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NOTE

A Result in Dual Ramsey Theory

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We present a result which is obtained by combining a result of Carlson with the Finitary Dual Ramsey Theorem of Graham–Rothschild. © 2002 Elsevier Science (USA)

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We start by introducing some notation.

We conform to the usual practice of identifying the least infinite ordinal ω with the set of non-negative integers.

Given $\alpha, \beta \leqslant \omega$, a *partition of* α *into* β *blocks* is an onto function $X : \alpha \to \beta$ such that $\min(X^{-1}(\{n\})) < \min(X^{-1}(\{m\}))$ whenever $n < m < \beta$. Thus, the blocks of X are ordered as their *leaders* (i.e., their least elements).

The *leader function* $\ell : (\alpha)^\beta \times \beta \to \alpha$ is defined by $\ell(X, m) := \min(X^{-1}(\{m\}))$. Hence, the function $m \mapsto \ell(X, m)$ enumerates the leaders of X in increasing order.

Given $X \in (\alpha)^\beta$ and $Y \in (\alpha)^\gamma$, where $\alpha, \beta, \gamma \leqslant \omega$, we let $Y \leqslant X$ if Y is *coarser* than X, i.e., each block of Y is a union of blocks of X.

Given $\alpha, \beta, \gamma \leqslant \omega$ and $X \in (\alpha)^\beta$, $(X)^\gamma := \{Y \in (\alpha)^\gamma : Y \leqslant X\}$.

Given $\alpha, \beta \leqslant \omega$ and $k < \omega$, $(\alpha)^\beta_k$ denotes the set of all $X \in (\alpha)^\beta$ such that

(a) $X^{-1}(\{n\})$ is finite if $k \leqslant n < \beta$, and

(b) $\max(X^{-1}(\{n\})) < \ell(X, n+1)$ if $k \leqslant n$ and $n + 1 < \beta$.

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Given $\alpha, \beta, \gamma \leqslant \omega$, $X \in (\alpha)^{\beta}$ and $k, m < \omega$ such that $k \leqslant \gamma$ and $m \leqslant \beta$, $(k, m, X)^{\gamma}$ is the set of all $Y \in (X)^{\gamma}$ such that

$$\{\ell(Y, i) : i < k\} \subseteq \{\ell(X, j) : j < m\}.$$

Note that $(0, m, X)^{\gamma} = (1, m, X)^{\gamma} = (X)^{\gamma}$ for all $m \leqslant \beta$.

The *amalgamation function* $\mathscr{A}$ is defined as follows: Given $X \in (\omega)^{\omega}$ and $t \in (p)^m$, where $0 < m \leqslant p < \omega$, $\mathscr{A}(t, X)$ is the partition of ω whose blocks are

$$\bigcup_{i \in t^{-1}(\{0\})} X^{-1}(\{i\}), \ldots, \bigcup_{i \in t^{-1}(\{m-1\})} X^{-1}(\{i\}), X^{-1}(\{p\}), X^{-1}(\{p+1\}), \ldots \ .$$

For $t \in (p)^m$, where $m \leqslant p < \omega$, let $O_t \coloneqq \{X \in (\omega)^{\omega} : X \restriction p = t\}$. We topologize $(\omega)^{\omega}$ by taking as basic open sets $\emptyset$ and O_t for $t \in \bigcup_{m \leqslant p < \omega} (p)^m$.

A function $F : (\omega)^{\omega} \to r$, where $1 \leqslant r < \omega$, is *clopen* if $F^{-1}(\{i\})$ is a clopen subset of $(\omega)^{\omega}$ for each $i < r$.

Our starting point is the following immediate consequence of the Dual Ellentuck Theorem [1, Theorem 4.1] of Carlson–Simpson.

Proposition 1. *Given $X \in (\omega)^{\omega}$ and a clopen $F : (\omega)^{\omega} \to r$, where $1 \leqslant r < \omega$, there is $Y \in (X)^{\omega}$ such that F is constant on $(Y)^{\omega}$.*

Even if every block of X is finite, there may not be any homogeneous Y having infinitely many finite blocks.

Proposition 2. *There is a clopen $F : (\omega)^{\omega} \to 2$ with the property that there is no $Y \in (\omega)^{\omega}$ such that F is constant on $(Y)^{\omega}$ and Y has infinitely many finite blocks.*

Proof. Define $F : (\omega)^{\omega} \to 2$ by stipulating that $F(X) = 0$ if and only if $X^{-1}(\{1\}) \cap \ell(X, 3) \subseteq \ell(X, 2)$. Obviously, F is clopen. Now suppose that there is $Y \in (\omega)^{\omega}$ such that Y has infinitely many finite blocks and F is constant on $(Y)^{\omega}$. Pick $Z \in (\omega)^{\omega}_1$ with $Z \leqslant Y$. Then F is constant on $(Z)^{\omega}$, which is clearly impossible. ∎

Carlson established a "specialized" version (Theorem 6.9 of [1], which follows from Theorem 2 of [2]) of the Dual Ellentuck Theorem that deals with partitions of ω having finitely many infinite blocks. Carlson's result immediately implies the following.

Proposition 3. *Given $k < \omega$, $X \in (\omega)^{\omega}_k$ and a clopen $F : (\omega)^{\omega} \to r$, where $1 \leqslant r < \omega$, there is $Y \in (\omega)^{\omega}_k \cap (k, k, X)^{\omega}$ such that F is constant on $(k, k, Y)^{\omega}$.*

The purpose of this paper is to present the combinatorial result which is obtained by combining Proposition 3 with the Finitary Dual Ramsey Theorem of Graham–Rothschild [3]. This last reads as follows.

PROPOSITION 4. *Suppose that* $1 \leqslant k \leqslant m < \omega$ *and* $1 \leqslant r < \omega$. *Then there is* $p < \omega$ *such that* $p \geqslant m$ *and the following holds: Given* $f : (p)^k \to r$, *there is* $s \in (p)^m$ *such that* f *is constant on* $(s)^k$.

We now state our result.

THEOREM. *Given* $1 < k < m < \omega$, $X \in (\omega)^\omega_k$ *and a clopen* $F : (\omega)^\omega \to r$, *where* $1 \leqslant r < \omega$, *there is* $Y \in (\omega)^\omega_m \cap (X)^\omega$ *such that* F *is constant on* $(k, m, Y)^\omega$.

Proof. Using Proposition 4, select $p \geqslant m$ so that every $f : (p)^k \to r$ is constant on $(s)^k$ for some $s \in (p)^m$. First we define $g : \bigcup_{i \leqslant p-k} (k-1+i)^{k-1} \to r$ and $Y_0, Y_1, \ldots, Y_{p-k}$ so that

(0) $Y_0 \in (\omega)^\omega_k \cap (k, k, X)^\omega$ and F takes the constant value $g(u)$ on $(k, k, \mathscr{A}(u, Y_0))^\omega$, where u is the unique element of $(k-1)^{k-1}$ (hence, $\mathscr{A}(u, Y_0) = Y_0$).

(1) $Y_1 \in (\omega)^\omega_{k+1} \cap (k+1, k+1, Y_0)^\omega$ and F takes the constant value $g(t)$ on $(k, k, \mathscr{A}(t, Y_1))^\omega$ for every $t \in (k)^{k-1}$.

(2) $Y_2 \in (\omega)^\omega_{k+2} \cap (k+2, k+2, Y_1)^\omega$ and F takes the constant value $g(t)$ on $(k, k, \mathscr{A}(t, Y_2))^\omega$ for every $t \in (k+1)^{k-1}$.

$\vdots$

$(p-k)$ $Y_{p-k} \in (\omega)^\omega_p \cap (p, p, Y_{p-k-1})^\omega$ and F takes the constant value $g(t)$ on $(k, k, \mathscr{A}(t, Y_{p-k}))^\omega$ for every $t \in (p-1)^{k-1}$.

For example, to define Y_3 and $g \restriction (k+2)^{k-1}$, proceed as follows. Let $t_0, t_1, \ldots, t_q$ be an enumeration of the elements of $(k+2)^{k-1}$. Applying Proposition 3 repeatedly, define T_j, Z_j and c_j for $j \leqslant q$ so that

(i) $T_j \in (\omega)^\omega_k$.

(ii) If $j = 0$, $T_j \in (k, k, \mathscr{A}(t_j, Y_2))^\omega$ and $Z_j \in (k+3, k+3, Y_2)^\omega$.

(iii) If $j > 0$, $T_j \in (k, k, \mathscr{A}(t_j, Z_{j-1}))^\omega$ and $Z_j \in (k+3, k+3, Z_{j-1})^\omega$.

(iv) F takes the constant value c_j on $(k, k, T_j)^\omega$.

(v) $\mathscr{A}(t_j, Z_j) = T_j$.

Then set $Y_3 = Z_q$ and $g(t_j) = c_j$ for every $j \leqslant q$.

Define $f : (p)^k \to r$ by $f(w) = g(w \restriction \ell(w, k-1))$. Set $W = Y_{p-k}$. Obviously, $W \in (\omega)_p^\omega \cap (X)^\omega$. Moreover, F takes the constant value $f(w)$ on $(k, k, \mathscr{A}(w \restriction \ell(w, k-1), W)^\omega$ for every $w \in (p)^k$. Let $s \in (p)^m$ be such that f is constant on $(s)^k$. Then $Y = \mathscr{A}(s, W)$ is as desired. ∎

The referee pointed out that the theorem and similar results can be derived from Theorems 10 and 11 of [2].

The theorem is optimal in the following sense:

PROPOSITION 5. *Suppose that* $1 < k < m < \omega$. *Then there is* $F : (\omega)^\omega \to 2$ *such that* $F^{-1}(\{0\})$ *is open and there is no* $Y \in (\omega)_m^\omega$ *with the property that* F *is constant on* $(k, m, Y)^\omega$.

Proof. Let $F(Y) = 0$ exactly when $Y(m) \notin \ell(Y, m+1)$. ∎

The theorem has the following finitary version, which is proved by arguing as for 3.2 in [1].

PROPOSITION 6. *Suppose that* $n \leqslant q \leqslant m < \omega$, $1 \leqslant k \leqslant m$, $n \leqslant k$ *and* $1 \leqslant r < \omega$. *Then there is* $p < \omega$ *such that* $p \geqslant m$ *and the following holds*: *Given* $f : (p)^k \to r$, *there is* $s \in (p)_q^m$ *such that* f *is constant on* $(n, q, s)^k$.

Proof. Assume that for every $p \geqslant m$ there is $f_p : (p)^k \to r$ such that for every $s \in (p)_q^m$, f_p is not constant on $(n, q, s)^k$. Define $F : (\omega)^\omega \to r$ by stipulating that $F(T) = f_{\ell(T,k)}(T \restriction \ell(T, k))$. Using the theorem (for $1 < n < q$) or Proposition 3 (otherwise), we find $Y \in (\omega)_q^\omega$ such that F is constant on $(n, q, Y)^\omega$. Set $p = \ell(Y, m)$ and $s = Y \restriction m$. Then $p \geqslant m$ and $s \in (p)_q^m$. Moreover, f_p is constant on $(n, q, s)^k$. Contradiction! ∎

When $n \in \{0, 1\}$ and $q \in \{m-1, m\}$, Proposition 6 reduces to the Finitary Dual Ramsey Theorem. When $n = k$ and $q \in \{m-1, m\}$, it reduces to the n-Parameter Set Theorem of Graham–Rothschild [3], which generalizes the Finitary Dual Ramsey Theorem.

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