mathcal{R}(X, a) \wedge \mathsf{lv}(X) = n]$, and $\mathsf{lv}(X) = n{+}1$ iff $X \subseteq \mathcal{R} \wedge (\forall x \in X)(\mathsf{lv}(x) \leq n) \wedge (\exists x \in X)(\mathsf{lv}(x) = n)$. Note that if $\mathsf{lv}(X) = n{+}1$ and $x \in X$, then x names an initial segment of words, or $\mathsf{lv}(x) \leq n$.

Lemma 26. *If $a \in \mathcal{R}_n$, then $\underline{\mathsf{bd}}\, a \in \mathcal{J}_0$ and $\underline{\mathsf{bd}}\, a \,\dot{\subseteq}\, \mathcal{J}_0$ and $\mathsf{lv}(a) \leq n$.*

Proof. By induction on n. We just consider the case where $a = \underline{\mathsf{j}}(b, g) \in \mathcal{R}_{n+1}$. Then, for each $x \,\dot{\in}\, b$, $gx \in \mathcal{R}_n$. Therefore $\{\underline{\mathsf{bd}}\, (gx) : x \in b\} \subseteq \mathcal{J}_0$ and $\mathsf{lv}(\underline{\mathsf{bd}}\, (gx)) \leq n$ by I.H. Now the claim follows by the definition of $\underline{\mathsf{bd}}\, a$ and $\mathsf{lv}(a)$. □

Definition 27. For $a \in \mathcal{J}_m$, let $X_{a,0} \doteq \underline{\mathsf{bd}}\, a$. For $0 \leq i \leq m$,

(i) $Y_{a,i} := \{x \in X_{a,i} : (x)_0 = \epsilon\}$,
(ii) If $i < m$, $X_{a,i+1} := \{y \,\dot{\in}\, x : x \in X_{a,i} \wedge (x)_0 > \epsilon\}$,
(iii) $Y_a := \bigcup_{0\leq i\leq m}\{(x)_1 : x \in Y_{a,i}\}$.

As the next lemma entails, $X_{a,i}$ contains only names, and, thus, $(x)_0 > \epsilon$ iff $(x)_0 \neq \epsilon$ for each $x \in X_{a,i}$, so that $X_{a,i+1} = \{y \,\dot{\in}\, x : x \in (X_{a,i} - Y_{a,i})\}$.

Lemma 28. *If $a \in \mathcal{J}_m \cap \mathcal{R}_n$, then for $0 \leq i \leq m$, $\mathsf{lv}(X_{a,i}) \leq m - i$, $X_{a,i} \subseteq \mathcal{J}_0$ and $\{(y)_1 : y \in Y_{a,i}\} \subseteq \mathsf{W}_{\mathbb{E}}$, $Y_a \subseteq \mathsf{W}_{\mathbb{E}}$.*

Proof. By main induction on m and side induction on n. □

Lemma 29. *For all names $a, b \in \mathcal{J}_m \cap \mathcal{R}_n$ and $w \in \mathsf{W}_{\mathbb{E}}$ and $g \in \mathsf{V}_{\mathbb{E}}$, the following holds:*

(i) $Y_{\underline{\mathsf{id}}} = \{\mathsf{i}_{\mathsf{w}}\epsilon\}$ and $Y_{\underline{\mathsf{i}_{\mathsf{w}}} w} = \{\underline{\mathsf{i}_{\mathsf{w}}} w\}$,
(ii) $Y_{\underline{\mathsf{all}}\, a} = Y_{\underline{\mathsf{dom}}\, a} = Y_{\underline{\mathsf{inv}}(a,g)} = Y_a$,
(iii) $Y_{\underline{\mathsf{int}}(a,b)} = Y_{\underline{\mathsf{un}}(a,b)} = Y_a \cup Y_b$,
(iv) $Y_{\underline{\mathsf{j}}(a,g)} = Y_a \cup \bigcup_{x \,\dot\in\, a} Y_{gx}$.

Proof. By main induction on m and side induction on n. (i)–(iii) are immediate. For (iv), note that by definition of $X_{\underline{\mathsf{j}}(a,g),0}$, $X_{\underline{\mathsf{j}}(a,g),0} = X_{a,0} \cup \bigcup_{x \,\dot\in\, a} X_{gx,0}$. Further, if $X_{\underline{\mathsf{j}}(a,g),i} = X_{a,i} \cup \bigcup_{x \,\dot\in\, a} X_{gx,i}$, then $Y_{\underline{\mathsf{j}}(a,g),i} = Y_{a,i} \cup \bigcup_{x \,\dot\in\, a} Y_{gx,i}$ and $X_{\underline{\mathsf{j}}(a,g),i+1} = X_{a,i+1} \cup \bigcup_{x \,\dot\in\, a} X_{gx,i}$, since

$$\left\{ y \,\dot\in\, z : z \in \bigcup_{x \,\dot\in\, a} X_{gx,i} \wedge (z)_0 > \epsilon \right\} = \bigcup_{x \,\dot\in\, a} \left\{ y \,\dot\in\, z : z \in X_{gx,i} \wedge (z)_0 > \epsilon \right\}.$$

If follows that for $0 \le i \le m{+}1$, $Y_{\underline{\mathsf{j}}(a,g),i} = Y_{a,i} \cup \bigcup_{x \,\dot\in\, a} Y_{gx,i}$. Therefore,

$$\begin{aligned} Y_{\underline{\mathsf{j}}(a,g)} &= \bigcup_{0 \le i \le m+1} \{(y)_1 : y \in Y_{a,i}\} \cup \bigcup_{0 \le i \le m+1} \left\{ (y)_1 : y \in \bigcup_{x \,\dot\in\, a} Y_{gx,i} \right\} \\ &= \bigcup_{0 \le i \le m+1} \{(y)_1 : y \in Y_{a,i}\} \cup \bigcup_{x \,\dot\in\, a} \bigcup_{0 \le i \le m+1} \{(y)_1 : y \in Y_{gx,i}\}. \end{aligned}$$

The claim follows. □

Lemma 30. *If $a \in \mathcal{R}_n$ and $Y_a \le w \in \mathsf{W}_{\mathbb{E}}$, then $a \doteq \{x : E_n(a, x, w)\}$.*

Proof. By induction on n. If $a \in \mathcal{R}_0$, then either $a = \underline{\mathsf{id}}$, in which case there is nothing to show, or $a = \underline{\mathsf{i}_{\mathsf{w}}} w$. In this case, $Y_a = \{w\}$, and since $(a)_1 = w$, $\underline{\mathsf{i}_{\mathsf{w}}} w \doteq \{x : E_0(a, x)\} = \{x : E_0(a, x, w)\}$. If e.g. $a = \underline{\mathsf{j}}(b, g) \in \mathcal{R}_{n+1}$, then $a \doteq \{(y, z) : E_n(b, y) \wedge E_n(gy, z)\}$, so by I.H. $a \doteq \{(y, z) : E_n(b, y, w) \wedge E_n(gy, z, w_y)\}$, for a word $w \ge Y_b$ and words $w_y \ge Y_{gy}$ $(y \,\dot\in\, b)$. Since E_n is positive, we also have $a \doteq \{(y, z) : E_n(b, y, v) \wedge E_n(gy, z, v)\}$, provided $v \ge Y_b \cup \bigcup_{y \,\dot\in\, b} Y_{gy}$. The claim follows by the definition of E_{n+1}. □

Lemma 31. *For each $a \in \mathcal{R}_n$, a names a simple type.*

Proof. If remains to show that Y_a is simple. Since $Y_a \subseteq \mathsf{W}_{\mathbb{E}}$ and $\mathbb{E}$ is a model of $\mathsf{PT}+$ (BP), Y_a is bounded, and the claim follows by Lemma 30. We show by induction on $0 \le i \le n$ that there are L formulas $B_i(u)$ and words w_i so that $X_{a,i} = \{x : B_i(x)\} = \{x : B_i^{w_i}(x)\}$. Thus, $X_{a,i}$ and $Y_{a,i}$ are simple, which implies that Y_a is simple. Since $X_{a,0} \in \mathcal{J}_0$, $B_0(u) := E_n(\underline{\mathsf{bd}}\, a, x)$ and $w_0 := \underline{\mathsf{bdo}}\,(\underline{\mathsf{bd}}\, a)$ is a valid choice by Lemma 25, the definition of $\underline{\mathsf{bd}}$ and Definition 27. So assume that $X_{a,i} = \{x : B_i^{w_i}(x)\}$. As $X_{a,i} \subseteq \mathcal{J}_0$, $\{\underline{\mathsf{bdo}}\, x : x \in X_{a,i}\} \subseteq \mathsf{W}_{\mathbb{E}}$ is simple and thus bounded by some $w_{i+1} \in \mathsf{W}_{\mathbb{E}}$. We may assume that $w_i \le_{\mathsf{W}} w_{i+1}$. Then, $X_{a,i+1}$ is

$$\{x : \exists z[B_i^{w_{i+1}}(z) \wedge \epsilon < (z)_0 \wedge E_n(z, x, w_{i+1})]\} = \{x : \exists z[B_i(z) \wedge \epsilon < (z)_0 \wedge E_n(z, x)]\}.$$

Hence, $B_{i+1}(u) := \exists z[B(z) \wedge \epsilon < (z)_0 \wedge E_n(z, x)]$ and w_{i+1} do the job. □

Theorem 32. *Let $\mathbb{E}'$ be a model of $\mathsf{PT}+$ (BP). Then, there is an elementary extension $\mathbb{E} \succ \mathbb{E}'$ so that there is an expansion $(\mathbb{E}, \mathcal{R})$ which is a model of PETJ whose types are exactly the simple classes of $\mathbb{E}$.*

Since PT_0 and $\mathsf{PT}+$ (BP) have the same provably terminating operations, we also have the following.

Corollary 33. *The theories PT_0, $\mathsf{PT}^{\dagger}_{\mathfrak{S}}+\mathsf{I}$, $\mathsf{PT}+$ (BP) and PETJ all have the same provably terminating operations on words.*

8. Concluding remarks

First, we have presented the theory $\mathsf{PT}_{\mathfrak{S}}$ which is PT with the somewhat technical induction schema (Σ^b_{W}-I) replaced by induction on notation for simple formulas. Then, we have introduced the boundedness principle (BP) which asserts that for each simple class X, $X \subseteq \mathsf{W} \to (\exists w \in \mathsf{W})(X \le w)$. We argued that each instance of (BP) is provable in the auxiliary theory $\mathsf{PT}^{\dagger}_{\mathfrak{S}}+\mathsf{I}$ that extends PT by the assertion that $\mathsf{exp} \notin (\mathsf{W} \to \mathsf{W})$ and an induction principle claiming that if $X \subseteq \mathsf{W}$ is simple and $Y = \{y : (\exists x \in X{\downarrow}) A(x, y)\}$ for some positive and W-free formula $A(u, v)$, then $\mathsf{Prog}_{\sqsubset}(Y) \to \mathsf{W} \subseteq Y$. A translation in the style of a realizability interpretation enabled us to verify that the provably total operations of $\mathsf{PT}^{\dagger}_{\mathfrak{S}}+\mathsf{I}$ and PT coincide.

In a second step we employed a non-monotone inductive definition over a structure for $\mathsf{PT}+$ (BP) to define a naming relation which enabled us to expand a model of $\mathsf{PT}+$ (BP) to a model of PETJ^-. Then, we proved that each model of $\mathsf{PT}+$ (BP) has an elementary extension above which the inductive definition closes off already at stage ω. The corresponding model $\mathcal{M}$ was readily seen to be a model of PET, as the extension of each type constructed without the use of join is simple. In fact, given a name b of such a type X, we could compute a word w so that $X = \{x : A^w(x)\}$ for some $A(u, v) \in \mathfrak{S}(v)$. A further argument was required to see that $\mathcal{M}$ also satisfies the join principle. Although we could not compute for each name b a word w so that $b \doteq \{x : A^w(x)\}$ for some $A(u, v) \in \mathfrak{S}(v)$, we were able to prove the existence of such words. For a name b of a type built using join, we have computed a name of a set $\underline{\mathsf{bd}}\, b$ of names of bounds so that all names x in the transitive closure of the type $\underline{\mathsf{bd}}\, b$ are in $\mathcal{J}_0$. As the bounding principle (BP) provides a simple definition of a type $\mathsf{j}(a, f)$ under the premise that fx is a name in $\mathcal{J}_0$ for each $x \dot{\in} a$, an iterative application of (BP) allowed us to find a simple definition of the type $\mathsf{j}(a, f)$.

In Spescha and Strahm [15], a realizability interpretation is performed to show that the provably terminating operations of $\mathsf{PETJ}^{\mathsf{i}}$ are the polytime functions on words. In their proof, a word w realizing the formula $\mathsf{R}(b)$ corresponds in essence to a word w so that $b \doteq \{x : A^w(x)\}$ for some $A(u, v) \in \mathfrak{S}(v)$. However, in order to compute such a word, they had to change to intuitionistic logic. Whether a realizability interpretation is also possible for PETJ with classical logic is still an open question.

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