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Representation theory of Neveu–Schwarz and Ramond algebras I: Verma modules

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Abstract

In this article, we study the structure of Verma modules of $N = 1$ super Virasoro algebras. As applications, we construct Bernstein-Gel'fand Gel'fand type resolutions. This article is the detailed and expanded version of Iohara and Koga (C. R. Acad. Sci. Paris Ser. I 328 (1999) 381).

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1. Introduction

It is known well that the Virasoro algebra plays significant roles both in theoretical physics and in mathematics. The basics of its representation theory, e.g., the structure of Verma modules over it, were studied by Feigin and Fuchs [FeFu1,FeFu2]. In [FeFu2], they studied not only the structure of Verma modules but also the structure of Fock modules (to be precise, representations realized on

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semi-infinite wedges). Its ($N = 1$)-super generalization, which is also important for its applications to theoretical physics, has two sector called the Neveu–Schwarz sector [NS] and the Ramond sector [R]. Although, the structure of Verma modules was partially studied in [As] for Neveu–Schwarz sector, there seems to be no reference where the structure of Verma modules for the Ramond sector was studied. In this paper, we will study the structure of Verma modules over the $N = 1$ super Virasoro algebras, and will construct the Bernstein–Gel'fand–Gel'fand (BGG) type resolutions of the irreducible highest weight modules. In the next paper [IK2], we will study the structure of Fock modules over $N = 1$ super Virasoro algebras.

One of the most crucial differences between the representation theory of non-super algebras and super algebras comes from the fact that a non-trivial morphism between Verma modules is not necessarily injective in the case of super algebras. Indeed, there are non-trivial and non-injective morphisms between Verma modules in the case of Ramond algebra, whereas in the case of Neveu–Schwarz algebra, any non-trivial morphism between Verma modules is a monomorphism (see Proposition 3.3). What is surprising is that we could obtain a complete description of the non-trivial and non-injective morphisms between Verma modules over the Ramond algebra in Theorem 4.4. Moreover, the structure of Verma modules over the $N = 1$ super Virasoro algebras is completely described in Theorems 4.1–4.3. The BGG type resolutions of the irreducible highest weight modules over them are obtained in Theorem 5.1 as an application.

This paper is organized as follows. In Section 2, we recall the definition of the $N = 1$ super Virasoro algebras and some formulae which will be used in the later section. In Section 3, we will study the basic properties of Verma modules. To be precise, the Shapovalov type determinant formula is recalled in Section 3.1. In Section 3.2, we will show that for each weight subspace, the maximal possible dimension of the space of singular vectors is 1 for the Neveu–Schwarz algebra (Proposition 3.1) and is 2 for the Ramond algebra (Proposition 3.2). The injectivity of a morphism between Verma modules will be studied in Proposition 3.3. In Section 4, more detailed structure of Verma modules will be presented. Indeed, the classification of highest weights of Verma modules will be managed in Section 4.1. In Sections 4.2–4.4, the structure of the Jantzen filtration of Verma modules over $N = 1$ super Virasoro algebras will be given. As an application, we obtain a detailed description of the structure of Verma modules in Section 4.5. In Section 5, the BGG type resolutions are constructed (Theorem 5.1), and the characters of the irreducible highest weight modules over the $N = 1$ super Virasoro algebras are computed as an application (Theorem 5.2). Finally, in Section 6, we will make a few remarks on the structure of a module for the Ramond algebra, what is sometimes called a Verma module in a literature. In Appendix A, we list up some data that will be used in a concrete computation.

2. Preliminary

In this section, we present our framework of the representation theory of $N = 1$ super Virasoro algebra.

In Section 2.1, we introduce all of the objects considered in this article. In Section 2.2, we consider some suitable categories of representations of $N=1$ super Virasoro algebra and state some general formulae.

2.1. Definitions

Here, we recall the objects that will be considered in this article, the $N=1$ super Virasoro algebras and Verma modules, etc.

The Lie superalgebras we are going to consider are the following:

Definition 2.1. The $N=1$ *super Virasoro algebras* Vir_ε $(\varepsilon=\frac{1}{2},0)$ are the Lie superalgebras

$$\mathrm{Vir}_\varepsilon := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \bigoplus_{m\in\varepsilon+\mathbb{Z}} \mathbb{C}G_m \oplus \mathbb{C}c,$$

which satisfy the following commutation relations:

$$\deg L_n = \bar{0}\ \ (n\in\mathbb{Z}), \quad \deg G_m = \bar{1}\ \ (m\in\varepsilon+\mathbb{Z}), \quad \deg c = \bar{0},$$

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\,\tfrac{1}{12}(m^3-m)c,$$

$$[G_m, L_n] = (m-\tfrac{1}{2}n)G_{m+n},$$

$$[G_m, G_n] = 2L_{m+n} + \delta_{m+n,0}\tfrac{1}{3}(m^2-\tfrac{1}{4})c,$$

$$[\mathrm{Vir}_\varepsilon, c] = \{0\}.$$

$\mathrm{Vir}_{\frac{1}{2}}$ and Vir_0 are called the *Neveu–Schwarz* and the *Ramond* algebra, respectively. Furthermore, Vir_ε is $\mathbb{Z}$-graded by setting

$$\mathfrak{h} := \mathbb{C}L_0 \oplus \mathbb{C}c$$

and

$$(\mathrm{Vir}_{\frac{1}{2}})_n := \begin{cases} \mathbb{C}L_{\frac{1}{2}n} & \text{if } n\in 2\mathbb{Z}\setminus\{0\}, \\ \mathbb{C}G_{\frac{1}{2}n} & \text{if } n\in 2\mathbb{Z}+1, \\ \mathfrak{h} & \text{if } n=0, \end{cases} \qquad (\mathrm{Vir}_0)_n := \begin{cases} \mathbb{C}L_n\oplus\mathbb{C}G_n & \text{if } n\neq 0, \\ \mathfrak{h}\oplus\mathbb{C}G_0 & \text{if } n=0. \end{cases}$$

By definition, Vir_ε satisfies the following decomposition:

$$\mathrm{Vir}_\varepsilon = \mathrm{Vir}_\varepsilon^{\bar{0}} \oplus \mathrm{Vir}_\varepsilon^{\bar{1}}, \quad \mathrm{Vir}_\varepsilon^{\bar{0}} := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}c, \quad \mathrm{Vir}_\varepsilon^{\bar{1}} := \bigoplus_{m\in\varepsilon+\mathbb{Z}} \mathbb{C}G_m.$$

Moreover, Vir_ε possesses the following triangular decomposition:

$$\mathrm{Vir}_\varepsilon = (\mathrm{Vir}_\varepsilon)_+ \oplus (\mathrm{Vir}_\varepsilon)_0 \oplus (\mathrm{Vir}_\varepsilon)_-, \quad (\mathrm{Vir}_\varepsilon)_\pm := \bigoplus_{\pm n \in \mathbb{Z}_{>0}} (\mathrm{Vir}_\varepsilon)_n.$$

Below, we define the objects that will be treated in this article.

Namely, we introduce Verma modules, irreducible highest weight modules of $(\mathfrak{g} :=)\mathrm{Vir}_\varepsilon$, pre-Verma modules and quasi-Verma modules for Vir_0.

For $\lambda \in \mathfrak{h}^*$ and $\tau \in \mathbb{Z}_2$, let

$$\mathbb{C}_\lambda^\tau := \mathbb{C}\mathbf{1}_\lambda^\tau$$

be the one-dimensional $\mathfrak{h}$-module defined as follows:

1. $\deg \mathbf{1}_\lambda^\tau = \tau$.
2. $h.\mathbf{1}_\lambda^\tau = \lambda(h)\mathbf{1}_\lambda^\tau$ for any $h \in \mathfrak{h}$.

The $\mathfrak{g}_0$-modules $P(\lambda;\tau)$, $Q(\lambda)$ (defined only for $\varepsilon = 0$) and $V(\lambda;\tau)$ are given as follows:

$$P(\lambda;\tau) := \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}_0} \mathbb{C}_\lambda^\tau,$$

$$Q(\lambda) := \mathbb{C}_\lambda^{\bar{0}} \oplus \mathbb{C}_\lambda^{\bar{1}},$$

$$V(\lambda;\tau) := \begin{cases} \mathbb{C}_\lambda^\tau & \text{if } \varepsilon = 0 \text{ and } \lambda(L_0) = \frac{1}{24}\lambda(c), \\ \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}_0} \mathbb{C}_\lambda^\tau & \text{otherwise,} \end{cases}$$

where the $\mathfrak{g}_0$-module structure on $V(\lambda;\tau)$ for $\varepsilon = 0$ and $\lambda(L_0) = \frac{1}{24}\lambda(c)$ is defined by

$$\mathfrak{g}_0^{\bar{1}}.\mathbf{1}_\lambda^\tau := 0,$$

and the $\mathfrak{g}_0$-module structure on $Q(\lambda)$ is defined by

$$G_0.\mathbf{1}_\lambda^\tau := (\lambda(G_0^2))^{\frac{1}{2}} \mathbf{1}_\lambda^{\bar{1}-\tau}.$$

Note that $P(\lambda;\tau)$, $Q(\lambda)$ and $V(\lambda;\tau)$ become $\mathfrak{g}_\geqslant := \mathfrak{g}_0 \oplus \mathfrak{g}_+$-modules via

$$\mathfrak{g}_+|_{P(\lambda;\tau)} = \mathfrak{g}_+|_{Q(\lambda)} = \mathfrak{g}_+|_{V(\lambda;\tau)} := 0.$$

Definition 2.2. 1. The Verma module $M_\varepsilon(\lambda;\tau)$ with highest weight $(\lambda,\tau) \in \mathfrak{h}^* \times \mathbb{Z}_2$ is defined by

$$M_\varepsilon(\lambda;\tau) = \mathrm{Ind}_{\mathfrak{g}_\geqslant}^{\mathfrak{g}} V(\lambda;\tau).$$

2. For $\varepsilon = 0$, the pre-Verma module $N(\lambda;\tau)$ with highest weight $(\lambda,\tau) \in \mathfrak{h}^* \times \mathbb{Z}_2$ is defined by

$$N(\lambda;\tau) = \mathrm{Ind}_{\mathfrak{g}_\geqslant}^{\mathfrak{g}} P(\lambda;\tau).$$

3. For $\varepsilon = 0$, the quasi-Verma module $\tilde{M}(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$ is defined by

$$\tilde{M}(\lambda) := \mathrm{Ind}_{\mathfrak{g}_{\geqslant}}^{\mathfrak{g}} Q(\lambda).$$

We define irreducible highest weight modules by the following proposition.

Definition-Proposition 2.1. *For each* $(\lambda; \tau) \in \mathfrak{h}^* \times \mathbb{Z}_2$, *there exists a unique maximal proper* $\mathbb{Z}_2$*-graded submodule* $J_\varepsilon(\lambda; \tau)$ *of* $M_\varepsilon(\lambda; \tau)$, *and hence we define the irreducible highest weight module* $L_\varepsilon(\lambda; \tau)$ *with highest weight* $(\lambda; \tau)$ *by*

$$L_\varepsilon(\lambda; \tau) := M_\varepsilon(\lambda; \tau)/J_\varepsilon(\lambda; \tau).$$

2.2. *Categories of* Vir_ε*-modules*

In this section, we set $\mathfrak{g} = \mathrm{Vir}_\varepsilon$ for simplicity. We first introduce some categories of left $\mathfrak{g}$-modules and some functors. Then, we will briefly review on some homological algebras that will be used in the later sections.

For a $\mathbb{Z}$-graded vector space V, let us denote the $\mathbb{Z}_2$-graded decomposition of V by

$$V = V^{\bar{0}} \oplus V^{\bar{1}},$$

where $V^{\bar{0}}$ and $V^{\bar{1}}$ signify the even (resp. the odd) parts of V. The dual V^* of V should be understood as

$$V^* := \mathrm{Hom}(V, \mathbb{C}^{1|0}),$$

where $\mathbb{C}^{1|0}$ is the $(1|0)$-dimensional vector space.

Definition 2.3.

1. Mod_ε is the abelian category of $\mathfrak{g}$-modules defined as follows:
   (i) $\mathrm{Ob}(\mathrm{Mod}_\varepsilon)$ consists of all left $\mathbb{Z}_2$-graded $\mathfrak{g}$-modules.
   (ii) $\mathrm{Hom}_{\mathrm{Mod}_\varepsilon}(V, W) := \mathrm{Hom}_{\mathfrak{g}}(V, W)$ for $V, W \in \mathrm{Ob}(\mathrm{Mod}_\varepsilon)$.
2. $\mathrm{Mod}_\varepsilon^{\mathbb{Z}_2}$ is a subcategory of Mod_ε defined as follows:
   (i) $\mathrm{Ob}(\mathrm{Mod}_\varepsilon^{\mathbb{Z}_2}) = \mathrm{Ob}(\mathrm{Mod}_\varepsilon)$.
   (ii) $\mathrm{Hom}_{\mathrm{Mod}_\varepsilon^{\mathbb{Z}_2}}(V, W) := \mathrm{Hom}_{\mathfrak{g}}^{\bar{0}}(V, W)$ for $V, W \in \mathrm{Ob}(\mathrm{Mod}_\varepsilon)$.

A subcategory $\mathscr{C}_{(\mathfrak{g},\mathfrak{h})}^{\mathbb{Z}_2}$ of $\mathrm{Mod}_\varepsilon^{\mathbb{Z}_2}$ is defined as follows:

Definition 2.4. Category $\mathscr{C}_{(\mathfrak{g},\mathfrak{h})}^{\mathbb{Z}_2}$ is the full subcategory of $\mathrm{Mod}_\varepsilon^{\mathbb{Z}_2}$ whose objects consist of the $\mathfrak{h}$ semi-simple $\mathbb{Z}_2$-graded $\mathfrak{g}$-modules M, i.e., M satisfies

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu \quad \text{and} \quad M_\mu = M_\mu^{\bar{0}} \oplus M_\mu^{\bar{1}},$$

where $M_\mu^\tau := M_\mu \cap M^\tau$.

For $M \in \mathrm{Ob}(\mathscr{C}^{\mathbb{Z}_2}_{(\mathfrak{g},\mathfrak{h})})$, we set

$$\mathscr{P}(M) := \{\mu \in \mathfrak{h}^* | M_\mu \neq \{0\}\}.$$

Definition 2.5. We say that $M \in \mathrm{Ob}(\mathscr{C}^{\mathbb{Z}_2}_{(\mathfrak{g},\mathfrak{h})})$ is $\mathfrak{h}$-diagonalizable if $\dim M_\mu < \infty$ for any $\mu \in \mathscr{P}(M)$.

For $\lambda \in \mathfrak{h}^*$, set

$$D(\lambda) := \{\mu \in \mathfrak{h}^* \mid (\mu - \lambda)(L_0) \in (1-\varepsilon)\mathbb{Z}_{\geqslant 0}, (\mu - \lambda)(c) = 0\}.$$

Two categories $\mathcal{O}$ and $\mathcal{O}^{\mathbb{Z}_2}$ are defined as follows:

Definition 2.6. 1. Let $\mathcal{O}$ be the full subcategory of $\mathrm{Mod}_{\mathfrak{g}}$ given by $M \in \mathrm{Ob}(\mathcal{O})$ if and only if $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$ with $\dim M_\mu < \infty$ and

$$\{\mu \in \mathfrak{h}^* | M_\mu \neq \{0\}\} \subset \bigcup_i D(\mu_i)$$

for finitely many $\{\mu_i\} \subset \mathfrak{h}^*$.

2. Category $\mathcal{O}^{\mathbb{Z}_2}$ is a full subcategory of $\mathscr{C}^{\mathbb{Z}_2}_{(\mathfrak{g},\mathfrak{h})}$ whose objects consist of the $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-modules M with the following properties: There exists finitely many elements $\{\mu_i\} \subset \mathfrak{h}^*$ such that

$$\mathscr{P}(M) \subset \bigcup_i D(\mu_i).$$

Next we recall two fundamental functors, the parity shift functor Π (cf. [Manin]) and the functor $(\cdot)^c$ of taking the contragradient dual defined on $\mathrm{Mod}^{\mathbb{Z}_2}_{\mathfrak{g}}$ and $\mathcal{O}^{\mathbb{Z}_2}$, respectively.

Definition 2.7.

1. The parity shift $\Pi:\ \mathrm{Mod}^{\mathbb{Z}_2}_{\varepsilon} \to \mathrm{Mod}^{\mathbb{Z}_2}_{\varepsilon}$ is the functor defined as follows: For $M = M^{\bar{0}} \oplus M^{\bar{1}} \in \mathrm{Ob}(\mathrm{Mod}^{\mathbb{Z}_2}_{\varepsilon})$,
   (i) $(\Pi M)^{\bar{0}} := M^{\bar{1}}$ and $(\Pi M)^{\bar{1}} := M^{\bar{0}}$ as $\mathbb{C}$-vector space and ΠM has the $\mathfrak{g}$-module structure given by

   $$g.(\Pi v) := (-1)^{\deg g}\Pi(g.v),$$

   where g is a homogeneous element of $\mathfrak{g}$ and Πv denotes the element of ΠM which corresponds to $v \in M$.

(ii) For $f \in \mathrm{Hom}^{\bar{0}}_{\mathfrak{g}}(M,N)$, define $f^{\Pi} \in \mathrm{Hom}^{\bar{0}}_{\mathfrak{g}}(\Pi M, \Pi N)$ by

$$f^{\Pi}(v) := \Pi f(\Pi v),$$

where $v \in \Pi M$.

2. $(\cdot)^c : \mathcal{O}^{\mathbb{Z}_2} \to \mathcal{O}^{\mathbb{Z}_2}$ is the functor defined as follows:
   (i) For $M \in \mathrm{Ob}(\mathcal{O}^{\mathbb{Z}_2})$, let $M^c := \bigoplus_{\mu \in \mathscr{P}(M)} (M_\mu)^*$ be the restricted dual of M. We regard M^c as a $\mathfrak{g}$-module via

$$(x.f)(v) := f(\sigma(x).v),$$

   where $f \in M^c$, $v \in M$ and $x \in U(\mathfrak{g})$ are homogeneous elements and σ is the anti-involution of $\mathfrak{g}$ satisfying $\sigma(\mathfrak{g}_n) = \mathfrak{g}_{-n}$ for $n \in \mathbb{Z}$.
   (ii) For $f \in \mathrm{Hom}^{\bar{0}}_{\mathfrak{g}}(M,N)$, we define $f^c := {}^t f \in \mathrm{Hom}^{\bar{0}}_{\mathfrak{g}}(N^c, M^c)$, where ${}^t f$ denotes the transpose of f.
   We call M^c the *contragredient dual* of M.

It is easy to see that the functors Π and $(\cdot)^c$ enjoy the following properties.

Lemma 2.1. 1. *Π is covariant and exact, and further $\Pi^2 = \mathbf{1}_{\mathrm{Mod}_{\mathfrak{g}}^{\mathbb{Z}_2}}$.*

2. *Let $\mathscr{C}$* be *$\mathscr{C}^{\mathbb{Z}_2}_{(\mathfrak{g},\mathfrak{h})}$ or $\mathcal{O}^{\mathbb{Z}_2}$. Then $\mathscr{C}$ is stable under Π, i.e., $\Pi : \mathscr{C} \to \mathscr{C}$ is well-defined.*

3. *$(\cdot)^c$ is a contravariant and exact functor, and further $((M)^c)^c \cong M$ for $M \in \mathrm{Ob}(\mathcal{O}^{\mathbb{Z}_2})$.*

Finally, we state two formulae which are well-known for Lie algebras.

Since the category $\mathscr{C}^{\mathbb{Z}^2}_{(\mathfrak{g},\mathfrak{h})}$ has enough projectives, $\mathrm{Ext}^p_{\mathscr{C}}$ is nothing but the relative extension bi-functor. For simplicity, let us set

$$P_\varepsilon(v;\tau) := \begin{cases} \mathbb{C}^\tau_v & \varepsilon = \frac{1}{2}, \\ P(v;\tau) & \varepsilon = 0, \end{cases} \qquad N_\varepsilon(v;\tau) := \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\geqslant}} P_\varepsilon(v;\tau).$$

Proposition 2.1 (E.g., Rocha-Caridi and Wallach [RW1]). *Let $N(v;\tau)$ be the pre-Verma module with highest weight $(v;\tau) \in \mathfrak{h}^* \times \mathbb{Z}_2$. Then, for $V \in \mathrm{Ob}(\mathscr{C}^{\mathbb{Z}^2}_{(\mathfrak{g},\mathfrak{h})})$, we have*

$$\mathrm{Ext}^p_{\mathscr{C}^{\mathbb{Z}^2}_{(\mathfrak{g},\mathfrak{h})}}(N_\varepsilon(v;\tau), V) \simeq \mathrm{Hom}^{\bar{0}}_{\mathfrak{h}}(\mathbb{C}^\tau_v, H^p(\mathfrak{g}_+, V)).$$

In particular,

$$\mathrm{Hom}_{\mathscr{C}^{\mathbb{Z}^2}_{(\mathfrak{g},\mathfrak{h})}}(N_\varepsilon(v;\tau), V) \simeq \mathrm{Hom}^{\bar{0}}_{\mathfrak{h}}(\mathbb{C}^\tau_v, V^{\mathfrak{g}_+}),$$

where $V^{\mathfrak{g}_+} := \{v \in V | \mathfrak{g}_+.v = 0\}$ is the $\mathfrak{g}_+$-invariant of V.

Proof. By the Frobenius reciprocity, we have

$$\begin{aligned}\mathrm{Ext}^p_{(\mathfrak{g},\mathfrak{h})}(N(v;\tau),V) &= \mathrm{Ext}^p_{(\mathfrak{g},\mathfrak{h})}(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_\geqslant} P_\varepsilon(v;\tau),V)\\ &\cong \mathrm{Ext}^p_{(\mathfrak{g}_\geqslant,\mathfrak{h})}(P_\varepsilon(v;\tau),V).\end{aligned}$$

Since the functor $\mathrm{Hom}^{\bar{0}}_{\mathfrak{g}_0}(P_\varepsilon(v;\tau),\cdot) = \mathrm{Hom}^{\bar{0}}_{\mathfrak{h}}(\mathbb{C}^\tau_v,\cdot)$ is exact, we get

$$\begin{aligned}\mathrm{Ext}^p_{(\mathfrak{g}_\geqslant,\mathfrak{h})}(P_\varepsilon(v;\tau),V) &\cong \mathrm{Hom}^{\bar{0}}_{\mathfrak{g}_0}(P_\varepsilon(v;\tau),H^p(\mathfrak{g}_+,V))\\ &= \mathrm{Hom}^{\bar{0}}_{\mathfrak{h}}(\mathbb{C}^\tau_v,H^p(\mathfrak{g}_+,V)). \qquad \square\end{aligned}$$

Remark 2.1. As the above proof indicates, the following formula does not hold in general:

$$\mathrm{Ext}^p_{(\mathfrak{g},\mathfrak{h})}(M_0(v;\tau),V) \cong \mathrm{Hom}^{\bar{0}}_{\mathfrak{g}_0}(V(v;\tau),H^p(\mathfrak{g}_+,V)).$$

The following relation between homologies and cohomologies is a corollary of Lemma 2.1.

Proposition 2.2. *Let V be an object of $\mathcal{O}^{\mathbb{Z}_2}$. Then, we have an isomorphism*

$$H^n(\mathfrak{g}_+,V^c) \simeq H_n(\mathfrak{g}_-,V)$$

as $\mathfrak{g}_0$-modules.

3. Verma modules I: basic properties

In this section, we will study some basic properties of Verma modules over the $N=1$ super Virasoro algebras.

For $(\lambda;\tau)\in\mathfrak{h}^*\times\mathbb{Z}_2$ satisfying $\lambda(L_0)=h$ and $\lambda(c)=z$, we denote $M_\varepsilon(\lambda;\tau)$, $N(\lambda;\tau)$ and $\tilde{M}(\lambda)$ by $M_\varepsilon(z,h;\tau)$, $N(z,h;\tau)$ and $\tilde{M}(z,h)$, respectively. Moreover, we may sometimes omit τ.

3.1. Determinant formulae

In this subsection, we recall the Shapovalov type determinant formulae for Verma modules. For simplicity, all of the formulations are given only for $\tau=\bar{0}\in\mathbb{Z}_2$.

Let σ be the anti-involution of $U(\mathrm{Vir}_\varepsilon)$ defined by $\sigma(L_n)=L_{-n}$ $(n\in\mathbb{Z})$, $\sigma(G_m)=G_{-m}$ $(m\in\varepsilon+\mathbb{Z})$ and $\sigma(c)=c$. We denote the projection $U(\mathrm{Vir}_\varepsilon)\to U((\mathrm{Vir}_\varepsilon)_0)$ with respect to the direct sum decomposition

$$U(\mathrm{Vir}_\varepsilon) = U((\mathrm{Vir}_\varepsilon)_0) \oplus (U(\mathrm{Vir}_\varepsilon)\mathrm{Vir}_\varepsilon^+ + \mathrm{Vir}_\varepsilon^- U(\mathrm{Vir}_\varepsilon))$$

by π, and the projection onto the even subspaces of $U((\mathrm{Vir}_\varepsilon)_0)$ by $\pi^{\bar{0}}$. Moreover, let ev_λ $(\lambda \in \mathfrak{h}^*)$ be the evaluation map defined by

$$\mathrm{ev}_\lambda : U((\mathrm{Vir}_\varepsilon)_0)^{\bar{0}} \cong \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}, \quad P \mapsto P(\lambda).$$

For $n \in (1-\varepsilon)\mathbb{Z}_{\geqslant 0}$, set

$$M_\varepsilon(z,h)_n := \{u \in M_\varepsilon(z,h) \mid L_0.u = (h+n)u\}.$$

Definition 3.1. 1. The bilinear form $\langle \, , \rangle$ defined by

$$\langle \, , \rangle : M_\varepsilon(z,h) \times M_\varepsilon(z,h) \to \mathbb{C},$$

$$(x.v_{z,h}, y.v_{z,h}) \mapsto \mathrm{ev}_{(z,h)}(\pi^{\bar{0}} \circ \pi(\sigma(x)y)),$$

where we set $v_{z,h} := 1 \otimes \mathbf{1}_{z,h}$, is called a *contravariant form* on $M_\varepsilon(z,h)$.

2. For each $n \in (1-\varepsilon)\mathbb{Z}_{\geqslant 0}$ and $\tau \in \mathbb{Z}_2$, choose a basis of $M_\varepsilon(z,h)_n^\tau$, and denote it by $\{v_1, v_2, \ldots, v_r\}$. Then, we set

$$\det\nolimits_\varepsilon(z,h)_n^\tau := \det(\langle v_i, v_j \rangle_{1 \leqslant i,j \leqslant r}).$$

The determinant $\det_\varepsilon(z,h)_n^\tau$ is called the *Shapovalov type determinant*.

The explicit form of the determinant $\det_\varepsilon(z,h)_n^\tau$ can be described as follows.

For $\varepsilon = 0, \frac{1}{2}$ and $n \in (1-\varepsilon)\mathbb{Z}_{\geqslant 0}$, let us set

$$p_\varepsilon(n) := (\tfrac{1}{2} + \varepsilon) \dim U((\mathrm{Vir}_\varepsilon)_-)_n,$$

where we set

$$U((\mathrm{Vir}_\varepsilon)_-)_n := \{u \in U((\mathrm{Vir}_\varepsilon)_-) \mid [L_0, u] = nu\}.$$

Then, the determinant $\det_\varepsilon(z,h)_n^\tau$ has the following expression:

Theorem 3.1. *For $\varepsilon = 0, \frac{1}{2}$, $\tau \in \mathbb{Z}_2$ and $n \in (1-\varepsilon)\mathbb{Z}_{>0}$ such that $M_\varepsilon(z,h)_n^\tau \neq \{0\}$, we have*

1. *if $\varepsilon = 0$ and $h = \frac{1}{24}z$, then*

$$\det\nolimits_0(z,h)_n^\tau \propto \prod_{\substack{\alpha,\beta \in \mathbb{Z}_{\geqslant 0}, \\ 1 \leqslant \alpha\beta \leqslant 2n, \\ \alpha-\beta \in 1+2\mathbb{Z}_{\geqslant 0}}} \Psi_{\alpha,\beta}(z)^{p_0(n-\frac{1}{2}\alpha\beta)},$$

where we set

$$\Psi_{\alpha,\beta}(z) = \alpha\beta z + \tfrac{3}{2}(2\alpha - \beta)(\alpha - 2\beta),$$

2. *otherwise we have*

$$\det_\varepsilon(z,h)^\tau_n \propto \left(h - \frac{1}{24}z\right)^{\delta_{\varepsilon,0}p_0(n)} \prod_{\substack{\alpha,\beta\in\mathbb{Z}_{\geqslant 0},\\ 1\leqslant\alpha\beta\leqslant 2n,\\ \alpha-\beta\in 1-2\varepsilon+2\mathbb{Z}_{\geqslant 0}}} \Phi_{\alpha,\beta;\varepsilon}(z,h)^{p_\varepsilon(n-\frac{1}{2}\alpha\beta)},$$

where we set

$$\Phi_{\alpha,\beta;\varepsilon}(z,h) = \begin{cases} [h + \frac{1}{24}(\alpha^2-1)(z-\frac{15}{2}) + \frac{1}{8}\{2(\alpha\beta-1)-(\frac{1}{2}-\varepsilon)\}] & \\ \quad\times[h + \frac{1}{24}(\beta^2-1)(z-\frac{15}{2}) + \frac{1}{8}\{2(\alpha\beta-1)-(\frac{1}{2}-\varepsilon)\}] & \text{if } \alpha>\beta, \\ \quad+\frac{1}{64}(\alpha^2-\beta^2)^2 & \\ h + \frac{1}{24}(z-\frac{3}{2})(\alpha^2-1) & \text{if } \alpha=\beta. \end{cases}$$

Proof (Sketch). For $\varepsilon = \frac{1}{2}$, the result is well-known (see, e.g., [KW]). For $\varepsilon = 0$, we may prove as follows. Notice that there is an isomorphism:

$$\tilde{M}(z,h) \cong \begin{cases} M_0(z,h) & h \neq \frac{1}{24}z, \\ M_0(z,h) \oplus \Pi M_0(z,h) & h = \frac{1}{24}z, \end{cases} \tag{1}$$

and via this isomorphism, one can naturally define a contravariant form, hence also the Shapovalov type determinants, on $\tilde{M}(z,h)$. Taking care of the degree of the determinants, we obtain the results in a way similar to [KW]. $\square$

Remark 3.1. 1. Regarding $\Phi_{\alpha,\beta;\varepsilon}(z,h)$ as a polynomial in h, its discriminant is zero if and only if $z \in \{\frac{3}{2}, \frac{27}{2}\}$.

2. The relation between $\Phi_{\alpha,\beta;0}$ and $\Psi_{\alpha,\beta}$ can be described as follows:

$$\Phi_{\alpha,\beta;0}\left(z, \frac{1}{24}z\right) = \frac{1}{24^2}\Psi_{\alpha,\beta}(z)^2.$$

3.2. Singular vector

In this subsection, we will compute the dimension of the vector subspace of $(\mathrm{Vir}_\varepsilon)_+$-invariants in $M_\varepsilon(z,h)_n$ for $n \in (1-\varepsilon)\mathbb{Z}_{\geqslant 0}$.

First, for the Neveu–Schwarz algebra $\mathrm{Vir}_{\frac{1}{2}}$, we have the following:

Proposition 3.1. *Let $z, h \in \mathbb{C}$ and $n \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$. Then, we have*

$$\dim \{M_{\frac{1}{2}}(z,h)_n\}^{(\mathrm{Vir}_{\frac{1}{2}})_+} \leqslant 1.$$

To prove this proposition, we introduce a basis of $M_{\frac{1}{2}}(z,h)_n$ as follows. For $j \in \frac{1}{2}\mathbb{Z}$, $i_j \in \frac{1}{2}\mathbb{Z}_{>1}$ and $\delta \in \{0,1\}$, set

$$x_j := \begin{cases} L_j & j \in \mathbb{Z}, \\ G_j & j \in \frac{1}{2} + \mathbb{Z}, \end{cases} \qquad m^{\delta}_{i_k,\dots,i_1} := x_{-i_k} \cdots x_{-i_1} (G_{-\frac{1}{2}})^{\delta} |z,h\rangle,$$

where $|z,h\rangle$ is a highest weight vector of $M_{\frac{1}{2}}(z,h)$. Then, the set

$$\mathscr{B}_n := \left\{ m^{\delta}_{i_k,\dots,i_1} \,\middle|\, \begin{array}{c} 1 \leqslant i_1 \leqslant \cdots \leqslant i_k, \delta \in \{0,1\}, \frac{1}{2}\delta + \sum_{j=1}^{k} i_j = n, \\ i_s \in \frac{1}{2} + \mathbb{Z} \Rightarrow i_s < i_{s+1} \end{array} \right\}$$

forms a basis of $M_{\frac{1}{2}}(z,h)_n$. We introduce a total order $<$ on the set $\mathscr{B}_n$ as follows:

$$m^{\delta}_{i_k,\dots,i_1} < m^{\delta'}_{j_l,\dots,j_1} \Leftrightarrow \exists s \text{ s.t. } i_1 = j_1, \dots, i_s = j_s \text{ and } i_{s+1} < j_{s+1}.$$

Now, we assume the existence of a singular vector $w_n \in M_{\frac{1}{2}}(z,h)_n \setminus \{0\}$. Let us express w_n by using elements of $\mathscr{B}_n$

$$w_n = \sum P^{\delta}_{i_k,\dots,i_1} m^{\delta}_{i_k,\dots,i_1}, \quad P^{\delta}_{i_k,\dots,i_1} \in \mathbb{C}.$$

The key step is the next lemma from which Proposition 3.1 follows, because of the triangularity of the equations characterizing a singular vector w_n given below.

Lemma 3.1. *Let $m^{\delta'}_{j_l,\dots,j_1}$ be an element of $\mathscr{B}_n$ such that $j_1 = \cdots = j_s = 1$, $j_{s+1} = j > 1$ for some non-negative integer $s < l$. Then, the coefficient of*

$$m^{\delta'}_{j_l,\dots,j_{s+2},\underbrace{1,\dots,1}_{s+1}}$$

in $x_{j-1}.w_n$ expanded with respect to the basis $\mathscr{B}_{n-j+1}$ looks as follows:

$$\alpha P^{\delta'}_{j_l,\dots,j_1} + \sum_{\substack{m^{\delta}_{i_k,\dots,i_1} \in \mathscr{B}_n \\ \text{s.t. } m^{\delta}_{i_k,\dots,i_1} < m^{\delta'}_{j_l,\dots,j_1}}} Q^{\delta}_{i_k,\dots,i_1} P^{\delta}_{i_k,\dots,i_1}$$

for some constants $\alpha \in \mathbb{C}^*$ and $Q^{\delta}_{i_k,\dots,i_1} \in \mathbb{C}$.

Proof. It is enough to show the following equivalence:

$$m^{\delta}_{i_k,\dots,i_1}\in\mathscr{B}_n \quad \text{s.t.} \quad \begin{cases} \text{(i)} & m^{\delta}_{i_k,\dots,i_1}\geqslant m^{\delta'}_{j_l,\dots,j_1}, \\ \text{(ii)} & (\text{the coeff. of } m^{\delta'}_{j_l,\dots,j_{s+2},\underbrace{1,\,\dots,\,1}_{s+1}} \text{ in } x_{j-1}.m^{\delta}_{i_k,\dots,i_1})\neq 0 \end{cases}$$

$$\Leftrightarrow m^{\delta}_{i_k,\dots,i_1}=m^{\delta'}_{j_l,\dots,j_1}.$$

($\Rightarrow$) Let t be a non-negative integer such that $i_1=\cdots=i_t=1$ and $i_{t+1}>1$. The first condition (i) implies $t\leqslant s$. Moreover, by direct computation

$$\begin{aligned} x_{j-1}.m^{\delta}_{i_k,\dots,i_1} &= [x_{j-1},x_{-i_k}\cdots x_{-i_{t+1}}](L_{-1})^t(G_{-\frac{1}{2}})^{\delta}.|z,h\rangle \\ &\quad \pm x_{-i_k}\cdots x_{-i_{t+1}}[x_{j-1},(L_{-1})^t](G_{-\frac{1}{2}})^{\delta}.|z,h\rangle \\ &\quad \pm x_{-i_k}\cdots x_{-i_{t+1}}(L_{-1})^t[x_{j-1},(G_{-\frac{1}{2}})^{\delta}].|z,h\rangle, \end{aligned}$$

we see that each term on the right-hand side satisfies the following:

A. Expressing the first term with respect to a basis $\mathscr{B}_{n-j+1}$, the powers of L_{-1} in each term with non-zero coefficient is at most $t+1$.
B. Expressing the second and the third terms with respect to a basis $\mathscr{B}_{n-j+1}$, the powers of L_{-1} in each term with non-zero coefficient is at most t.

Thus, we get $t=s$. Next, we will show $i_m=j_m$ for $m\geqslant s$. Take a positive integer $u\geqslant s+1$ such that $i_u=\cdots=i_{s+1}=i>1$, $i_{u+1}>i_u$. Then, by direct calculation

$$\begin{aligned} &[x_{j-1},x_{-i_k}\cdots x_{-i_{s+1}}](L_{-1})^s(G_{-\frac{1}{2}})^{\delta}.|z,h\rangle \\ &\quad = [x_{j-1},x_{-i_k}\cdots x_{-i_{u+1}}](x_{-i})^{u-s}(L_{-1})^s(G_{-\frac{1}{2}})^{\delta}.|z,h\rangle \\ &\qquad \pm x_{-i_k}\cdots x_{-i_{u+1}}[x_{j-1},(x_{-i})^{u-s}](L_{-1})^s(G_{-\frac{1}{2}})^{\delta}.|z,h\rangle, \end{aligned}$$

the assumption $i_{u+1}>i_u$ implies that expanding the first term of the right-hand side with respect to $\mathscr{B}_{n-j+1}$, we have

$$\text{each term with non-zero coeff. } \gneqq m^{\delta'}_{j_l,\dots,j_{s+2},\underbrace{1,\,\dots,\,1}_{s+1}}.$$

Hence, by condition (ii), it follows that $i_{s+1}=i=j=j_{s+1}$. Similarly, we can prove $i_m=j_m$ for any $m>s+1$.

($\Leftarrow$) Let u, s be positive integers such that $j_{u+1} > j_u = \cdots = j_{s+1} = j > j_s = \cdots = j_1 = 1$. Then, a straightforward computation shows that

$$\text{The coeff. of } m^{\delta'}_{j_l,\ldots,j_{s+2},\underbrace{1,\ldots,1}_{s+1}} \text{ in } x_{j-1}.m^{\delta'}_{j_l,\ldots,j_1} = \begin{cases} (u-s)(2j-1) & j \in \mathbb{Z}, \\ \pm 2 & j \in \frac{1}{2} + \mathbb{Z}, \end{cases}$$

which leads to the conclusion. $\square$

Second, for Ramond algebra Vir_0, we can state as follows. Recall that $M_0(z,h)$ is $\mathbb{Z}$-graded via

$$M_0(z,h) = \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} M_0(z,h)_n, \quad M_0(z,h)_n := \{u \in M_0(z,h) \mid L_0.u = (h+n)u\}.$$

Proposition 3.2. *Let $z, h \in \mathbb{C}$, $\sigma \in \mathbb{Z}_2$ and $n \in \mathbb{Z}_{>0}$.*

1. *If $h \neq \frac{1}{24}z$, then we have* $\dim \{M_0(z,h)^\sigma_n\}^{(\mathrm{Vir}_0)_+} \leqslant 2$.
2. *If $h = \frac{1}{24}z$, then we have* $\dim \{M_0(z,h)^\sigma_n\}^{(\mathrm{Vir}_0)_+} \leqslant 1$.

Since the proof of this proposition can be managed by essentially the same way as for the Neveu–Schwarz algebra, we will state a lemma corresponding to Lemma 3.1, in particular for $h \neq \frac{1}{24}z$, and indicate the only statement that has to be proved. The case $h = \frac{1}{24}z$ can be obtained simply by setting $\mathscr{B}^\sigma_n := \mathscr{B}^{\sigma;0}_n$ in the discussion given below.

For $j \in \mathbb{Z}$, $i_j \in \mathbb{Z}_{>0}$, $\varepsilon, \varepsilon_j \in \mathbb{Z}_2$ and $\delta \in \{0,1\}$, set

$$x^\varepsilon_j := \begin{cases} L_j, & \varepsilon = \bar{0}, \\ G_j, & \varepsilon = \bar{1}, \end{cases} \quad m^\delta_{(i_k,\varepsilon_k),\ldots,(i_1,\varepsilon_1)} = x^{\varepsilon_k}_{-i_k} \cdots x^{\varepsilon_1}_{-i_1} (G_0)^\delta |z,h\rangle.$$

Then, for $\sigma \in \mathbb{Z}_2$ and $\delta \in \{0,1\}$, setting

$$\mathscr{B}^{\sigma;\delta}_n := \left\{ m^\delta_{(i_k,\varepsilon_k),\ldots,(i_1,\varepsilon_1)} \;\middle|\; \begin{array}{c} 1 \leqslant i_1 \leqslant \cdots \leqslant i_k, \varepsilon_1, \ldots, \varepsilon_k \in \mathbb{Z}_2, \sum_{j=1}^k i_j = n \\ \sum_{j=1}^k \varepsilon_j = \sigma, \ \varepsilon_s = \bar{1} \Rightarrow i_s < i_{s+1} \end{array} \right\},$$

$\mathscr{B}^\sigma_n := \mathscr{B}^{\sigma;0}_n \cup \mathscr{B}^{\bar{1}-\sigma;1}_n$ forms a basis of $M_0(z,h)^\sigma_n$. We define a total order $<_\sigma$ on the set $\mathscr{B}^\sigma_n$ as follows. First, we introduce a total order $\prec$ on the set $\{L_j, G_j \mid j \in \mathbb{Z}_{<0}\}$ by

$$L_{-1} \prec G_{-1} \prec L_{-2} \prec G_{-2} \prec \cdots.$$

Second, the order $<_\sigma$ is defined by

$$m^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)} <_\sigma m^{\delta'}_{(j_l,\eta_l),\dots,(j_1,\eta_1)}$$
$$\Leftrightarrow \exists s \text{ s.t. } x^{\varepsilon_1}_{-i_1} = x^{\eta_1}_{-j_1}, \dots, x^{\varepsilon_s}_{-i_s} = x^{\eta_s}_{-j_s} \text{ and } x^{\varepsilon_{s+1}}_{-i_{s+1}} \prec x^{\eta_{s+1}}_{-j_{s+1}}.$$

Now, we assume the existence of a singular vector $w_n \in M_0(z,h)^{\sigma}_n \setminus \{0\}$. Expressing w_n by using elements of $\mathscr{B}^{\sigma}_n$

$$w_n = \sum P^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)} m^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)}, \quad P^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)} \in \mathbb{C},$$

we have the following lemma:

Lemma 3.2. *Let $m^{\delta'}_{(j_l,\eta_l),\dots,(j_1,\eta_1)}$ be an element of $\mathscr{B}^{\sigma;\delta'}_n$ satisfying $x^{\eta_1}_{-j_1} = \cdots = x^{\eta_{s-1}}_{-j_{s-1}} = L_{-1}$, $x^{\eta_s}_{-j_s} \in \{L_{-1}, G_{-1}\}$ and $x^{\eta_{s+1}}_{-j_{s+1}} = x^{\eta}_{-j} \succ G_{-1}$ for some $s<l$. Then, the coefficient of*

$$m^{\delta'}_{(j_l,\eta_l),\dots,(j_{s+2},\eta_{s+2}),(1,\eta_s),(1,\bar{0})^s}$$

in $x^{\eta}_{j-1}.w_n$ expanded with respect to the basis $\mathscr{B}^{\sigma-\eta+\delta'\bar{1}}_{n-j+1}$ looks as follows:

$$\alpha P^{\delta'}_{(j_l,\eta_l),\dots,(j_1,\eta_1)} + \sum_{\substack{m^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)} \in \mathscr{B}^{\sigma+\delta'\bar{1}}_n \text{ s.t.} \\ m^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)} <_\sigma m^{\delta'}_{(j_l,\eta_l),\dots,(j_1,\eta_1)}}} Q^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)} P^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)}$$

for some constants $\alpha \in \mathbb{C}^*$ and $Q^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)} \in \mathbb{C}$.

Proof. It is enough to prove the following equivalence:

$$m^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)} \in \mathscr{B}^{\sigma+\delta'\bar{1}}_n$$
$$\text{s.t.} \begin{cases} \text{(i)} & m^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)\sigma} \geqslant m^{\delta'}_{(j_l,\eta_l),\dots,(j_1,\eta_1)}, \\ \text{(ii)} & (\text{the coeff. of } m^{\delta'}_{(j_l,\eta_l),\dots,(j_{s+2},\eta_{s+2}),(1,\eta_s),(1,\bar{0})^s} \text{ in } x^{\eta}_{j-1}.m^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)}) \neq 0 \end{cases}$$
$$\Leftrightarrow m^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)} = m^{\delta'}_{(j_l,\eta_l),\dots,(j_1,\eta_1)}.$$

We leave the detail of its proof to the reader. □

Now, for each $\delta \in \{0,1\}$, one can show that the solutions of the equations $\{x^{\eta}_{j-1}.w_n = 0\}$ in Lemma 3.2 are parametrized by $(P^{\delta}_{(1,\bar{0})^n}, P^{1-\delta}_{(1,\bar{1}),(1,\bar{0})^{n-1}}) \in \mathbb{C}^2$, hence we have proved Proposition 3.2.

Remark 3.2. In general, we have

$$\dim\{M_0(z,h)_n^{\bar{0}}\}^{(\mathrm{Vir}_0)_+} = \dim\{M_0(z,h)_n^{\bar{1}}\}^{(\mathrm{Vir}_0)_+}$$

for any $n \in \mathbb{Z}_{>0}$. Indeed, since $G_0^2.S|z,h\rangle = (h+n-\frac{1}{24}z)S|z,h\rangle$ for $S|z,h\rangle \in \{M_0(z,h)_n^{\sigma}\}^{(\mathrm{Vir}_0)_+}$ $(\sigma \in \mathbb{Z}_2)$, one can prove the following:

1. If $h+n-\frac{1}{24}z \neq 0$, then

$$G_0.S|z,h\rangle \in \{M_0(z,h)_n^{\bar{1}-\sigma}\}^{(\mathrm{Vir}_0)_+} \setminus \{0\}.$$

2. If $h+n-\frac{1}{24}z = 0$, then

$$SG_0.|z,h\rangle \in \{M_0(z,h)_n^{\bar{1}-\sigma}\}^{(\mathrm{Vir}_0)_+} \setminus \{0\}.$$

Moreover, the above correspondence is clearly one-to-one.

3.3. Injectivity

In this subsection, we will see the injectivity of a non-trivial morphism between Verma modules. Namely, we will prove the following proposition:

Proposition 3.3. *Let $h, h' \in \mathbb{C}$ such that $h \neq h'$.*

1. *Any non-trivial $\mathbb{Z}_2$-graded $\mathrm{Vir}_{\frac{1}{2}}$-homomorphism $M_{\frac{1}{2}}(z,h') \to M_{\frac{1}{2}}(z,h)$ is injective.*
2. *Let $\varphi : M_0(z,h') \to M_0(z,h)$ be a non-trivial $\mathbb{Z}_2$-graded Vir_0-homomorphism. Assume that $h \neq \frac{1}{24}z$.*
   (i) *If we have $\dim\{M_0(z,h)_{h'-h}^{\tau}\}^{(\mathrm{Vir}_0)_+} = 1$ for any $\tau \in \mathbb{Z}_2$, then any such φ is injective.*
   (ii) *If $\dim\{M_0(z,h)_{h'-h}^{\tau}\}^{(\mathrm{Vir}_0)_+} = 2$ for any $\tau \in \mathbb{Z}_2$, there exists such an injective map φ.*
3. *If $h = \frac{1}{24}z$, then a non-trivial $\mathbb{Z}_2$-graded homomorphism $\varphi : M_0(z,h') \to \tilde{M}(z,h)$ is injective if and only if $\mathrm{Im}\,\varphi \not\subset M_0(z,h;\tau)$ for any $\tau \in \mathbb{Z}_2$.*

Note that by Remark 3.2, it is equivalent to say 'for some $\tau \in \mathbb{Z}_2$' instead of 'for any $\tau \in \mathbb{Z}_2$' in the second statements (i.e., the case $h \neq \frac{1}{24}z$,) of Proposition 3.3.

Let us turn to the proof of Proposition 3.3. The first part of the above proposition follows from the fact that $U((\mathrm{Vir}_{\frac{1}{2}})_-)$ is an integral domain by Theorem 2.7 of [AL]. Next, we prove the second part of Proposition 3.3. Suppose that $h \neq \frac{1}{24}z$. We may assume that $M_0(z,\tilde{h}) = M_0(z,\tilde{h};\bar{0})$ for $\tilde{h} = h, h'$ without loss of generality. Let $|z,h\rangle$

(resp. $|z, h'\rangle$) be an even highest weight vector. First, we have the following technical lemma:

Lemma 3.3. *Suppose we have* $\dim\{M_0(z,\tilde{h})_k^{\bar{0}}\}^{(\mathrm{Vir}_0)_+} = 1$, *and let* $(AG_0 + B)|z, \tilde{h}\rangle$ *be an even singular vector of* $M_0(z,\tilde{h})_k$ *which has the following expression with respect to the basis* $\mathscr{B}_k^{\bar{0}}$:

$$A = c_A G_{-1} L_{-1}^{k-1} + d_A G_{-2} L_{-1}^{k-2} + \cdots,$$

$$B = c_B L_{-1}^{k} + d_B L_{-2} L_{-1}^{k-2} + \cdots.$$

Moreover, assume that $\tilde{h} - \frac{1}{24}z \neq 0, -k$. *Then, we have*

$$4(\tilde{h} - \tfrac{1}{24}z)c_A^2 + 4(\tilde{h} - \tfrac{1}{24}z)c_A c_B - k c_B^2 = 0.$$

Proof. By assumption, $G_0(AG_0 + B)G_0|z, \tilde{h}\rangle$ is a non-trivial even singular vector which is proportional to $(AG_0 + B)|z, \tilde{h}\rangle$. Since we have

$$G_0(AG_0 + B)G_0|z, \tilde{h}\rangle = \{(\tilde{h} - \tfrac{1}{24}z)([G_0, A] + B) + (-(\tilde{h} - \tfrac{1}{24}z)A + [G_0, B])G_0\}|z, \tilde{h}\rangle,$$

comparing the coefficient of $L_{-1}^{k}|z, \tilde{h}\rangle$ and $G_{-1}L_{-1}^{k-1}G_0|z, \tilde{h}\rangle$, we obtain the formula. □

Let $\varphi : M_0(z, h') \to M_0(z, h)$ be a non-trivial homomorphism, and set $\varphi(|z, h'\rangle) = (PG_0 + Q)|z, h\rangle$. We may assume that $M_0(z, h')$ is reducible, since otherwise the map φ is automatically injective by definition. In particular, we have $h' - \frac{1}{24}z \neq 0$ in this case (see Section 4). Furthermore, we assume that $\operatorname{Ker}\varphi \neq \{0\}$, and lead to a contradiction.

Let m be the smallest positive integer such that $\operatorname{Ker}\varphi \cap M_0(z, h')_m \neq \{0\}$, and take a non-zero element $(XG_0 + Y)|z, h'\rangle \in \operatorname{Ker}\varphi \cap M_0(z, h')_m^{\bar{0}}$. It is clear that $(XG_0 + Y)|z, h'\rangle$ is a singular vector. Moreover, since we have

$$\varphi((XG_0 + Y)|z, h'\rangle) = (XG_0 + Y)(PG_0 + Q)|z, h\rangle = 0$$

by definition, we have the following equations in $U((\mathrm{Vir}_0)_-)$:

$$X[G_0, P] + XQ + YP = 0, \tag{2}$$

$$X[G_0, Q] + YQ - \lambda XP = 0, \tag{3}$$

where λ is defined by $\lambda = h - \frac{1}{24}z$. Expanding P and Q with respect to the basis $\mathscr{B}_n^{\bar{1}}$ (resp. $\mathscr{B}_n^{\bar{0}}$)

$$P = c_P G_{-1} L_{-1}^{n-1} + d_P G_{-2} L_{-1}^{n-2} + \cdots,$$

$$Q = c_Q L_{-1}^{n} + d_Q L_{-2} L_{-1}^{n-2} + \cdots,$$

where we set $n := h' - h$, and expanding X and Y with respect to the basis $\mathscr{B}_m^{\bar{1}}$ (resp. $\mathscr{B}_m^{\bar{0}}$)

$$X = c_X G_{-1} L_{-1}^{m-1} + d_X G_{-2} L_{-1}^{m-2} + \cdots,$$

$$Y = c_Y L_{-1}^{m} + d_Y L_{-2} L_{-1}^{m-2} + \cdots,$$

we deduce the following relations from Eqs. (2) and (3):

The coeff. of $G_{-1}L_{-1}^{m+n-1}$ in (2) : (A) $\quad 2c_X c_P + c_X c_Q + c_Y c_P = 0,$

The coeff. of L_{-1}^{m+n} in (3) : (B) $\quad c_Y c_Q = 0.$

From relation (B), we conclude that either $c_Y = 0$ or $c_Q = 0$ holds.

First, let us consider the case $\dim \{M_0(z,h)_n^{\tau}\}^{(\mathrm{Vir}_0)_+} = 1$ for $\tau \in \mathbb{Z}_2$.

Suppose $c_Y = 0$. From relation (A), we have $c_X(2c_P + c_Q) = 0$. If $c_X = 0$, then we get $(XG_0 + Y)|z,h'\rangle = 0$ by Lemma 3.2 which says that φ is injective. Now, if $2c_P + c_Q = 0$, then Lemma 3.3 implies that

$$\left(h - \frac{1}{24}z + n\right)c_Q^2 = 0.$$

By assumption, we have $c_Q = c_P = 0$ which implies that $(PG_0 + Q)|z,h\rangle = 0$ by Lemma 3.2. This contradicts the non-triviality of φ. Next, suppose $c_Q = 0$. Similar argument shows that $(PG_0 + Q)|z,h\rangle = 0$ which is impossible.

Second, we consider the case $\dim \{M_0(z,h)_n^{\tau}\}^{(\mathrm{Vir}_0)_+} = 2$ for $\tau \in \mathbb{Z}_2$. In this case, Lemma 3.2 ensures that the elements of $\{M_0(z,h)_n^{\bar{0}}\}^{(\mathrm{Vir}_0)_+}$ are parametrized by $(c_P, c_Q) \in \mathbb{C}^2$. Now, if φ enjoys the properties

$$c_Q \neq 0, \quad 2c_P + c_Q \neq 0,$$

then relations (A) and (B) imply that $c_X = c_Y = 0$. Then, by Lemma 3.2, it follows that $(XG_0 + Y)|z,h'\rangle = 0$ which says that φ is injective.

Let us prove the third statement. Suppose that $\mathrm{Im}\,\varphi \subset M_0(z,h;\sigma)$ for some $\sigma \in \mathbb{Z}_2$. Let $f(\tau)$ be the generating function of the dimension of weight subspaces of $M_0(z,h)$ (here and after, we omit σ), i.e.,

$$f(\tau) := \sum_{n \geqslant 0} p_0(n) q^n \quad (q = e^{2\pi i \tau}).$$

Then, $f(\tau)$ can be expressed in terms of the Dedekind eta function $\eta(\tau) = q^{\frac{1}{24}} \prod_{n \in \mathbb{Z}_{>0}} (1 - q^n)$ as

$$f(\tau) = \frac{\eta(2\tau)}{\eta(\tau)^2}.$$

Since the asymptotic behavior of $\eta(\tau)$ at the cusp 0 is known to be given by

$$\eta(\tau) \underset{\tau \downarrow 0}{\sim} (-i\tau)^{-\frac{1}{2}} e^{-\frac{\pi i}{12\tau}},$$

one can deduce the following asymptotic behavior:

$$p_0(n) \sim \frac{1}{8n} e^{\pi\sqrt{n}} \quad (n \to \infty) \tag{4}$$

as an application of a Tauberian theorem (see, e.g., [In, Theorem 2]). Set $h' - h = k$. Then, we have

$$\lim_{n\to\infty} \frac{\dim M_0(z,h)_{n+k}}{\dim M_0(z,h')_n} = \lim_{n\to\infty} \frac{p_0(n+k)}{2p_0(n)} = \frac{1}{2},$$

which implies that φ cannot be injective.

Now, for each $\tau \in \mathbb{Z}_2$, let $|z\rangle^\tau$ be a highest weight vector of $M_0(z, \frac{1}{24}z; \tau)$, and regard them as elements of $\tilde{M}(z, \frac{1}{24}z)$ via an isomorphism (1). Let $\varphi : M_0(z,h') \to \tilde{M}(z, \frac{1}{24}z)$ be a non-trivial homomorphism, and set $\varphi(|z,h'\rangle) = Q|z\rangle^{\bar{0}} + [G_0, Q]|z\rangle^{\bar{1}}$. As in the proof of the second assertion, we may assume that $M_0(z,h')$ is reducible. Suppose that $\mathrm{Ker}\,\varphi \neq \{0\}$, and lead to a contradiction.

Let m be the smallest positive integer such that $\mathrm{Ker}\,\varphi \cap M_0(z,h')_m \neq \{0\}$, and take a non-zero element $(XG_0 + Y)|z,h'\rangle \in \mathrm{Ker}\,\varphi \cap M_0(z,h')^{\bar{0}}_m$ which is a singular vector by definition. Since we have

$$\varphi((XG_0 + Y)|z,h'\rangle) = (XG_0 + Y)(Q|z\rangle^{\bar{0}} + [G_0, Q]|z\rangle^{\bar{1}}) = 0$$

by assumption, each coefficient of $|z\rangle^\tau$ gives the following equations in $U((\mathrm{Vir}_0)_-)$:

$$X[G_0, Q] + YQ = 0, \tag{5}$$

$$(h' - \tfrac{1}{24}z)XQ + Y[G_0, Q] = 0. \tag{6}$$

Expand X and Y with respect to the basis $\mathscr{B}^{\bar{0}}_m$:

$$X = c_X G_{-1} L_{-1}^{m-1} + d_X G_{-2} L_{-1}^{m-2} + \cdots,$$

$$Y = c_Y L_{-1}^{m} + d_Y L_{-2} L_{-1}^{m-2} + \cdots.$$

Comparing the coefficient of L_{-1}^{m+n} in (5), we get $c_Y c_Q = 0$. If $c_Q = 0$, then we have $Q = 0$ by Lemma 3.2, which contradicts the non-triviality of φ. Hence, we have $c_Y = 0$. Next, from the coefficient of $G_{-1}L_{-1}^{m+n-1}$ in (6), we see that $(h' - \frac{1}{24}z)c_X c_Q = 0$. Thus, we get $c_X = 0$. Therefore, again by Lemma 3.2, we conclude $X = Y = 0$ which is a contradiction. □

Remark 3.3. Under the assumption of Proposition 3.3, 2, (ii), set

$$\varphi(|z,h'\rangle) = (PG_0 + Q)|z,h\rangle \quad (P, Q \in U((\mathrm{Vir}_0)_-)),$$

$$P = c_P G_{-1} L_{-1}^{h'-h-1} + \cdots, \quad Q = c_Q L_{-1}^{h'-h} + \cdots.$$

The above proof implies that φ is injective if and only if

$$c_Q \neq 0 \wedge 2c_P + c_Q \neq 0. \tag{7}$$

4. Verma modules II: structure theorem

In this section, we will study the structure of Verma modules by means of the Jantzen filtration.

4.1. Classification of weights

In this subsection, we will classify all weights by means of the determinant formula.

It can be easily shown that the zeros of each factor $\Phi_{\alpha,\beta;\varepsilon}(z,h)$ (see Theorem 3.1 for the definition) can be described as follows.

Lemma 4.1. *For $\alpha, \beta \in \mathbb{Z}_{\geqslant 0}$ and $\varepsilon = 0, \frac{1}{2}$ satisfying $\beta - \alpha \in 1 - 2\varepsilon + 2\mathbb{Z}_{\geqslant 0}$, we have*

$$Z_{\alpha,\beta;\varepsilon} := \{(z,h) \in \mathbb{C}^2 \mid \Phi_{\alpha,\beta;\varepsilon}(z,h) = 0\} = \{(z(t), h_{\alpha,\beta;\varepsilon}(t)) \mid t \in \mathbb{C}^*\},$$

where we set

$$z(t) := \tfrac{15}{2} - 3(t + t^{-1}),$$

$$h_{\alpha,\beta;\varepsilon}(t) := \tfrac{1}{8}(\alpha^2 - 1)t - \tfrac{1}{4}(\alpha\beta - 1) + \tfrac{1}{8}(\beta^2 - 1)t^{-1} + \tfrac{1}{16}(1 - 2\varepsilon).$$

For $(\alpha, \beta) \in \mathbb{Z}^2$, we define $h_{\alpha,\beta;\varepsilon}(t)$ by the above formula. First, we note that if we set

$$z_{P,Q} := \frac{15}{2} - 3\left(\frac{P}{Q} + \frac{Q}{P}\right), \quad h_{P,Q;m} := \frac{1}{8PQ}\{m^2 - (P-Q)^2\} + \frac{1}{16}(1 - 2\varepsilon),$$

for $P, Q \in \mathbb{C}^*$ and $m \in \mathbb{C}$, then we have

$$\Phi_{\alpha,\beta;\varepsilon}(z_{P,Q}, h_{P,Q;m}) = \begin{cases} \frac{1}{(8PQ)^2}(m + P\alpha - Q\beta)(m - P\alpha + Q\beta) & \\ \quad \times (m + Q\alpha - P\beta)(m - Q\alpha + P\beta) & \text{if } \alpha \neq \beta, \\ \frac{1}{8PQ}(m + (P-Q)\alpha)(m - (P-Q)\alpha) & \text{if } \alpha = \beta. \end{cases}$$

Remark that the four lines $P\alpha - Q\beta = \pm m,\ Q\alpha - P\beta = \pm m$ in the (α, β)-plane degenerate if and only if $m = 0$ or $\frac{P}{Q} \in \{\pm 1\}$.

We will classify the set of lattice points on these lines by the following principle. To enumerate all of the lattice points on these four lines in the region $\{(\alpha, \beta) | \alpha \geqslant \beta > 0\}$, we consider the next three types. Let $l_{z_{P,Q}, h_{P,Q;m}}$ be one of four lines $P\alpha - Q\beta = \pm m$, $Q\alpha - P\beta = \pm m$.

Type 1: $m \neq 0$ and $\frac{P}{Q} \in \mathbb{Q} \setminus \{\pm 1\}$.

Type 2: $m \neq 0$ and $\frac{P}{Q} = 1$, or $m = 0$ and $\frac{P}{Q} \in \mathbb{Q}_{>0}$.

Type 3: $m \neq 0$ and $\frac{P}{Q} = -1$, or $m = 0$ and $\frac{P}{Q} \in \mathbb{Q}_{<0}$.

Note that in the case $m = 0$ for Type 3, there is no lattice point on $l_{z_{P,Q}, h_{P,Q;m}}$ in the first quadrant. For each case, the following equivalence holds:

$$\exists (\alpha, \beta) \in (\mathbb{Z}_{>0})^2 \text{ s.t } \alpha \geqslant \beta \wedge \alpha - \beta \in 1 - 2\varepsilon + 2\mathbb{Z} \wedge \Phi_{\alpha,\beta;\varepsilon}(z_{P,Q}, h_{P,Q;m}) = 0$$

$$\Leftrightarrow \begin{cases} \exists (\alpha, \beta) \in l_{z_{P,Q}, h_{P,Q;m}} \cap \mathbb{Z}^2 \text{ s.t. } \alpha\beta > 0 \wedge \alpha - \beta \in 1 - 2\varepsilon + 2\mathbb{Z} & \text{for Type 1,} \\ \exists (\alpha, \beta) \in l_{z_{P,Q}, h_{p,Q;m}} \cap (\mathbb{Z}_{>0})^2 \text{ s.t. } \alpha - \beta \in 1 - 2\varepsilon + 2\mathbb{Z} & \text{for Type 2,} \\ \exists (\alpha, \beta) \in l_{z_{P,Q}, h_{P,Q;m}} \cap (\mathbb{Z}_{>0})^2 \text{ s.t. } \alpha - \beta \in 1 - 2\varepsilon + 2\mathbb{Z}_{\geqslant 0} & \text{for Type 3.} \end{cases}$$

Hence, we have only to consider the lattice points on one of the four lines $P\alpha - Q\beta = \pm m,\ Q\alpha - P\beta = \pm m$ in each case.

If $\frac{P}{Q} \notin \mathbb{Q}$, then $l_{z,h}$ contains at most one lattice point. Here and after, we denote $l_{z,h}$ instead of $l_{z_{P,Q}, h_{P,Q;m}}$ for simplicity. Hence, in this case, we have

Class V: $l_{z,h}$ contains no lattice point.

Class I: $\frac{P}{Q} \notin \mathbb{Q}$ and $l_{z,h}$ contains one lattice point.

Next, we will consider the case

$$\frac{P}{Q} \in \mathbb{Q} \quad \text{s.t.} \quad P \in \mathbb{Z}, \quad Q \in \mathbb{Z}_{>0}, \quad P - Q \in 2\mathbb{Z}, \quad \left(\tfrac{1}{2}(P - Q), Q\right) = 1.$$

Class R^+: $\frac{P}{Q} \in \mathbb{Q}_{>0}$ $(P, Q \in \mathbb{Z}_{>0})$, and $l_{z,h}$ contains at least one point.

First, we assume that $(P, Q) \neq (1, 1)$, i.e., $z \neq \frac{3}{2}$, and $m \neq 0$. Let $l_{z,h}$ be one of two lines $P\alpha - Q\beta = \pm m$ such that if (α, β) is a lattice point of $l_{z,h}$ with the smallest possible $\alpha\beta$, then $\alpha, \beta > 0$. Set

$$l_{z,h} \cap \{(\mathbb{Z}_{>0})^2 \cup (\mathbb{Z}_{<0})^2\} = \left\{ (\alpha_i, \beta_i)\ (i \in \mathbb{Z}_{>0}) \,\middle|\, \begin{matrix} \alpha_j - \beta_j \in 1 - 2\varepsilon + 2\mathbb{Z}, \\ \alpha_j\beta_j \leqslant \alpha_{j+1}\beta_{j+1}, \end{matrix} \ \forall j \in \mathbb{Z}_{>0} \right\}.$$

By definition, we have $1 \leqslant \alpha_1 \leqslant Q$ or $1 \leqslant \beta_1 \leqslant P$. For such (α_1, β_1), we define $\alpha_1',\ \beta_1',\ \delta_\alpha,\ \delta_\beta,\ \delta \in \mathbb{Z}_{\geqslant 0}$ by

$$\begin{cases} \alpha_1 = \alpha_1' + \delta_\alpha Q & (0 \leqslant \alpha_1' < Q), \\ \beta_1 = \beta_1' + \delta_\beta P & (0 \leqslant \beta_1' < P), \end{cases} \quad \delta := \max\{\delta_\alpha, \delta_\beta\}.$$

Then, one can check that the lattice points (α_i, β_i) $(i \in \mathbb{Z}_{>0})$ can be expressed as follows:

$$(\alpha_i, \beta_i) = \begin{cases} (\alpha_1, \beta_1) + \frac{1}{2}(i-1)(Q,P) & (i \equiv 1 \bmod 2), \\ (\alpha_1, \beta_1) - (\frac{1}{2}i + \delta)(Q,P) & (i \equiv 0 \bmod 2). \end{cases}$$

Set $\tilde{P} := \frac{P}{(P,Q)}$, $\tilde{Q} := \frac{Q}{(P,Q)}$. Then, by direct computations, one can verify the following relations:

(1) If $\alpha_1 \not\equiv 0 \bmod \tilde{Q}$ or $\beta_1 \not\equiv 0 \bmod \tilde{P}$, then we have
$\alpha_1\beta_1 < \alpha_2\beta_2 < \alpha_3\beta_3 < \alpha_4\beta_4 < \cdots$.

(2) Assume that $\alpha_1 \equiv 0 \bmod \tilde{Q}$ and $\beta_1 \equiv 0 \bmod \tilde{P}$. We set $(\alpha_1, \beta_1) = (i\tilde{Q}, j\tilde{P})$ with $(i,j) \neq (1,1)$.
  (2.1) If $(P,Q) = 1$, then we have $\alpha_1\beta_1 = \alpha_2\beta_2 < \alpha_3\beta_3 = \alpha_4\beta_4 < \cdots$.
  (2.2) If $(P,Q) = 2$, then in addition, we assume $(i,j) \neq (2,2)$.
    (2.2.1) If $i + j \equiv 0 \bmod 2$, then we have
    $\alpha_1\beta_1 = \alpha_2\beta_2 < \alpha_3\beta_3 = \alpha_4\beta_4 < \cdots$.
    (2.2.2) If $i + j \equiv 1 \bmod 2$, then we have
    $\alpha_1\beta_1 < \alpha_2\beta_2 < \alpha_3\beta_3 < \alpha_4\beta_4 < \cdots$.

Next, we assume that $(P,Q) = (1,1)$ or $m = 0$. Let $l_{z,h}$ be the line $P\alpha - Q\beta = m$. Set

$$l_{z,h} \cap (\mathbb{Z}_{>0})^2 = \left\{ (\alpha_i, \beta_i) \ (i \in \mathbb{Z}_{>0}) \,\middle|\, \begin{matrix} \alpha_j - \beta_j \in 1 - 2\varepsilon + 2\mathbb{Z}, \\ \alpha_j\beta_j \leqslant \alpha_{j+1}\beta_{j+1}, \end{matrix} \ \forall j \in \mathbb{Z}_{>0} \right\}.$$

Now, if we set

$$(\alpha_1, \beta_1) = \begin{cases} (m+1, 1) & \text{if } \frac{P}{Q} = 1, \\ (Q, P) & \text{if } m = 0 \text{ and } \varepsilon = \frac{1}{2}, \\ (\tilde{Q}, \tilde{P}) & \text{if } m = 0, (P,Q) = 2 \text{ and } \varepsilon = 0, \end{cases}$$

then the lattice points (α_i, β_i) $(i \in \mathbb{Z}_{>0})$ can be parametrized as follows:

$$(\alpha_i, \beta_i) = (\alpha_1, \beta_1) + (i-1)(Q,P).$$

In this case, one can easily check the following relations:
(3) $\alpha_1\beta_1 < \alpha_2\beta_2 < \alpha_3\beta_3 < \alpha_4\beta_4 < \cdots$.

Class R^-: $\frac{P}{Q} \in \mathbb{Q}_{<0}$ $(P \in \mathbb{Z}_{<0}, Q \in \mathbb{Z}_{>0})$, and $l_{z,h}$ contains at least one point.

First, we consider the case $(P,Q) \neq (-1,1)$, i.e., $z \neq \frac{27}{2}$. We may assume $m < 0$, and let $l_{z,h}$ be the line defined by $P\alpha - Q\beta = m$. Set

$$l_{z,h} \cap (\mathbb{Z}_{>0})^2 = \left\{ (\alpha_i, \beta_i) \ (i \in \mathbb{Z}_{>0}) \,\middle|\, \begin{matrix} \alpha_j - \beta_j \in 1 - 2\varepsilon + 2\mathbb{Z}, \\ \alpha_j\beta_j \geqslant \alpha_{j+1}\beta_{j+1}, \end{matrix} \ \forall j \right\}.$$

Then, by definition, (α_1, β_1) is one of the nearest lattice points on $l_{z,h}$ to the point $(\frac{m}{2P}, -\frac{m}{2Q})$. One can check that the lattice points (α_i, β_i) $(i \in \mathbb{Z}_{>0})$ have the following expression:

$$(\alpha_i, \beta_i) = \begin{cases} (\alpha_1, \beta_1) + \frac{i}{2}(Q, P) & (i \equiv 0 \bmod 2), \\ (\alpha_1, \beta_1) - \frac{i-1}{2}(Q, P) & (i \equiv 1 \bmod 2), \end{cases} \quad \alpha_1 \leqslant \frac{m}{2P},$$

$$(\alpha_i, \beta_i) = \begin{cases} (\alpha_1, \beta_1) - \frac{i}{2}(Q, P) & (i \equiv 0 \bmod 2), \\ (\alpha_1, \beta_1) + \frac{i-1}{2}(Q, P) & (i \equiv 1 \bmod 2), \end{cases} \quad \alpha_1 > \frac{m}{2P}.$$

In fact, one can check that the following hold:

(1) If $\alpha_1 \neq \frac{m}{2P}$ and $\frac{1}{2}(\alpha_1 + \alpha_2) \neq \frac{m}{2P}$, then we have
$\alpha_1\beta_1 > \alpha_2\beta_2 > \alpha_3\beta_3 > \cdots$.
(2) If $\alpha_1 \neq \frac{m}{2P}$ and $\frac{1}{2}(\alpha_1 + \alpha_2) = \frac{m}{2P}$, then we have
$\alpha_1\beta_1 = \alpha_2\beta_2 > \alpha_3\beta_3 = \alpha_4\beta_4 > \cdots$.
(3) If $\alpha_1 = \frac{m}{2P}$, then we have
$\alpha_1\beta_1 > \alpha_2\beta_2 = \alpha_3\beta_3 > \alpha_4\beta_4 = \cdots$.

Second, we consider the case $(P, Q) = (-1, 1)$. Let $l_{z,h}$ be the line $\alpha + \beta = m(>0)$. Set

$$\begin{aligned} & l_{z,h} \cap \{(\alpha, \beta) \in (\mathbb{Z}_{>0})^2 \mid \alpha \geqslant \beta\} \\ &= \left\{ (\alpha_i, \beta_i) \ (i \in \mathbb{Z}_{>0}) \,\middle|\, \begin{array}{c} \alpha_j - \beta_j \in 1 - 2\varepsilon + 2\mathbb{Z}, \\ \alpha_j \geqslant \beta_j \text{ and } \alpha_j\beta_j \geqslant \alpha_{j+1}\beta_{j+1}, \end{array} \ \forall j \right\}. \end{aligned}$$

Hence, if we set $(\alpha_1, \beta_1) = (\lceil \frac{m+1}{2} \rceil, \lceil \frac{m}{2} \rceil)$, where $\lceil x \rceil$ signifies the maximal integer satisfying $\lceil x \rceil \leqslant x$, then the lattice points (α_i, β_i) $(i \in \mathbb{Z}_{>0})$ can be parametrized as follows:

$$(\alpha_i, \beta_i) = (\alpha_1, \beta_1) + (i-1)(1, -1).$$

In fact, one can immediately check the following relations:
(4) $\alpha_1\beta_1 > \alpha_2\beta_2 > \alpha_3\beta_3 > \alpha_4\beta_4 > \cdots$.

Remark 4.1. In Class R^+, there are infinitely many lattice points (α_i, β_i) on $l_{z,h}$ in the first and the third quadrants for each (z, h). On the contrary, in Class R^-, there are only finitely many lattice points (α_i, β_i) on $l_{z,h}$ in the first quadrant for each (z, h).

Now, we will classify weights following the recipe explained above. Our classification is given in a way that is suitable to describe the embedding diagrams among Verma modules.

For a weight belonging to Class V, we have nothing to do. Next, for a weight belonging to Class I, Lemma 4.1 with $t \in \mathbb{C} \setminus \mathbb{Q}$ already gives us a parametrization. Hence, the rest of our task is to classify the weights belonging to Class $R^{\pm}$.

Let p, q be positive integers satisfying $p - q \in 2\mathbb{Z}$, $(\frac{p-q}{2}, q) = 1$. In the rest of this subsection, we will fix the central charge as follows:

$$z\left(\pm\frac{p}{q}\right) = \frac{15}{2} \mp 3\left(\frac{p}{q} + \frac{q}{p}\right)$$

for Class $R^{\pm}$, respectively. We first introduce the sets

$$K^{\pm}_{p,q} := \left\{ (r, \pm s) \in \mathbb{Z}^2 \,\middle|\, \begin{matrix} 0 < r < q \\ 0 < s < p \end{matrix},\ rp + sq \leqslant pq \right\} \cup \left\{ (r, \pm s) \in \mathbb{Z}^2 \,\middle|\, \begin{matrix} r = q \\ 0 < s \leqslant p \end{matrix} \right\} \cup \left\{ (r, \pm s) \in \mathbb{Z}^2 \,\middle|\, \begin{matrix} 0 \leqslant r < q \\ s = p \end{matrix} \right\},$$

which parametrize the sequences of weights. For each $(r, s) \in K^{\pm}_{p,q}$ and $i \in \mathbb{Z}$, we define the L_0-weights $h_{i;\varepsilon}$ as follows:

$$h_{i;\varepsilon} := \begin{cases} h_{(i-1)q+r,-s;\varepsilon}(\pm\frac{p}{q}) & i \equiv 1 \bmod 2, \\ h_{iq+r,s;\varepsilon}(\pm\frac{p}{q}) & i \equiv 0 \bmod 2. \end{cases}$$

It is easy to check that for each L_0-weight belonging to Class $R^{\pm}$, there exist $(r, s) \in K^{\pm}_{p,q}$ and $i \in \mathbb{Z}$ such that $h = h_{i;\varepsilon}$.

Note that these L_0-weights are generically distinct, but some of them may coincide for some $(r, s) \in K^{\pm}_{p,q}$. According to the degeneration of these weights, we will classify them into following cases:

	Neveu-Schwarz ($\varepsilon = \frac{1}{2}$)	Ramond ($\varepsilon = 0$)
Case $1^{\pm}$	$0 < r < q \ \wedge\ 0 < \pm s < p$	$0 < r < q \ \wedge\ 0 < \pm s < p$ $(r, \pm s) \neq (\frac{q}{2}, \frac{p}{2})$
Case $2^{\pm}$	$r = q \ \wedge\ 0 < \pm s < p$	
Case $3^{\pm}$	$0 < r < q \ \wedge\ \pm s = p$	
Case $4.1^{\pm}$	$r = q \ \wedge\ \pm s = p$	
Case $4.2^{\pm}$	$r = 0 \ \wedge\ \pm s = p$ $(p, q) = 2$	$r = 0 \ \wedge\ \pm s = p$ $(p, q) = 1$
Case $5^{\pm}$		$r = \frac{q}{2} \ \wedge \pm s = \frac{p}{2}$

Remark 4.2. In the above classification, the L_0-weights $h_{i;\varepsilon}$ for Case $1^{\pm}$ are all distinct, and the rest of the cases have the following degeneration:

Case $2^{\pm}$	$h_{-(i+1);\varepsilon} = h_{i;\varepsilon}\ (i \in \mathbb{Z}_{\geqslant 0})$
Case $3^{\pm}$	$h_{2i;\varepsilon} = h_{2i-1;\varepsilon}\ (i \in \mathbb{Z})$
Case $4.1^{\pm}$	$h_{-2i-1;\varepsilon} = h_{-2i;\varepsilon} = h_{2i;\varepsilon} = h_{2i-1;\varepsilon}\ (i \in \mathbb{Z}_{\geqslant 0})$
Case $4.2^{\pm}$	$h_{-2i;\varepsilon} = h_{-2i-1;\varepsilon} = h_{2i+1;\varepsilon} = h_{2i+2;\varepsilon}\ (i \in \mathbb{Z}_{\geqslant 0})$
Case $5^{\pm}$	$h_{i;\varepsilon} = h_{-i;\varepsilon}\ (i \in \mathbb{Z}_{\geqslant 0})$

Thus, we may assume that the range of i in $h_{i;\varepsilon}$ for each case is given by the following table:

Case	1^{σ}	2^{σ}	3^{σ}	4.1^{σ}	4.2^{σ}	5^{σ}
Range $(\sigma = +)$	$\mathbb{Z}$	$\mathbb{Z}_{\geqslant 0}$	$\{(-1)^{i-1}i\}_{i\in\mathbb{Z}_{\geqslant 0}}$	$2\mathbb{Z}_{\geqslant 0}$	$2\mathbb{Z}_{\leqslant 0}$	$\mathbb{Z}_{\geqslant 0}$
Range $(\sigma = -)$	$\mathbb{Z}\setminus\{0\}$	$\mathbb{Z}_{>0}$	$\{(-1)^{i-1}i\}_{i\in\mathbb{Z}_{>0}}$	$2\mathbb{Z}_{>0}$	$2\mathbb{Z}_{<0}$	$\mathbb{Z}_{>0}$

Remark 4.3. For Case 1^+ and Case 5^+, the irreducible highest weight modules $L_\varepsilon(z, h_{0;\varepsilon})$ are called *minimal* by physicists.

4.2. Character sum formulae

In this subsection, we will describe the character sum formulae of the Jantzen filtration of Verma modules whose highest weight belongs to the Class $\mathrm{R}^{\pm}$.

For $(z,h)\in\mathbb{C}^2$, we define the Jantzen filtration of $M_\varepsilon(z,h)$

$$M_\varepsilon(z,h) = M_\varepsilon(z,h)(0) \supset M_\varepsilon(z,h)(1) \supset M_\varepsilon(z,h)(2) \supset \cdots$$

as follows. Let $\mathscr{A}$ be a polynomial ring $\mathbb{C}[T]$. We take the lift $(\tilde{z},\tilde{h})\in\mathscr{A}^2$ of a weight (z,h) as follows:

$$(\tilde{z},\tilde{h}) := \begin{cases} (z+T, h+\frac{1}{24}T) & \text{if } \varepsilon = 0 \text{ and } h = \frac{1}{24}z, \\ (z, h+T) & \text{otherwise, if } z \neq \frac{3}{2}, \frac{27}{2}, \\ (z+T, h) & \text{otherwise.} \end{cases}$$

For $k\in\mathbb{Z}_{>0}$, we set

$$M_\varepsilon(z,h)(k) := \{v|_{T=0} \mid \langle v, w\rangle \in T^k\mathscr{A} \quad \forall w \in M(\tilde{z},\tilde{h})\}.$$

Note that regarding T as a complex parameter, $(\tilde{z},\tilde{h})$ defines a line, and the lift here is so chosen that the line $(\tilde{z},\tilde{h})$ intersects with the curves $Z_{\alpha,\beta,\varepsilon}$ transversally.

For $V\in\mathrm{Ob}(\mathcal{O})$, let ch V be the formal character of V

$$\mathrm{ch}\ V := \sum_{\lambda\in\mathfrak{h}^*} (\dim V_\lambda)e^\lambda,$$

where $V = \bigoplus V_\lambda$ is the weight space decomposition. As an application of Theorem 3.1 and Section A, we can compute the character sum

$$\sum_{l>0} \mathrm{ch}\, M_\varepsilon(z,h)(l),$$

and the results are summarized in the following lemma:

Lemma 4.2. *Let us fix* $p, q \in \mathbb{Z}_{>0}$ *satisfying* $p-q \in 2\mathbb{Z}$, $(\frac{p-q}{2}, q) = 1$.

1. *Class* R^+: $z = z(\frac{p}{q})$,
   (I) *Case* 1^+: $i \in \mathbb{Z}$,

$$\sum_{l>0} \mathrm{ch}\, M_\varepsilon(z,h_{i;\varepsilon})(l) = \sum_{\substack{k\in\mathbb{Z}\\ |k|>|i|\\ k-i\equiv 1 \bmod 2}} \mathrm{ch}\, M_\varepsilon(z,h_{k;\varepsilon}).$$

   (II) *Case* 2^+: $i \in \mathbb{Z}_{\geqslant 0}$,

$$\sum_{l>0} \mathrm{ch}\, M_\varepsilon(z,h_{i;\varepsilon})(l) = \sum_{k>i} \mathrm{ch}\, M_\varepsilon(z,h_{k;\varepsilon}).$$

   (III) *Case* 3^+: $i \in \mathbb{Z}_{\geqslant 0}$,

$$\sum_{l>0} \mathrm{ch}\, M_\varepsilon(z,h_{(-1)^{i-1}i;\varepsilon})(l) = \sum_{k>i} \mathrm{ch}\, M_\varepsilon(z,h_{(-1)^{k-1}k;\varepsilon}).$$

   (IV) *Case* 4.1^+: $i \in \mathbb{Z}_{\geqslant 0}$,
      i. $(p,q) \neq (1,1)$ *and* $i>0$,

$$\sum_{l>0} \mathrm{ch}\, M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(l) = 2\sum_{k>i} \mathrm{ch}\, M_{\frac{1}{2}}(z,h_{2k;\frac{1}{2}}).$$

      ii. $(p,q) = (1,1)$ *or* $i=0$,

$$\sum_{l>0} \mathrm{ch}\, M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(l) = \sum_{k>i} \mathrm{ch}\, M_{\frac{1}{2}}(z,h_{2k;\frac{1}{2}}).$$

   (V) *Case* 4.2^+: $i \in \mathbb{Z}_{\geqslant 0}$,
      i. $(p,q) \neq (1,1)$,

$$\sum_{l>0} \mathrm{ch}\, M_\varepsilon(z,h_{-2i;\varepsilon})(l) = 2\sum_{k>i} \mathrm{ch}\, M_\varepsilon(z,h_{-2k;\varepsilon}).$$

ii. $(p,q)=(1,1)$,

$$\sum_{l>0} \mathrm{ch}\ M_0(z,h_{-2i;0})(l) = \sum_{k>i} \mathrm{ch}\ M_0(z,h_{-2k;0}).$$

(VI) *Case* 5^+: $i \in \mathbb{Z}_{\geqslant 0}$,

$$\sum_{k>0} \mathrm{ch}\ M_0(z,h_{i;0})(k) = 2^{1-\delta_{i,0}} \sum_{\substack{k>i \\ k-i\equiv 1 \bmod 2}} \mathrm{ch}\ M_0(z,h_{k;0}).$$

2. *Class* R^-: $z = z(-\frac{p}{q})$

(I) *Case* 1^-: $i \in \mathbb{Z} \setminus \{0\}$,

$$\sum_{l>0} \mathrm{ch}\ M_\varepsilon(z,h_{i;\varepsilon})(l) = \sum_{\substack{k\in\mathbb{Z} \\ |k|<|i| \\ k-i\equiv 1 \bmod 2}} \mathrm{ch}\ M_\varepsilon(z,h_{k;\varepsilon}).$$

(II) *Case* 2^-: $i \in \mathbb{Z}_{>0}$,

$$\sum_{l>0} \mathrm{ch}\ M_\varepsilon(z,h_{i;\varepsilon})(l) = \sum_{0\leqslant k<i} \mathrm{ch}\ M_\varepsilon(z,h_{k;\varepsilon}).$$

(III) *Case* 3^-: $i \in \mathbb{Z}_{>0}$,

$$\sum_{l>0} \mathrm{ch}\ M_\varepsilon(z,h_{(-1)^{i-1}i;\varepsilon})(l) = \sum_{0\leqslant k<i} \mathrm{ch}\ M_\varepsilon(z,h_{(-1)^{k-1}k;\varepsilon}).$$

(IV) *Case* 4.1^-: $i \in \mathbb{Z}_{>0}$,

i. $(p,q)\neq(1,1)$,

$$\sum_{l>0} \mathrm{ch}\ M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(l) = 2\sum_{0<k<i} \mathrm{ch}\ M_{\frac{1}{2}}(z,h_{2k;\frac{1}{2}}) + \mathrm{ch}\ M_{\frac{1}{2}}(z,h_{0;\frac{1}{2}}).$$

ii. $(p,q)=(1,1)$,

$$\sum_{l>0} \mathrm{ch}\ M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(l) = \sum_{0\leqslant k<i} \mathrm{ch}\ M_{\frac{1}{2}}(z,h_{2k;\frac{1}{2}}).$$

(V) *Case* 4.2^-: $i \in \mathbb{Z}_{>0}$,

i. $(p,q)\neq(1,1)$,

$$\sum_{l>0} \mathrm{ch}\ M_\varepsilon(z,h_{-2i;\varepsilon})(l) = 2\sum_{0\leqslant k<i} \mathrm{ch}\ M_\varepsilon(z,h_{-2k;\varepsilon}).$$

ii. $(p,q)=(1,1)$,

$$\sum_{l>0} \text{ch } M_0(z,h_{-2i;0})(l) = \sum_{0\leqslant k<i} \text{ch } M_0(z,h_{-2k;0}).$$

(VI) *Case* 5^-: $i\in\mathbb{Z}_{>0}$,

$$\sum_{l>0} \text{ch } M_0(z,h_{i;0})(l) = 2 \sum_{\substack{0\leqslant k<i\\ k-i\equiv 1 \bmod 2}} \text{ch } M_0(z,h_{k;0}).$$

Remark 4.4. Let $\alpha,\beta\in\mathbb{Z}_{>0}$ such that $\alpha-\beta\in 1-2\varepsilon+2\mathbb{Z}$, and suppose $(z,h)=(z(t_0),h_{\alpha,\beta;\varepsilon}(t_0))\in\mathbb{C}^2$ for some $t_0\in\mathbb{C}^*$. We can compute the character sum of the quotient module

$$N_\varepsilon(z,h) := M_\varepsilon(z,h)/M_\varepsilon(z,h+\tfrac{1}{2}\alpha\beta)$$

in a way similar to [RW2], and the result looks as follows.

Suppose a singular vector in $M_\varepsilon(z,h)$ of level $\frac{1}{2}\alpha\beta$ lies not in $M_\varepsilon(z,h)(k+1)$ but in $M_\varepsilon(z,h)(k)$. Then, we have

$$\sum_{l>0} \text{ch } N_\varepsilon(z,h)(l) = \sum_{l>0} \text{ch } M_\varepsilon(z,h)(l) - \Big\{k\text{ch } M_\varepsilon(z,h+\tfrac{1}{2}\alpha\beta) + \sum_{l>0} \text{ch } M_\varepsilon(z,h+\tfrac{1}{2}\alpha\beta)(l)\Big\}.$$

4.3. Embedding diagrams

In this subsection, we will describe monomorphisms between Verma modules. Namely, we first compute the dimension of the space of $(\text{Vir}_\varepsilon)_+$-invariants in each weight subspace that are useful to construct embedding diagrams. Second, as an application, we construct embedding diagrams among Verma modules. For simplicity, let us fix $\tau=\bar{0}\in\mathbb{Z}_2$ in this subsection.

Since, we have already proved $\dim\{M_{\frac{1}{2}}(z,h)_n\}^{(\text{Vir}_{\frac{1}{2}})_+}\leqslant 1$ in Proposition 3.1 for Neveu–Schwarz algebra $\text{Vir}_{\frac{1}{2}}$, we have only to study for the Ramond algebra Vir_0.

First, let us check the next proposition:

Proposition 4.1 (Case $1^\pm, 2^\pm$ and $3^\pm$). *For* $\sigma\in\mathbb{Z}_2$ *and* $k,i\in\mathbb{Z}_{\geqslant 0}$ *satisfying* $\pm(k-i)>0$, *we have the following*:

1. *Case* $1^\pm$: $h\in\{h_{\pm k;0}\}$,

$$\dim\{M_0(z,h_{\pm i;0})^\sigma_{h-h_{\pm i;0}}\}^{(\text{Vir}_0)_+}=1.$$

2. *Case* $2^{\pm}$: $h = h_{k;0}$,

$$\dim\{M_0(z,h_{i;0})^{\sigma}_{h-h_{i;0}}\}^{(\mathrm{Vir}_0)_+} = 1.$$

3. *Case* $3^{\pm}$: $h = h_{(-1)^{k-1}k;0}$,

$$\dim\{M_0(z,h_{(-1)^{i-1}i;0})^{\sigma}_{h-h_{(-1)^{i-1}i;0}}\}^{(\mathrm{Vir}_0)_+} = 1.$$

Proof. It is enough to prove this proposition for $k = i \pm 1$. For Case $1^{\pm}$ with $h = \min\{h_{k;0}, h_{-k;0}\}$, Cases $2^{\pm}$ and $3^{\pm}$, the statements immediately follow from Lemma 4.2 and the fact that $M_0(z,h)(1)$ is the maximal proper submodule of $M_0(z,h)$. Hence, we have only to prove the case $h = \max\{h_{k;0}, h_{-k;0}\}$ for Case $1^{\pm}$. Let us take $\tilde{h}_i \in \{h_{i;0}, h_{-i;0}\}$, and set

$$h_+ \coloneqq \max\{h_{k;0}, h_{-k;0}\}, \quad h_- \coloneqq \min\{h_{k;0}, h_{-k;0}\}.$$

By Theorem 3.1, one can easily check that

$$\dim\{M_0(z,\tilde{h}_i)^{\tau}_{h_+-\tilde{h}_i}\}^{(\mathrm{Vir}_0)_+} \neq 0 \quad (\tau \in \mathbb{Z}_2).$$

Moreover, since the Verma module $M_0(z,h_-)$ does not contain any singular vector of L_0-weight$= h_+$ by the determinant formula (Theorem 3.1), the statement follows from Lemma 4.2. □

Second, let us analyze Case $4.2^{\pm}$. In this case, we have the following proposition:

Proposition 4.2 (Case $4.2^{\pm}$). *For* $\sigma \in \mathbb{Z}_2$ *and* $k, i \in \mathbb{Z}_{\leqslant 0}$ *satisfying* $\mp(k-i) > 0$, *we have*

$$\dim\{M_0(z,h_{2i;0})^{\sigma}_{h_{2k;0}-h_{2i;0}}\}^{(\mathrm{Vir}_0)_+} = 1.$$

Proof. Here, we will prove only for Case 4.2^{+}, since one can prove Case 4.2^{-} by a similar method.

Since we can easily check $\dim\{M_0(z,h_{2i;0})^{\sigma}_{h_{2k;0}-h_{2i;0}}\}^{(\mathrm{Vir}_0)_+} \neq 0$, we assume that $\dim\{M_0(z,h_{2i;0})^{\sigma}_{h_{2k;0}-h_{2i;0}}\}^{(\mathrm{Vir}_0)_+} = 2$, and lead to a contradiction. Moreover, it is enough to prove this only for $k = i+1$.

Let $v_1, v_2 \in M_0(z,h_{2i;0})_{h_{2i-2;0}-h_{2i;0}} \setminus \{0\}$ be singular vectors such that $\mathbb{C}v_1 \neq \mathbb{C}v_2$. Set,

$$\begin{aligned}
V_j(z,h_{2i;0}) &\coloneqq U(\mathrm{Vir}_0).v_j \quad (j=1,2)\\
&= \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} V_j(z,h_{2i;0})_n, \quad V_j(z,h_{2i;0})_n \coloneqq \{u \mid L_0.u = (h_{2i;0}+n)u\},\\
I &\coloneqq V_1(z,h_{2i;0}) \cap V_2(z,h_{2i;0})\\
&= \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} I_n, \quad I_n \coloneqq \{u \mid L_0.u = (h_{2i;0}+n)u\}.
\end{aligned}$$

By Proposition 3.3 (actually from its proof), we may assume $V_j(z, h_{2i;0}) \cong M_0(z, h_{2i-2;0})$ $(j = 1, 2)$ without loss of generality. Set

$$N_0(z, h_{2i;0}) := M_0(z, h_{2i;0})/V_1(z, h_{2i;0}),$$

and let π be the canonical projection

$$\pi : M_0(z, h_{2i;0}) \twoheadrightarrow N_0(z, h_{2i;0}).$$

We denote the Jantzen filtration of the module $N_0(z, h_{2i;0})$ by

$$N_0(z, h_{2i;0}) \supset N_0(z, h_{2i;0})(1) \supset N_0(z, h_{2i;0})(2) \supset \cdots$$

(cf. Remark 4.4). For each $l \in \mathbb{Z}_{>0}$, $N_0(z, h_{2i;0})(l)$ clearly has the weight space decomposition

$$N_0(z, h_{2i;0})(l) = \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} N_0(z, h_{2i;0})(l)_n, \quad N_0(z, h_{2i;0})(l)_n := \{u \mid L_0.u = (h_{2i;0} + n)u\}.$$

Set $\overline{V_2} := \pi(V_2(z, h_{2i;0}))$ and $(\overline{V_2})_n := \pi(V_2(z, h_{2i;0})_n)$ for each $n \in \mathbb{Z}_{\geqslant 0}$. Then, one can check the following formulae:

$$\sum_{l>0} \mathrm{ch}\, N_0(z, h_{2i;0})(l) = \mathrm{ch}\, M_0(z, h_{2i-2;0}), \tag{8}$$

$$\mathrm{ch}\, \overline{V_2} = \mathrm{ch}\, M_0(z, h_{2i-2;0}) - \mathrm{ch}\, I, \tag{9}$$

$$N_0(z, h_{2i;0})(1) \supset \overline{V_2}. \tag{10}$$

In fact, (8) follows from Remark 4.4 and Lemma 4.2, (9) follows from the definition and (10) follows from a well-known fact on the Jantzen filtration. Moreover, we have

$$I_{h-h_{2i;0}} = \{0\} \quad \text{if } h < h_{2i-4;0}, \tag{11}$$

since $I \subset M_0(z, h_{2i-2;0})$ is a proper submodule. Hence, (8)–(10) imply

($\natural$) For $h < h_{2i-4;0}$, we have

$$N_0(z, h_{2i;0})(l)_{h-h_{2i;0}} = \begin{cases} (\overline{V_2})_{h-h_{2i;0}} & l = 1, \\ \{0\} & l > 1. \end{cases}$$

Eq. (11) also implies that $I_{h_{2i-4;0}-h_{2i;0}} \subset \{M_0(z, h_{2i;0})_{h_{2i-4;0}-h_{2i;0}}\}^{(\mathrm{Vir}_0)_+}$, hence we get $\dim I_{h_{2i-4;0}-h_{2i;0}} \in \{0, 2, 4\}$ by taking the parity into account. Therefore, we will discuss each case separately.

The case: $\dim I_{h_{2i-4;0}-h_{2i;0}} = 0$: In this case, it follows from (9) that

$$\dim\, (\overline{V_2})_{h_{2i-4;0}-h_{2i;0}} = \dim M_0(z, h_{2i-2;0})_{h_{2i-4;0}-h_{2i-2;0}}. \tag{12}$$

On the other hand, since we have $\{V_2(z,h_{2i;0})_{h_{2i-4;0}-h_{2i;0}}\}^{(\mathrm{Vir}_0)_+}\neq\{0\}$ by definition, it follows that $\{(\overline{V_2})_{h_{2i-4;0}-h_{2i;0}}\}^{(\mathrm{Vir}_0)_+}\neq\{0\}$ by assumption, which implies $N_0(z,h_{2i;0})(2)_{h_{2i-4;0}-h_{2i;0}}\neq\{0\}$. Thus, we have

$$\begin{aligned}\sum_{l>0}\dim N_0(z,h_{2i;0})(l)_{h_{2i-4;0}-h_{2i;0}} &\gneqq \dim N_0(z,h_{2i;0})(1)_{h_{2i-4;0}-h_{2i;0}}\\ &\geqslant \dim(\overline{V_2})_{h_{2i-4;0}-h_{2i;0}} \quad (\because (10))\\ &= \dim M_0(z,h_{2i-2;0})_{h_{2i-4;0}-h_{2i-2;0}} \quad (\because (12))\end{aligned}$$

which contradicts (8).

The case: $\dim I_{h_{2i-4;0}-h_{2i;0}}=2$: In this case, it follows from Lemma 3.2 and (11) that

$$\{V_j(z,h_{2i;0})_{h_{2i-4;0}-h_{2i;0}}\}^{(\mathrm{Vir}_0)_+}\subset I_{h_{2i-4;0}-h_{2i;0}} \quad (j=1,2). \tag{13}$$

First, we assume that $N_0(z,h_{2i;0})(1)_{h_{2i-4;0}-h_{2i;0}}=(\overline{V_2})_{h_{2i-4;0}-h_{2i;0}}$. Under this assumption, we can show the following:

$$\text{(A)} \quad N_0(z,h_{2i;0})(l)_{h_{2i-4;0}-h_{2i;0}}=\{0\} \quad \text{for } l>1.$$

Indeed, we can prove as follows. Let us assume that $N_0(z,h_{2i;0})(2)_{h_{2i-4;0}-h_{2i;0}}\neq\{0\}$. Then, it follows that $\overline{V_2}\cap N_0(z,h_{2i;0})(2)_{h_{2i-4;0}-h_{2i;0}}\neq\{0\}$. Set $W:=V_2(z,h_{2i;0})\cap\pi^{-1}\{N_0(z,h_{2i;0})(2)\}$ and $W_n:=W\cap V_2(z,h_{2i;0})_n$ for each $n\in\mathbb{Z}_{\geqslant 0}$. By definition, we have

$$W_{h_{2i-4;0}-h_{2i;0}}\neq\{0\}.$$

On the other hand, since $V_2(z,h_{2i;0})\cong M_0(z,h_{2i-2;0})$, it follows that $W_{h-h_{2i;0}}=\{0\}$ for $h<h_{2i-4;0}$. Hence, we have

$$W_{h_{2i-4;0}-h_{2i;0}}\subset\{V_2(z,h_{2i;0})_{h_{2i-4;0}-h_{2i;0}}\}^{(\mathrm{Vir}_0)_+}$$

and

$$W_{h_{2i-4;0}-h_{2i;0}}\not\subset V_1(z,h_{2i;0})_{h_{2i-4;0}-h_{2i;0}}$$

which contradicts (13). Thus, we proved (A).

Now, it follows from (A) that

$$\begin{aligned}\dim(\overline{V_2})_{h_{2i-4;0}-h_{2i;0}} &= \sum_{l>0}\dim N_0(z,h_{2i;0})(l)_{h_{2i-4;0}-h_{2i;0}} \quad (\because \text{ by assumption})\\ &= \dim M_0(z,h_{2i;0})_{h_{2i-4;0}-h_{2i;0}} \quad (\because\ (8)).\end{aligned}$$

On the other hand, since $\dim I_{h_{2i-4;0}-h_{2i;0}}=2$ by assumption, (9) implies

$$\dim(\overline{V_2})_{h_{2i-4;0}-h_{2i;0}}=\dim M_0(z,h_{2i;0})_{h_{2i-4;0}-h_{2i;0}}-2$$

which is impossible. Thus, we conclude that

$$N_0(z, h_{2i;0})(1)_{h_{2i-4;0}-h_{2i;0}} \supsetneq (\overline{V_2})_{h_{2i-4;0}-h_{2i;0}}.$$

Now, it follows from (♮) that there exists a subsingular vector

$$u \in N_0(z, h_{2i;0})(1)_{h_{2i-4;0}-h_{2i;0}} \setminus (\overline{V_2})_{h_{2i-4;0}-h_{2i;0}}.$$

In fact, one can say more:

(B) For any $w \in \pi^{-1}(u) \cap M_0(z, h_{2i;0})_{h_{2i-4;0}-h_{2i;0}}$, w is a singular vector.

This can be proved as follows. Since u is an element of $N_0(z, h_{2i;0})(1)$, it follows that w is an element of $M_0(z, h_{2i;0})(1)$. Moreover, if $w \in M_0(z, h_{2i;0})(2)$, then w is automatically a singular vector, since we have

$$M_0(z, h_{2i;0})(2)_{h-h_{2i;0}} = \{0\} \quad \text{for } h < h_{2i-4;0} \tag{14}$$

by assumption from the very beginning. Thus, we may assume that $w \notin M_0(z, h_{2i;0})(2)$. Assume that w is not a singular vector. Then, (14) implies

$$(\mathrm{Vir}_0)_+ . \{w + M_0(z, h_{2i;0})(2)\} \neq \{0\}$$

as an element of $M_0(z, h_{2i;0})(1)/M_0(z, h_{2i;0})(2)$, where the latter space is semi-simple. But, by the definition of u, it follows that $w + M_0(z, h_{2i;0})(2)$ is a non-zero subsingular vector which is impossible. Hence, w is a singular vector, and (B) is proved.

Now, (B) implies

$$w \in \{V_2(z, h_{2i;0})\}^{(\mathrm{Vir}_0)_+} \quad \text{and} \quad w \notin V_1(z, h_{2i;0})$$

which contradicts (13).

The case: $\dim I_{h_{2i-4;0}-h_{2i;0}} = 4$: In this case, we have

$$\dim L_0(z, h_{2i-2;0})_{h_{2i-4;0}-h_{2i-2;0}} = \dim M_0(z, h_{2i-2;0})_{h_{2i-4;0}-h_{2i-2;0}} - 4.$$

Indeed, it follows from (11) and the assumption that

$$\dim \{M_0(z, h_{2i-2;0})_{h_{2i-4;0}-h_{2i-2;0}}\}^{(\mathrm{Vir}_0)_+} = 4,$$

hence we get

$$\dim L_0(z, h_{2i-2;0})_{h_{2i-4;0}-h_{2i-2;0}} \leqslant \dim M_0(z, h_{2i-2;0})_{h_{2i-4;0}-h_{2i-2;0}} - 4.$$

But, since we have

$$\dim L_0(z, h_{2i-2;0})_{h-h_{2i-2;0}} = \dim M_0(z, h_{2i-2;0})_{h-h_{2i-2;0}} \quad (h < h_{2i-4;0})$$

by the determinant formula (Theorem 3.1), any element of $M_0(z,h_{2i-2;0})_{h_{2i-4;0}-h_{2i-2;0}}$ that lies in a proper submodule should be a singular vector, which proves the equality. Thus, (9) implies that

$$\dim(\overline{V_2})_{h-h_{2i-2;0}} = \dim L_0(z,h_{2i-2;0})_{h-h_{2i-2;0}} \quad (h \leqslant h_{2i-4;0}). \tag{15}$$

Now, the same argument as in the previous case (in particular, the proof of (B)) shows that

$$N_0(z,h_{2i;0})(1)_{h-h_{2i;0}} = (\overline{V_2})_{h-h_{2i;0}} \quad (h \leqslant h_{2i-4;0}).$$

This together with (8) and (9) ensures that

$$\dim N_0(z,h_{2i;0})(2)_{h_{2i-4;0}-h_{2i;0}} > 0.$$

But, this is impossible, since (♮) implies that $N_0(z,h_{2i;0})(2)$ is a proper submodule of $N_0(z,h_{2i;0})(1)$ which contradicts (15).

Therefore, we have proved that $\dim\{M_0(z,h_{2i;0})^{\sigma}_{h_{2k;0}-h_{2i;0}}\}^{(\mathrm{Vir}_0)_+} = 2$ cannot be happened. □

Third, we will analyze Case $5^{\pm}$. In this case, we have the following proposition:

Proposition 4.3 (Case $5^{\pm}$). *For $\sigma \in \mathbb{Z}_2$, $i \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}_{\geqslant 0}$ satisfying $\pm(k-i) > 0$, we have*

$$\dim\{M_0(z,h_{i;0})^{\sigma}_{h_{k;0}-h_{i;0}}\}^{(\mathrm{Vir}_0)_+} = \begin{cases} 2 & \text{if } k>0, \\ 1 & \text{if } k=0. \end{cases}$$

Proof. For $n \in \mathbb{Z}_{\geqslant 0}$ and $l \in \mathbb{Z}_{>0}$, set

$$M_0(z,h_{i;0})(l)_n := \{u \in M_0(z,h_{i;0})(l) \mid L_0.u = (h_{i;0}+n)u\}.$$

Then, by the determinant formula (Theorem 3.1), we see that

$$\dim\{M_0(z,h_{i;0})^{\sigma}_{h_{i\pm1;0}-h_{i;0}}\}^{(\mathrm{Vir}_0)_+} > 0.$$

We first prove the case $k>0$. In this case, it is enough to prove the proposition for the case $k = i \pm 1$, since the other cases can be obtained by using iterations of injective morphisms. Assuming $\dim\{M_0(z,h_{i;0})^{\sigma}_{h_{i\pm1;0}-h_{i;0}}\}^{(\mathrm{Vir}_0)_+} = 1$, we will lead to a contradiction. By Lemma 4.2, we conclude $M_0(z,h_{i;0})(1)^{\sigma}_{h_{i\pm1;0}-h_{i;0}} = M_0(z,h_{i;0})(2)^{\sigma}_{h_{i\pm1;0}-h_{i;0}}$ which implies $M_0(z,h_{i\pm1;0}) \subset M_0(z,h_{i;0})(2)$. Since the determinant formula (Theorem 3.1) implies $\{M_0(z,h_{i\pm1;0})^{\sigma}_{h_{i\pm2;0}-h_{i\pm1;0}}\}^{(\mathrm{Vir}_0)_+} \neq \{0\}$, we have

$M_0(z, h_{i\pm 2;0}) \subset M_0(z, h_{i;0})(3)$. Hence, we obtain

$$\sum_{l>0} \dim M_0(z, h_{i;0})(l)^{\sigma}_{h_{i\pm 2;0} - h_{i;0}} \geqslant 2 \dim M_0(z, h_{i\pm 1;0})^{\sigma}_{h_{i\pm 2;0} - h_{i\pm 1;0}} + 1.$$

But this is impossible by Lemma 4.2.

Second, assume $k = 0$. Note that this is possible only for Case 5^-. Let us first prove the case $i = 1$. In this case, Lemma 4.2 implies

$$\sum_{l>0} \mathrm{ch}\, M_0(z, h_{1;0})(l) = 2\mathrm{ch}\, M_0(z, h_{0;0}).$$

Since $h_{0;0} = \frac{1}{24} z$ and $M_0(z, h_{0;0})$ is irreducible in this case, it is easy to see that

$$M_0(z, h_{1;0})(1) \cong \tilde{M}(z, \tfrac{1}{24} z),$$

which proves the proposition. For $i > 1$, we have only to prove the case $i = 2$ by the same reason as in the case $k > 0$. Since, we have already proved

$$\dim \{M_0(z, h_{2;0})^{\sigma}_{h_{1;0} - h_{2;0}}\}^{(\mathrm{Vir}_0)_+} = 2$$

for each $\sigma \in \mathbb{Z}_2$, we have only to show that any two submodules of $M_0(z, h_{2;0})$ which are isomorphic to $M_0(z, h_{1;0})$ have a non-trivial intersection, since then it should be isomorphic to $\tilde{M}(z, \frac{1}{24} z)$, and there cannot be other singular vectors by Proposition 3.2. Assume the contrary, and let W be the sum of two such submodules, and let

$$W = \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} W_n, \quad W_n := \{w \in W \mid L_0.w = (h_{1;0} + n)w\}$$

be its weight space decomposition. By assumption, we have $W \cong M_0(z, h_{1;0})^{\oplus 2}$ which implies $\dim W_n = 4p_0(n)$. But since we have

$$\lim_{n \to \infty} \frac{\dim W_n}{\dim M_0(z, h_{2;0})_{h_{1;0} - h_{2;0} + n}} = \lim_{n \to \infty} \frac{2p_0(n)}{p_0(n + h_{1;0} - h_{2;0})} = 2$$

by (4), this is a contradiction. Thus, we completed the proof. □

Therefore, we conclude that except for Case $5^{\pm}$, any non-trivial homomorphism between Verma modules is injective.

Now, let us construct embedding diagrams among Verma modules as an application of Propositions 4.1–4.3.

Definition 4.1. We define symbols $[h], [\tilde{h}]$ and $\rightarrowtail$ as follows:

1. For $h \in \mathbb{C}$ we set

$$[h] := M_\varepsilon(z,h), \quad [\tilde{h}] := \tilde{M}(z,h),$$

   where we omit indicating ε and z. Indeed, we use these symbols only when they are clear from the context.
2. When a map $V \to W$ of Vir_ε-modules is injective, we express this fact symbolically as follows:

$$V \rightarrowtail W.$$

If a weight (z,h) belongs to Class V, then $M_\varepsilon(z,h)$ is irreducible and we have nothing to do. If a weight (z,h) belongs to Class I, then there exists $\alpha, \beta \in \mathbb{Z}_{>0}$ and $t \in \mathbb{C} \setminus \mathbb{Q}$ such that $\alpha - \beta \in 1 - 2\varepsilon + 2\mathbb{Z}_{\geqslant 0}$ and $z = z(t)$ and $h = h_{\alpha,\beta;\varepsilon}(t)$ (see Lemma 4.1). In this case, the Verma module $M_\varepsilon(z, h + \frac{1}{2}\alpha\beta)$ is irreducible, and hence we obtain the following embedding diagram, Fig. 1.

If a weight (z,h) belongs to the Class $\mathrm{R}^{\pm}$, then we obtain the embedding diagrams, which are commutative, by Propositions 3.3, 4.1–4.3. We will describe them for Classes R^+ and R^- separately as shown in Figs. 2 and 3:

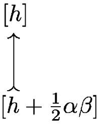

Fig. 1. Embedding diagram for Class I.

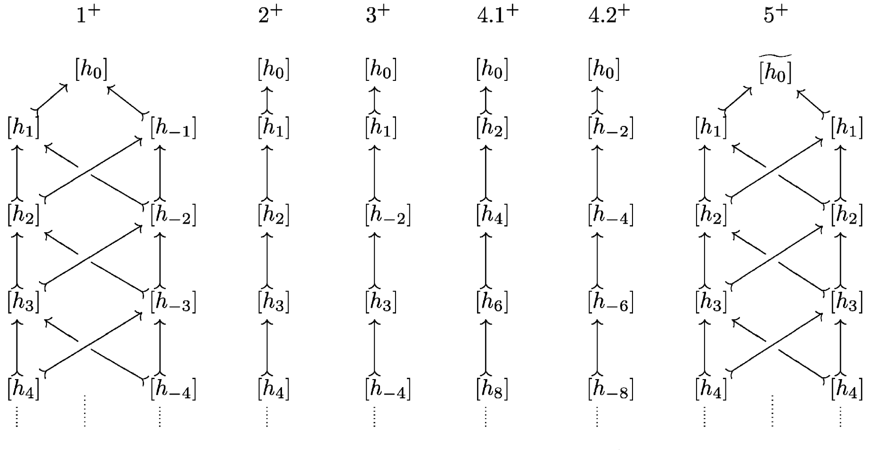

Fig. 2. Embedding diagrams for Class R^+.

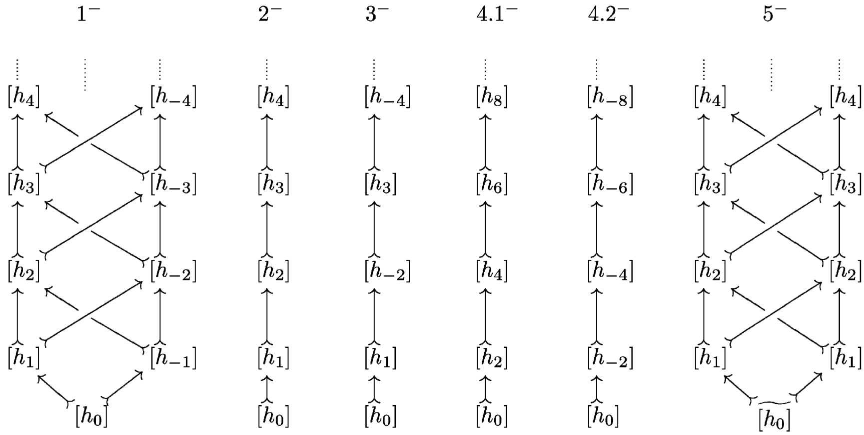

Fig. 3. Embedding diagrams for Class R$^-$.

Note that the embedding diagrams for Case $5^{\pm}$ are constructed as follows. Let $|z,h\rangle$ be an even highest weight vector of the Verma module $M_0(z,h)$. For $i \in \mathbb{Z}_{>0}$, let us take two linearly independent singular vectors, say $S_{i\mp1,1}$ and $S_{i\mp1,2}$, of the Verma module $M_0(z,h_{i\mp1;0})$ (resp. $\tilde{M}(z,h_{0;0})$ for $i=1$ in Case 5^+) with level $h_{i\mp1;0}-h_{i;0}$ satisfying conditions (7) (resp. the conditions in Proposition 3.3, 3). Then, we define the homomorphisms

$$\varphi^{\pm}_{i,j} : M_0(z,h_{i;0}) \to \begin{cases} \tilde{M}(z,h_{0;0}) & \text{if } i=1 \text{ in Case } 5^+, \\ M_0(z,h_{i\mp1;0}) & \text{otherwise,} \end{cases} \qquad (j=1,2),$$

by $\varphi^{\pm}_{i,j}(|z,h\rangle) := S_{i\mp1;j}$. Then, by Proposition 3.3 and Remark 3.3, the morphisms $\varphi^{\pm}_{i,j}$ are monomorphisms. The embedding diagrams for Case $5^{\pm}$ are constructed by using such morphisms $\varphi^{\pm}_{i,j}$.

4.4. Structure of Jantzen filtration

In this subsection, we will describe the Jantzen filtration of $M_\varepsilon(z,h)$ explicitly. Here, when we have an injective map $\varphi : M_\varepsilon(z,h') \to M_\varepsilon(z,h)$, we will identify $\operatorname{Im}\varphi$ with $M_\varepsilon(z,h')$.

The results can be summarized as follows:

Theorem 4.1. *In each case, the submodules* $M_\varepsilon(z,h)(k)$ $(k \in \mathbb{Z}_{>0})$ *can be explicitly described as follows*:

1. *Class* V:

$$M_\varepsilon(z,h)(k) = \{0\}.$$

2. *Class* I: *In this case,* $(z,h)=(z(t),h_{\alpha,\beta;\varepsilon}(t))$ *for some* $t\in\mathbb{C}\setminus\mathbb{Q}$ *and* $\alpha,\beta\in\mathbb{Z}_{>0}$ *satisfying* $\alpha-\beta\in 1-2\varepsilon+2\mathbb{Z}_{\geqslant 0}$.

$$M_\varepsilon(z,h)(k)=\begin{cases} M_\varepsilon(z,h+\frac{1}{2}\alpha\beta) & k=1,\\ \{0\} & k>1.\end{cases}$$

3. *Class* $\mathrm{R}^\pm$: *Let* $p,q\in\mathbb{Z}_{>0}$ *satisfying* $p-q\in 2\mathbb{Z}$, $(\frac{p-q}{2},q)=1$. *Set* $z=z(\pm\frac{p}{q})$.
   (I) *Case* $1^\pm$: $i\in\mathbb{Z}(\setminus\{0\}$ *for Case* $1^-)$,

$$M_\varepsilon(z,h_{i;\varepsilon})(k)=M_\varepsilon(z,h_{|i|+k;\varepsilon})+M_\varepsilon(z,h_{-|i|-k;\varepsilon})\quad (Case\ 1^+),$$

$$M_\varepsilon(z,h_{i;\varepsilon})(k)=\begin{cases} \sum_{\substack{l\in\mathbb{Z}\\ |l|=|i|-k}} M_\varepsilon(z,h_{l;\varepsilon}) & k\leqslant i\\ \{0\} & k>i\end{cases}\quad (Case 1^-).$$

   (II) *Case* $2^\pm$: $i\in\mathbb{Z}_{\geqslant 0}(\setminus\{0\}$ *for Case* $2^-)$,

$$M_\varepsilon(z,h_{i;\varepsilon})(k)=M_\varepsilon(z,h_{i+k;\varepsilon})\quad (Case\ 2^+),$$

$$M_\varepsilon(z,h_{i;\varepsilon})(k)=\begin{cases} M_\varepsilon(z,h_{i-k;\varepsilon}) & k\leqslant i\\ \{0\} & k>i\end{cases}\quad (Case\ 2^-).$$

   (III) *Case* $3^\pm$: $i\in\mathbb{Z}_{\geqslant 0}(\setminus\{0\}$ *for Case* $3^-)$,

$$M_\varepsilon(z,h_{(-1)^{i-1}i;\varepsilon})(k)=M_\varepsilon(z,h_{(-1)^{i+k-1}(i+k);\varepsilon})\quad (Case\ 3^+),$$

$$M_\varepsilon(z,h_{(-1)^{i-1}i;\varepsilon})(k)=\begin{cases} M_\varepsilon(z,h_{(-1)^{i-k-1}(i-k);\varepsilon}) & k\leqslant i\\ \{0\} & k>i\end{cases}\quad (Case\ 3^-).$$

   (IV) *Case* $4.1^\pm$: $i\in\mathbb{Z}_{\geqslant 0}(\setminus\{0\}$ *for Case* $4.1^-)$,
      i. *The case* $(p,q)\neq(1,1)$ *and* $i>0$,

$$M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(k)=M_{\frac{1}{2}}(z,h_{2i+2\lceil\frac{k+1}{2}\rceil;\frac{1}{2}})\quad (Case\ 4.1^+)$$

$$M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(k)=\begin{cases} M_{\frac{1}{2}}(z,h_{2i-2\lceil\frac{k+1}{2}\rceil;\frac{1}{2}}) & k<2i\\ \{0\} & k\geqslant 2i\end{cases}\quad (Case\ 4.1^-).$$

ii. *The case* $(p,q)=(1,1)$ *or* $i=0$,

$$M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(k)=M_{\frac{1}{2}}(z,h_{2(i+k);\frac{1}{2}})\quad (\textit{Case } 4.1^{+}),$$

$$M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(k)=\begin{cases} M_{\frac{1}{2}}(z,h_{2(i-k);\frac{1}{2}}) & k\leqslant i \\ \{0\} & k>i \end{cases}\quad (\textit{Case } 4.1^{-}).$$

(V) *Case* $4.2^{\pm}$: $i\in\mathbb{Z}_{\geqslant 0}(\setminus\{0\}$ *for Case* $4.2^{-})$,

i. *The case* $(p,q)\neq(1,1)$ *and* $i>0$,

$$M_{\varepsilon}(z,h_{-2i;\varepsilon})(k)=M_{\varepsilon}(z,h_{-2i-2\lceil\frac{k+1}{2}\rceil;\varepsilon})\quad (\textit{Case } 4.2^{+}),$$

$$M_{\varepsilon}(z,h_{-2i;\varepsilon})(k)=\begin{cases} M_{\varepsilon}(z,h_{-2i+2\lceil\frac{k+1}{2}\rceil;\varepsilon}) & k\leqslant 2i \\ \{0\} & k>2i \end{cases}\quad (\textit{Case } 4.2^{-}).$$

ii. *The case* $(p,q)=(1,1)$ *or* $i=0$,

$$M_{\varepsilon}(z,h_{-2i;\varepsilon})(k)=M_{\varepsilon}(z,h_{-2(i+k);\varepsilon})\quad (\textit{Case } 4.2^{+}),$$

$$M_{\varepsilon}(z,h_{-2i;\varepsilon})(k)=\begin{cases} M_{\varepsilon}(z,h_{-2(i-k);\varepsilon}) & k\leqslant i \\ \{0\} & k>i \end{cases}\quad (\textit{Case } 4.2^{-}).$$

(VI) *Case* $5^{\pm}$: $i\in\mathbb{Z}_{\geqslant 0}(\setminus\{0\}$ *for Case* $5^{-})$,

$$M_0(z,h_{i;0})(k)=\begin{cases} \operatorname{Im}\varphi_k & i=0 \\ M_0(z,h_{i+k;0})_{(1)}+M_0(z,h_{i+k;0})_{(2)} & i>0 \end{cases}\quad (\textit{Case } 5^{+}),$$

$$M_0(z,h_{i;0})(k)=\begin{cases} M_0(z,h_{i-k;0})_{(1)}+M_0(z,h_{i-k;0})_{(2)} & k<i \\ \tilde{M}(z,h_{0;0}) & k=i \\ \{0\} & k>i \end{cases}\quad (\textit{Case } 5^{-}),$$

where $\varphi_k : M_0(z,h_{k;0})\to M_0(z,h_{0;0})$ *are non-trivial homomorphisms, and* $M_0(z,h_{i\pm k;0})_{(j)}$ $(j=1,2)$ *are two distinct submodules that are both isomorphic to* $M_0(z,h_{i\pm k;0})$.

The proof of this theorem is an application of Lemma 4.2, Figs. 2 and 3.

If the weight $(z,h)\in\mathbb{C}^2$ belongs to Class V, then the Verma module $M_\varepsilon(z,h)$ is irreducible, and hence in this case, we get the result.

If the weight $(z,h)\in\mathbb{C}^2$ belongs to Class I, then there exists $\alpha,\beta\in\mathbb{Z}_{>0}$ and $t\in\mathbb{C}\setminus\mathbb{Q}$ such that $\alpha-\beta\in 1-2\varepsilon+2\mathbb{Z}_{\geqslant 0}$ and $z=z(t)$ and $h=h_{\alpha,\beta;\varepsilon}(t)$ (see Lemma 4.1). In this case, its character sum can be easily computed, and the result is

$$\sum_{k>0} \mathrm{ch}\; M_\varepsilon(z,h)(k) = \mathrm{ch}\; M_\varepsilon(z,h+\tfrac{1}{2}\alpha\beta).$$

Since the maximal proper submodule of $M_\varepsilon(z,h)$ is isomorphic to $M_\varepsilon(z,h+\frac{1}{2}\alpha\beta)$, we get the result.

Now, let us analyze the case when the weight $(z,h)\in\mathbb{C}^2$ belongs to Class $\mathrm{R}^\pm$. Cases $1^\pm$–$3^\pm$, $4.1^\pm$ ($(p,q)=(1,1)$ or $i=0$) and Case $4.2^\pm$ ($(p,q)=(1,1)$) can be studied in the same way as in [Mal], and the results follow. Let us consider Case $4.1^\pm$ ($(p,q)\neq(1,1)$ and $i>0$) and Case $4.2^\pm$ ($(p,q)\neq(1,1)$). In this case, we have

$$\dim\{M_\varepsilon(z,h)^\tau_n\}^{(\mathrm{Vir}_\varepsilon)_+}\leqslant 1 \quad (n\in(1-\varepsilon)\mathbb{Z}_{\geqslant 0},\tau\in\mathbb{Z}_2)$$

by Propositions 3.1 and 4.2. Therefore, we will prove only Case 4.1^+ since the other cases can be proved by the same manner.

For $i,k\in\mathbb{Z}_{>0}$, take a non-zero element $v_k^{(i)}\in\{M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})_{h_{2(i+k);\frac{1}{2}}-h_{2i;\frac{1}{2}}}\}^{(\mathrm{Vir}_{\frac{1}{2}})_+}$, and set

$$N_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(k) := U(\mathrm{Vir}_{\frac{1}{2}})v_k^{(i)}, \quad \bar{M}_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}}) := M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})/N_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(1).$$

Remark that $N_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(k)\cong M_{\frac{1}{2}}(z,h_{2(i+k);\frac{1}{2}})$. First, we will show that $\bar{M}_{\frac{1}{2}}(z,h_{2l;\frac{1}{2}})$ is irreducible for $l\in\mathbb{Z}_{>0}$. In fact, let

$$\bar{M}_{\frac{1}{2}}(z,h_{2l;\frac{1}{2}})\supset\bar{M}_{\frac{1}{2}}(z,h_{2l;\frac{1}{2}})(1)\supset\bar{M}_{\frac{1}{2}}(z,h_{2l;\frac{1}{2}})(2)\supset\cdots$$

be Jantzen filtration of $\bar{M}_{\frac{1}{2}}(z,h_{2l;\frac{1}{2}})$. Since, $v_1^{(l)}\in M_{\frac{1}{2}}(z,h_{2l;\frac{1}{2}})(2)$ by Lemma 4.2, we have

$$\begin{aligned}\sum_{k>0}\mathrm{ch}\;\bar{M}_{\frac{1}{2}}(z,h_{2l;\frac{1}{2}})(k) &= \sum_{k>0}\mathrm{ch}\; M_{\frac{1}{2}}(z,h_{2l;\frac{1}{2}})(k)\\ &\quad-\left\{2\mathrm{ch}\; N_{\frac{1}{2}}(z,h_{2l;\frac{1}{2}})(1)+\sum_{k>0}\mathrm{ch}\; M_{\frac{1}{2}}(z,h_{2(l+1);\frac{1}{2}})(k)\right\}\\ &=0,\end{aligned}$$

by Remark 4.4 and Lemma 4.2, which implies that the module $\bar{M}_{\frac{1}{2}}(z,h_{2l;\frac{1}{2}})$ is irreducible.

Now, we will prove

$$M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(2k-1)=M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(2k)=N_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(k)$$

by induction on $k\in\mathbb{Z}_{>0}$. The case $k=1$ immediately follows by $v_1^{(i)}\in M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(2)$ and the irreducibility of $\bar{M}_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})$. Suppose we have proved up to the case $k-1$. By Lemma 4.2 and the induction hypothesis, we have

$$\sum_{l=2k-1}^{\infty}\operatorname{ch} M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(l)=2\sum_{l=k}^{\infty}\operatorname{ch} M_{\frac{1}{2}}(z,h_{2(i+l);\frac{1}{2}}).$$

This formula and Proposition 3.1 imply $v_k^{(i)}\in M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(2k)\subset N_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(k-1)$ by assumption. Thus, we get $N_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(k)\subset M_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(2k)$. Since $N_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(k-1)/N_{\frac{1}{2}}(z,h_{2i;\frac{1}{2}})(k)\cong\bar{M}_{\frac{1}{2}}(z,h_{2(i+k-1);\frac{1}{2}})$ is irreducible, we get the result. This completes the above induction.

Case $5^{\pm}$ can be also studied by the same way as [Mal] except for Case 5^{+} $(i=0)$. For Case 5^{+} $(i=0)$, we first apply the same technique as [Mal] to the module $\tilde{M}_0(z,h_{0;0})$. This is possible by the existence of the embedding diagram (see Fig. 2). Now isomorphism (1) yields the result.

4.5. Structure of Verma modules

In this subsection, we will summarize some basic properties of Verma modules $M_{\varepsilon}(z,h)$, in particular, a refinement of results in Section 3.

First, the following theorem follows from the results obtained in Section 4.4.

Theorem 4.2. *Any submodule of a Verma module is generated by singular vectors.*

Therefore, it is useful to have some informations on singular vectors. Summarizing Propositions 3.1, 3.2, 4.1–4.3, we obtain the following theorem:

Theorem 4.3. *The dimension of the space of* $(\mathrm{Vir}_{\varepsilon})_{+}$*-invariants in each weight subspace satisfies the following*:

1. *For the Neveu–Schwarz algebra* (*i.e.*, $\varepsilon=\frac{1}{2}$), *we have*

$$\dim\{M_{\frac{1}{2}}(z,h)_n\}^{(\mathrm{Vir}_{\frac{1}{2}})_{+}}\leqslant 1\quad\forall n\in\tfrac{1}{2}\mathbb{Z}_{>0}.$$

2. *For the Ramond algebra* (*i.e.*, $\varepsilon = 0$), *we have*
   (i) *If the highest weight* $(z,h)\in\mathbb{C}^2$ *belongs to Case* $5^{\pm}$, *for* $i,k\in\mathbb{Z}_{>0}$ *satisfying* $\pm(k-i)>0$, *we have*

$$\dim\{M_0(z,h_{i;0})^{\tau}_{h_{k;0}-h_{i;0}}\}^{(\mathrm{Vir}_0)_+}=2\quad \tau\in\mathbb{Z}_2,$$

   (ii) *otherwise*, *we have*

$$\dim\{M_0(z,h)^{\tau}_{n}\}^{(\mathrm{Vir}_0)_+}\leqslant 1\quad \forall n\in\mathbb{Z}_{>0}\ \tau\in\mathbb{Z}_2.$$

Motivated by Remark 3.3, we will study Case $5^{\pm}$. For $i\in\mathbb{Z}_{\geqslant 0}$, let $|z,h\rangle$ be an even highest weight vector of $M_0(z,h)$.

Lemma 4.3 (Case $5^{\pm}$). *For* $i,j\in\mathbb{Z}_{\geqslant 0}$ *satisfying* $\pm(j-i)>0$, *let* $(P_jG_0+Q_j)|z,h_{i;0}\rangle\in M_0(z,h_{i;0})_{h_{j;0}-h_{i;0}}$ *be an even singular vector with the expansion with respect to the basis* $\mathscr{B}^{\bar{0}}_{h_{j;0}-h_{i;0}}$

$$P_j=c_{P_j}G_{-1}L_{-1}^{h_{j;0}-h_{i;0}-1}+\cdots,\quad Q_j=c_{Q_j}L_{-1}^{h_{j;0}-h_{i;0}}+\cdots.$$

1. *For* $k\in\mathbb{Z}_{>0}$ *satisfying* $\pm(k-j)>0$, *any even singular vector* $(P_kG_0+Q_k)|z,h_{i;0}\rangle$ *of level* $h_{k;0}-h_{i;0}$ *which is contained in the submodule* $U(\mathrm{Vir}_0).(P_jG_0+Q_j)|z,h_{i;0}\rangle$ *satisfying* $2c_{P_j}+c_{Q_j}=0$ (*resp.* $c_{Q_j}=0$) *satisfies* $2c_{P_k}+c_{Q_k}=0$ (*resp.* $c_{Q_k}=0$).
2. *For Case* 5^-, *a singular vector* $(P_0G_0+Q_0)|z,h_{i;0}\rangle$ *satisfies* $2c_{P_0}+c_{Q_0}=0$.

Before going into the proof, let us explain this lemma graphically (Fig. 4).

Both the first and the second series describe the singular vectors of the Verma modules with highest L_0-weight $h_{i;0}$. The number h with the polynomial F at the bottom signifies the singular vector with L_0-weight h satisfying the equation $F=0$. The arrow $h\to h'$ with the polynomial F at the bottom means that the singular vector with L_0-weight h' is generated by the singular vector with L_0-weight h both satisfying the equation $F=0$.

Proof. First, we prove 1 of this lemma. Let $\varphi: M_0(z,h_{j;0})\to M_0(z,h_{i;0})$ be a Vir_0-homomorphism defined by $\varphi(|z,h_{j;0}\rangle)=(P_jG_0+Q_j)|z,h_{i;0}\rangle$. By Theorem 4.2, there exists an even singular vector $(X_kG_0+Y_k)|z,h_{j;0}\rangle\in M(z,h_{j;0})$ with the expansion with respect to the basis $\mathscr{B}^{\bar{0}}_{h_{k;0}-h_{j;0}}$

$$X_k=c_{X_k}G_{-1}L_{-1}^{h_{k;0}-h_{j;0}-1}+\cdots,\quad Y_k=c_{Y_k}L_{-1}^{h_{k;0}-h_{j;0}}+\cdots$$

such that $\varphi((X_kG_0+Y_k)|z,h_{j;0}\rangle)=(P_kG_0+Q_k)|z,h_{i;0}\rangle$. Since, we have

$$(X_kG_0+Y_k)(P_jG_0+Q_j)|z,h_{i;0}\rangle=(P_kG_0+Q_k)|z,h_{i;0}\rangle$$

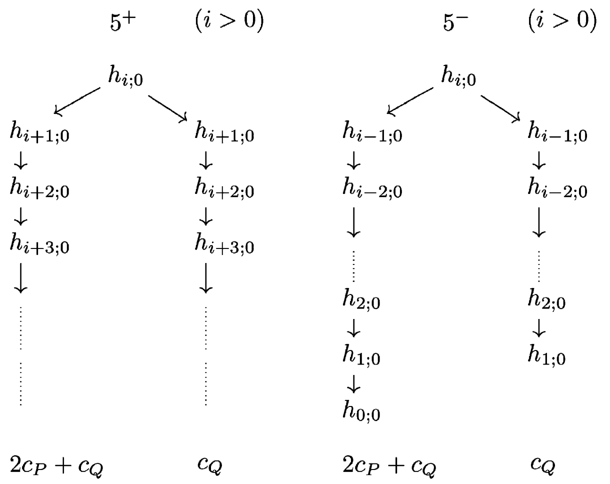

Fig. 4. Singular vectors of Verma modules for Case $5^{\pm}$.

by definition, we get

$$c_{P_k} = 2c_{X_k}c_{P_j} + c_{X_k}c_{Q_j} + c_{Y_k}c_{P_j}, \quad c_{Q_k} = c_{Y_k}c_{Q_j}. \tag{16}$$

Let us study (16) by case-by-case.

1. The case $2c_{P_j} + c_{Q_j} = 0$: In this case, (16) reduce to $c_{P_k} = c_{Y_k}c_{P_j}$ and $c_{Q_k} = c_{Y_k}c_{Q_j}$ which imply $c_{Y_k} \neq 0$ and $2c_{P_k} + c_{Q_k} = 0$.
2. The case $c_{Q_j} = 0$: In this case, (16) reduce to $c_{P_k} = (2c_{X_k} + c_{Y_k})c_{P_j}$ and $c_{Q_k} = 0$ which imply $2c_{X_k} + c_{Y_k} \neq 0$ and $c_{Q_k} = 0$.

This completes the proof of 1.

To prove 2 of this lemma, we first show that G_0 annihilates $(P_0G_0 + Q_0)|z, h_{i;0}\rangle$. In fact, suppose $G_0(PG_0 + Q)|z, h_{i;0}\rangle \neq 0$. Then, by Remark 3.2, it follows that $G_0(P_0G_0 + Q_0)G_0|z, h_{i;0}\rangle$ is a non-zero even singular vector which is G_0-invariant. But, this contradicts Proposition 4.3. Now, since we have

$$G_0.(P_0G_0 + Q_0)|z, h_{i;0}\rangle = \{(2c_{P_0} + c_{Q_0})L_{-1}^{h_{i;0}-h_{0;0}}G_0 + \cdots\}\,|\,z, h_{i;0}\rangle = 0,$$

the result follows. □

Next, we will summarize the results concerning the injectivity of morphisms between Verma modules. Theorem 4.3 and Proposition 3.3 imply that the only non-trivial cases are Case $5^{\pm}$ for Ramond algebra Vir_0. Let $|z, h\rangle$ be an even highest weight vector of $M_\varepsilon(z, h)$. Then, by Proposition 3.3, Remark 3.3 and (the proof of) Lemma 4.3, we obtain the following theorem:

Theorem 4.4. *Let $h, h' \in \mathbb{C}$ such that $h \neq h'$.*

1. *Any non-trivial $\mathbb{Z}_2$-graded $\mathrm{Vir}_{\frac{1}{2}}$-homomorphism $M_{\frac{1}{2}}(z,h') \to M_{\frac{1}{2}}(z,h)$ is monomorphism.*
2. *Let $\varphi : M_0(z,h') \to M_0(z,h)$ be a non-trivial $\mathbb{Z}_2$-graded Vir_0-homomorphism. Assume that $h \neq \frac{1}{24} z$.*
   (I) *If h and h' do not belong to Case $5^{\pm}$, then φ is a monomorphism.*
   (II) *Suppose $h = h_{i;0}$ and $h' = h_{j;0}$ for some $i, j \in \mathbb{Z}_{\geqslant 0}$ satisfying $\pm(j-i) > 0$ belong to Case $5^{\pm}$. For $k \in \mathbb{Z}_{\geqslant 0}$ satisfying $\pm(k-j) > 0$, let $(P_k^{(*)} G_0 + Q_k^{(*)})|z, h_{*;0}\rangle$ $(* = i, j)$ be an even singular vector with the expansion with respect to the basis $\mathscr{B}^{\bar{0}}_{h_{k;0} - h_{*;0}}$*

$$P_k^{(*)} = c_{P_k^{(*)}} G_{-1} L_{-1}^{h_{k;0} - h_{*;0} - 1} + \cdots, \quad Q_k^{(*)} = c_{Q_k^{(*)}} L_{-1}^{h_{k;0} - h_{*;0}} + \cdots.$$

In particular, we take

$$\varphi(|z, h_{j;0}\rangle) = (P_j^{(i)} G_0 + Q_j^{(i)})|z, h_{i;0}\rangle,$$

with the expansion as above.

i. *φ is a monomorphism if and only if $c_{P_j^{(i)}}$ and $c_{Q_j^{(i)}}$ satisfy both $2c_{P_j^{(i)}} + c_{Q_j^{(i)}} \neq 0$ and $c_{Q_j^{(i)}} \neq 0$. For $* \in \{i, j\}$, set*

$$V^{(*)}_{2c_P + c_Q} := \bigoplus_{\substack{k \in \mathbb{Z}_{\geqslant 0} \\ \pm(k-j) > 0}} \mathbb{C}(P_k^{(*)} G_0 + Q_k^{(*)})|z, h_{*;0}\rangle \quad 2c_{P_k^{(*)}} + c_{Q_k^{(*)}} = 0,$$

$$V^{(*)}_{c_Q} := \bigoplus_{\substack{k \in \mathbb{Z}_{>0} \\ \pm(k-j) > 0}} \mathbb{C}(P_k^{(*)} G_0 + Q_k^{(*)})|z, h_{*;0}\rangle \quad c_{Q_k^{(*)}} = 0.$$

ii. *In case $2c_{P_j^{(i)}} + c_{Q_j^{(i)}} = 0$, we have*

$$\{(\mathrm{Im}\,\varphi)^{\bar{0}}\}^{(\mathrm{Vir}_0)_+} = V^{(i)}_{2c_P + c_Q}, \quad \{(\mathrm{Ker}\,\varphi)^{\bar{0}}\}^{(\mathrm{Vir}_0)_+} = V^{(j)}_{c_Q}.$$

iii. *In case $c_{Q_j^{(i)}} = 0$, we have*

$$\{(\mathrm{Im}\,\varphi)^{\bar{0}}\}^{(\mathrm{Vir}_0)_+} = V^{(i)}_{c_Q}, \quad \{(\mathrm{Ker}\,\varphi)^{\bar{0}}\}^{(\mathrm{Vir}_0)_+} = V^{(j)}_{2c_P + c_Q}.$$

3. *For* $h = \frac{1}{24}z$, *we have the following*:
   (I) *A non-trivial* $\mathbb{Z}_2$*-graded* Vir_0*-homomorphism* $\varphi : M_0(z, h') \to \tilde{M}(z, h)$ *is a monomorphism if and only if* $\mathrm{Im}\,\varphi \not\subset M_0(z, h; \tau)$ *for any* $\tau \in \mathbb{Z}_2$.
   (II) *For* $h' = h_{j;0}$ *with* $j \in \mathbb{Z}_{>0}$, *let* $\varphi : M_0(z, h') \to M_0(z, h)$ *be a non-trivial* $\mathbb{Z}_2$*-graded* Vir_0*-homomorphism. Then,*

$$\{(\mathrm{Ker}\,\varphi)^{\bar{0}}\}^{(\mathrm{Vir}_0)_+} = V_{c_Q}^{(j)}.$$

Let i, j be positive integers satisfying $\pm(j - i) > 0$. The non-injective morphisms can be illustrated as shown in Figs. 5–7.

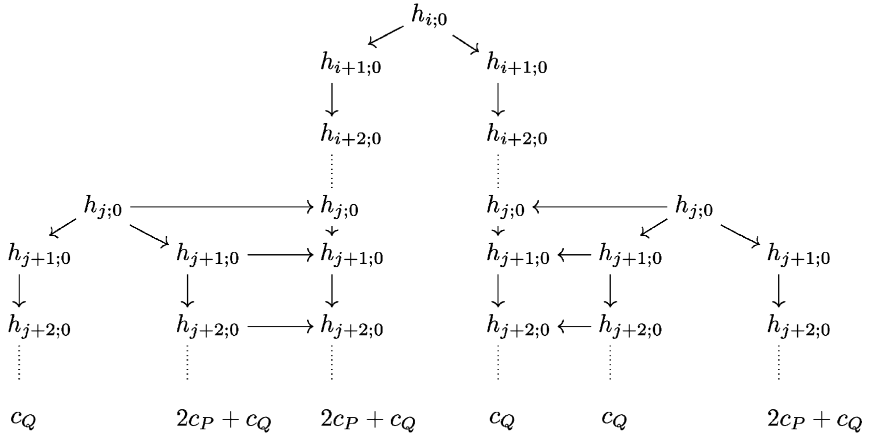

Fig. 5. Non-injective maps for Case 5^+ $(i \neq 0)$.

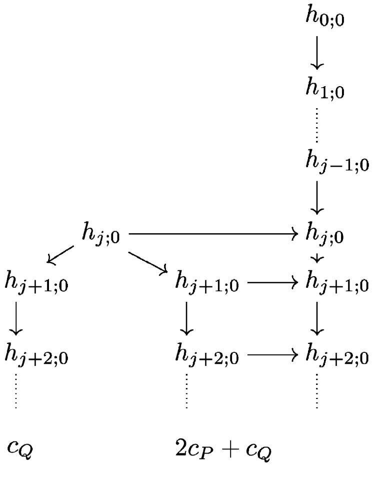

Fig. 6. Non-injective maps for Case 5^+ $(i = 0)$.

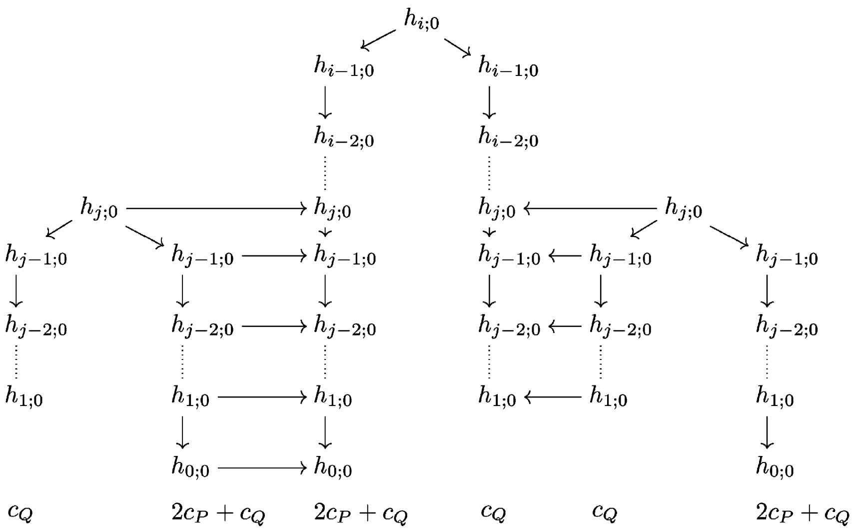

Fig. 7. Non-injective maps for Case 5^-.

5. Bernstein–Gel'fand–Gel'fand resolution

In this section, we will construct the so-called Bernstein–Gel'fand–Gel'fand type (BGG type, for short) resolution as an application of Theorem 4.1. Then, we will compute the characters of irreducible highest weight Vir_ε-modules.

5.1. Bernstein–Gel'fand–Gel'fand resolution

In this subsection, we will construct the strong BGG type resolutions for irreducible highest weight Vir_ε modules.

For $(z,h),(z,h')\in\mathfrak{h}^* = (\mathrm{Vir}_\varepsilon^{\bar{0}} \cap (\mathrm{Vir}_\varepsilon)_0)^* \cong \mathbb{C}^2$ which do not belong to Case $5^\pm$, let

$$\pi^\varepsilon_{z,h} : M_\varepsilon(z,h) \to L_\varepsilon(z,h)$$

be the canonical projection, and

$$\iota^\varepsilon_z(h',h) : M_\varepsilon(z,h') \to M_\varepsilon(z,h)$$

be a non-trivial Vir_ε-module homomorphism. By Theorem 4.4, any such $\iota^\varepsilon_z(h',h)$ is a monomorphism and is unique up to a scalar. For $(z,h),(z,h')\in\mathfrak{h}^*$ which belong to Case $5^\pm$ and $F\in\{2c_P + c_Q, c_Q\}$, let

$$\iota^0_z(h',h)_F : M_0(z,h') \to M_0(z,h)$$

be a non-trivial non-injective map defined as follows. For $\tilde{h}\in\{h,h'\}$, let $|z,\tilde{h}\rangle$ be an even highest weight vector of $M_0(z,\tilde{h})$, and set

$$\iota_z^0(h',h)_F(|z,h'\rangle) \coloneqq (PG_0+Q)|z,h\rangle,$$

where $P,Q\in U((\mathrm{Vir}_0)_-)$ have the following expansion with respect to the basis $\mathscr{B}^{\bar{0}}_{h'-h}$

$$P = c_P G_{-1}L_{-1}^{h'-h-1}+\cdots,\quad Q = c_Q L_{-1}^{h'-h}+\cdots.$$

Then for each F, we define $\iota_z^0(h',h)_F$ so that the $(c_P,c_Q)\in\mathbb{C}^2$ satisfies $F=0$. In particular, for $h=\frac{1}{24}z$, we let

$$\iota_z^0(h',h): M_0(z,h')\to M_0(z,h)$$

be a non-trivial Vir_0-module homomorphism, and for $h'=\frac{1}{24}z$, we let

$$\iota_z^0(h',h): \tilde{M}_0(z,h')\to M_0(z,h)$$

be a non-trivial Vir_0-module homomorphism. Such Vir_0-module homomorphisms surely exist, and are unique up to a scalar by Theorem 4.4.

The results can be stated as follows:

Theorem 5.1. *In each case, the following sequence is exact.*

1. *Class* V:

$$0\to M_\varepsilon(z,h)\overset{d_0}{\to} L_\varepsilon(z,h)\to 0,$$

where we set

$$d_0 \coloneqq \pi^\varepsilon_{z,h}.$$

2. *Class* I: *In this case,* $(z,h)=(z(t),h_{\alpha,\beta;\varepsilon}(t))$ *for some* $t\in\mathbb{C}\setminus\mathbb{Q}$ *and* $\alpha,\beta\in\mathbb{Z}_{>0}$ *satisfying* $\alpha-\beta\in 1-2\varepsilon+2\mathbb{Z}_{\geqslant 0}$.

$$0\to M_\varepsilon(z,h+\tfrac{1}{2}\alpha\beta)\overset{d_1}{\to} M_\varepsilon(z,h)\overset{d_0}{\to} L_\varepsilon(z,h)\to 0,$$

where we set

$$d_1 \coloneqq \iota_z^\varepsilon(h+\tfrac{1}{2}\alpha\beta,h),\quad d_0 \coloneqq \pi^\varepsilon_{z,h}.$$

3. *Class* $\mathrm{R}^{\pm}$: *Let* $p, q \in \mathbb{Z}_{>0}$ *satisfying* $p-q \in 2\mathbb{Z}$, $(\frac{p-q}{2}, q) = 1$. *Set* $z = z(\pm\frac{p}{q})$.

(I) *Case* 1^+: $i \in \mathbb{Z}$,

$$\cdots \stackrel{d_{k+1}}{\to} M_\varepsilon(z, h_{|i|+k;\varepsilon}) \oplus M_\varepsilon(z, h_{-|i|-k;\varepsilon}) \stackrel{d_k}{\to} \cdots$$

$$\stackrel{d_2}{\to} M_\varepsilon(z, h_{|i|+1;\varepsilon}) \oplus M_\varepsilon(z, h_{-|i|-1;\varepsilon}) \stackrel{d_1}{\to} M_\varepsilon(z, h_{i;\varepsilon}) \stackrel{d_0}{\to} L_\varepsilon(z, h_{i;\varepsilon}) \to 0,$$

where we set

$$d_k(x,y) := (\iota_z^\varepsilon(h_{|i|+k;\varepsilon}, h_{|i|+k-1;\varepsilon})(x) + \iota_z^\varepsilon(h_{-|i|-k;\varepsilon}, h_{|i|+k-1;\varepsilon})(y),$$
$$- \ \iota_z^\varepsilon(h_{|i|+k;\varepsilon}, h_{-|i|-k+1;\varepsilon})(x) - \iota_z^\varepsilon(h_{-|i|-k;\varepsilon}, h_{-|i|-k+1;\varepsilon})(y)) \quad k>1,$$

$$d_k(x,y) := \iota_z^\varepsilon(h_{|i|+1;\varepsilon}, h_{i;\varepsilon})(x) + \iota_z^\varepsilon(h_{-|i|-1;\varepsilon}, h_{i;\varepsilon})(y) \quad k=1,$$

$$d_k(x) := \pi^\varepsilon_{z,h_{i;\varepsilon}}(x) \quad k=0.$$

(II) *Case* 1^-: $i \in \mathbb{Z} \setminus \{0\}$,

$$0 \to M_\varepsilon(z, h_{0;\varepsilon}) \stackrel{d_i}{\to} M_\varepsilon(z, h_{1;\varepsilon}) \oplus M_\varepsilon(z, h_{-1;\varepsilon}) \stackrel{d_{i-1}}{\to} \cdots$$

$$\stackrel{d_2}{\to} M_\varepsilon(z, h_{|i|-1;\varepsilon}) \oplus M_\varepsilon(z, h_{-|i|+1;\varepsilon}) \stackrel{d_1}{\to} M_\varepsilon(z, h_{i;\varepsilon}) \stackrel{d_0}{\to} L_\varepsilon(z, h_{i;\varepsilon}) \to 0,$$

where we set

$$d_k(x) := \iota_z^\varepsilon(h_{0;\varepsilon}, h_{1;\varepsilon})(x) + \iota_z^\varepsilon(h_{0;\varepsilon}, h_{1;\varepsilon})(x) \quad k=i,$$

$$d_k(x,y) := (\iota_z^\varepsilon(h_{|i|-k;\varepsilon}, h_{|i|-k+1;\varepsilon})(x) + \iota_z^\varepsilon(h_{-|i|+k;\varepsilon}, h_{|i|-k+1;\varepsilon})(y),$$
$$- \ \iota_z^\varepsilon(h_{|i|-k;\varepsilon}, h_{-|i|+k-1;\varepsilon})(x) - \iota_z^\varepsilon(h_{-|i|+k;\varepsilon}, h_{-|i|+k-1;\varepsilon})(y)) \quad 1<k<i,$$

$$d_k(x,y) := \iota_z^\varepsilon(h_{|i|-1;\varepsilon}, h_{i;\varepsilon})(x) + \iota_z^\varepsilon(h_{-|i|+1;\varepsilon}, h_{i;\varepsilon})(y) \quad k=1,$$

$$d_k(x) := \pi^\varepsilon_{z,h_{i;\varepsilon}}(x) \quad k=0.$$

(III) *Case* $2^{\pm}$: $i \in \mathbb{Z}_{\geqslant 0}$ ($\setminus\{0\}$ *for Case* 2^-),

$$0 \to M_\varepsilon(z, h_{i\pm 1;\varepsilon}) \stackrel{d_1}{\to} M_\varepsilon(z, h_{i;\varepsilon}) \stackrel{d_0}{\to} L_\varepsilon(z, h_{i;\varepsilon}) \to 0,$$

where we set

$$d_1 := \imath_z^{\varepsilon}(h_{i\pm 1;\varepsilon}, h_{i;\varepsilon}), \quad d_0 := \pi^{\varepsilon}_{z,h_{i;\varepsilon}}.$$

(IV) *Case* $3^{\pm}$: $i \in \mathbb{Z}_{\geqslant 0}(\setminus\{0\}$ *for Case* $3^-)$,

$$0 \to M_{\varepsilon}(z, h_{(-1)^i(i\pm 1);\varepsilon}) \overset{d_1}{\to} M_{\varepsilon}(z, h_{(-1)^{i-1}i;\varepsilon}) \overset{d_0}{\to} L_{\varepsilon}(z, h_{(-1)^{i-1}i;\varepsilon}) \to 0,$$

where we set

$$d_1 := \imath_z^{\varepsilon}(h_{(-1)^i(i\pm 1);\varepsilon}, h_{(-1)^{i-1}i;\varepsilon}), \quad d_0 := \pi^{\varepsilon}_{z,h_{(-1)^{i-1}i;\varepsilon}}.$$

(V) *Case* $4.1^{\pm}$: $i \in \mathbb{Z}_{\geqslant 0}(\setminus\{0\}$ *for Case* $4.1^-)$,

$$0 \to M_{\frac{1}{2}}(z, h_{2(i\pm 1);\frac{1}{2}}) \overset{d_1}{\to} M_{\frac{1}{2}}(z, h_{2i;\frac{1}{2}}) \overset{d_0}{\to} L_{\frac{1}{2}}(z, h_{2i;\frac{1}{2}}) \to 0,$$

where we set

$$d_1 := \imath_z^{\frac{1}{2}}(h_{2(i\pm 1);\frac{1}{2}}, h_{2i;\frac{1}{2}}), \quad d_0 := \pi^{\frac{1}{2}}_{z,h_{2i;\frac{1}{2}}}.$$

(VI) *Case* $4.2^{\pm}$: $i \in \mathbb{Z}_{\geqslant 0}(\setminus\{0\}$ *for Case* $4.2^-)$,

$$0 \to M_{\varepsilon}(z, h_{-2(i\pm 1);\varepsilon}) \overset{d_1}{\to} M_{\varepsilon}(z, h_{-2i;\varepsilon}) \overset{d_0}{\to} L_{\varepsilon}(z, h_{-2i;\varepsilon}) \to 0,$$

where we set

$$d_1 := \imath_z^{\varepsilon}(h_{-2(i\pm 1);\varepsilon}, h_{-2i;\varepsilon}), \quad d_0 := \pi^{\varepsilon}_{z,h_{-2i;\varepsilon}}.$$

(VII) *Case* 5^+: $i \in \mathbb{Z}_{>0}$,

$$\cdots \overset{d_{k+1}}{\to} M_0(z, h_{i+k;0})^{\oplus 2} \overset{d_k}{\to} \cdots$$

$$\overset{d_2}{\to} M_0(z, h_{i+1;0})^{\oplus 2} \overset{d_1}{\to} M_0(z, h_{i;0}) \overset{d_0}{\to} L_0(z, h_{i;0}) \to 0,$$

where we set

$$d_k(x, y) := (\imath_z^0(h_{i+k;0}, h_{i+k-1;0})_{c_Q}(x), \imath_z^0(h_{i+k;0}, h_{i+k-1;0})_{2c_P+c_Q}(y))$$

$$k > 1 \wedge k \equiv 0 \bmod 2,$$

$$d_k(x,y) \coloneqq (\iota_z^0(h_{i+k;0}, h_{i+k-1;0})_{2c_P+c_Q}(x), \iota_z^0(h_{i+k;0}, h_{i+k-1;0})_{c_Q}(y)) \quad k>1 \wedge k \equiv 1 \bmod 2,$$

$$d_k(x,y) \coloneqq \iota_z^0(h_{i+1;0}, h_{i;0})_{2c_p+c_Q}(x) + \iota_z^0(h_{i+1;0}, h_{i;0})_{c_Q}(y) \quad k=1,$$

$$d_k(x) \coloneqq \pi^0_{z,h_{i;0}}(x) \quad k=0,$$

$i = 0$,

$$\cdots \xrightarrow{d_{k+1}} M_0(z,h_{k;0}) \xrightarrow{d_k} \cdots$$

$$\xrightarrow{d_2} M_0(z,h_{1;0}) \xrightarrow{d_1} M_0(z,h_{0;0}) \xrightarrow{d_0} L_0(z,h_{0;0}) \to 0,$$

where we set

$$d_k \coloneqq \begin{cases} \iota_z^0(h_{k;0}, h_{k-1;0})_{c_Q} & k>1 \wedge k \equiv 0 \bmod 2, \\ \iota_z^0(h_{k;0}, h_{k-1;0})_{2c_P+c_Q} & k>1 \wedge k \equiv 1 \bmod 2, \\ \iota_z^0(h_{1;0}, h_{0;0}) & k=1, \\ \pi^0_{z,h_{0;0}} & k=0. \end{cases}$$

(VIII) *Case* 5^-: $i \in \mathbb{Z} \setminus \{0\}$,

$$0 \to M_0(z,h_{0;0}) \xrightarrow{d_i} M_0(z,h_{1;0})^{\oplus 2} \xrightarrow{d_{i-1}} \cdots$$

$$\xrightarrow{d_2} M_0(z,h_{|i|-1;0})^{\oplus 2} \xrightarrow{d_1} M_0(z,h_{i;0}) \xrightarrow{d_0} L_0(z,h_{i;0}) \to 0,$$

where we set

$$d_k(x) \coloneqq (0, \iota_z^0(h_{0;0}, h_{1;0})_{2c_P+c_Q}(x)) \quad k=i \wedge i \equiv 0 \bmod 2,$$

$$d_k(x) \coloneqq (\iota_z^0(h_{0;0}, h_{1;0})_{2c_P+c_Q}(x), 0) \quad k=i \wedge i \equiv 1 \bmod 2,$$

$$d_k(x,y) \coloneqq (\iota_z^0(h_{i-k;0}, h_{i-k+1;0})_{c_Q}(x), \iota_z^0(h_{i-k;0}, h_{i-k+1;0})_{2c_P+c_Q}(y)) \quad 1<k<i \wedge k \equiv 0 \bmod 2,$$

$$d_k(x,y) \coloneqq (\iota_z^0(h_{i-k;0}, h_{i-k+1;0})_{2c_P+c_Q}(x), \iota_z^0(h_{i-k;0}, h_{i-k+1;0})_{c_Q}(y)) \quad 1<k<i \wedge k \equiv 1 \bmod 2,$$

$$d_k(x,y) := \iota_z^0(h_{1;0},h_{0;0})_{2c_P+c_Q}(x)+\iota_z^0(h_{1;0},h_{0;0})_{c_Q}(y) \quad k=1,$$

$$d_k(x) := \pi^0_{z,h_{i;0}}(x) \quad k=0.$$

Remark 5.1. For Case 5^+, the BGG type resolution of the module $L_0(z,h_{0;0})\oplus \Pi L_0(z,h_{0;0})$

$$\cdots \stackrel{d_{k+1}}{\to} M_0(z,h_{k;0})^{\oplus 2} \stackrel{d_k}{\to} \cdots \stackrel{d_2}{\to} M_0(z,h_{1;0})^{\oplus 2} \stackrel{d_1}{\to}$$

$$M_0(z,h_{0;0})\oplus \Pi M_0(z,h_{0;0}) \cong \tilde{M}(z,h_{0;0}) \stackrel{d_0}{\to} L_0(z,h_{0;0})\oplus \Pi L_0(z,h_{0;0}) \to 0,$$

given in [IK1], which is important for its application to physics, can be constructed as follows. Let $(\mathscr{C}_*,\tilde{d}_*)$ be the BGG type resolution of $L_0(z,h_{0;0})$ given in Theorem 5.1. Then, the above complex is just $(\mathscr{C}_*\oplus \Pi\mathscr{C}_*,(\tilde{d}_*,\tilde{d}_*^{\Pi}))$.

Besides Case $5^{\pm}$, we can prove Theorem 5.1 by the same argument as in [Mal] since we have Theorem 4.1. Theorem 5.1 for Case $5^{\pm}$ and Remark 5.1 are simple corollaries of Theorems 4.1 and 4.4. The details are left to the reader.

5.2. *Characters*

In this subsection, we will compute the characters of the irreducible highest weight modules $L_\varepsilon(z,h)$. In particular, we will also compute the super-character of the minimal series for the Neveu–Schwarz algebra $\mathrm{Vir}_{\frac{1}{2}}$. The modular transformations of the characters of minimal series representations will be given as an application.

For $M\in \mathrm{Ob}(\mathscr{O})$ such that $c|_M = z\,\mathrm{id}$, we denote the weight space decomposition of M by

$$M=\bigoplus_{h\in\mathbb{C}} M_h, \quad M_h := \{u\in M | L_0.u = hu\}.$$

Recall that the so-called normalized character of M is defined by

$$\mathrm{tr}_M q^{L_0-\frac{1}{24}c} := \sum_{h\in\mathbb{C}} (\dim M_h) q^{h-\frac{1}{24}z},$$

and the normalized super-character of M is defined by

$$\mathrm{str}_M q^{L_0-\frac{1}{24}c} := \sum_{h\in\mathbb{C}} (\mathrm{sdim}\, M_h) q^{h-\frac{1}{24}z},$$

where $q=e^{2\pi i\tau}\in\mathbb{C}$ satisfying $0<|q|<1$. Note that the normalized (super-)character is an additive function on $\mathscr{O}$, i.e., a function defined on the Grothendieck group of $\mathscr{O}$.

The normalized characters of irreducible highest weight Vir_ε-modules can be expressed in terms of holomorphic modular forms of weight $\frac{1}{2}$, the Dedekind η-function $\eta(\tau)$ and the classical theta functions $\Theta_{n,m}(\tau)$. Let us briefly recall their definition. The Dedekind η-function $\eta(\tau)$ is defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n\in\mathbb{Z}_{>0}} (1-q^n),$$

and for $m\in\frac{1}{2}\mathbb{Z}_{>0}$ and $n\in\mathbb{Z}/2m\mathbb{Z}$ the classical theta functions $\Theta_{n,m}(\tau)$ are defined by

$$\Theta_{n,m}(\tau) = \sum_{k\in\mathbb{Z}+\frac{n}{2m}} e^{2\pi i m k^2 \tau}.$$

By the Poincaré–Birkhoff–Witt theorem, the normalized character of the Verma module $M_\varepsilon(z,h)$ can be expressed as

$$\mathrm{tr}_{M_\varepsilon(z,h)} q^{L_0-\frac{1}{24}c} = 2^{-\delta_{\varepsilon,0}\delta_{h,\frac{1}{24}z}} q^{h-\frac{1}{24}z-\frac{1}{8}\varepsilon} \varphi_\varepsilon(\tau), \tag{17}$$

where $\varphi_\varepsilon(\tau)$ is given by

$$\varphi_\varepsilon(\tau) := \begin{cases} \dfrac{\eta(\tau)}{\eta(\frac{1}{2}\tau)\eta(2\tau)}, & \varepsilon = \frac{1}{2}, \\ 2\dfrac{\eta(2\tau)}{\eta(\tau)^2}, & \varepsilon = 0. \end{cases}$$

Now, by the Euler–Poincaré principle, we obtain the normalized characters of the irreducible highest weight modules by Theorem 5.1 and (17) as follows:

Theorem 5.2. *For each case, the normalized character* $\mathrm{tr}_{L_\varepsilon(z,h)} q^{L_0-\frac{1}{24}z}$ *is given as follows*:

1. *Class V*:

$$2^{-\delta_{\varepsilon,0}\delta_{h,\frac{1}{24}z}} q^{h-\frac{1}{24}z-\frac{1}{8}\varepsilon} \varphi_\varepsilon(\tau).$$

2. *Class* I: *In this case*, $(z,h) = (z(t), h_{\alpha,\beta;\varepsilon}(t))$ *for some* $t\in\mathbb{C}\setminus\mathbb{Q}$ *and* $\alpha,\beta\in\mathbb{Z}_{>0}$ *satisfying* $\alpha-\beta\in 1-2\varepsilon+2\mathbb{Z}_{\geqslant 0}$,

$$(q^{\frac{1}{8}(\alpha t^{\frac{1}{2}}-\beta t^{-\frac{1}{2}})^2} - q^{\frac{1}{8}(\alpha t^{\frac{1}{2}}+\beta t^{-\frac{1}{2}})^2})\varphi_\varepsilon(\tau).$$

3. *Class* R^+: *Let* $p,q\in\mathbb{Z}_{>0}$ *satisfying* $p-q\in 2\mathbb{Z}$, $(\frac{p-q}{2},q)=1$. *Set* $z = z(\frac{p}{q})$.

(I) *Case* 1^+: $h = h_{i;\varepsilon}$ $(i \in \mathbb{Z})$,

$$(-1)^i[\Theta_{\frac{1}{2}(rp-sq),\frac{1}{2}pq}(\tau) - \Theta_{\frac{1}{2}(rp+sq),\frac{1}{2}pq}(\tau) - r_i(\tau)]\varphi_\varepsilon(\tau),$$

where we set

$$r_i(\tau) = \sum_{\substack{k \in \mathbb{Z} \\ s.t.\ |k| \leqslant |i|, \\ k \neq i}} (-1)^k q^{\frac{1}{8pq}\{2\lceil \frac{k}{2} \rceil pq + rp - (-1)^k sq\}}.$$

(II) *Case* 2^+: $h = h_{i;\varepsilon}$ $(i \in \mathbb{Z}_{\geqslant 0})$,

$$[q^{\frac{1}{8pq}\{(2\lceil \frac{i}{2} \rceil + 1)pq - (-1)^i sq\}^2} - q^{\frac{1}{8pq}\{(2\lceil \frac{i+1}{2} \rceil + 1)pq + (-1)^i sq\}^2}]\varphi_\varepsilon(\tau).$$

(III) *Case* 3^+: $h = h_{(-1)^{i-1}i;\varepsilon}$ $(i \in \mathbb{Z}_{\geqslant 0})$,

$$[q^{\frac{1}{8pq}\{(2\lceil \frac{(-1)^{i-1}i}{2} \rceil - (-1)^i)pq + rp\}^2} - q^{\frac{1}{8pq}\{(2\lceil \frac{(-1)^i(i+1)}{2} \rceil + (-1)^i)pq + rp\}^2}]\varphi_\varepsilon(\tau).$$

(IV) *Case* 4.1^+: $h = h_{2i;\varepsilon}$ $(i \in \mathbb{Z}_{\geqslant 0})$,

$$[q^{\frac{1}{2}pqi^2} - q^{\frac{1}{2}pq(i+1)^2}]\varphi_{\frac{1}{2}}(\tau).$$

(V) *Case* 4.2^+: $h = h_{-2i;\varepsilon}$ $(i \in \mathbb{Z}_{\geqslant 0})$,

$$[q^{\frac{1}{8}pq(2i+1)^2} - q^{\frac{1}{8}pq(2i+3)^2}]\varphi_\varepsilon(\tau).$$

(VI) *Case* 5^+: $h = h_{i;0}$ $(i \in \mathbb{Z}_{\geqslant 0})$,

$$(-1)^i 2^{-\delta_{i,0}}[\Theta_{0,\frac{1}{2}pq}(\tau) - \Theta_{\frac{1}{2}pq,\frac{1}{2}pq}(\tau) - r_i(\tau)]\varphi_\varepsilon(\tau),$$

where we set

$$r_i(\tau) = \sum_{\substack{k \in \mathbb{Z} \\ s.t.\ |k| \leqslant i, \\ k \neq i}} (-1)^k q^{\frac{1}{8}pqk^2}.$$

4. *Class* R$^-$: *Let* $p, q \in \mathbb{Z}_{>0}$ *satisfying* $p - q \in 2\mathbb{Z}$, $(\frac{p-q}{2}, q) = 1$. *Set* $z = z(-\frac{p}{q})$.

(I) *Case* 1^-: $h = h_{i;\varepsilon}$ $(i \in \mathbb{Z} \setminus \{0\})$,

$$\left[\sum_{\substack{k \in \mathbb{Z} \\ s.t.\ |k| \leqslant |i|, \\ k \neq -i}} (-1)^{k-i} q^{-\frac{1}{8pq}\{2\lceil \frac{k}{2} \rceil pq + rp + (-1)^k sq\}^2} \right] \varphi_\varepsilon(\tau).$$

(II) *Case* 2^-: $h = h_{i;\varepsilon}$ $(i \in \mathbb{Z}_{>0})$,

$$[q^{-\frac{1}{8pq}\{(2\lceil \frac{i}{2} \rceil + 1)pq + (-1)^i sq\}^2} - q^{-\frac{1}{8pq}\{(2\lceil \frac{i-1}{2} \rceil + 1)pq - (-1)^i sq\}^2}] \varphi_\varepsilon(\tau).$$

(III) *Case* 3^-: $h = h_{(-1)^{i-1} i;\varepsilon}$ $(i \in \mathbb{Z}_{>0})$,

$$[q^{-\frac{1}{8pq}\{(2\lceil \frac{(-1)^{i-1} i}{2} \rceil - (-1)^i) pq + rp\}^2} - q^{-\frac{1}{8pq}\{(2\lceil \frac{(-1)^i (i-1)}{2} \rceil + (-1)^i) pq + rp\}^2}] \varphi_\varepsilon(\tau).$$

(IV) *Case* 4.1^-: $h = h_{2i;\varepsilon}$ $(i \in \mathbb{Z}_{>0})$,

$$[q^{-\frac{1}{2} pqi^2} - q^{-\frac{1}{2} pq(i-1)^2}] \varphi_{\frac{1}{2}}(\tau).$$

(V) *Case* 4.2^-: $h = h_{-2i;\varepsilon}$ $(i \in \mathbb{Z}_{>0})$,

$$[q^{-\frac{1}{8} pq(2i+1)^2} - q^{-\frac{1}{8} pq(2i-1)^2}] \varphi_\varepsilon(\tau).$$

(VI) *Case* 5^-: $h = h_{i;0}$ $(i \in \mathbb{Z}_{>0})$,

$$\left[\sum_{\substack{k \in \mathbb{Z} \\ s.t.\ |k| \leqslant |i|, \\ k \neq -i}} (-1)^{k-i} q^{-\frac{1}{8} pqk^2} \right] \varphi_0(\tau).$$

Similarly, we can compute the normalized super-characters of the irreducible highest weight modules. For the Ramond algebra Vir_0, they are trivial. In fact, we have

$$\mathrm{str}_{M_0(z,h;\bar{0})} q^{L_0 - \frac{1}{24} c} = \delta_{h, \frac{1}{24} z} q^{h - \frac{1}{24} z},$$

which implies that

$$\mathrm{str}_{L_0(z,h;\bar{0})} q^{L_0 - \frac{1}{24}c} = \delta_{h,\frac{1}{24}z}.$$

Thus, we will write them down for the Neveu–Schwarz algebra $\mathrm{Vir}_{\frac{1}{2}}$ below. In addition, the only results that will be used later in this paper are those for minimal series representations, so we will stick to this case. Recall that the highest weight of the minimal series representations are parametrized by the set

$$\mathring{K}_{p,q} := \left\{ (r,s) \in \mathbb{Z}^2 \,\middle|\, \begin{matrix} 0<r<q \\ 0<s<p \end{matrix}, rp + sq \leqslant pq \right\} \subset K^+_{p,q},$$

for a fixed central charge $z = z_{p,q}$ where $p, q \in \mathbb{Z}_{>0}$ satisfy $p - q \in 2\mathbb{Z}$ and $(\frac{p-q}{2}, q) = 1$ (cf. Remark 4.3). For each $\varepsilon \in \{\frac{1}{2}, 0\}$, we define a subset of $\mathring{K}_{p,q}$ by

$$\mathring{K}^{\varepsilon}_{p,q} := \{(r,s) \in \mathring{K}_{p,q} \mid r - s \in 1 - 2\varepsilon + 2\mathbb{Z}\}.$$

Clearly, we have

$$\mathring{K}_{p,q} = \mathring{K}^0_{p,q} \sqcup \mathring{K}^{\frac{1}{2}}_{p,q}.$$

Set

$$\psi(\tau) := \frac{\eta(\frac{1}{2}\tau)}{\eta(\tau)^2}.$$

Theorem 5.3 (Neveu–Schwarz only). *For Case* 1^+ *and* $h = h_{0;\frac{1}{2}}$ *with a fixed* $(r,s) \in \mathring{K}^{\frac{1}{2}}_{p,q}$, *the normalized super-character* $\mathrm{str}_{L_{\frac{1}{2}}(z,h)} q^{L_0 - \frac{1}{24}c}$ *is given by*

$$\begin{aligned}&[\Theta_{rp-sq,2pq}(\tau) + (-1)^{pq}\Theta_{rp-sq+2pq,2pq}(\tau) \\ &\quad - (-1)^{rs}\{\Theta_{rp+sq,2pq}(\tau) + (-1)^{pq}\Theta_{rp+sq+2pq,2pq}(\tau)\}]\psi(\tau).\end{aligned}$$

Next, we will describe how the characters of the minimal series transform under the action of $\mathrm{SL}(2,\mathbb{Z})$. We fix $p, q \in \mathbb{Z}_{>0}$ satisfying $p - q \in 2\mathbb{Z}$ and $(\frac{p-q}{2}, q) = 1$. For $\varepsilon \in \{\frac{1}{2}, 0\}$ and $(r,s) \in \mathring{K}^{\varepsilon}_{p,q}$, we set

$$\chi^{\varepsilon}_{r,s}(\tau) := \{2(1-\varepsilon)\}^{\delta_{r,\frac{q}{2}}\delta_{s,\frac{p}{2}} - 1} \mathrm{tr}_{L_{\varepsilon}(z_{p,q}, h_{0;\varepsilon})} q^{L_0 - \frac{1}{24}c},$$

and for $\varepsilon = \frac{1}{2}$ we set

$$\tilde{\chi}_{r,s}(\tau) := \mathrm{str}_{L_{\frac{1}{2}}(z_{p,q},h_{0;\frac{1}{2}})} q^{L_0 - \frac{1}{24}c}.$$

By using the following modular transformation laws,

$$\eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau), \quad \eta(-\tfrac{1}{\tau}) = \sqrt{\frac{\tau}{i}}\,\eta(\tau),$$

$$\Theta_{n,m}(\tau+1) = e^{\frac{\pi i n^2}{2m}}\Theta_{n,m}(\tau), \quad \Theta_{n,m}(-\tfrac{1}{\tau}) = \sqrt{\frac{\tau}{2mi}} \sum_{n' \in \mathbb{Z}/2m\mathbb{Z}} e^{-\frac{\pi i n n'}{m}}\Theta_{n',m}(\tau),$$

$$\text{for } m \in \mathbb{Z}_{>0}, \ n \in \mathbb{Z}/2m\mathbb{Z},$$

we obtain the next formulae:

Proposition 5.1. *Modular transformations of the* (*super*-)*characters of the minimal series are given as follows*:

1. *For* $\tau \mapsto \tau + 1$, *we have*

$$\chi^{\frac{1}{2}}_{r,s}(\tau+1) = e^{\pi i\{\frac{(rp-sq)^2}{4pq} - \frac{1}{8}\}}\tilde{\chi}_{r,s}(\tau), \ \ \tilde{\chi}_{r,s}(\tau+1) = e^{\pi i\{\frac{(rp-sq)^2}{4pq} - \frac{1}{8}\}}\chi^{\frac{1}{2}}_{r,s}(\tau),$$

$$\chi^0_{r,s}(\tau+1) = e^{\pi i\frac{(rp-sq)^2}{4pq}}\chi^0_{r,s}(\tau).$$

2. *For* $\tau \mapsto -\frac{1}{\tau}$, *we have*

$$\chi^{\frac{1}{2}}_{r,s}(-\tfrac{1}{\tau}) = \sum_{(r',s') \in \mathring{K}^{\frac{1}{2}}_{p,q}} S^{\frac{1}{2},\frac{1}{2}}_{(r,s),(r',s')}\chi^{\frac{1}{2}}_{r',s'}(\tau),$$

$$\sqrt{2}\chi^0_{r,s}(-\tfrac{1}{\tau}) = \sum_{(r',s') \in \mathring{K}^{\frac{1}{2}}_{p,q}} S^{0,\frac{1}{2}}_{(r,s),(r',s')}\tilde{\chi}_{r',s'}(\tau),$$

$$\tilde{\chi}_{r,s}(-\tfrac{1}{\tau}) = \sum_{(r',s') \in \mathring{K}^{0}_{p,q}} S^{\frac{1}{2},0}_{(r,s),(r',s')}\sqrt{2}\chi^0_{r',s'}(\tau),$$

where we set

$$S^{\frac{1}{2},\frac{1}{2}}_{(r,s),(r',s')} = \frac{4}{\sqrt{pq}} \sin \frac{\pi r r'}{2q}(p-q) \sin \frac{\pi s s'}{2p}(p-q),$$

$$S^{0,\frac{1}{2}}_{(r,s),(r',s')} = (-1)^{\frac{1}{2}(r'-s')} \frac{4}{\sqrt{pq}} \sin \frac{\pi r r'}{2q}(p-q) \sin \frac{\pi s s'}{2p}(p-q),$$

$$S^{\frac{1}{2},0}_{(r,s),(r',s')} = \begin{cases} (-1)^{\frac{1}{2}(r-s)} \dfrac{2}{\sqrt{pq}} \sin \dfrac{\pi r r'}{2q}(p-q) \sin \dfrac{\pi s s'}{2p}(p-q) & \textit{if } (r',s') = (\frac{q}{2},\frac{p}{2}), \\ (-1)^{\frac{1}{2}(r-s)} \dfrac{4}{\sqrt{pq}} \sin \dfrac{\pi r r'}{2q}(p-q) \sin \dfrac{\pi s s'}{2p}(p-q) & \textit{otherwise.} \end{cases}$$

6. Structure of $N(z, \frac{1}{24}z)$

In this section, we will study a property of a pre-Verma module $N(z,h)$, what is usually called a Verma module, satisfying $h = \frac{1}{24}z$. Indeed, we have

$$N(z,h) \cong \tilde{M}(z,h) \quad \text{if and only if} \quad h \neq \frac{1}{24}z.$$

Hence, this is the only non-trivial case that has to be studied.

Notice that $N(z,h)$ is $\mathbb{Z}$-graded via

$$N(z,h) = \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} N(z,h)_n, \quad N(z,h)_n := \{u \in N(z,h) \mid L_0.u = (h+n)u\}.$$

Let us recall the determinant formulae for pre-Verma modules. For $n \in \mathbb{Z}_{\geqslant 0}$ and $\tau \in \mathbb{Z}_2$, let $\widehat{\det}(z,h)^\tau_n$ be the Shapovalov type determinant defined for $N(z,h)^\tau_n$. Then, we have

Lemma 6.1 (E.g. Kac and Wakimoto [KW]).

$$\widehat{\det}(z,h)^\tau_n \propto \left(h - \frac{1}{24}z\right)^{\frac{1}{2}(p_0(n) + (-1)^{\delta_{\tau,\bar{0}}}\delta_{n,0})} \prod_{\substack{\alpha,\beta \in \mathbb{Z}_{\geqslant 0}, \\ 1 \leqslant \alpha\beta \leqslant 2n, \\ \alpha - \beta \in 1 + 2\mathbb{Z}_{\geqslant 0}}} \Phi_{\alpha,\beta;0}(z,h)^{p_0(n - \frac{1}{2}\alpha\beta)}.$$

Remark 6.1. Let $p, q \in 2\mathbb{Z}_{>0}$ satisfying $(\frac{p-q}{2}, q) = 1$, and set

$$z_{p,q} := z\left(\frac{p}{q}\right), \quad h_0 := -\frac{(p-q)^2}{8pq}.$$

Note that the weight $(z_{p,q}, h_0)$ belongs to Case 5^+, and $h_0 = h_{0;0}$ there. For any $k \in \mathbb{Z}_{>0}$, the curves $Z_{((k+\frac{1}{2})p,(k+\frac{1}{2})q)}$ (see Section A) are tangent to the line $h - \frac{1}{24}z = 0$ at the point $(z_{p,q}, h_0)$.

6.1. Subsingular vectors

In this subsection, we will study some properties of $N(z, \frac{1}{24}z)$ concerning subsingular vectors.

First, we will prove the following proposition:

Proposition 6.1 (Cf. Iohara and Koga [IK1]). *Let $n \in \mathbb{Z}_{>0}$. Then, we have the following*:

1. $\dim \{N(z, \frac{1}{24}z)_n^{\bar{0}}\}^{(\mathrm{Vir}_0)_+} = \dim \{N(z, \frac{1}{24}z)_n^{\bar{1}}\}^{(\mathrm{Vir}_0)_+} \leqslant 1.$
2. *If* $\dim \{N(z, \frac{1}{24}z)_n^{\sigma}\}^{(\mathrm{Vir}_0)_+} = 1$ *for* $\sigma \in \mathbb{Z}_2$, *then there is a subsingular vector in* $N(z, \frac{1}{24}z)_n^{\sigma}$ *which is not a singular vector.*

Since, for $\tau \in \mathbb{Z}_2$, there exists the following exact sequence:

$$\{0\} \to M_0\left(z, \frac{1}{24}z; \bar{1} - \tau\right) \xrightarrow{\iota} N\left(z, \frac{1}{24}z; \tau\right) \xrightarrow{\pi} M_0\left(z, \frac{1}{24}z; \tau\right) \to \{0\}, \tag{18}$$

the next lemma together with Proposition 3.2 implies Proposition 6.1.

For $z \in \mathbb{C}$, let $|z\rangle$ be a highest weight vector of $N(z, \frac{1}{24}z)$.

Lemma 6.2. *For $n \in \mathbb{Z}_{\geqslant 0}$, $\sigma \in \mathbb{Z}_2$, let X, Y be elements of $U((Vir_0)_-)$ satisfying*

$$(XG_0 + Y).|z\rangle \in \{N(z, \tfrac{1}{24}z)_n^{\sigma}\}^{(\mathrm{Vir}_0)_+}.$$

Then, we have $Y = 0$.

Proof. We define $n^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)}$ simply by replacing $|z,h\rangle$ with $|z\rangle$ in the definition of $m^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)}$, and define $\mathscr{B}_n^{\sigma;\delta}$ and $\mathscr{B}_n^{\sigma}$ by using $n^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)}$ in place of $m^{\delta}_{(i_k,\varepsilon_k),\dots,(i_1,\varepsilon_1)}$. Introducing the total order $<_{\sigma}$ on $\mathscr{B}_n^{\sigma}$ by the same manner, one can show an assertion similar to Lemma 3.2. (We omit the detail, since one just has to replace m's with n's.)

We first show that if $\dim \{N(z, \frac{1}{24}z)_n^{\sigma}\}^{(\mathrm{Vir}_0)_+} \neq 0$, then there exists $X' \in U((\mathrm{Vir}_0)_-)^{\bar{1}-\sigma} \setminus \{0\}$ such that $X'G_0|z\rangle \in \{N(z, \frac{1}{24}z)_n^{\sigma}\}^{(\mathrm{Vir}_0)_+}$. Let X, Y be elements of $U((\mathrm{Vir}_0)_-)$ such that $(XG_0 + Y)|z\rangle \in \{N(z, \frac{1}{24}z)_n^{\sigma}\}^{(\mathrm{Vir}_0)_+}$. If $Y = 0$, then the assertion is clear. If $Y \neq 0$, then it is easy to see that $(XG_0 + Y)G_0|z\rangle = YG_0|z\rangle$ is also a singular vector in $N(z, \frac{1}{24}z)_n^{\bar{1}-\sigma}$. Since the action of G_0 on $N(z, \frac{1}{24}z)_n$ is semi-simple and invertible, we see that the vector $[G_0, Y]G_0|z\rangle$ is a non-zero singular vector in $N(z, \frac{1}{24}z)^\sigma_n$. Let us denote $XG_0 + Y$ by S.

Expanding X and X' with respect to the basis $\mathscr{B}^\sigma_n$

$$X = c_X G_{-1}L_{-1}^{n-1} + \cdots, \quad X' = c_{X'} G_{-1}L_{-1}^{n-1} + \cdots,$$

we set

$$\tilde{S} := \tilde{X}G_0 + \tilde{Y} = S - \frac{c_X}{c_{X'}} X'G_0 \quad (\tilde{X}, \tilde{Y} \in U((\mathrm{Vir}_0)_-)).$$

It is clear that $\tilde{S}|z\rangle$ is an element of $\{N(z, \frac{1}{24}z)^\sigma_n\}^{(\mathrm{Vir}_0)_+}$. Let us show that $\tilde{S} = 0$. Expanding $\tilde{X}$ and $\tilde{Y}$ with respect to the basis $\mathscr{B}^\sigma_n$

$$\tilde{X} = c_{\tilde{X}} G_{-1}L_{-1}^{n-1} + d_{\tilde{X}} G_{-2}L_{-1}^{n-2} + \cdots,$$

$$\tilde{Y} = c_{\tilde{Y}} L_{-1}^{n} + d_{\tilde{Y}} L_{-2}L_{-1}^{n-2} + \cdots,$$

we remark that $c_{\tilde{X}} = 0$ by definition. Moreover, by direct calculation, we have

$$\text{The coeff. of } G_{-1}L_{-1}^{n-2}|z\rangle \text{ in } G_1\tilde{S}|z\rangle: \quad \tfrac{3}{2}n(n-1)c_{\tilde{Y}} + 2d_{\tilde{Y}} = 0.$$

$$\text{The coeff. of } L_{-1}^{n-1}|z\rangle \text{ in } L_1\tilde{S}|z\rangle: \quad \tfrac{1}{12}n(z + 6(n-1))c_{\tilde{Y}} + 3d_{\tilde{Y}} = 0.$$

Now, if $z = 0$ and $n = 1$, then one can check Lemma 6.2 by direct computation. For other cases, since one can easily check $z \neq 15(n-1)$, we see that $c_{\tilde{Y}} = 0$. By a statement similar to Lemma 3.2, we conclude that $\tilde{S} = 0$ which proves Lemma 6.2. □

Second, let us see the behavior of subsingular vectors. For $\tau, \sigma \in \mathbb{Z}_2$, let $\mathrm{pr}^{\tau,\sigma}$ be the projection $N(z, \frac{1}{24}z; \tau) \twoheadrightarrow M_0(z, \frac{1}{24}z; \sigma)$ with respect to the decomposition

$$N(z, \tfrac{1}{24}z; \tau) = M_0(z, \tfrac{1}{24}z; \bar{0}) \oplus M_0(z, \tfrac{1}{24}z; \bar{1})$$

as a vector space.

Proposition 6.2. *For* $l \in \mathbb{Z}_{>0}$, *let* u *be a non-zero vector in* $\pi^{-1}(\{M_0(z, \frac{1}{24}z; \tau)_{h_{l;0} - \frac{1}{24}z}\}^{(\mathrm{Vir}_0)_+})$ *satisfying* $\mathrm{pr}^{\tau, \bar{1}-\tau}(u) \in \iota(M_0(z, \frac{1}{24}z; \bar{1} - \tau)(l-1))$. *Then, we have*

$$(\mathrm{Vir}_0)_+ . u \subset M_0(z, \tfrac{1}{24}z; \bar{1} - \tau)(l-1).$$

Proof. Set $\mathscr{C} := \mathscr{C}^{\mathbb{Z}^2}_{(\mathfrak{g},\mathfrak{h})}$ and

$$M_0(z;\tau)[l] := M_0(z, \tfrac{1}{24}z; \tau)/M_0(z, \tfrac{1}{24}z; \tau)(l) \quad (\tau \in \mathbb{Z}_2, l \in \mathbb{Z}_{>0}).$$

It is enough to show the following formulae: For $k > l > 0$,

$$\mathrm{Ext}^1_{\mathscr{C}}(M_0(z, \tfrac{1}{24}z; \tau)(k), M_0(z; \bar{1} - \tau)[l]) \cong 0.$$

By Theorems 4.1, 4.2 and Lemma 4.3, we get the next short exact sequence

$$0 \to L_0(z, h_{l-1;0}; \bar{1} - \tau) \to M_0(z; \bar{1} - \tau)[l] \to M_0(z; \bar{1} - \tau)[l-1] \to 0 \quad (l \in \mathbb{Z}_{>1}),$$

from which we see that the following sequence is exact:

$$\begin{aligned}&\mathrm{Ext}^1_{\mathscr{C}}(M_0(z, \tfrac{1}{24}z; \tau)(k), L_0(z, h_{l-1;0}; \bar{1} - \tau))\\ &\quad\to \mathrm{Ext}^1_{\mathscr{C}}(M_0(z, \tfrac{1}{24}z; \tau)(k), M_0(z, \tfrac{1}{24}z; \bar{1} - \tau)[l])\\ &\quad\to \mathrm{Ext}^1_{\mathscr{C}}(M_0(z, \tfrac{1}{24}z; \tau)(k), M_0(z, \tfrac{1}{24}z; \bar{1} - \tau)[l-1]).\end{aligned}$$

Since, we have

$$M_0(z;\tau)[1] \cong L_0(z, \tfrac{1}{24}z; \tau),$$

we reduce the proof to show that

$$\mathrm{Ext}^1_{\mathscr{C}}(M_0(z, \tfrac{1}{24}z; \tau)(k), L_0(z, h_{l-1;0})) \cong 0. \tag{19}$$

Now, if we set $N_k := U(\mathrm{Vir}_0) V^{(k)}_{2c_P + c_Q}$, then we get the short exact sequence

$$0 \to N_k \to M_0(z, h_{k;0}) \to M_0(z, \tfrac{1}{24}z; \tau)(k) \to 0$$

which implies that the following sequence is exact:

$$\begin{aligned}&\mathrm{Hom}_{\mathscr{C}}(N_k, L_0(z, h_{l-1;0}))\\ &\quad\to \mathrm{Ext}^1_{\mathscr{C}}(M_0(z, \tfrac{1}{24}z; \tau)(k), L_0(z, h_{l-1;0}))\\ &\quad\to \mathrm{Ext}^1_{\mathscr{C}}(M_0(z, h_{k;0}), L_0(z, h_{l-1;0})).\end{aligned}$$

Observe that N_k is a highest weight Vir_0-module with highest weight $h_{k+1;0}$ by Lemma 4.3 from which it follows that

$$\mathrm{Hom}_{\mathscr{C}}(N_k, L_0(z, h_{l-1;0})) \cong 0.$$

Moreover, since we have

$$\mathrm{Ext}^1_{\mathscr{C}}(M_0(z,h_{k;0}),L_0(z,h_{l-1;0}))\cong 0$$

by Propositions 2.1, 2.2 and Theorem 5.1, we get (19). □

6.2. *Jantzen filtration of* $N(z,\frac{1}{24}z)$

In this subsection, we study Jantzen filtration of pre-Verma module whose highest weight belongs to case 5^+. Let us fix $z\in\mathbb{C}$ and $\tau\in\mathbb{Z}_2$.

As in Section 4.2, we define the Jantzen filtration of $N(z,\frac{1}{24}z;\tau)$

$$N(z,\tfrac{1}{24}z;\tau)=N(z,\tfrac{1}{24}z;\tau)(0)\supset N(z,\tfrac{1}{24}z;\tau)(1)\supset N(z,\tfrac{1}{24}z;\tau)(2)\supset\cdots$$

by taking a lift $(\tilde{z},\tilde{h})\in\mathscr{A}^2$ satisfying

1. $\tilde{z}-z=O(T)$, $\tilde{h}-h=O(T)$ $\tilde{h}-\frac{1}{24}\tilde{z}=O(T)$, or
2. $\tilde{z}-z=O(T)$, $\tilde{h}-h=O(T)$ $\tilde{h}-\frac{1}{24}\tilde{z}=O(T^2)$.

Then, by Remark 6.1 and Section A, we obtain the following character sum formula for each case:

$$\sum_{k>0}\mathrm{ch}\,N(z,\tfrac{1}{24}z;\tau)(k)=\mathrm{ch}\,M_0(z,h_{0;0})+\sum_{k>0}\mathrm{ch}\,M_0(z,h_{2k-1;0})\quad\text{for }1,\tag{20}$$

$$\sum_{k>0}\mathrm{ch}\,N(z,\tfrac{1}{24}z;\tau)(k)=2\,\mathrm{ch}\,M_0(z,h_{0;0})+2\sum_{k>0}\mathrm{ch}\,M_0(z,h_{2k-1;0})\quad\text{for }2.\tag{21}$$

Concerning the detailed structure of the Jantzen filtration, we have the following conjecture:

Conjecture 1. *Let* $k\in\mathbb{Z}_{>0}$.

1. *For* 1, *one has*

$$N(z,\tfrac{1}{24}z;\tau)(k)=D_k,$$

where D_k *is the submodule of* $N(z,\frac{1}{24}z;\tau)$ *which satisfies the next short exact sequence*:

$$0\to M_0(z,\tfrac{1}{24}z;\bar{1}-\tau)(2(k-1))\to D_k\to M_0(z,\tfrac{1}{24}z;\tau)(2k-1)\to 0.$$

2. *For* 2, *one has*

$$N(z,\tfrac{1}{24}z;\tau)(k)=\begin{cases}\pi^{-1}(M_0(z,\frac{1}{24}z;\tau)(1)) & k=1,\\ E_k & k>1,\end{cases}$$

where E_k *is the submodule of* $N(z, \frac{1}{24}z; \tau)$ *which satisfies the next short exact sequence*:

$$0 \to M_0(z, \tfrac{1}{24}z; \bar{1} - \tau)(k-2) \to E_k \to M_0(z, \tfrac{1}{24}z; \tau)(k) \to 0.$$

Appendix A. Lattice points (α_k, β_k) on the Line $l_{z,h}$

In this section, we will supplement some data used in Section 3.5. Namely, we will list up the following data:

1. The choice of the line $l_{z,h}$ we made.
2. The list of lattice points (α_k, β_k).
3. The relations among the weights under consideration and the lattice points.

In particular, we will also indicate the type of a weight if it is not of Type 1.

Class $\mathbf{R}^+$

Case 1^+: $(r,s) \in \{(a,b) \in \mathbb{Z}^2 | \begin{smallmatrix} 0<a<q \\ 0<b<p \end{smallmatrix}, pa + qb \leqslant pq\}$, $h = h_{i;\varepsilon}$ $(i \in \mathbb{Z})$, $\delta \in \{0,1\}$, $(-1)^{i\delta}(rp - sq) < 0$: $p\alpha - q\beta = (-1)^{i\delta}(2\lceil \frac{i}{2} \rceil pq + rp - (-1)^i sq)$,

$$(\alpha_1, \beta_1) = (-1)^{i\delta}(r, (-1)^i s) + \left\{ \left(\left\lceil \frac{|i|+1}{2} \right\rceil + (-1)^{i\delta} \left\lceil \frac{i}{2} \right\rceil \right) q, \left(\left\lceil \frac{|i|+1}{2} \right\rceil - (-1)^{i\delta} \left\lceil \frac{i}{2} \right\rceil \right) p \right\},$$

$$(\alpha_k, \beta_k) = \begin{cases} (\alpha_1, \beta_1) + \frac{1}{2}(k-1)(q,p) & k \equiv 1 \bmod 2, \\ (\alpha_1, \beta_1) - (\frac{1}{2}k + |i|)(q,p) & k \equiv 0 \bmod 2, \end{cases}$$

$$h_{i;\varepsilon} + \tfrac{1}{2}\alpha_k\beta_k = h_{-(-1)^{i\delta+k}(|i| + 2\lceil \frac{k-1}{2} \rceil + 1);\varepsilon}.$$

Case 2^+: $r = q$ and $0 < s < p$, $h = h_{i;\varepsilon}(i \in \mathbb{Z}_{\geqslant 0})$:
$p\alpha - q\beta = (2\lceil \frac{i}{2} \rceil + 1)pq - (-1)^i sq$,

$$(\alpha_1, \beta_1) = \left((i+1)q, \frac{1-(-1)^i}{2} p + (-1)^i s \right),$$

$$(\alpha_k, \beta_k) = \begin{cases} (\alpha_1, \beta_1) + \frac{1}{2}(k-1)(q,p) & k \equiv 1 \bmod 2, \\ (\alpha_1, \beta_1) - (\frac{1}{2}k + i + 1)(q,p) & k \equiv 0 \bmod 2, \end{cases}$$

$$h_{i;\varepsilon} + \tfrac{1}{2}\alpha_k\beta_k = h_{i+k;\varepsilon}.$$

Case 3^+: $0<r<q$ and $s=p$, $h=h_{(-1)^{i-1}i;\varepsilon}(i\in\mathbb{Z}_{\geqslant 0})$:
$p\alpha-q\beta=-(2\lceil \frac{i}{2}\rceil+1)pq+(-1)^i rp$,

$$(\alpha_1,\beta_1)=\left(\frac{1-(-1)^i}{2}q+(-1)^i r,(i+1)p\right),$$

$$(\alpha_k,\beta_k)=\begin{cases}(\alpha_1,\beta_1)+\frac{1}{2}(k-1)(q,p) & k\equiv 1 \bmod 2,\\ (\alpha_1,\beta_1)-(\frac{1}{2}k+i+1)(q,p) & k\equiv 0 \bmod 2,\end{cases}$$

$$h_{(-1)^{i-1}i;\varepsilon}+\frac{1}{2}\alpha_k\beta_k=h_{(-1)^{i+k-1}(i+k);\varepsilon}.$$

Case 4.1^+ (Neveu–Schwarz only): $r=q$ and $s=p$, $h=h_{2i;\frac{1}{2}}$ $(i\in\mathbb{Z}_{\geqslant 0})$,

1. $h=h_{2i;\frac{1}{2}}$ $(i>0)$ and $(p,q)\neq(1,1)$: $p\alpha-q\beta=2ipq$,

$$(\alpha_1,\beta_1)=((2i+1)q,p),$$

$$(\alpha_k,\beta_k)=\begin{cases}(\alpha_1,\beta_1)+\frac{1}{2}(k-1)(q,p) & k\equiv 1 \bmod 2,\\ (\alpha_1,\beta_1)-(\frac{1}{2}k+i+1)(q,p) & k\equiv 0 \bmod 2,\end{cases}$$

$$h_{2i;\frac{1}{2}}+\frac{1}{2}\alpha_k\beta_k=h_{2i+k;\frac{1}{2}}.$$

2. $h=h_{0;\frac{1}{2}}$ and $(p,q)\neq(1,1)$ (Type 2): $p\alpha-q\beta=0$,

$$(\alpha_k,\beta_k)=k(q,p),$$

$$h_{0;\frac{1}{2}}+\frac{1}{2}\alpha_k\beta_k=h_{2k;\frac{1}{2}}.$$

3. $h=h_{2i;\frac{1}{2}}$ and $(p,q)=(1,1)$ (Type 2): $\alpha-\beta=2i$,

$$(\alpha_k,\beta_k)=(2i+1,1)+(k-1)(1,1),$$

$$h_{2i;\frac{1}{2}}+\frac{1}{2}\alpha_k\beta_k=h_{2i+2k;\frac{1}{2}}.$$

Case 4.2$^+$: $r=0$ and $s=p$, $h=h_{-2i;\varepsilon}$ $(i\in\mathbb{Z}_{\geqslant 0})$,

1. $h=h_{-2i;\varepsilon}$ and $(p,q)\neq(1,1)$: $p\alpha-q\beta=(2i+1)pq$,

$$(\alpha_1,\beta_1)=((2i+2)q,p),$$

$$(\alpha_k,\beta_k)=\begin{cases}(\alpha_1,\beta_1)+\frac{1}{2}(k-1)(q,p) & k\equiv 1 \bmod 2,\\ (\alpha_1,\beta_1)-(\frac{1}{2}k+2i+2)(q,p) & k\equiv 0 \bmod 2,\end{cases}$$

$$h_{-2i;\varepsilon}+\tfrac{1}{2}\alpha_k\beta_k=h_{-2(i+\lceil\frac{k+1}{2}\rceil);\varepsilon}.$$

2. (Ramond only):
   $h=h_{-2i;0}$ and $(p,q)=(1,1)$ (Type 2): $\alpha-\beta=2i+1$,

$$(\alpha_k,\beta_k)=(2i+2,1)+(k-1)(1,1),$$

$$h_{-2i;0}+\tfrac{1}{2}\alpha_k\beta_k=h_{-2i-2k;0}.$$

Case 5$^+$ (Ramond only): $r=\frac{q}{2}$ and $s=\frac{p}{2}$, $h=h_{i;0}$ $(i\in\mathbb{Z}_{\geqslant 0})$,

1. $h=h_{i;0}$ $(i>0)$: $p\alpha-q\beta=ipq$,

$$(\alpha_1,\beta_1)=((i+\tfrac{1}{2})q,(i+\tfrac{1}{2})p),$$

$$(\alpha_k,\beta_k)=\begin{cases}(\alpha_1,\beta_1)+\frac{1}{2}(k-1)(q,p) & k\equiv 1 \bmod 2,\\ (\alpha_1,\beta_1)-(\frac{1}{2}k+i)(q,p) & k\equiv 0 \bmod 2,\end{cases}$$

$$h_{i;0}+\tfrac{1}{2}\alpha_k\beta_k=h_{i+2\lceil\frac{k-1}{2}\rceil+1;0}.$$

2. $h=h_{0;0}$ (Type 2): $p\alpha-q\beta=0$,

$$(\alpha_k,\beta_k)=(k-\tfrac{1}{2})(q,p),$$

$$h_{0;0}+\tfrac{1}{2}\alpha_k\beta_k=h_{2k-1;0}.$$

Class R$^-$

Case 1$^-$: $(r,s)\in\{(a,-b)\in\mathbb{Z}^2|\,{}^{0<a<p}_{0<b<q},pa+qb\leqslant pq\}$,

$h=h_{i;\varepsilon}$ $(i\in\mathbb{Z}\setminus\{0\})$, $\delta\in\{0,1\}$, $(-1)^\delta(rp+sq)>0$:

$p\alpha + q\beta = 2|\lceil \frac{i}{2} \rceil| pq + (\operatorname{sgn} i)(rp + (-1)^i sq)$,

$$(\alpha_1, \beta_1) = \left((\operatorname{sgn} i) r + \left| \left\lceil \frac{i}{2} \right\rceil \right| q, (\operatorname{sgn} i)(-1)^i s + \left| \left\lceil \frac{i}{2} \right\rceil \right| p \right),$$

$$(\alpha_k, \beta_k) = \begin{cases} (\alpha_1, \beta_1) + \frac{1}{2}(\operatorname{sgn} i)(-1)^{i\delta}(k-1)(q, -p) & k \equiv 1 \bmod 2, \\ (\alpha_1, \beta_1) - \frac{1}{2}(\operatorname{sgn} i)(-1)^{i\delta} k(q, -p) & k \equiv 0 \bmod 2, \end{cases}$$

$$h_{i;\varepsilon} + \tfrac{1}{2}\alpha_k \beta_k = h_{(-1)^{i\delta+k-1}(k - \frac{1+(-1)^{i+k}}{2});\varepsilon} \quad 1 \leqslant k \leqslant |i|.$$

Case 2^-: $r = q$ and $0 < -s < p$, $h = h_{i;\varepsilon}$ $(i \in \mathbb{Z}_{>0})$:
$p\alpha + q\beta = (2\lceil \frac{i}{2} \rceil + 1)pq + (-1)^i sq$,

$$(\alpha_1, \beta_1) = \left(\left\lceil \frac{i+1}{2} \right\rceil q, \left\lceil \frac{i+1}{2} \right\rceil p + (-1)^i (p + s) \right),$$

$$(\alpha_k, \beta_k) = \begin{cases} (\alpha_1, \beta_1) - \frac{1}{2}(-1)^i (k-1)(q, -p) & k \equiv 1 \bmod 2, \\ (\alpha_1, \beta_1) + \frac{1}{2}(-1)^i k(q, -p) & k \equiv 0 \bmod 2, \end{cases}$$

$$h_{i;\varepsilon} + \tfrac{1}{2}\alpha_k \beta_k = h_{k-1;\varepsilon} \quad 1 \leqslant k \leqslant i.$$

Case 3^-: $0 < r < q$ and $-s = p$, $h = h_{(-1)^{i-1}i;\varepsilon}$ $(i \in \mathbb{Z}_{>0})$:
$p\alpha + q\beta = (2\lceil \frac{i}{2} \rceil + 1)pq - (-1)^i rp$,

$$(\alpha_1, \beta_1) = \left(\left\lceil \frac{i+1}{2} \right\rceil q + (-1)^i (q - r), \left\lceil \frac{i+1}{2} \right\rceil p \right),$$

$$(\alpha_k, \beta_k) = \begin{cases} (\alpha_1, \beta_1) + \frac{1}{2}(-1)^i (k-1)(q, -p) & k \equiv 1 \bmod 2, \\ (\alpha_1, \beta_1) - \frac{1}{2}(-1)^i k(q, -p) & k \equiv 0 \bmod 2, \end{cases}$$

$$h_{(-1)^{i-1}i;\varepsilon} + \tfrac{1}{2}\alpha_k \beta_k = h_{(-1)^k (k-1);\varepsilon} \quad 1 \leqslant k \leqslant i.$$

Case 4.1^- (Neveu–Schwarz only): $r = q$ and $-s = p$, $h = h_{2i;\frac{1}{2}}$ $(i \in \mathbb{Z}_{>0})$,

1. $h = h_{2i;\frac{1}{2}}$ and $(p,q) \neq (1,1)$: $p\alpha + q\beta = 2ipq$,

$$(\alpha_1, \beta_1) = i(q,p),$$

$$(\alpha_k, \beta_k) = \begin{cases} (\alpha_1, \beta_1) - \frac{1}{2}(k-1)(q,-p) & k \equiv 1 \bmod 2, \\ (\alpha_1, \beta_1) + \frac{1}{2}k(q,-p) & k \equiv 0 \bmod 2, \end{cases}$$

$$h_{2i;\frac{1}{2}} + \tfrac{1}{2}\alpha_k\beta_k = h_{i+2\lceil\frac{k-1}{2}\rceil+1;0}.$$

2. $h = h_{2i;\frac{1}{2}}$ and $(p,q) = (1,1)$ (Type 3): $\alpha + \beta = 2i$,

$$(\alpha_k, \beta_k) = i(1,1) + (k-1)(1,-1),$$

$$h_{2i;\frac{1}{2}} + \tfrac{1}{2}\alpha_k\beta_k = h_{2k-2;\frac{1}{2}} \quad 1 \leqslant k \leqslant i.$$

Case 4.2$^-$: $r = 0$ and $-s = p$, $h = h_{-2i;\varepsilon}$ $(i \in \mathbb{Z}_{>0})$,

1. $h = h_{-2i;\varepsilon}$ and $(p,q) \neq (1,1)$: $p\alpha + q\beta = (2i+1)pq$,

$$(\alpha_1, \beta_1) = (iq, (i+1)p),$$

$$(\alpha_k, \beta_k) = \begin{cases} (\alpha_1, \beta_1) - \frac{1}{2}(k-1)(q,-p) & k \equiv 1 \bmod 2, \\ (\alpha_1, \beta_1) + \frac{1}{2}k(q,-p) & k \equiv 0 \bmod 2, \end{cases}$$

$$h_{-2i;\varepsilon} + \tfrac{1}{2}\alpha_k\beta_k = h_{-2\lceil\frac{k-1}{2}\rceil;\varepsilon} \quad 1 \leqslant k \leqslant 2i.$$

2. (Ramond only):
   $h = h_{-2i;0}$ and $(p,q) = (1,1)$ (Type 3): $\alpha + \beta = 2i+1$.

$$(\alpha_k, \beta_k) = (i+1, i) + (k-1)(1,-1),$$

$$h_{-2i;0} + \tfrac{1}{2}\alpha_k\beta_k = h_{-2k+2;0} \quad 1 \leqslant k \leqslant i.$$

Case 5$^-$ (Ramond only): $r = \frac{q}{2}$ and $-s = \frac{p}{2}$, $h = h_{i;0}$ $(i \in \mathbb{Z}_{>0})$:

$p\alpha + q\beta = ipq,$

$$(\alpha_1, \beta_1) = \left(\left(\left\lceil \frac{i-1}{2} \right\rceil + \frac{1}{2}\right)q, \left(\left\lceil \frac{i}{2} \right\rceil + \frac{1}{2}\right)p\right),$$

$$(\alpha_k, \beta_k) = \begin{cases} (\alpha_1, \beta_1) - \frac{1}{2}(k-1)(q, -p) & k \equiv 1 \bmod 2, \\ (\alpha_1, \beta_1) + \frac{1}{2}k(q, -p) & k \equiv 0 \bmod 2, \end{cases}$$

$$h_{i;0} + \tfrac{1}{2}\alpha_k\beta_k = h_{2\lceil \frac{k}{2} - \frac{1+(-1)^i}{4} \rceil + \frac{1+(-1)^i}{2};0} \quad 1 \leqslant k \leqslant i.$$

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