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New textures for the lepton mass matrices

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Abstract

We study predictive textures for the lepton mass matrices in which the charged-lepton mass matrix has either four or five zero matrix elements while the neutrino Majorana mass matrix has, respectively, either four or three zero matrix elements. We find that all the viable textures of these two kinds share many predictions: the neutrino mass spectrum is inverted, the sum of the light-neutrino masses is close to 0.1 eV, the Dirac phase δ in the lepton mixing matrix is close to either 0 or π, and the mass term responsible for neutrinoless double-beta decay lies in between 12 and 22 meV.

1. Introduction

The origin of neutrino masses, the reasons behind their smallness, and the structure of lepton mixing are still unanswered questions. There has been a great deal of theoretical work in this area, trying to provide answers based on such diverse ideas as, for instance, seesaw mechanisms, radiative generation of the neutrino masses, Abelian and non-Abelian symmetries imposed on the leptonic sector, and 'textures' for the leptonic mass matrices. In the past few years, a wealth of experimental data concerning neutrino oscillations—in particular the recent confirmation [1–3] of a non-zero value for the mixing angle θ_{13}—became available, allowing theorists to test their

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models and discard those that do not conform to the experimental discoveries. Here, we shall consider new textures for the leptonic mass matrices and investigate what the most recent and stringent phenomenological data say about them and their predictive power.

In this paper we work in the context of a model with three light neutrinos which are Majorana particles. The lepton mass terms are given by

$$\mathcal{L}_{\text{mass}} = -\bar{\ell}_L M_\ell \ell_R - \bar{\ell}_R M_\ell^\dagger \ell_L + \frac{1}{2}\left(\nu^T C^{-1} M_\nu \nu - \bar{\nu} M_\nu^* C \bar{\nu}^T\right), \tag{1}$$

where C is the charge-conjugation matrix in Dirac space. The three light-neutrino fields in the column–vector ν are left-handed. The neutrino mass matrix M_ν acts in flavour space and is symmetric. Let the two mass matrices be bi-diagonalized as

$$U_L^\dagger M_\ell U_R = \text{diag}(m_e, m_\mu, m_\tau), \tag{2a}$$

$$U_\nu^T M_\nu U_\nu = \text{diag}(m_1, m_2, m_3), \tag{2b}$$

where U_L, U_R, and U_ν are 3×3 unitary matrices. Then, the lepton mixing matrix is

$$U = U_L^\dagger U_\nu. \tag{3}$$

Even though M_ℓ is more fundamental, in practice we only need to consider

$$H \equiv M_\ell M_\ell^\dagger, \tag{4}$$

since it is its diagonalization that fixes the matrix U_L which appears in U:

$$U_L^\dagger H U_L = \text{diag}\left(m_e^2, m_\mu^2, m_\tau^2\right). \tag{5}$$

Let $\mathcal{M}_\nu$ denote the neutrino mass matrix in the basis where the charged-lepton mass matrix is diagonal. Then,

$$\mathcal{M}_\nu = U_L^T M_\nu U_L. \tag{6}$$

There have been many attempts at using non-Abelian symmetries to constrain lepton mixing [4–21]. This is usually done with the goal of obtaining ‘mass-independent schemes’, wherein the constraints on U do not depend on the values of the lepton masses. However, those attempts appear to have reached their limits [22]. A simpler avenue, at least in group-theoretical terms, is provided by Abelian symmetries. In appropriate bases for the lepton and Higgs fields, they enforce ‘texture zeros’ in the lepton mass matrices, but they cannot enforce relationships among their nonzero matrix elements. In the pioneering work of Ref. [23], M_ℓ was assumed to be diagonal, hence to have six zero matrix elements, while M_ν had two zero matrix elements. This was later generalized to the situation wherein M_ℓ is diagonal and M_ν^{-1} has two zero matrix elements [24]; mixed situations in which both M_ν and M_ν^{-1} have one zero matrix element, while M_ℓ remains diagonal, were considered in Ref. [25].

In this work we propose new textures for the lepton mass matrices which are in principle as predictive as the ones considered in Refs. [23–25]. Let (m, n) denote a class of textures with m nonzero matrix elements in M_ℓ and n nonzero matrix elements in M_ν.[1] Then, the textures

[1] M_ν is symmetric because it is a Majorana mass matrix. Hence, only six out of its nine matrix elements are independent. The integer n denotes the number of *independent* matrix elements which do not vanish; if some of those elements are off-diagonal, then the actual number of nonzero entries in M_ν is larger than n.

mentioned at the end of the previous paragraph are in the $(3, 4)$ class. In this paper we consider predictive, viable textures in the $(4, 3)$ and $(5, 2)$ classes. Those textures are in principle just as predictive as the ones in class $(3, 4)$; each of them has *eight* degrees of freedom—seven moduli and one rephasing-invariant phase—in the matrices H and M_ν. Those eight degrees of freedom are meant to fit *ten* observables—the three charged-lepton masses $m_{e,\mu,\tau}$, the three neutrino masses $m_{1,2,3}$, the three lepton mixing angles $\theta_{12,13,23}$, and the Dirac phase δ. (We do not care about the Majorana phases in U because they are not observable in neutrino oscillations. However, we shall specify the predictions of our textures for the mass term responsible for neutrinoless double-beta decay, $m_{\beta\beta} \equiv |(\mathcal{M}_\nu)_{ee}|$.) So, in principle each texture yields *two* predictions, which may conveniently be taken to be one prediction for the overall scale of the neutrino masses and one prediction for $\cos\delta$.

It has long been known [26] that *any* mass-matrix texture, in particular any set of matrices M_ℓ and M_ν with some zero matrix elements, can be implemented in a suitable extension of the Standard Model of the electroweak interactions, furnished with both additional scalar multiplets and appropriate Abelian symmetries. We rely on this fact to assert that all the textures in this paper may be implemented in renormalizable models. However, we shall not attempt here to construct a specific model for any of the textures; we also do not attempt to search for the simplest model which might justify any given texture [27].

We emphasize that all the textures will be analyzed in this paper only at the 'classical' level, *i.e.* we shall neglect both quantum corrections to the mass matrices and renormalization-group effects.

The texture-zero approach for the mass matrices pursued in this paper is inherently limited in its scope and objectives. Even if it were found that the experimental data fully agree with the predictions of some texture, we would not be sure that the mass matrices indeed have that texture, because there are many sets of mass matrices leading to the same observables—in particular, any two sets of mass matrices connected among themselves through a weak-basis transformation lead to the same observables. Further studies would be necessary in order to identify specific models that lead to mass matrices with that texture and also to identify other observable predictions of those models, *viz.* extra particles and interactions that they may feature. So, the study of textures may be looked upon as just the first part of a longer search for models of 'new physics'. Still, that study has some relevance in itself, since it may suggest the most likely ranges for some observables—for instance, knowing whether the phase δ is more likely to be large or small—and which correlations among observables may be expected and are enforceable through well-defined renormalizable models.

This paper is organized as follows. In Section 2 we derive all the viable $(5, 2)$ textures and briefly survey their predictions. We do the same for $(4, 3)$ textures in Section 3. A listing of all the viable textures that we have found, and a summary of their predictions, is provided in Section 4.

2. $(5, 2)$ textures

Since all three charged leptons are massive, the determinant of M_ℓ cannot vanish. Therefore, through an appropriate permutation of the columns of M_ℓ—this permutation changes U_R but does not change U_L, hence it leaves U invariant—one may always obtain the $(1, 1)$, $(2, 2)$, and $(3, 3)$ matrix elements of M_ℓ to be nonzero. Since in a $(5, 2)$ texture M_ℓ has five nonzero matrix elements, there are then $(6 \times 5)/2 = 15$ possibilities:

$$M_\ell \sim \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \tag{7a}$$

$$M_\ell \sim \begin{pmatrix} \times & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & \times \end{pmatrix}, \tag{7b}$$

$$M_\ell \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}; \tag{7c}$$

$$M_\ell \sim \begin{pmatrix} \times & 0 & \times \\ 0 & \times & 0 \\ 0 & \times & \times \end{pmatrix}, \tag{8a}$$

$$M_\ell \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ \times & 0 & \times \end{pmatrix}, \tag{8b}$$

$$M_\ell \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ \times & \times & \times \end{pmatrix}; \tag{8c}$$

$$M_\ell \sim \begin{pmatrix} \times & \times & 0 \\ 0 & \times & \times \\ 0 & 0 & \times \end{pmatrix}, \tag{9a}$$

$$M_\ell \sim \begin{pmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ 0 & \times & \times \end{pmatrix}, \tag{9b}$$

$$M_\ell \sim \begin{pmatrix} \times & 0 & 0 \\ \times & \times & \times \\ 0 & 0 & \times \end{pmatrix}; \tag{9c}$$

$$M_\ell \sim \begin{pmatrix} \times & \times & 0 \\ 0 & \times & 0 \\ \times & 0 & \times \end{pmatrix}, \tag{10a}$$

$$M_\ell \sim \begin{pmatrix} \times & 0 & \times \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \tag{10b}$$

$$M_\ell \sim \begin{pmatrix} \times & \times & \times \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}; \tag{10c}$$

$$M_\ell \sim \begin{pmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ \times & 0 & \times \end{pmatrix}; \tag{11}$$

$$M_\ell \sim \begin{pmatrix} \times & \times & 0 \\ 0 & \times & 0 \\ 0 & \times & \times \end{pmatrix}; \tag{12}$$

$$M_\ell \sim \begin{pmatrix} \times & 0 & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{pmatrix}. \tag{13}$$

Eqs. (8) are equivalent[2] because they all lead to $H_{12} = 0$, hence to the same constraints on U_L and on U. Similarly, Eqs. (9) feature $H_{13} = 0$ and Eqs. (10) have $H_{23} = 0$. Also, $(H^{-1})_{23} = 0$ for Eq. (11), $(H^{-1})_{13} = 0$ for Eq. (12), and $(H^{-1})_{12} = 0$ for Eq. (13).

In a $(5, 2)$ texture M_ν has only two nonzero matrix elements. Leaving aside possible reorderings of the rows and columns of M_ν, there are only four possibilities:

$$M_\nu \sim \begin{pmatrix} 0 & \times & \times \\ \times & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}, \tag{14a}$$

$$M_\nu \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{14b}$$

$$M_\nu \sim \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}, \tag{14c}$$

$$M_\nu \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & \times \end{pmatrix}. \tag{14d}$$

Both Eqs. (14a) and (14c) lead to two degenerate neutrinos and are therefore incompatible with experiment.

With Eq. (14b) lepton mixing originates fully in M_ℓ; indeed, one then has $U = U_L^\dagger$ but for possible reorderings of the columns of U. For two physical neutrinos ν_i and ν_j with $i \neq j$,

$$\begin{aligned} H_{ij} &= m_e^2 U_{ei}^* U_{ej} + m_\mu^2 U_{\mu i}^* U_{\mu j} + m_\tau^2 U_{\tau i}^* U_{\tau j} \\ &= \left(m_\mu^2 - m_e^2\right) U_{\mu i}^* U_{\mu j} + \left(m_\tau^2 - m_e^2\right) U_{\tau i}^* U_{\tau j}. \end{aligned} \tag{15}$$

So,

$$H_{ij} = 0 \quad \Rightarrow \quad -\frac{U_{\tau i}^* U_{\tau j}}{U_{\mu i}^* U_{\mu j}} = \frac{m_\mu^2 - m_e^2}{m_\tau^2 - m_e^2} \approx \frac{m_\mu^2}{m_\tau^2} \approx \frac{1}{280}. \tag{16}$$

Similarly,

$$\left(H^{-1}\right)_{ij} = 0 \quad \Rightarrow \quad -\frac{U_{ei}^* U_{ej}}{U_{\mu i}^* U_{\mu j}} = \frac{m_\mu^{-2} - m_\tau^{-2}}{m_e^{-2} - m_\tau^{-2}} \approx \frac{m_e^2}{m_\mu^2} \approx \frac{1}{43\,000}. \tag{17}$$

Phenomenologically, there are no two columns i and j of U such that either $|(U_{\tau i} U_{\tau j})/(U_{\mu i} U_{\mu j})|$ or $|(U_{ei} U_{ej})/(U_{\mu i} U_{\mu j})|$ are allowed to be so much smaller than unity as indicated by Eqs. (16)

[2] It may easily be demonstrated that, through unitary redefinitions of the right-handed charged leptons, one may transform any one of Eqs. (8) into any other of them.

and (17). Therefore, with Eq. (14b) either $H_{ij} = 0$ or $(H^{-1})_{ij} = 0$ are phenomenologically forbidden for $i \neq j$. If, together with Eq. (14b), the form of M_ℓ is as in one of Eqs. (7), then lepton mixing would only be 2×2, which is also incompatible with experiment. Therefore, Eq. (14b) must be excluded, just as Eqs. (14a) and (14c).

We shall therefore concentrate on Eq. (14d). With that form for M_ν, *one neutrino is massless; this is one of the predictions of viable* $(5, 2)$ *textures*.

If M_ν is as in Eq. (14d) while M_ℓ is as in one of Eqs. (7), then the matrix U has one vanishing matrix element. This contradicts the phenomenology. Therefore, we may exclude Eqs. (7) and concentrate exclusively on the other possibilities for M_ℓ. As we have seen, they can be subsumed in six different possibilities: $H_{12} = 0$, $H_{13} = 0$, $H_{23} = 0$, $(H^{-1})_{23} = 0$, $(H^{-1})_{13} = 0$, and $(H^{-1})_{12} = 0$.

Let

$$A \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{18}$$

Then, the permutation group of three objects is represented by

$$S_3 = \{A, B, ABA, AB, BA, A^2\}. \tag{19}$$

Let Z be any of the six matrices in S_3. Those matrices are orthogonal, hence $Z^{-1} = Z^T$. Interchanging the rows and columns of M_ν is equivalent to making $M_\nu \to Z M_\nu Z^T$. But $U_\nu \to Z U_\nu$ when this happens. Therefore $U \to U_L^\dagger Z U_\nu$. This is equivalent to letting $U_L \to Z^\dagger U_L$ or $H \to Z^\dagger H Z$, which means a reordering of the rows and columns of H.

So, a reordering of the rows and columns of M_ν is equivalent to an analogous reordering of the rows and columns of H. Therefore, instead of considering separately each of the three conditions $H_{12} = 0$, $H_{13} = 0$, and $H_{23} = 0$, one may consider only the condition $H_{12} = 0$ provided one allows for all the possible reorderings of the rows and columns of M_ν. We thus conclude that there are twelve potentially viable $(5, 2)$ textures:

$$H_{12} = 0 \quad \text{and} \quad M_\nu \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & \times \end{pmatrix}, \tag{20a}$$

$$H_{12} = 0 \quad \text{and} \quad M_\nu \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & 0 \\ \times & 0 & \times \end{pmatrix}, \tag{20b}$$

$$H_{12} = 0 \quad \text{and} \quad M_\nu \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & 0 \end{pmatrix}, \tag{20c}$$

$$H_{12} = 0 \quad \text{and} \quad M_\nu \sim \begin{pmatrix} 0 & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{20d}$$

$$H_{12} = 0 \quad \text{and} \quad M_\nu \sim \begin{pmatrix} \times & \times & 0 \\ \times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{20e}$$

$$H_{12} = 0 \quad \text{and} \quad M_\nu \sim \begin{pmatrix} \times & 0 & \times \\ 0 & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}, \tag{20f}$$

together with the six textures that result from substituting $H_{12} = 0$ by $(H^{-1})_{12} = 0$ in each of Eqs. (20).

With either $H_{12} = 0$ or $(H^{-1})_{12} = 0$, the matrix H can always be made real through a rephasing, *i.e.* there is always a diagonal unitary matrix Y_ℓ such that

$$Y_\ell^* H Y_\ell = H_{\text{real}} \tag{21}$$

has real matrix elements. The real and symmetric matrix H_{real} is diagonalized by an orthogonal matrix O_ℓ:

$$O_\ell^T H_{\text{real}} O_\ell = \operatorname{diag}\left(m_e^2, m_\mu^2, m_\tau^2\right). \tag{22}$$

Since either H_{real} or its inverse has one vanishing matrix element, it contains only five degrees of freedom; three of them correspond to the three charged-lepton masses and the remaining two are implicitly contained in O_ℓ. Thus, O_ℓ is not fully general—a general 3×3 orthogonal matrix has three degrees of freedom, not just two.

Similarly, phases may be withdrawn from the matrix M_ν in Eq. (14d):

$$Y_\nu \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & f e^{i\phi} \\ 0 & f e^{i\phi} & r e^{i\rho} \end{pmatrix} Y_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & f \\ 0 & f & r \end{pmatrix}, \tag{23}$$

where f and r are non-negative real and $Y_\nu = \operatorname{diag}(e^{i\xi}, e^{i(\rho/2-\phi)}, e^{-i\rho/2})$, the phase ξ being arbitrary. Therefore, the lepton mixing matrix always ends up being

$$U = O_\ell^T X O_{\text{b}}, \tag{24}$$

where O_{b} is the real, orthogonal matrix that diagonalizes the real version of M_ν while X is a diagonal unitary matrix containing *only one* phase. This is because the arbitrariness of the phase ξ in Y_ν allows one to absorb one phase in X.

Let us define

$$\delta \equiv \Delta m_{\text{sol}}^2 \equiv m_2^2 - m_1^2, \qquad \Delta \equiv \Delta m_{\text{atm}}^2 \equiv \left| m_3^2 - m_1^2 \right|, \qquad \varepsilon \equiv \frac{\delta}{\Delta} \approx \frac{1}{30}. \tag{25}$$

With a massless neutrino there are two possibilities for the neutrino mass spectrum: either it is 'normal' (which we call "case n"), and then[3]

$$\frac{m_1}{\sqrt{\Delta}} = 0, \qquad \frac{m_2}{\sqrt{\Delta}} = \sqrt{\varepsilon}, \qquad \frac{m_3}{\sqrt{\Delta}} = 1, \qquad \frac{m_1 + m_2 + m_3}{\sqrt{\Delta}} = 1 + \sqrt{\varepsilon}; \tag{26}$$

or it is 'inverted' (which we call "case i"), and then

$$\frac{m_1}{\sqrt{\Delta}} = 1, \qquad \frac{m_2}{\sqrt{\Delta}} = \sqrt{1+\varepsilon}, \qquad \frac{m_3}{\sqrt{\Delta}} = 0, \qquad \frac{m_1 + m_2 + m_3}{\sqrt{\Delta}} = 1 + \sqrt{1+\varepsilon}. \tag{27}$$

Notice that

[3] We use in this paper the quantity $\sqrt{\Delta} \approx 0.5$ eV as the unit for all neutrino masses.

$$\frac{m_1 + m_2 + m_3}{\sqrt{\Delta}} \approx 1 \quad \text{in case n, but} \tag{28a}$$

$$\frac{m_1 + m_2 + m_3}{\sqrt{\Delta}} \approx 2 \quad \text{in case i.} \tag{28b}$$

Suppose the initial M_ν was as in Eq. (14d). Then, after withdrawing phases from it, we would have, in case n

$$M_\nu \to M_{\rm n} = \sqrt{\Delta} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon^{1/4} \\ 0 & \varepsilon^{1/4} & 1 - \sqrt{\varepsilon} \end{pmatrix}, \tag{29}$$

while, in case i

$$M_\nu \to M_{\rm i} = \sqrt{\Delta} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (1+\varepsilon)^{1/4} \\ 0 & (1+\varepsilon)^{1/4} & \sqrt{1+\varepsilon} - 1 \end{pmatrix}. \tag{30}$$

The diagonalization of the real matrices M_n and M_i proceeds as $O_{\rm n}^T M_{\rm n} O_{\rm n} = \sqrt{\Delta}\, {\rm diag}(0, -\sqrt{\varepsilon}, 1)$ and $O_{\rm i}^T M_{\rm i} O_{\rm i} = \sqrt{\Delta}\, {\rm diag}(-1, \sqrt{1+\varepsilon}, 0)$, with[4]

$$O_{\rm n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1+\sqrt{\varepsilon}}} & \frac{\varepsilon^{1/4}}{\sqrt{1+\sqrt{\varepsilon}}} \\ 0 & -\frac{\varepsilon^{1/4}}{\sqrt{1+\sqrt{\varepsilon}}} & \frac{1}{\sqrt{1+\sqrt{\varepsilon}}} \end{pmatrix}, \tag{31a}$$

$$O_{\rm i} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{(1+\varepsilon)^{1/4}}{\sqrt{1+\sqrt{1+\varepsilon}}} & \frac{1}{\sqrt{1+\sqrt{1+\varepsilon}}} & 0 \\ -\frac{1}{\sqrt{1+\sqrt{1+\varepsilon}}} & \frac{(1+\varepsilon)^{1/4}}{\sqrt{1+\sqrt{1+\varepsilon}}} & 0 \end{pmatrix}, \tag{31b}$$

We see that the mixing angle $\theta_{\rm b}$ appears in $O_{\rm b}$. If b is n, then

$$\cos\theta_{\rm n} = \sqrt{\frac{1}{1+\sqrt{\varepsilon}}} \equiv c_{\rm n}, \qquad \sin\theta_{\rm n} = \sqrt{\frac{\sqrt{\varepsilon}}{1+\sqrt{\varepsilon}}} \equiv s_{\rm n}. \tag{32}$$

If b is i, then

$$\cos\theta_{\rm i} = \sqrt{\frac{1}{1+\sqrt{1+\varepsilon}}} \equiv c_{\rm i}, \qquad \sin\theta_{\rm i} = \sqrt{\frac{\sqrt{1+\varepsilon}}{1+\sqrt{1+\varepsilon}}} \equiv s_{\rm i}. \tag{33}$$

Since $\varepsilon \sim 1/30$ is small, $\theta_{\rm n} \sim 20^\circ$ is smallish. On the other hand, $\theta_{\rm i}$ is very close to 45 degrees, *viz.* almost maximal.

It turns out that, because the mixing angle $\theta_{\rm n}$ is so small, case n is not much different from the one, treated in Eqs. (15)–(17), in which lepton mixing originates fully in M_ℓ. Because of this, *a normal neutrino mass spectrum does not work with* $(5,2)$ *textures*.

For case i, one may write down the six possible forms of the lepton mixing matrices. They are

[4] The remarkable and desirable properties of mass matrices like $M_{\rm n}$ and $M_{\rm i}$ have been noticed long ago [28].

$$\frac{M_{\rm i}}{\sqrt{\Delta}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (1+\varepsilon)^{1/4} \\ 0 & (1+\varepsilon)^{1/4} & \sqrt{1+\varepsilon}-1 \end{pmatrix} \quad \Rightarrow \quad U = O_\ell^T \begin{pmatrix} 0 & 0 & 1 \\ s_{\rm i} e^{i\aleph} & c_{\rm i} e^{i\aleph} & 0 \\ -c_{\rm i} & s_{\rm i} & 0 \end{pmatrix}, \tag{34a}$$

$$\frac{M_{\rm i}}{\sqrt{\Delta}} = \begin{pmatrix} 0 & (1+\varepsilon)^{1/4} & 0 \\ (1+\varepsilon)^{1/4} & \sqrt{1+\varepsilon}-1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad U = O_\ell^T \begin{pmatrix} s_{\rm i} e^{i\aleph} & c_{\rm i} e^{i\aleph} & 0 \\ -c_{\rm i} & s_{\rm i} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{34b}$$

$$\frac{M_{\rm i}}{\sqrt{\Delta}} = \begin{pmatrix} \sqrt{1+\varepsilon}-1 & 0 & (1+\varepsilon)^{1/4} \\ 0 & 0 & 0 \\ (1+\varepsilon)^{1/4} & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad U = O_\ell^T \begin{pmatrix} -c_{\rm i} & s_{\rm i} & 0 \\ 0 & 0 & 1 \\ s_{\rm i} e^{i\aleph} & c_{\rm i} e^{i\aleph} & 0 \end{pmatrix}, \tag{34c}$$

$$\frac{M_{\rm i}}{\sqrt{\Delta}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{1+\varepsilon}-1 & (1+\varepsilon)^{1/4} \\ 0 & (1+\varepsilon)^{1/4} & 0 \end{pmatrix} \quad \Rightarrow \quad U = O_\ell^T \begin{pmatrix} 0 & 0 & 1 \\ -c_{\rm i} & s_{\rm i} & 0 \\ s_{\rm i} e^{i\aleph} & c_{\rm i} e^{i\aleph} & 0 \end{pmatrix}, \tag{34d}$$

$$\frac{M_{\rm i}}{\sqrt{\Delta}} = \begin{pmatrix} 0 & 0 & (1+\varepsilon)^{1/4} \\ 0 & 0 & 0 \\ (1+\varepsilon)^{1/4} & 0 & \sqrt{1+\varepsilon}-1 \end{pmatrix} \quad \Rightarrow \quad U = O_\ell^T \begin{pmatrix} s_{\rm i} e^{i\aleph} & c_{\rm i} e^{i\aleph} & 0 \\ 0 & 0 & 1 \\ -c_{\rm i} & s_{\rm i} & 0 \end{pmatrix}, \tag{34e}$$

$$\frac{M_{\rm i}}{\sqrt{\Delta}} = \begin{pmatrix} \sqrt{1+\varepsilon}-1 & (1+\varepsilon)^{1/4} & 0 \\ (1+\varepsilon)^{1/4} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad U = O_\ell^T \begin{pmatrix} -c_{\rm i} & s_{\rm i} & 0 \\ s_{\rm i} e^{i\aleph} & c_{\rm i} e^{i\aleph} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{34f}$$

In the forms for U in Eqs. (34), the matrix O_ℓ is the real orthogonal matrix that diagonalizes H according to Eq. (22). The matrix O_ℓ contains two degrees of freedom because H satisfies either $H_{12} = 0$ or $(H^{-1})_{12} = 0$. The matrix U depends on three degrees of freedom: one of them is the phase $\aleph$ and the other two are contained in O_ℓ. So, there is one non-trivial constraint on U.

For the mass term responsible for neutrinoless double-beta decay one finds the formula

$$\frac{m_{\beta\beta}}{\sqrt{\Delta}} = \left|(O_\ell)_{i1}\right|\left|(\sqrt{1+\varepsilon}-1)(O_\ell)_{i1} + 2(1+\varepsilon)^{1/4} e^{i\aleph} (O_\ell)_{j1}\right|, \tag{35}$$

where the values for the indices i and j are given in Table 1.

One should note that Eqs. (34d)–(34f) correspond to Eqs. (34a)–(34c), respectively, after an interchange between ν_μ and ν_τ. This interchange is equivalent, in the standard parametrization of U, to the transformations $\cos\theta_{23} \leftrightarrow \sin\theta_{23}$ and $\cos\delta \to -\cos\delta$. In so far as the extant phenomenological data are approximately invariant under $\cos\theta_{23} \leftrightarrow \sin\theta_{23}$, one may anticipate that the predictions of Eqs. (34d)–(34f) for $\cos\delta$ will be approximately symmetric to the corresponding predictions of Eqs. (34a)–(34c).

We have found numerically that all six Eqs. (34) are able to fit U provided $H_{12} = 0$, but they are unable to achieve that fit when $(H^{-1})_{12} = 0$. Furthermore, the predictions of Eqs. (34a)–(34c) (with $H_{12} = 0$) are all very similar (but not really identical) among themselves. In all those cases, one must have a rather large solar mixing angle, $\sin^2\theta_{12} \gtrsim 0.3$. The prediction of Eqs. (34a)–(34c) for the Dirac phase is $\cos\delta \lesssim -0.6$, while Eqs. (34d)–(34f)

Table 1
The indices i and j to be used in Eq. (35).

Equation for U	(34a)	(34b)	(34c)	(34d)	(34e)	(34f)
i	3	2	1	2	3	1
j	2	1	3	3	1	2

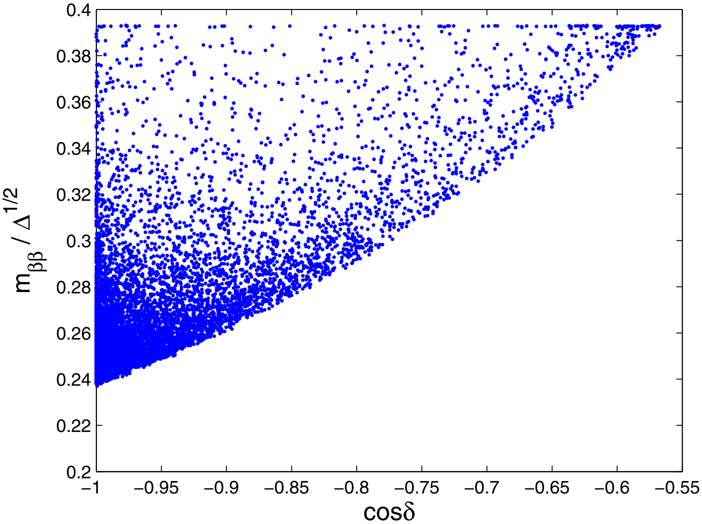

Fig. 1. $m_{\beta\beta}/\sqrt{\Delta}$ *vs* $\cos\delta$ in a numeric scan for models with (5, 2) textures.

make the symmetric prediction $\cos\delta \gtrsim 0.6$. The prediction for neutrinoless double-beta decay is $0.24 \lesssim m_{\beta\beta}/\sqrt{\Delta} \lesssim 0.4$. We can see these predictions displayed in Fig. 1, in which we plot $m_{\beta\beta}/\sqrt{\Delta}$ against $\cos\delta$. Each point in the plot corresponds to some definite values for the parameters of the model—neutrino oscillation observables, phase $\aleph$, and two entries of the matrix H. For definiteness, these predictions are based on the use of the phenomenological 3σ data in Ref. [29]; other phenomenological fits to the data—see Refs. [30] and [31]—can hardly yield much too different results.

3. (4, 3) textures

Since there are no massless charged leptons, the determinant of M_ℓ must be nonzero. Therefore, after an adequate reordering of the rows and columns of M_ℓ,

$$M_\ell = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & t_3 & t_4 \end{pmatrix}. \tag{36}$$

Therefore,

$$H = \begin{pmatrix} |t_1|^2 & 0 & 0 \\ 0 & |t_2|^2 & |t_2 t_3| e^{i\gamma} \\ 0 & |t_2 t_3| e^{-i\gamma} & |t_3|^2 + |t_4|^2 \end{pmatrix}, \tag{37}$$

where $\gamma \equiv \arg(t_2 t_3^*)$. From Eq. (5), the columns of U_L are the normalized eigenvectors of H. It is clear from Eq. (37) that one of the eigenvalues of H is $|t_1|^2$ and the corresponding normalized eigenvector is $(1, 0, 0)^T$. Therefore, either

$$U_L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & e^{i\gamma}\sin\theta \\ 0 & -e^{-i\gamma}\sin\theta & \cos\theta \end{pmatrix} X, \quad \text{or} \tag{38a}$$

$$U_L = \begin{pmatrix} 0 & 1 & 0 \\ \cos\theta & 0 & e^{i\gamma}\sin\theta \\ -e^{-i\gamma}\sin\theta & 0 & \cos\theta \end{pmatrix} X, \quad \text{or} \tag{38b}$$

$$U_L = \begin{pmatrix} 0 & 0 & 1 \\ \cos\theta & e^{i\gamma}\sin\theta & 0 \\ -e^{-i\gamma}\sin\theta & \cos\theta & 0 \end{pmatrix} X, \tag{38c}$$

where X is a diagonal unitary matrix containing the phases of the eigenvectors of H; those phases are meaningless. Eq. (38a) holds if $|t_1| = m_e$, Eq. (38b) holds if $|t_1| = m_\mu$, and Eq. (38c) holds if $|t_1| = m_\tau$. The angle θ is fixed by

$$\tan 2\theta = \frac{2|t_2 t_3|}{|t_3|^2 + |t_4|^2 - |t_2|^2}. \tag{39}$$

We assume that only three out of the six independent matrix elements of M_ν are nonzero. Therefore there are $(6 \times 5 \times 4)/3! = 20$ possible forms for M_ν. They are

$$M_\nu \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \tag{40}$$

$$M_\nu \sim \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{41}$$

$$M_\nu \sim \begin{pmatrix} \times & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & 0 \end{pmatrix}, \tag{42}$$

$$M_\nu \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & 0 \end{pmatrix}, \tag{43}$$

$$M_\nu \sim \begin{pmatrix} \times & \times & 0 \\ \times & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}, \tag{44}$$

$$M_\nu \sim \begin{pmatrix} \times & 0 & \times \\ 0 & 0 & 0 \\ \times & 0 & \times \end{pmatrix}, \tag{45}$$

$$M_\nu \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & \times \end{pmatrix}, \tag{46}$$

$$M_\nu \sim \begin{pmatrix} 0 & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \tag{47}$$

$$M_\nu \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & \times \end{pmatrix}, \tag{48}$$

$$M_\nu \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}, \tag{49}$$

$$M_\nu \sim \begin{pmatrix} \times & \times & \times \\ \times & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}, \tag{50}$$

$$M_\nu \sim \begin{pmatrix} \times & \times & 0 \\ \times & 0 & \times \\ 0 & \times & 0 \end{pmatrix}, \tag{51}$$

$$M_\nu \sim \begin{pmatrix} \times & 0 & \times \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix}, \tag{52}$$

$$M_\nu \sim \begin{pmatrix} 0 & \times & \times \\ \times & \times & 0 \\ \times & 0 & 0 \end{pmatrix}, \tag{53}$$

$$M_\nu \sim \begin{pmatrix} 0 & \times & 0 \\ \times & \times & \times \\ 0 & \times & 0 \end{pmatrix}, \tag{54}$$

$$M_\nu \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & \times & \times \\ \times & \times & 0 \end{pmatrix}, \tag{55}$$

$$M_\nu \sim \begin{pmatrix} 0 & \times & \times \\ \times & 0 & 0 \\ \times & 0 & \times \end{pmatrix}, \tag{56}$$

$$M_\nu \sim \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \times \end{pmatrix}, \tag{57}$$

$$M_\nu \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & \times \end{pmatrix}, \tag{58}$$

$$M_\nu \sim \begin{pmatrix} 0 & \times & \times \\ \times & 0 & \times \\ \times & \times & 0 \end{pmatrix}. \tag{59}$$

Let Z be any of the six matrices in the group S_3 of Eq. (19). Those matrices are orthogonal, hence $Z^{-1} = Z^T$. Interchanging the rows and columns of M_ν is equivalent to making $M_\nu \to Z M_\nu Z^T$. But $U_\nu \to Z U_\nu$ when this happens. Therefore $U \to U_L^\dagger Z U_\nu$. This is equivalent to letting $U_L \to Z^\dagger U_L$, which corresponds to a reordering of the rows of U_L. We conclude that, provided one allows for a reordering of the rows of the three possibilities for U_L in Eqs. (38), one is free to avoid considering separately two matrices M_ν which differ only by an interchange of their rows and columns. In this way, out of the 20 forms for M_ν in Eqs. (40)–(59), one only needs to consider the following six:

$$M_\nu \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \tag{60a}$$

$$M_\nu \sim \begin{pmatrix} \times & 0 & \times \\ 0 & 0 & 0 \\ \times & 0 & \times \end{pmatrix}, \tag{60b}$$

$$M_\nu \sim \begin{pmatrix} \times & \times & 0 \\ \times & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}, \tag{60c}$$

$$M_\nu \sim \begin{pmatrix} \times & \times & \times \\ \times & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}, \tag{60d}$$

$$M_\nu \sim \begin{pmatrix} \times & \times & 0 \\ \times & 0 & \times \\ 0 & \times & 0 \end{pmatrix}, \tag{60e}$$

$$M_\nu \sim \begin{pmatrix} 0 & \times & \times \\ \times & 0 & \times \\ \times & \times & 0 \end{pmatrix}. \tag{60f}$$

The first three of these forms for M_ν are excluded when taken in conjunction with the U_L matrices in Eqs. (38). Indeed, Eq. (60a) leads to U with four zero matrix elements; either Eq. (60b) or Eq. (60c) lead to U with one zero matrix element; and both those situations are phenomenologically excluded. The only viable forms of M_ν are those that give rise to genuine 3×3 mixing in M_ν, *viz.* Eqs. (60d)–(60f).

One may, without lack of generality, assume the three nonzero matrix elements of M_ν to be real and positive, because, for each of the three matrices in Eqs. (60d)–(60f), there is a diagonal unitary matrix $Y_\psi = \mathrm{diag}(e^{i\psi_1}, e^{i\psi_2}, e^{i\psi_3})$ such that $Y_\psi M_\nu Y_\psi$ is real and has positive nonzero matrix elements. We may thus write the matrices

$$M_A = \begin{pmatrix} a & d & b \\ d & 0 & 0 \\ b & 0 & 0 \end{pmatrix}, \qquad M_B = \begin{pmatrix} a & b & 0 \\ b & 0 & d \\ 0 & d & 0 \end{pmatrix}, \qquad M_C = \begin{pmatrix} 0 & a & b \\ a & 0 & d \\ b & d & 0 \end{pmatrix}, \tag{61}$$

where a, b, and d are positive. One has $M_\nu = Y_\psi^* M_K Y_\psi^*$, where K may be either A, B, or C.

The matrix M_K is diagonalized by the orthogonal matrix O_K:

$$O_K^T M_K O_K = \mathrm{diag}(\mu_1, \mu_2, \mu_3). \tag{62}$$

The real numbers μ_k ($k = 1, 2, 3$) are the eigenvalues of M_K; $|\mu_k| = m_k$ are the neutrino masses.

From Eq. (2b),

$$U_\nu = Y_\psi O_K Y'; \tag{63}$$

the matrix Y' is a diagonal unitary matrix which affects the transformation $\mu_k \to m_k$ in the following way: $Y'_{kk} = 1$ if $\mu_k > 0$ and $Y'_{kk} = i$ if $\mu_k < 0$. So, from Eq. (3), $U = U_L^\dagger Y_\psi O_K Y'$, where U_L is either one of the matrices in Eqs. (38) or one of them with the rows interchanged.

The matrix $U_L^\dagger Y_\psi$ contains four phases—one phase γ in $U_L^\dagger$ and three phases $\psi_{1,2,3}$ in Y_ψ. One may pull three of those phases to the left-hand side of $U_L^\dagger$, leaving at its right-hand side only one phase—let χ denote it. Suppose for instance that Eq. (38a) holds, then

$$
\begin{aligned}
U_L^\dagger Y_\psi &= X^* \begin{pmatrix} e^{i\psi_1} & 0 & 0 \\ 0 & e^{i\psi_2}\cos\theta & -e^{i(\psi_3+\gamma)}\sin\theta \\ 0 & e^{i(\psi_2-\gamma)}\sin\theta & e^{i\psi_3}\cos\theta \end{pmatrix} \\
&= X^* \operatorname{diag}\left(e^{i\psi_1}, e^{i\psi_2}, e^{i(\psi_2-\gamma)}\right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -e^{i\chi}\sin\theta \\ 0 & \sin\theta & e^{i\chi}\cos\theta \end{pmatrix} \\
&= X' \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -e^{i\chi}\sin\theta \\ 0 & \sin\theta & e^{i\chi}\cos\theta \end{pmatrix},
\end{aligned} \tag{64}
$$

where $\chi \equiv \psi_3 - \psi_2 + \gamma$. The matrix $X' \equiv X^* \operatorname{diag}(e^{i\psi_1}, e^{i\psi_2}, e^{i(\psi_2-\gamma)})$ contains unphysical phases.

Thus, there are 18 possible forms for U in $(4, 3)$ textures. Let Kp denote those 18 forms, where K may be either A, B, or C. If $K = B$, then the number p may be $1, 2, \ldots, 9$:

$$
U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -e^{i\chi}\sin\theta \\ 0 & \sin\theta & e^{i\chi}\cos\theta \end{pmatrix} O_B Y' \quad (\text{form } B1); \tag{65a}
$$

$$
U = \begin{pmatrix} 0 & \cos\theta & -e^{i\chi}\sin\theta \\ 1 & 0 & 0 \\ 0 & \sin\theta & e^{i\chi}\cos\theta \end{pmatrix} O_B Y' \quad (\text{form } B2); \tag{65b}
$$

$$
U = \begin{pmatrix} 0 & \cos\theta & -e^{i\chi}\sin\theta \\ 0 & \sin\theta & e^{i\chi}\cos\theta \\ 1 & 0 & 0 \end{pmatrix} O_B Y' \quad (\text{form } B3); \tag{65c}
$$

$$
U = \begin{pmatrix} 0 & 1 & 0 \\ \cos\theta & 0 & -e^{i\chi}\sin\theta \\ \sin\theta & 0 & e^{i\chi}\cos\theta \end{pmatrix} O_B Y' \quad (\text{form } B4); \tag{65d}
$$

$$
U = \begin{pmatrix} \cos\theta & 0 & -e^{i\chi}\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & e^{i\chi}\cos\theta \end{pmatrix} O_B Y' \quad (\text{form } B5); \tag{65e}
$$

$$
U = \begin{pmatrix} \cos\theta & 0 & -e^{i\chi}\sin\theta \\ \sin\theta & 0 & e^{i\chi}\cos\theta \\ 0 & 1 & 0 \end{pmatrix} O_B Y' \quad (\text{form } B6); \tag{65f}
$$

$$
U = \begin{pmatrix} 0 & 0 & 1 \\ \cos\theta & -e^{i\chi}\sin\theta & 0 \\ \sin\theta & e^{i\chi}\cos\theta & 0 \end{pmatrix} O_B Y' \quad (\text{form } B7); \tag{65g}
$$

$$
U = \begin{pmatrix} \cos\theta & -e^{i\chi}\sin\theta & 0 \\ 0 & 0 & 1 \\ \sin\theta & e^{i\chi}\cos\theta & 0 \end{pmatrix} O_B Y' \quad (\text{form } B8); \tag{65h}
$$

$$
U = \begin{pmatrix} \cos\theta & -e^{i\chi}\sin\theta & 0 \\ \sin\theta & e^{i\chi}\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} O_B Y' \quad (\text{form } B9). \tag{65i}
$$

The real orthogonal matrix O_B diagonalizes the real symmetric matrix M_B, see Eqs. (61) and (62).

When one interchanges the second and third rows and columns in the matrix M_A one obtains the same matrix with b and d interchanged; this is just a meaningless renaming of parameters. Similarly, any permutation of the rows and columns of M_C is equivalent to a renaming of the parameters a, b, and d. Therefore, there are nine more possible forms for U:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -e^{i\chi}\sin\theta \\ 0 & \sin\theta & e^{i\chi}\cos\theta \end{pmatrix} O_A Y' \quad \text{(form } A1\text{)}; \tag{66a}$$

$$U = \begin{pmatrix} 0 & \cos\theta & -e^{i\chi}\sin\theta \\ 1 & 0 & 0 \\ 0 & \sin\theta & e^{i\chi}\cos\theta \end{pmatrix} O_A Y' \quad \text{(form } A2\text{)}; \tag{66b}$$

$$U = \begin{pmatrix} 0 & \cos\theta & -e^{i\chi}\sin\theta \\ 0 & \sin\theta & e^{i\chi}\cos\theta \\ 1 & 0 & 0 \end{pmatrix} O_A Y' \quad \text{(form } A3\text{)}; \tag{66c}$$

$$U = \begin{pmatrix} 0 & 1 & 0 \\ \cos\theta & 0 & -e^{i\chi}\sin\theta \\ \sin\theta & 0 & e^{i\chi}\cos\theta \end{pmatrix} O_A Y' \quad \text{(form } A4\text{)}; \tag{66d}$$

$$U = \begin{pmatrix} \cos\theta & 0 & -e^{i\chi}\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & e^{i\chi}\cos\theta \end{pmatrix} O_A Y' \quad \text{(form } A5\text{)}; \tag{66e}$$

$$U = \begin{pmatrix} \cos\theta & 0 & -e^{i\chi}\sin\theta \\ \sin\theta & 0 & e^{i\chi}\cos\theta \\ 0 & 1 & 0 \end{pmatrix} O_A Y' \quad \text{(form } A6\text{)}; \tag{66f}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -e^{i\chi}\sin\theta \\ 0 & \sin\theta & e^{i\chi}\cos\theta \end{pmatrix} O_C Y' \quad \text{(form } C1\text{)}; \tag{66g}$$

$$U = \begin{pmatrix} 0 & \cos\theta & -e^{i\chi}\sin\theta \\ 1 & 0 & 0 \\ 0 & \sin\theta & e^{i\chi}\cos\theta \end{pmatrix} O_C Y' \quad \text{(form } C2\text{)}; \tag{66h}$$

$$U = \begin{pmatrix} 0 & \cos\theta & -e^{i\chi}\sin\theta \\ 0 & \sin\theta & e^{i\chi}\cos\theta \\ 1 & 0 & 0 \end{pmatrix} O_C Y' \quad \text{(form } C3\text{)}. \tag{66i}$$

In Eqs. (65) and (66) the angle θ and the phase χ are free parameters, to be adjusted in order to obtain a good fit of U. The diagonal unitary matrix Y' is in practice irrelevant for phenomenology.

For the parameter of neutrinoless double-beta decay $m_{\beta\beta}$ one easily derives the formulae

$$m_{\beta\beta} = 0 \quad \text{for forms } A2,\ A3,\ A4,\ B4,\ B7,\ \text{and } C1; \tag{67a}$$

$$m_{\beta\beta} = a \quad \text{for forms } A1 \text{ and } B1; \tag{67b}$$

$$m_{\beta\beta} = a\cos^2\theta \quad \text{for forms } B5 \text{ and } B6; \tag{67c}$$

$$m_{\beta\beta} = d\sin 2\theta \quad \text{for forms } B2,\ B3,\ C2,\ \text{and } C3; \tag{67d}$$

$$m_{\beta\beta} = \left| a\cos^2\theta - b e^{-i\chi}\sin 2\theta \right| \quad \text{for forms } A5,\ A6,\ B8,\ \text{and } B9. \tag{67e}$$

3.1. Forms $A1$–6

We first consider the matrix M_A in the first Eq. (61) and its diagonalizing matrix O_A. It is convenient to define $f \equiv \sqrt{b^2+d^2}$ and the angle φ:

$$\cos\varphi = \frac{b}{f}, \qquad \sin\varphi = \frac{d}{f}. \tag{68}$$

The matrix M_A has vanishing determinant. Therefore, *one neutrino is massless* and Eqs. (26)–(28) apply. In case n,

$$M_A = \sqrt{\Delta}\begin{pmatrix} 1-\sqrt{\varepsilon} & \varepsilon^{1/4}\sin\varphi & \varepsilon^{1/4}\cos\varphi \\ \varepsilon^{1/4}\sin\varphi & 0 & 0 \\ \varepsilon^{1/4}\cos\varphi & 0 & 0 \end{pmatrix}, \tag{69a}$$

$$O_A = \begin{pmatrix} 0 & \frac{\varepsilon^{1/4}}{\sqrt{1+\sqrt{\varepsilon}}} & \frac{1}{\sqrt{1+\sqrt{\varepsilon}}} \\ \cos\varphi & -\frac{1}{\sqrt{1+\sqrt{\varepsilon}}}\sin\varphi & \frac{\varepsilon^{1/4}}{\sqrt{1+\sqrt{\varepsilon}}}\sin\varphi \\ -\sin\varphi & -\frac{1}{\sqrt{1+\sqrt{\varepsilon}}}\cos\varphi & \frac{\varepsilon^{1/4}}{\sqrt{1+\sqrt{\varepsilon}}}\cos\varphi \end{pmatrix}. \tag{69b}$$

In case i,

$$M_A = \sqrt{\Delta}\begin{pmatrix} \sqrt{1+\varepsilon}-1 & (1+\varepsilon)^{1/4}\sin\varphi & (1+\varepsilon)^{1/4}\cos\varphi \\ (1+\varepsilon)^{1/4}\sin\varphi & 0 & 0 \\ (1+\varepsilon)^{1/4}\cos\varphi & 0 & 0 \end{pmatrix}, \tag{70a}$$

$$O_A = \begin{pmatrix} \frac{1}{\sqrt{1+\sqrt{1+\varepsilon}}} & \frac{(1+\varepsilon)^{1/4}}{\sqrt{1+\sqrt{1+\varepsilon}}} & 0 \\ -\frac{(1+\varepsilon)^{1/4}}{\sqrt{1+\sqrt{1+\varepsilon}}}\sin\varphi & \frac{1}{\sqrt{1+\sqrt{1+\varepsilon}}}\sin\varphi & \cos\varphi \\ -\frac{(1+\varepsilon)^{1/4}}{\sqrt{1+\sqrt{1+\varepsilon}}}\cos\varphi & \frac{1}{\sqrt{1+\sqrt{1+\varepsilon}}}\cos\varphi & -\sin\varphi \end{pmatrix}. \tag{70b}$$

It is clear from Eqs. (66a)–(66f) that one row of U must coincide, but for the phases contained in Y', with a row of O_A. But, no row of the matrix O_A in Eq. (69b) may possibly coincide with a row of U, therefore *case n is excluded*. This is because:

1. The first row of O_A in Eq. (69b) contains a zero matrix element, while no matrix element of U vanishes.
2. In the second and third rows of Eq. (69b), the second entry is larger in modulus than the third entry by a factor $\varepsilon^{-1/4} \approx 2.4$; this factor is much too small for what is observed in the first row of U and much too large for what is observed in the second and third rows of U.

Coming to case i, either the second row or the third row of O_A in Eq. (70b) may coincide with either the second row or the third row of U. This is because those rows of O_A feature a first entry which is larger in modulus than the second entry by a factor $(1+\varepsilon)^{1/4} \approx 1$; this is compatible with what occurs in either the second or third row of U. Therefore, *models* $A5, 6$ *are viable* (although with some deviation from the mean values of the mixing angles) in case i.

Numerically, we have found that the form $A5$ for U works (with the 3σ data of Ref. [29]) provided $\cos\delta \geq 0.55$ when $\sin^2\theta_{23} = 0.64$; for θ_{23} in the first octant $\cos\delta$ must be even closer to 1, in particular $\cos\delta \geq 0.92$ for $\sin^2\theta_{23} = 0.40$. The mixing angle θ_{12} must also be relatively large:

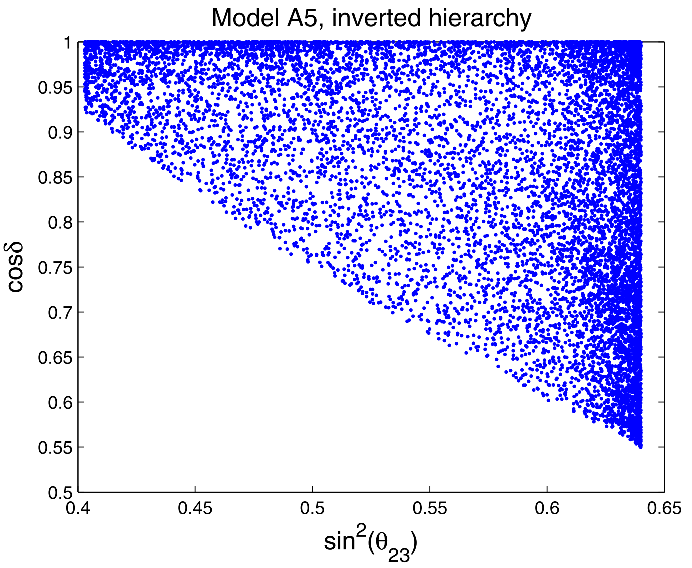

Fig. 2. $\cos\delta$ *vs* $\sin^2\theta_{23}$ for form $A5$. The numeric scan was made using the 3σ data of Ref. [29].

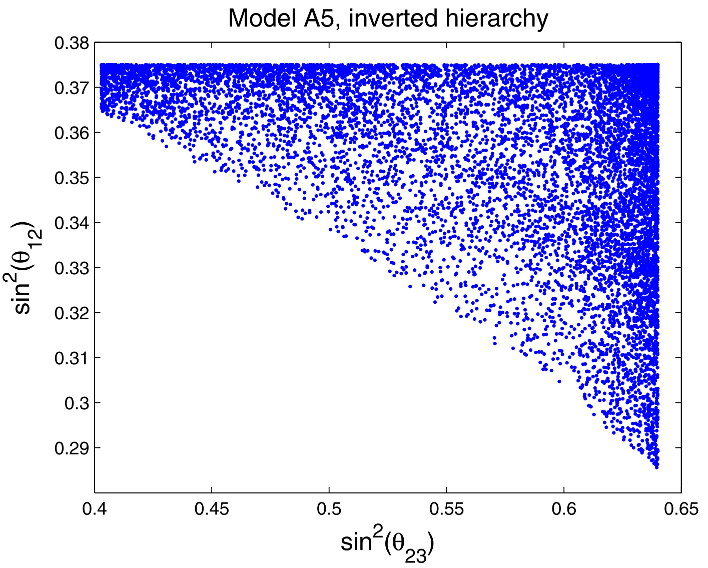

Fig. 3. $\sin^2\theta_{12}$ *vs* $\sin^2\theta_{23}$ for form $A5$. The numeric scan was made using the 3σ data of Ref. [29].

$\sin^2\theta_{12} > 0.285$ for $\sin^2\theta_{23} = 0.64$ and $\sin^2\theta_{12} > 0.365$ for $\sin^2\theta_{23} = 0.40$. These correlations between the angles θ_{12}, θ_{23} and the phase δ can be appreciated in Figs. 2, 3. For form $A6$ of U the results are analogous to those of form $A5$, except that $\cos\delta$ is negative instead of positive and θ_{23} is preferred to be in the first octant instead of in the second one.

Neutrinoless double beta decay is governed by $0.25 < m_{\beta\beta}/\sqrt{\Delta} < 0.33$ in forms $A5, 6$.

3.2. Forms $C1$–3

We next consider the matrix M_C in the third Eq. (61) and its diagonalizing matrix O_C, *cf.* Eq. (62) with $K = C$. Since the trace of M_C is zero, $\mu_1 + \mu_2 + \mu_3 = 0$. Also, $\mu_1\mu_2\mu_3 = 2abd > 0$ and $\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 = -a^2 - b^2 - d^2 < 0$. Therefore, the largest μ_k in absolute value is positive and the other two μ_k are negative. Thus, in case n

$$\frac{\mu_1}{\sqrt{\Delta}} = -\sqrt{\frac{-1 - \varepsilon + 2\sqrt{1 - \varepsilon + \varepsilon^2}}{3}}, \tag{71a}$$

$$\frac{\mu_2}{\sqrt{\Delta}} = -\sqrt{\frac{-1 + 2\varepsilon + 2\sqrt{1 - \varepsilon + \varepsilon^2}}{3}}, \tag{71b}$$

$$\frac{\mu_3}{\sqrt{\Delta}} = \sqrt{\frac{2 - \varepsilon + 2\sqrt{1 - \varepsilon + \varepsilon^2}}{3}}; \tag{71c}$$

while in case i

$$\frac{\mu_1}{\sqrt{\Delta}} = -\sqrt{\frac{1 - \varepsilon + 2\sqrt{1 + \varepsilon + \varepsilon^2}}{3}}, \tag{72a}$$

$$\frac{\mu_2}{\sqrt{\Delta}} = \sqrt{\frac{1 + 2\varepsilon + 2\sqrt{1 + \varepsilon + \varepsilon^2}}{3}}, \tag{72b}$$

$$\frac{\mu_3}{\sqrt{\Delta}} = -\sqrt{\frac{-2 - \varepsilon + 2\sqrt{1 + \varepsilon + \varepsilon^2}}{3}}. \tag{72c}$$

Therefore,

$$\frac{m_1 + m_2 + m_3}{\sqrt{\Delta}} \approx \frac{4 - \varepsilon}{\sqrt{3}} \quad \text{in case n}, \tag{73a}$$

$$\frac{m_1 + m_2 + m_3}{\sqrt{\Delta}} \approx 2 + \varepsilon \quad \text{in case i}. \tag{73b}$$

According to Eqs. (66g)–(66i), if the PMNS matrix is of form Ci ($i = 1, 2, 3$) then its ith row coincides, in the moduli of its matrix elements, with the first row of O_C. It follows from Eq. (62) that

$$(M_C)_{11} = \sum_{j=1}^{3} \mu_j \left[(O_C)_{1j}\right]^2 = 0. \tag{74}$$

Eq. (74), together with the normalization of the first row of O_C, yield

$$\left[(O_C)_{11}\right]^2 = \frac{\mu_2 + (\mu_3 - \mu_2)[(O_C)_{13}]^2}{\mu_2 - \mu_1}, \tag{75a}$$

$$\left[(O_C)_{12}\right]^2 = \frac{-\mu_1 + (\mu_1 - \mu_3)[(O_C)_{13}]^2}{\mu_2 - \mu_1}. \tag{75b}$$

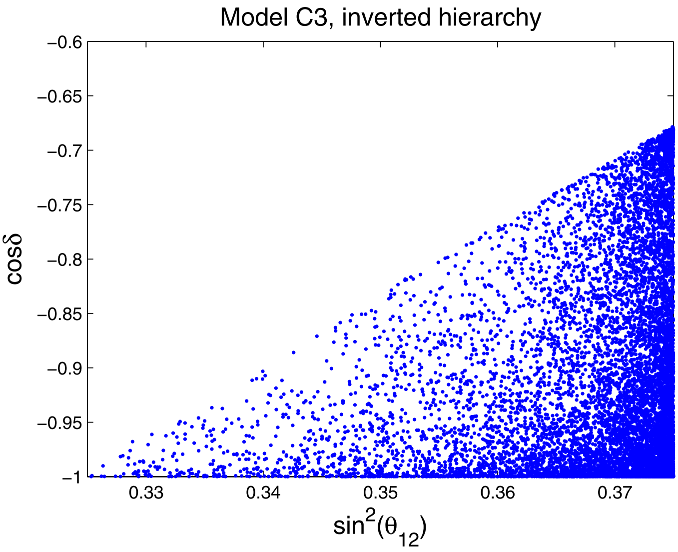

Fig. 4. $\cos\delta$ *vs* $\sin^2\theta_{12}$ for form $C3$. The numeric scan made using the 3σ data of Ref. [29].

Thus, when U has the form Ci,

$$\frac{|U_{i1}|^2}{|U_{i2}|^2} = \frac{\mu_2 + (\mu_3 - \mu_2)|U_{i3}|^2}{-\mu_1 + (\mu_1 - \mu_3)|U_{i3}|^2}. \tag{76}$$

One may use the expressions of $\mu_{1,2,3}$ in either Eqs. (71) or Eqs. (72)—for cases n and i, respectively—together with $|U_{i3}|^2$ to compute $|U_{i1}/U_{i2}|^2$ through Eq. (76). One can in this way find out for which values of ε and of the parameters of U the form Ci agrees with experiment. We have found that form $C1$ is incompatible with the phenomenology, while both forms $C2$ and $C3$ are viable, but only for the case of an inverted hierarchy. Form $C2$ predicts $\cos\delta \gtrsim 0.67$ while form $C3$ predicts $\cos\delta \lesssim -0.67$; both forms predict $0.24 \le m_{\beta\beta}/\sqrt{\Delta} \le 0.34$; furthermore, these forms only work for $\sin^2\theta_{12} \gtrsim 0.325$, *cf.* Fig. 4.

3.3. Forms B1–9

The mass matrix M_B in the second Eq. (61) is of 'Fritzsch type' [32]. The exact diagonalization of a Fritzsch mass matrix has been known for a long time [33]. The use of Fritzsch-type mass matrices in the lepton sector has been proposed before [34].

With M_B the neutrino masses are not fixed. In case n, m_1 is the smallest neutrino mass, $m_2 = (m_1^2 + \delta)^{1/2}$, $m_3 = (m_1^2 + \Delta)^{1/2}$, and

$$a = m_3 - m_2 + m_1, \tag{77a}$$

$$b = \sqrt{\frac{(m_3 - m_2)(m_3 + m_1)(m_2 - m_1)}{a}}, \tag{77b}$$

$$d = \sqrt{\frac{m_3 m_2 m_1}{a}}. \tag{77c}$$

Then,

$$O_B = \begin{pmatrix} -\sqrt{\frac{(m_3+m_1)(m_2-m_1)m_1}{(m_3-m_1)a(m_2+m_1)}} & \sqrt{\frac{(m_3-m_2)m_2(m_2-m_1)}{(m_3+m_2)a(m_2+m_1)}} & \sqrt{\frac{m_3(m_3-m_2)(m_3+m_1)}{(m_3+m_2)(m_3-m_1)a}} \\ \sqrt{\frac{(m_3-m_2)m_1}{(m_3-m_1)(m_2+m_1)}} & -\sqrt{\frac{(m_3+m_1)m_2}{(m_3+m_2)(m_2+m_1)}} & \sqrt{\frac{m_3(m_2-m_1)}{(m_3+m_2)(m_3-m_1)}} \\ \sqrt{\frac{m_3(m_3-m_2)m_2}{(m_3-m_1)a(m_2+m_1)}} & \sqrt{\frac{(m_3+m_1)m_3m_1}{(m_3+m_2)a(m_2+m_1)}} & \sqrt{\frac{m_2(m_2-m_1)m_1}{(m_3+m_2)(m_3-m_1)a}} \end{pmatrix}. \tag{78}$$

If the matrix U has form Bp, then one of its rows coincides, in the moduli of its matrix elements, with a row of O_B. Considering the absolute values of the matrix elements in the third column of O_B, one finds that none of them can be equal to either $\sin\theta_{23}\cos\theta_{13}$ or $\cos\theta_{23}\cos\theta_{13}$—they are either too large or too small for that. On the other hand, either $(O_B)_{23}$ or $(O_B)_{33}$ may coincide with $\sin\theta_{13}$. However, whenever this happens the other two matrix elements in the corresponding row of O_B are practically equal in absolute value, which means that θ_{12} would be close to maximal, contradicting phenomenology. We thus conclude that the forms $B1$–9 for U are not viable in case n.

In case i, m_3 is the smallest neutrino mass, $m_1 = (m_3^2 + \Delta)^{1/2}$, $m_2 = (m_3^2 + \Delta + \delta)^{1/2}$, and

$$a = m_2 - m_1 + m_3, \tag{79a}$$

$$b = \sqrt{\frac{(m_2 - m_1)(m_2 + m_3)(m_1 - m_3)}{a}}, \tag{79b}$$

$$d = \sqrt{\frac{m_2 m_1 m_3}{a}}. \tag{79c}$$

Moreover,

$$O_B = \begin{pmatrix} \sqrt{\frac{(m_2-m_1)m_1(m_1-m_3)}{(m_2+m_1)a(m_1+m_3)}} & \sqrt{\frac{m_2(m_2+m_3)(m_2-m_1)}{(m_2+m_1)(m_2-m_3)a}} & -\sqrt{\frac{(m_2+m_3)(m_1-m_3)m_3}{(m_2-m_3)a(m_1+m_3)}} \\ -\sqrt{\frac{(m_2+m_3)m_1}{(m_2+m_1)(m_1+m_3)}} & \sqrt{\frac{m_2(m_1-m_3)}{(m_2+m_1)(m_2-m_3)}} & \sqrt{\frac{(m_2-m_1)m_3}{(m_2-m_3)(m_1+m_3)}} \\ \sqrt{\frac{(m_2+m_3)m_2m_3}{(m_2+m_1)a(m_1+m_3)}} & \sqrt{\frac{m_1(m_1-m_3)m_3}{(m_2+m_1)(m_2-m_3)a}} & \sqrt{\frac{m_2(m_2-m_1)m_1}{(m_2-m_3)a(m_1+m_3)}} \end{pmatrix}. \tag{80}$$

In this case one finds that either the first or the third row of O_B are suitable to fit either the second or the third row of U; this means that the forms $B2$, $B3$, $B8$, and $B9$ of U are viable.

Forms $B2$ and $B8$ predict a positive $\cos\delta$: $\cos\delta > 0.37$ for form $B2$ and $\cos\delta > 0.58$ for form $B8$. They both privilege higher-than-average θ_{12} and θ_{23}—both angles are not allowed to be simultaneously below their best-fit values. The overall scale of the neutrino masses is given by $2.023 \le (m_1 + m_2 + m_3)/\sqrt{\Delta} \le 2.050$ (2.047) for form $B2$ ($B8$); neutrinoless $\beta\beta$ decay is governed by $0.24 \le m_{\beta\beta}/\sqrt{\Delta} \le 0.47, 0.41$ for forms $B2$ and $B8$, respectively. We also find some broad correlation between these mass ratios and $\sin^2\theta_{23}$, as may be seen in Figs. 5 and 6 for the form $B8$. Similar correlations occur for form $B2$, but there the variation is opposite: $(m_1 + m_2 + m_3)/\sqrt{\Delta}$ increases with $\sin^2\theta_{23}$ and $m_{\beta\beta}/\sqrt{\Delta}$ decreases.

Forms $B3$ and $B9$ of U are similar to forms $B2$ and $B8$, respectively, with the interchange $\nu_\mu \leftrightarrow \nu_\tau$. Therefore the predictions are broadly similar, only $\cos\delta$ is predicted to be negative instead of positive and θ_{23} is expected to be small, *viz.* in the first octant, rather than large. Again, there are hints of correlations of mass parameters with $\sin^2\theta_{23}$, similar to those found for models $B2$ and $B8$, but they now appear reversed: for model $B3$, $(m_1 + m_2 + m_3)/\sqrt{\Delta}$ decreases with $\sin^2\theta_{23}$ and $m_{\beta\beta}/\sqrt{\Delta}$ increases (the exact opposite of what occurred for model $B2$). Likewise, the behaviour of these correlations for model $B9$ is the opposite of what occurred for model $B8$.

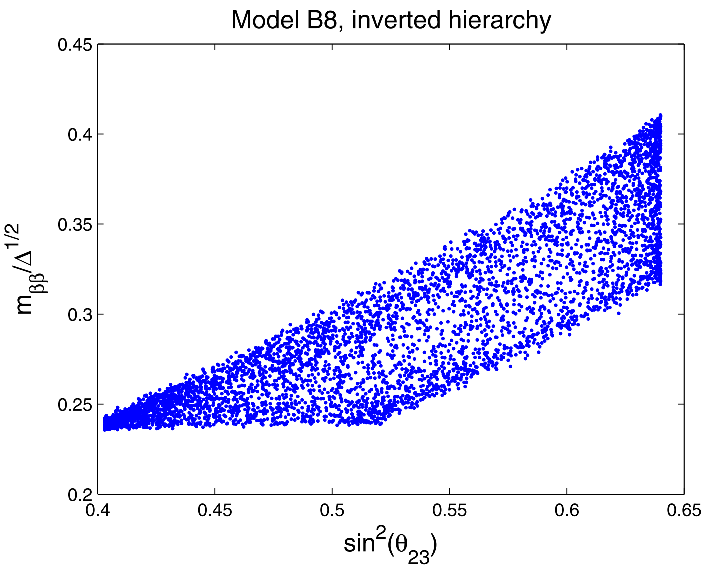

Fig. 5. $m_{\beta\beta}/\sqrt{\Delta}$ *vs* $\sin^2\theta_{23}$ in form $B8$. The numeric scan was made by using the 3σ data of Ref. [29].

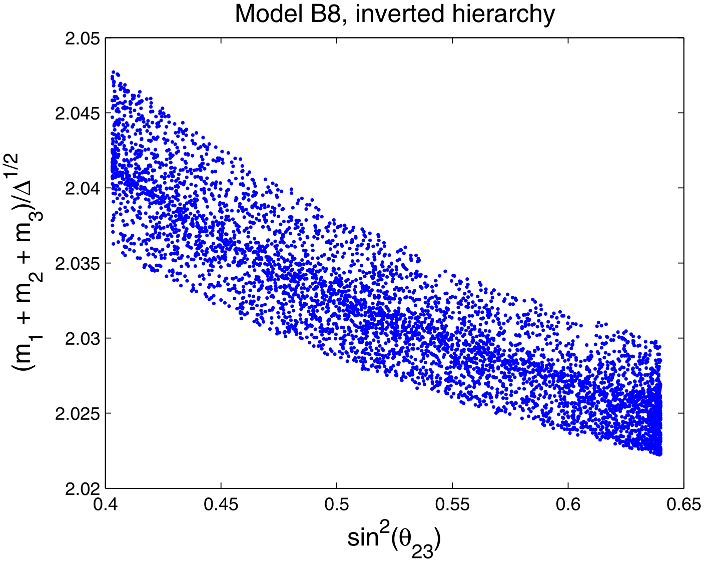

Fig. 6. $(m_1 + m_2 + m_3)/\sqrt{\Delta}$ *vs* $\sin^2\theta_{23}$ for form $B8$. The numeric scan was made by using the 3σ data of Ref. [29].

4. Synopsis

In this paper we have found that there are six $(5, 2)$ textures that are still viable: they are listed in Eqs. (20), wherein $H_{12} = 0$ is the consequence of one of the M_ℓ textures in Eqs. (8). As for $(4, 3)$ textures, there are eight of them which agree with present-day phenomenology; the corresponding forms the lepton mixing matrix are given in Eqs. (65b), (65c), (65h), (65i), (66e), (66f), (66h), and (66i); the corresponding textures of M_ν are those in Eqs. (60d)–(60f).[5]

Even though there such a large variety of viable textures, the same cannot be said about the ensuing predictions, which are broadly similar for all of them: all the viable textures only tolerate

- an inverted neutrino mass spectrum,
- an overall scale of the neutrino masses given by $(m_1 + m_2 + m_3)/\sqrt{\Delta}$ in the range [2.0, 2.1],
- $\cos\delta$ far away from 0, *i.e.* close to either $+1$ or -1, and
- neutrinoless double-beta decay governed by $m_{\beta\beta}/\sqrt{\Delta} \in [0.24, 0.48]$.

We thus conclude that texture-zero models of the (5, 2) and (4, 3) varieties are quite monotonous in their predictive power.

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References

[1] Y. Abe, et al., DOUBLE-CHOOZ Collaboration, Indication for the disappearance of reactor electron antineutrinos in the Double Chooz experiment, Phys. Rev. Lett. 108 (2012) 131801, arXiv:1112.6353 [hep-ex].

[2] F.P. An, et al., DAYA-BAY Collaboration, Observation of electron-antineutrino disappearance at Daya Bay, Phys. Rev. Lett. 108 (2012) 171803, arXiv:1203.1669 [hep-ex].

[3] J.K. Ahn, et al., RENO Collaboration, Observation of reactor electron antineutrino disappearance in the RENO experiment, Phys. Rev. Lett. 108 (2012) 191802, arXiv:1204.0626 [hep-ex].

[4] C.S. Lam, Determining horizontal symmetry from neutrino mixing, Phys. Rev. Lett. 101 (2008) 121602, arXiv:0804.2622 [hep-ph].

[5] C.S. Lam, The unique horizontal symmetry of leptons, Phys. Rev. D 78 (2008) 073015, arXiv:0809.1185 [hep-ph].

[6] W. Grimus, L. Lavoura, P.O. Ludl, Is S_4 the horizontal symmetry of tri-bimaximal lepton mixing? J. Phys. G 36 (2009) 115007, arXiv:0906.2689 [hep-ph].

[7] S.F. Ge, D.A. Dicus, W.W. Repko, $\mathbb{Z}_2$ symmetry prediction for the leptonic Dirac *CP* phase, Phys. Lett. B 702 (2011) 220, arXiv:1104.0602 [hep-ph].

[8] H.J. He, F.R. Yin, Common origin of μ–τ and *CP* breaking in neutrino seesaw, baryon asymmetry, and hidden flavor symmetry, Phys. Rev. D 84 (2011) 033009, arXiv:1104.2654 [hep-ph].

[9] R. de A. Toorop, F. Feruglio, C. Hagedorn, Discrete flavour symmetries in light of T2K, Phys. Lett. B 703 (2011) 447, arXiv:1107.3486 [hep-ph].

[10] S.F. Ge, D.A. Dicus, W.W. Repko, Residual symmetries for neutrino mixing with a large θ_{13} and nearly maximal δ_D, Phys. Rev. Lett. 108 (2012) 041801, arXiv:1108.0964 [hep-ph].

[11] R. de, A. Toorop, F. Feruglio, C. Hagedorn, Finite modular groups and lepton mixing, Nucl. Phys. B 858 (2012) 437, arXiv:1112.1340 [hep-ph].

[12] H.J. He, X.J. Xu, Octahedral symmetry with geometrical breaking: new prediction for neutrino mixing angle θ_{13} and *CP* violation, Phys. Rev. D 86 (2012) 111301, arXiv:1203.2908 [hep-ph].

[5] Some of our textures had already been presented, although from a different vantage point, in Ref. [35]. *All* of our textures have been independently derived in a recent paper [36].

[13] D. Hernandez, A.Yu. Smirnov, Lepton mixing and discrete symmetries, Phys. Rev. D 86 (2012) 053014, arXiv:1204.0445 [hep-ph].

[14] C.S. Lam, Finite symmetry of leptonic mass matrices, Phys. Rev. D 87 (2013) 013001, arXiv:1208.5527 [hep-ph].

[15] D. Hernandez, A.Yu. Smirnov, Discrete symmetries and model-independent patterns of lepton mixing, Phys. Rev. D 87 (2013) 053005, arXiv:1212.2149 [hep-ph].

[16] M. Holthausen, K.S. Lim, M. Lindner, Lepton mixing patterns from a scan of finite discrete groups, Phys. Lett. B 721 (2013) 61, arXiv:1212.2411 [hep-ph].

[17] B. Hu, Neutrino mixing and discrete symmetries, Phys. Rev. D 87 (2013) 033002, arXiv:1212.2819 [hep-ph].

[18] D. Hernandez, A.Yu. Smirnov, Relating neutrino masses and mixings by discrete symmetries, Phys. Rev. D 88 (2013) 093007, arXiv:1304.7738 [hep-ph].

[19] S.F. King, T. Neder, A.J. Stuart, Lepton mixing predictions from $\Delta(6n^2)$ family symmetry, Phys. Lett. B 726 (2013) 312, arXiv:1305.3200 [hep-ph].

[20] C. Hagedorn, A. Meroni, L. Vitale, Mixing patterns from the groups $\Sigma(n\phi)$, J. Phys. A 47 (2014) 055201, arXiv:1307.5308 [hep-ph].

[21] L. Lavoura, P.O. Ludl, Residual $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetries and lepton mixing, Phys. Lett. B 731 (2014) 331, arXiv:1401.5036 [hep-ph].

[22] R.M. Fonseca, W. Grimus, Classification of lepton mixing matrices from residual symmetries, J. High Energy Phys. 1409 (2014) 033, arXiv:1405.3678 [hep-ph].

[23] P.H. Frampton, S.L. Glashow, D. Marfatia, Zeroes of the neutrino mass matrix, Phys. Lett. B 536 (2002) 79, arXiv:hep-ph/0201008;
See also Z.Z. Xing, Texture zeros and Majorana phases of the neutrino mass matrix, Phys. Lett. B 530 (2002) 159, arXiv:hep-ph/0201151;
Z.Z. Xing, A full determination of the neutrino mass spectrum from two-zero textures of the neutrino mass matrix, Phys. Lett. B 539 (2002) 85, arXiv:hep-ph/0205032.

[24] L. Lavoura, Zeros of the inverted neutrino mass matrix, Phys. Lett. B 609 (2005) 317, arXiv:hep-ph/0411232.

[25] S. Dev, S. Verma, S. Gupta, R.R. Gautam, Neutrino mass matrices with a texture zero and a vanishing minor, Phys. Rev. D 81 (2010) 053010, arXiv:1003.1006 [hep-ph].

[26] W. Grimus, A.S. Joshipura, L. Lavoura, M. Tanimoto, Symmetry realization of texture zeros, Eur. Phys. J. C 36 (2004) 227, arXiv:hep-ph/0405016.

[27] R.G. Felipe, H. Serôdio, Abelian realization of phenomenological two-zero neutrino textures, Nucl. Phys. B 886 (2014) 75, arXiv:1405.4263 [hep-ph].

[28] S. Weinberg, in: A Festschrift for I.I. Rabi, Annals of the New York Academy of Sciences, 1978.

[29] D.V. Forero, M. Tórtola, J.W.F. Valle, Neutrino oscillations refitted, arXiv:1405.7540 [hep-ph].

[30] G.L. Fogli, E. Lisi, A. Marrone, D. Montanino, A. Palazzo, A.M. Rotunno, Global analysis of neutrino masses, mixings, and phases: entering the era of leptonic *CP* violation searches, Phys. Rev. D 86 (2012) 013012, arXiv:1205.5254 [hep-ph].

[31] M.C. Gonzalez-Garcia, M. Maltoni, J. Salvado, T. Schwetz, Global fit to three neutrino mixing: critical look at present precision, J. High Energy Phys. 1212 (2012) 123, arXiv:1209.3023 [hep-ph];
See also M.C. Gonzalez-Garcia, M. Maltoni, T. Schwetz, Updated fit to three neutrino mixing: status of leptonic *CP* violation, arXiv:1409.5439 [hep-ph].

[32] H. Fritzsch, Weak interaction mixing in the six-quark theory, Phys. Lett. B 73 (1978) 317;
See also H. Fritzsch, Quark masses and flavour mixing, Nucl. Phys. B 155 (1979) 189.

[33] L.F. Li, Comments on the derivation of the mixing angles, Phys. Lett. B 84 (1979) 461.

[34] Z.Z. Xing, Hierarchical neutrino masses and large mixing angles from the Fritzsch texture of lepton mass matrices, Phys. Lett. B 550 (2002) 178, arXiv:hep-ph/0210276;
Z.Z. Xing, S. Zhou, Seesaw invariance of Fritzsch-like neutrino mass matrices, leptogenesis, and lepton flavor violation, Phys. Lett. B 606 (2005) 145, arXiv:hep-ph/0411044.

[35] R. Barbieri, T. Hambye, A. Romanino, Natural relations among physical observables in the neutrino mass matrix, J. High Energy Phys. 0303 (2003) 017, arXiv:hep-ph/0302118.

[36] P.O. Ludl, W. Grimus, A complete survey of texture zeros in the lepton mass matrices, J. High Energy Phys. 1407 (2014) 090, arXiv:1406.3546 [hep-ph].