Asymptotically minimax bias estimation of the correlation coefficient for bivariate independent component distributions

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For bivariate independent component distributions, the asymptotic bias of the correlation coefficient estimators based on principal component variances is derived. This result allows to design an asymptotically minimax bias (in the Huber sense) estimator of the correlation coefficient, namely, the trimmed correlation coefficient, for contaminated bivariate normal distributions. The limit cases of this estimator are the sample, median and MAD correlation coefficients, the last two simultaneously being the most $B$- and $V$-robust estimators. In contaminated normal models, the proposed estimators dominate both in bias and in efficiency over the sample correlation coefficient on small and large samples.

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1. Introduction

Consider the problem of estimation of the correlation coefficient $\rho$ between random variables $X$ and $Y$ from the observed sample $(x_1, y_1), \ldots, (x_n, y_n)$. Its classical estimator is given by the sample correlation coefficient

$$r = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\left(\sum_{i=1}^{n} (x_i - \bar{x})^2 \sum_{i=1}^{n} (y_i - \bar{y})^2\right)^{1/2}},$$

where $\bar{x} = n^{-1} \sum_{i=1}^{n} x_i$ and $\bar{y} = n^{-1} \sum_{i=1}^{n} y_i$ are the sample means.

On the one hand, the sample correlation coefficient $r$ is a statistical counterpart of the correlation coefficient $\rho = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X) \text{var}(Y)}}$, where $\text{var}(X)$ and $\text{var}(Y)$ are the variances and $\text{cov}(X, Y)$ the covariance of $X$ and $Y$. On the other hand, it is the efficient maximum likelihood estimator of $\rho$ for the bivariate normal distribution $N(x; \mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$, where the parameters $\mu_1$ and $\mu_2$ are the means, $\sigma_1$ and $\sigma_2$ are the standard deviations of $X$ and $Y$, respectively.

In [5], it is shown that $r$ is extremely sensitive to the presence of outliers in the data, and hence it is necessary to use its robust counterparts. For instance, in the gross error model [17]

$$f(x, y) = (1 - \varepsilon)N(x, y; 0, 0, 1, 1, \rho) + \varepsilon N(x, y; \mu_1, \mu_2, k, k, \rho'), \quad 0 \leq \varepsilon < 1,$$

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for estimating the correlation coefficient $\rho = 0.9$ of the main bulk of the data under the contamination with $\epsilon = 0.1$, $\mu_1 = 0, \mu_2 = 0, k = 3$ and $\rho' = -0.99$, the expectation of $r$ is $E(r) = -0.055$, thus even the sign of $r$ is wrong.

At present there appear two principal approaches to robust estimation, the minimax method of quantitative robustness [7] and the method of qualitative robustness based on influence functions [6]. According to the first of these methods, the least informative (favorable) distribution minimizing Fisher information over a given class of distributions is determined with the subsequent use of the maximum likelihood estimator for this distribution. With the second method, an estimator with an assigned influence function whose type of behavior determines the qualitative robustness properties of the estimation procedure, e.g. its sensitivity to large outliers in the data, their rounding off, etc., is designed.

In the pioneer works on robust estimation of the correlation coefficient, [5,4], the approach based on influence functions is exploited to propose several robust measures of correlation. In [7], robust estimation of correlation is mostly treated basing on the minimax approach. In particular, it is shown that the quadrant (sign) correlation coefficient, i.e., the sample correlation coefficient between the signs of observation deviations from their medians [1], is asymptotically minimax with respect to bias at the mixture $F = (1 - \epsilon)G + \epsilon H$, where $G$ and $H$ are centrosymmetric distributions in $\mathbb{R}^2$ (a density $f(x, y)$ is centrosymmetric with respect to the origin $(0, 0)$ if the relation $f(-x, -y) = f(x, y)$ holds for all $(x, y) \in \mathbb{R}^2$).

Robust estimators of the correlation coefficient can be roughly classified as follows [15]: (i) direct robust counterparts of the sample correlation coefficient; (ii) nonparametric measures of correlation; (iii) techniques built on robust estimation of the principal variable variances (dubbed as classical measures of correlation to the rest of observations.

In [5,4,15], the performance of the typical estimators from those groups was thoroughly examined in gross error model (2), and it was found out that robust estimators based on the principal variable variances (dubbed as $r^*$ in [4]) e.g. the trimmed, median and MAD correlation coefficients, proved to be highly robust.

In particular, Shevlyakov and Vilchekvskii [16] establish that the trimmed, median and MAD correlation coefficients are the minimax variance (in the Huber sense) estimators of the correlation coefficient for $\epsilon$-contaminated bivariate normal distributions.

Monte Carlo studies of various robust estimators of correlation demonstrate that bias sometimes seems to be a more informative characteristic of estimation quality than variance [4,15].

In this paper, we extend our former results on the minimax variance robust estimation of correlation [16] onto the minimax bias robust estimation showing that the trimmed, median and MAD correlation coefficients are also the minimax bias estimators of the correlation coefficient for $\epsilon$-contaminated bivariate normal distributions. Moreover, the median and MAD correlation coefficient are simultaneously the most $B$- and $V$-robust estimators of the correlation coefficient—this result was announced in [13].

2. Main result

2.1. Preliminaries: estimators based on principal variable variances and bivariate independent component distributions

For the sake of clarity, first we briefly enlist basic results from [16] necessary for our further constructions. Consider the identity for the correlation coefficient [5]

$$\rho = (\text{var}(U) - \text{var}(V))/(\text{var}(U) + \text{var}(V)), \quad (3)$$

where $U = (X/\sigma_1 + Y/\sigma_2)/\sqrt{2}, V = (X/\sigma_1 - Y/\sigma_2)/\sqrt{2}$ are the principal variables such that $\text{cov}(U, V) = 0, \text{var}(U) = 1 + \rho, \text{var}(V) = 1 - \rho$.

Basing on this identity, we introduce the class of estimators for the correlation coefficient $\rho$

$$\hat{\rho} = (\widehat{S_U^2} - \widehat{S_V^2})/(\widehat{S_U^2} + \widehat{S_V^2}), \quad (4)$$

where $\widehat{S}$ is the sample estimator of a scale functional $S_X$: $\widehat{S_{aX+b}} = |a|\widehat{S_X}$.

Below, we focus on Huber’s $M$-estimators of scale for $S_X$ implicitly defined by the equation $\int \chi (x/S_X) \ dF(x) = 0$, where $\chi$ is a score function, usually even $\chi (-x) = \chi (x)$ [7]. The following choices of a score function $\chi$ with the corresponding $M$-estimators $\widehat{S}$ of scale and the estimators $\hat{\rho}$ of correlation given by formula (4) are of our particular interest.

Example 1. An $M$-estimator of scale with $\chi (x) = x^2 - 1$ is the standard deviation yielding the structure of the sample correlation coefficient $r$ in formula (4).

Example 2. An $M$-estimator of scale with $\chi (x) = \text{sign}(|x| - 1)$ is the median absolute deviation $\widehat{S} = \text{MAD} x = \text{med} |x| - \text{med} x|$ yielding the following asymptotically equivalent estimators [15]: the median correlation coefficient

$$r_{med} = (\text{med}^2 |u| - \text{med}^2 |v|)/(|\text{med}^2 |u| + \text{med}^2 |v|) \quad (5)$$

and the MAD correlation coefficient

$$r_{MAD} = (\text{MAD}^2 u - \text{MAD}^2 v)/(\text{MAD}^2 u + \text{MAD}^2 v), \quad (6)$$
where $u$ and $v$ are the robust principal variables

$$
u = \frac{x - \text{med } x}{\sqrt{2 \text{ MAD } x}} + \frac{y - \text{med } y}{\sqrt{2 \text{ MAD } y}}, \quad \nu = \frac{x - \text{med } x}{\sqrt{2 \text{ MAD } x}} - \frac{y - \text{med } y}{\sqrt{2 \text{ MAD } y}}. \quad (7)$$

**Example 3.** The minimax variance $M$-estimator of scale for the $\varepsilon$-contaminated normal distribution is defined by the following choice of a score function

$$\chi(x) = \begin{cases} x^2 - 1 & (|x| < x_0), \\ x^2 - 1 & (x_0 \leq |x| \leq x_1), \\ x^1 - 1 & (|x| > x_1). \end{cases}$$

The trimming thresholds $x_0(\varepsilon)$ and $x_1(\varepsilon)$ depend on the contamination parameter $\varepsilon$, so that the obtained estimator of scale is asymptotically equivalent to the trimmed standard deviation (see [7, pp. 120–122]). The limit cases of this estimator are the standard deviation with $\varepsilon = 0$ and the median absolute deviation as $\varepsilon \to 1$. In this case, the corresponding estimator of $\rho$ is the trimmed correlation coefficient [16]:

$$r_{\text{trim}} = \left( \frac{\sum_{i=1}^{n-n_2} u_i^2 - \sum_{i=n_2+1}^{n-n_2} v_i^2}{\left( \sum_{i=1}^{n-n_2} u_i^2 + \sum_{i=n_2+1}^{n-n_2} v_i^2 \right)} \right), \quad (8)$$

where $u_i\nu$ and $v_i\nu$ are the $i$th order statistics of the squared robust principal variables.

Now formula (8) yields the following limit cases: (i) the sample correlation coefficient $r$ with $n_1 = n_2 = 0$ and with the sample means for location and the standard deviations for scale in its inner structure; (ii) the median correlation coefficient $r_{\text{med}}$ with $n_1 = n_2 = 0.5(n - 1)$.

Consider the class of bivariate independent component distribution densities [2,16] with unknown but equal variances (the parameters of location of the random variables $X$ and $Y$ are assumed known: $\mu_1 = \mu_2 = 0$)

$$f(x, y) = \frac{1}{\sigma \sqrt{1 + \rho}} g \left( \frac{u}{\sigma \sqrt{1 + \rho}} \right) \frac{1}{\sigma \sqrt{1 - \rho}} g \left( \frac{v}{\sigma \sqrt{1 - \rho}} \right), \quad (9)$$

where $\sigma$ is the standard deviation; $\rho$ is the correlation coefficient, $u$ and $v$ are the principal variables $u = (x + y)/\sqrt{2}$, $v = (x - y)/\sqrt{2}$; $g(x)$ is a symmetric density belonging to a class $g$.

The class (9) contains the standard bivariate normal density $f(x, y) = N(x, y; 0, 0, 1, 1, \rho)$ when $g(x) = \varphi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.

Now we formulate the idea of introducing class (9): for any pair $(X, Y)$, the transformation $U = X + Y$, $V = X - Y$ gives the correlated random principal variables $(U, V)$, independent for distributions (9). Thus, estimation of their scales $\hat{S}_U = \sigma \sqrt{1 + \rho}$ and $\hat{S}_V = \sigma \sqrt{1 - \rho}$ solves the problem of estimation of correlation between $X$ and $Y$, since $\rho = (S_U^2 - S_V^2)/(S_U^2 + S_V^2)$.

**Theorem 1.** Assume that the following regularity conditions imposed on symmetric densities $g$ and scores $\chi$ hold:

- (g1) A density $g$ is twice continuously differentiable and satisfies $g(x) > 0$ for all $x$ in $\mathbb{R}$.
- (g2) Fisher information for scale $I(g)$ satisfies $0 < I(g) < \infty$.
(χ1) A score function χ is well-defined and continuous on \( \mathbb{R} \setminus C(\chi) \), where the set \( C(\chi) \) of discontinuity points of χ(⋅) is finite. In each point of \( C(\chi) \) there exist finite left and right limits of χ which are different. Also χ(−x) = χ(x) if (−x, x) \( \subset \mathbb{R} \setminus C(\chi) \), and there exists \( d > 0 \) such that χ(x) ≤ 0 on (0, d) and χ(x) ≥ 0 on (d, ∞).

(χ2) The set \( D(\chi) \) of points in which χ is continuous but in which \( \chi^* \) is not defined or not continuous is finite.

(χ3) \( \int x \chi(x)g(x) \, dx = 0 \) and \( \int x^2 \chi(x)g(x) \, dx < \infty \).

(χ4) \( 0 < \int x \chi^2(x)g(x) \, dx < \infty \).

Then, in the class of bivariate independent component distributions (9), the asymptotic bias of estimator (4) has the form

\[
E(\hat{\rho}_n) - \rho = b_n + o(1/n)
\]

where \( n^{-1} V(\chi, g) \) is the asymptotic variance (11) of M-estimators of scale.

**Proof.** Conditions (g1)–(χ4) are sufficient for consistency and asymptotic normality of the estimator \( \hat{\rho}_n \) (4) with variance (11) [16]. This result follows from the asymptotic normality of the M-estimators of scale \( \hat{S}_U \) and \( \hat{S}_V \) such that

\[
\text{var}(\hat{S}_U) = \frac{S_U^2}{n} V(\chi, g) + o(1/n), \quad \text{var}(\hat{S}_V) = \frac{S_V^2}{n} V(\chi, g) + o(1/n),
\]

where \( S_U^2 = \sigma^2(1 + \rho) \), \( S_V^2 = \sigma^2(1 - \rho) \).

Then formula (12) is obtained by a direct routine calculation by using the asymptotic expansion for the expectation of the fraction of asymptotically normal random variables \( \xi_n \) and \( \eta_n \) [3, 27.7–8]

\[
E\left( \frac{\xi_n}{\eta_n} \right) = \frac{E(\xi_n)}{E(\eta_n)} - \frac{1}{E^2(\eta_n)} \text{cov}(\xi_n, \eta_n) + \frac{E(\xi_n)}{E^3(\eta_n)} \text{var}(\eta_n) + o(1/n),
\]

where \( \xi_n = \hat{S}_U - S_U^2 \) and \( \eta_n = \hat{S}_V + S_V^2 \).

By the independence of \( \hat{S}_U \) and \( \hat{S}_V \), we have the following components of (14):

\[
E(\xi_n) = S_U^2 - S_V^2 + \text{var}(\hat{S}_U) - \text{var}(\hat{S}_V) + o(1/n),
\]

\[
E(\eta_n) = S_U^2 + S_V^2 + \text{var}(\hat{S}_U) + \text{var}(\hat{S}_V) + o(1/n),
\]

\[
\text{var}(\eta_n) = 4 \{ \text{var}(\hat{S}_U) \text{var}(\hat{S}_V) + S_U^2 \text{var}(\hat{S}_V) \} + o(1/n),
\]

\[
\text{cov}(\xi_n, \eta_n) = 4 \{ \text{var}(\hat{S}_U) - S_U^2 \text{var}(\hat{S}_V) \} + o(1/n).
\]

By substituting these components together with formulas (13) into (14), we obtain (12). □

First of all, we briefly comment on regularity conditions. In the literature, the conditions imposed on distribution densities and score functions take different forms depending on the pursued goals: in general, one may strengthen the conditions on densities and weaken those on scores, and vice versa (various suggestions can be found in [7,6,8,9]).

In this paper, we follow [6, pp. 125, 139] using a balanced set of conditions. The requirement of symmetry is restrictive but necessary for Huber’s minimax theory [7]. The conditions (g2), (χ1) and (χ2) define smooth densities and allow for a finite number of points of discontinuity for scores and their derivatives. The conditions (g1), (χ3) and (χ4) requiring the existence of Fisher information \( I(g) \) and other integrals are used for the proofs of consistency (the first equation of (χ3) provides consistency of M-estimators of scale) and asymptotic normality of M-estimators in robust statistics [7,6].

Formula (12) for asymptotic bias is similar to formula (11) for asymptotic variance also having two factors: the first depends only on \( \rho \), the second \( n^{-1} V(\chi, g) \) is the asymptotic variance of M-estimators of scale. Thus, most results on robust estimation of scale [7,6] can be directly applied to robust estimation of the correlation coefficient of bivariate independent component distributions. All further results are based on Theorem 1 and Huber’s results on minimax variance estimation of scale [7, pp. 120–121].

Finally, note that formula (12) for asymptotic bias can be used for correcting estimator’s bias. In this case, a new estimator is obtained, generally with a smaller bias.

### 2.3. Minimax bias estimators

Consider the class \( \mathcal{F}_\varepsilon \) of \( \varepsilon \)-contaminated bivariate independent component densities (9)

\[
\mathcal{F}_\varepsilon = \{ f : f(x, y) \geq (1 - \varepsilon) N(x, y; 0, 0, 1, 1, \rho), 0 \leq \varepsilon < 1 \}
\]

with the correlation coefficient \( \rho \). To estimate \( \rho \), we apply formula (4), in which the M-estimator of scale \( \hat{S} \) is defined by the choice of a score function \( \chi \).
Then the problem of minimax bias estimation of $\rho$ can be written as follows

$$\min_{\chi} \max_{f \in F_\varepsilon} n b_n(\chi, f) = \min_{\chi} \max_{g \in G_\varepsilon} n b_n(\chi, g) = n b_n(\chi^*, g^*),$$  \hspace{1cm} (16)

where $(\chi^*, g^*)$ is an optimal saddle-point pair $(\chi^*, g^*)$ solving problem (16); $G_\varepsilon$ is the class of $\gamma$-contaminated univariate standard normal distributions

$$G_\varepsilon = \{ g: g(x) \geq (1 - \gamma) \varphi(x), 0 \leq \gamma < 1 \}$$  \hspace{1cm} (17)

with $\varepsilon = 2\gamma - \gamma^2$ (for details, see [16]). In other words, due to the structure of formula (12), the problem of minimax bias estimation of $\rho$ is equivalent to the problem of minimax variance estimation of a scale parameter. Since the latter problem is solved in [7], we directly arrive at the following result.

**Theorem 2.** In the class $F_\varepsilon$ of $\varepsilon$-contaminated bivariate independent component distribution densities, the minimax bias estimator of $\rho$ is the trimmed correlation coefficient (8), where the numbers $n_1 = n_1(\varepsilon)$ and $n_2 = n_2(\varepsilon)$ of the trimmed smallest and greatest order statistics $u_{(i)}$ and $v_{(i)}$ depend on the value of the contamination parameter $\varepsilon$ through the auxiliary parameter $\gamma = 1 - \sqrt{1 - \varepsilon}$. The precise character of the dependencies $n_1 = n_1(\gamma)$ and $n_2 = n_2(\gamma)$ can be found in [7, 5.6].

The proof is based on Theorem 1 literally repeating the proof of Theorem 2 in [16].

Thus, the trimmed correlation coefficient and its limit cases, the median and MAD correlation coefficients, are simultaneously asymptotically minimax bias and variance estimators of the correlation coefficient.

**Corollary 1.** Under the aforementioned regularity conditions (g1)–(χ4) imposed on score functions $\chi$ and densities $g$, the most B- and $V$-robust estimators of the correlation coefficient in the sense of Theorems 9 and 10 [6, pp. 142–143] are given by estimator (4) with the optimal score $X_{\text{MAD}}(x) = \text{sign}(x) - 1$ in Eqs. (10), i.e., by the median and MAD correlation coefficients.

Since the median and MAD correlation coefficients have the maximum values of their breakdown points equal to 1/2 [15], we can regard these estimators as correlation analogues to such classical robust estimators of location and scale as the sample median and the absolute median.

Concluding this section, we write down the least favorable distribution density $f^*(x, y)$ at which the minimax is attained.

**Corollary 2.** From Example 2, formula (9) and Huber’s results on minimax variance estimation of scale [7, pp. 120–121] it follows that the least favorable bivariate independent component distribution density $f^*$ over the class $F_\varepsilon$ and the corresponding maximum likelihood score $\chi^*$ are as follows

$$f^*(x, y) = \frac{1}{\sigma \sqrt{1 + \rho^2}} g^* \left( \frac{x + y}{\sqrt{2} \sigma \sqrt{1 + \rho}} \right) \frac{1}{\sigma \sqrt{1 - \rho^2}} g^* \left( \frac{x - y}{\sqrt{2} \sigma \sqrt{1 - \rho}} \right),$$  \hspace{1cm} (18)

where

$$g^*(x) = \begin{cases} (1 - \gamma) \varphi(x_0) \left( \frac{x_0}{|x|} \right)^{\gamma} & (|x| < x_0), \\ (1 - \gamma) \varphi(x) & (x_0 \leq |x| \leq x_1), \\ (1 - \gamma) \varphi(x_1) \left( \frac{x_1}{|x|} \right)^{\gamma} & (|x| > x_1); \end{cases}$$

$$\chi^*(x) = -\sqrt[3]{\frac{f^*(x, y)}{f^*(x)}} - 1 = \begin{cases} x_0^2 - 1 & (|x| < x_0), \\ x^2 - 1 & (x_0 \leq |x| \leq x_1), \\ x_1^2 - 1 & (|x| > x_1). \end{cases}$$

### 3. Monte Carlo comparative study

In Tables 1–3, we exhibit experimental results (50,000 trials) on the comparative performance of the proposed and classical estimators on small ($n = 20$) and large ($n = 1000$) samples at the normal and $\varepsilon$-contaminated normal distribution (2); all the chosen competitors except one possess asymptotically optimal properties.

To provide unbiasedness of estimation, the quadrant correlation coefficient $r_q$ is transformed as follows: $r_q = \sin \left( \frac{\pi}{2} q \right)$, where $q = \left( (n_1 + n_2) - (n_2 + n_4) / n \right)$, $n_i$ is the number of observations in the $i$th quadrant, $n = \sum n_i$, and the quadrants are defined using the component-wise medians as the origin [10].

The median and MAD correlation coefficients are defined by formulas (5) and (6), respectively. Next, we use two versions of the trimmed correlation coefficient $r_{trim} (8)$: the first $r_{trim1}$ is chosen as the asymptotically minimax bias and variance estimator over the class of $\varepsilon$-contaminated normal distributions with $\varepsilon = 0.1$ and the corresponding values of trimming thresholds $n_1 = 0$ and $n_2 = [0.06 n]$ [16]; the second $r_{trim2}$ is taken symmetrically and harder trimmed with $n_1 = n_2 = [0.2 (n - 1)]$.

The best performances in table rows are boldfaced, the next to them values are starred.
robust counterparts of the sample correlation coefficient at which the minimax property of performance is inferior to the performance of those estimators. This can be explained by the choice of the class of direct estimators given in [14]. However, this topic deserves a separate consideration.

The use of those estimators in formula (4) seems to be prospective: some preliminary results in this direction are given in [14]. However, this topic deserves a separate consideration.

### Table 1
Normal distribution: $\rho = 0.9$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$E(r)$</th>
<th>$r_Q$</th>
<th>$r_{trim1}$</th>
<th>$r_{trim2}$</th>
<th>$r_{MAD}$</th>
<th>$r_{med}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.895</td>
<td>0.857</td>
<td>0.892</td>
<td>0.887</td>
<td>0.851</td>
<td>0.833</td>
</tr>
<tr>
<td>$n$ var($r$)</td>
<td>0.048</td>
<td>0.360</td>
<td>0.057</td>
<td>0.081</td>
<td>0.294</td>
<td>0.316</td>
</tr>
<tr>
<td>1000</td>
<td>0.900</td>
<td>0.899</td>
<td>0.909</td>
<td>0.900</td>
<td>0.899</td>
<td>0.899</td>
</tr>
<tr>
<td>$n$ var($r$)</td>
<td>0.036</td>
<td>0.233</td>
<td>0.046</td>
<td>0.058</td>
<td>0.100</td>
<td>0.101</td>
</tr>
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</table>

### Table 2
Symmetric scale contamination: $\varepsilon = 0.1, \rho = 0.9, \rho' = -0.9, k = 10$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$E(r)$</th>
<th>$r_Q$</th>
<th>$r_{trim1}$</th>
<th>$r_{trim2}$</th>
<th>$r_{MAD}$</th>
<th>$r_{med}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>-0.332</td>
<td>0.709</td>
<td>0.228</td>
<td>0.772</td>
<td>0.838</td>
<td>0.795*</td>
</tr>
<tr>
<td>$n$ var($r$)</td>
<td>8.650</td>
<td>0.880</td>
<td>9.820</td>
<td>2.118</td>
<td>0.320</td>
<td>0.430*</td>
</tr>
<tr>
<td>1000</td>
<td>-0.747</td>
<td>0.779</td>
<td>0.863</td>
<td>0.884</td>
<td>0.888</td>
<td>0.887*</td>
</tr>
<tr>
<td>$n$ var($r$)</td>
<td>1.403</td>
<td>0.653</td>
<td>0.776</td>
<td>0.078</td>
<td>0.122*</td>
<td>0.125</td>
</tr>
</tbody>
</table>

### Table 3
Asymmetric shift contamination: $\varepsilon = 0.1, \rho = 0.9, \rho' = -0.9, \mu_1 = 0, \mu_2 = 3, k = 1$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$E(r)$</th>
<th>$r_Q$</th>
<th>$r_{trim1}$</th>
<th>$r_{trim2}$</th>
<th>$r_{MAD}$</th>
<th>$r_{med}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.572</td>
<td>0.778</td>
<td>0.726</td>
<td>0.825</td>
<td>0.831</td>
<td>0.799</td>
</tr>
<tr>
<td>$n$ var($r$)</td>
<td>1.562</td>
<td>0.667</td>
<td>1.016</td>
<td>0.357</td>
<td>0.388*</td>
<td>0.439</td>
</tr>
<tr>
<td>1000</td>
<td>0.536</td>
<td>0.834</td>
<td>0.856</td>
<td>0.881</td>
<td>0.894</td>
<td>0.886*</td>
</tr>
<tr>
<td>$n$ var($r$)</td>
<td>1.717</td>
<td>0.475</td>
<td>0.188</td>
<td>0.093</td>
<td>0.114*</td>
<td>0.141</td>
</tr>
</tbody>
</table>

### 4. Discussion

**Normal distribution.** From Table 1 it follows that

1. on small and large samples, the best is the sample correlation coefficient $r$ both with respect to bias and variance;
2. the median and MAD correlation coefficients are close to each other in performance; however, $r_{MAD}$ is slightly better than $r_{med}$, especially on small samples;
3. on large samples, estimator’s biases can be neglected, but not their variances;
4. the best estimator among the chosen set of robust alternatives to the sample correlation coefficient is the asymptotically minimax trimmed correlation coefficient $r_{trim1}$.

**Contaminated normal distributions.** From Tables 2 to 3 it follows that

1. the sample correlation coefficient is catastrophically bad under contamination;
2. on small samples, the MAD estimator dominates over the others in respect to bias both for symmetric scale and asymmetric shift contaminations;
3. on large samples, the MAD, median and trimmed correlation coefficients are the best in bias confirming their asymptotic minimax bias properties with $r_{trim2}$ being superior in variance;
4. The minimax approach generates a rather soft trimming procedure ($n_1 = 0, n_2 = [0.06n]$ for $\varepsilon = 0.1$) in $r_{trim1}$ not sufficient to effectively resist heavy contamination, especially on small samples, so in the case of really heavy contamination, one should use either a greater level of trimming like in $r_{trim2}$ (the trimming with $n_1 = 0.2(n - 1)$, $n_2 = [0.2(n - 1)]$ approximately corresponds to the minimax solution for $\varepsilon = 0.4$) or the limit ($\varepsilon \to 1$) minimax bias and variance, the most $B$- and $V$-robust MAD correlation coefficient.

As it is aforementioned in Section 1, the quadrant correlation coefficient $r_Q$ is also an asymptotically minimax bias estimator of the correlation coefficient [7] like $r_{med}$ and $r_{MAD}$. Nevertheless, as it follows from Tables 1 to 3, its overall performance is inferior to the performance of those estimators. This can be explained by the choice of the class of direct robust counterparts of the sample correlation coefficient at which the minimax property of $r_Q$ is established [7]—the class of estimators based on principal variable variances is more advantageous than the competing class. However, we may recommend the quadrant coefficient $r_Q$ as a moderate robust alternative to the sample correlation coefficient $r$ both due to its low-complexity and to its finite sample binomial distribution [1].

Although the median and MAD correlation coefficient are asymptotically equivalent, the MAD correlation coefficient performs evidently better on small samples. Thus, under heavy contamination, we recommend to use the most $B$- and $V$-robust MAD estimator both on small and large samples. It would not be out of place to note the paper by Ma and Genton [11], in which the MAD correlation construction is effectively used for robust estimation of dispersion matrices.

Finally, note that estimators based on robust principal variables have a certain reserve for enhancing their efficiency: all the obtained results are based exclusively on Huber’s $M$-estimators of scale, whereas there are robust and highly efficient estimators of scale of the different genesis, e.g., several alternatives to the MAD estimator of scale proposed by Rousseeuw and Croux [12]. The use of those estimators in formula (4) seems to be prospective: some preliminary results in this direction are given in [14]. However, this topic deserves a separate consideration.
References