Comp. & Maths. with Appls. Vol. 9, No. 3, pp. 443–445, 1983
Printed in Great Britain.
0097–4943/83/030443–03$03.00/0
Pergamon Press Ltd.

PARTIAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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(*Received May* 1982)

Abstract—Hyperbolic nonconservative partial differential equations, such as the Von Foerster system, in which boundary conditions may depend upon the dependent variable (integral boundary conditions, for example) are solved by an approximation method based on similar work of the author for (nonlinear stochastic) ordinary differential equations.

In certain problems of physics modelled by nonconservative hyperbolic partial differential equations, the boundary conditions may depend upon the dependent variable. The Von Foerster system, for example, is given by

$$\alpha_1 \frac{\partial u(x,t)}{\partial x} + \alpha_2 \frac{\partial u(x,t)}{\partial t} = -\mu(x,t,u) f_1(u) \tag{1}$$

$$u(x,0) = \phi(x)$$

$$u(0,t) = f_2(u(x,t))$$

where α_1, α_2 are arbitrary constants and f_1, f_2 are arbitrary functions of u. Let's write the right side as

$$-\sum_{i=1}^{n} b_i(x,t) N_i(u) + g(x,t)$$

where $N_i(u)$ are nonlinear functions in u. For this specific problem—let's take $\alpha_1 = \alpha_2 = 1$ for convenience, and consider one term in the sum; now write $L_x = \partial/\partial x$ and $L_t = \partial/\partial t$ as convenient linear operators, then we have

$$L_x u + L_t u + b(x,t) N(u) = g(x,t). \tag{2}$$

We can as well let L_x, L_t represent more general operators, e.g. $L_x = \partial^2/\partial x^2$, $L_t = a(x,t)\partial/\partial t$ and $N(u)$ could depend not only on u but some derivatives of u, and g, b, and coefficients in L_x, L_t could be stochastic, but we will limit ourselves in this discussion to the specified problem in (2) with the first order partials. Solving for $L_x u$ and $L_t u$ in turn,

$$L_x u = g - L_t u - bN(u)$$
$$L_t u = g - L_x u - bN(u).$$

Defining the inverses L_x^{-1}, L_t^{-1},

$$L_1^{-1} L_x u = L_x^{-1} g - L_x^{-1} L_t u - L_x^{-1} bN(u)$$
$$L_t^{-1} L_t u = L_t^{-1} g - L_t^{-1} L_x u - L_t^{-1} bN(u).$$

Thus

$$[L_x^{-1}L_x + L_t^{-1}L_t]u = [L_x^{-1} + L_t^{-1}]g - [L_x^{-1}L_t + L_t^{-1}L_x]u - [L_x^{-1} + L_t^{-1}]bN(u). \tag{3}$$

Examine $L_t^{-1}L_tu$. This is $u(x, t) - u(x, 0)$ for the specific case $L_t = \partial/\partial t$ here. Similarly, $L_x^{-1}L_xu = u(x, t) - u(0, t)$. Hence the left side is $2u(x, t) + u(x, 0) + u(0, t)$ so we can write

$$u(x, t) = (1/2)\{u(x, 0) + [L_x^{-1} + L_t^{-1}]g - [L_x^{-1}L_t + L_t^{-1}L_x]u - [L_x^{-1} + L_t^{-1}]bN(u) + u(0, t)\}.$$

The solution u is decomposed into components Σu_i to be determined assuming

$$u_0 = (1/2)\{u(x, 0) + [L_x^{-1} + L_t^{-1}]g\} \tag{4}$$

noting that $u(0, t)$ is not included in u_0 as it would be if it were not dependent on u but is put on the right side.

We now have

$$u(x, t) = u_0 - (1/2)\{[L_x^{-1}L_t + L_t^{-1}L_x]u - [L_x^{-1} + L_t^{-1}]bN(u) + f_2(u(x, t))\} \tag{5}$$

where $f_2(u) = \int_0^\infty \gamma(x, t)u(x, t)\,dx$ is given, $N(u)$ is, of course, a given nonlinearity, and u_0 is given by (4). We parametrize (5) as

$$u = u_0 - \lambda(1/2)\{[L_x^{-1}L_t + L_t^{-1}L_x]u - \lambda[L_x^{-1} + L_t^{-1}]bN(u) + \lambda\int_0^\infty \gamma(x, t)u(x, t)\,dx\}$$

and assume temporarily $u = \sum_{n=0}^{\infty} \lambda^n u_n$ where λ will later be set equal to one; it is only an aid in collecting terms. Then $N(u) = f(u(\lambda))$ which we set equal to $\sum_{n=0}^{\infty} A_n\lambda^n$ assuming analyticity. If the solution u exists, $u = F^{-1}g$ since the entire left side of (2) can be written Fu where F is a nonlinear operator. Thus, the decomposition of u means $u = \sum_{n=0}^{\infty} \lambda^n F_n^{-1}g$. Hence

$$\sum_{n=0}^{\infty} \lambda^n F_n^{-1}g = u_0 - \lambda(1/2)[L_x^{-1}L_t + L_t^{-1}L_x]\sum_{n=0}^{\infty} \lambda^n F_n^{-1}g - \lambda(1/2)[L_x^{-1} + L_t^{-1}]b\sum \lambda^n A_n + \lambda(1/2)\int_0^\infty \gamma(x, t)\sum_{n=0}^{\infty} \lambda^n F_n^{-1}g\,dx.$$

Consequently, $F_0^{-1}g = u_0$ and

$$F_1^{-1}g = -(1/2)[L_x^{-1}L_t + L_t^{-1}L_x]F_0^{-1}g - (1/2)[L_x^{-1} + L_t^{-1}]bA_0 + (1/2)\int_0^\infty \gamma(x, t)F_0^{-1}g\,dx \text{ etc.}$$

or

$$u_1 = -(1/2)[L_x^{-1}L_t + L_t^{-1}L_x]u_0 - (1/2)[L_x^{-1} + L_t^{-1}]bA_0 + \int_0^\infty \gamma(x, t)u_0(x, t)\,dx$$

$$u_2 = -(1/2)[L_x^{-1}L_t + L_t^{-1}L_x]u_1 - (1/2)[L_x^{-1} + L_t^{-1}]bA_1 + (1/2)\int_0^\infty \gamma(x, t)u_1(x, t)\,dx \text{ etc.}$$

It remains then to discuss the A_n. For simple (non-differential) operators N in $N(u)$ it is quite easy to find the A_n (and the techniques can be extended even to $N(u, u', \ldots, u^{(n)})$ if needed). A_0 will depend only on u_0, A_1 will depend on u_0, u_1, A_2 will depend on u_0, u_1, u_2, etc., and can be found from

$$A_n = (1/n!)D^n f|_{\lambda=0}$$

where $D = d/d\lambda = (du/d\lambda)(d/du)$ since $f = f(u)$ and $u = u(\lambda)$. If we define the polynomials

$$h_n(u_0) = \frac{d^n}{du^n} f(u(\lambda))|_{\lambda=0}$$

then

$$A_0 = h_0(u_0)$$

$$A_1 = h_1(u_0)u_1$$

$$A_2 = (1/2!)\{h_2(u_0)u_1^2 + 2h_1(u_0)u_2\}$$

$$A_3 = (1/3!)\{h_3(u_0)u_1^3 + 6h_2(u_0)u_1u_2 + 6h_1(u_0)u_3\}$$

$$A_4 = (1/4!)\{h_4(u_0)u_1^4 + 12h_3(u_0)u_1^2u_2 + h_2(u_0)[12u_2^2 + 24u_1u_3] + 24h_1(u_0)u_4\} \text{ etc.}$$

The A_n have now been calculated to A_{10} and offer no difficulty to go higher. A_n for more complicated nonlinear functions involving derivatives have also been calculated and will be published along with various applications such as Van der Pol and Duffing oscillators and soliton equations.

The solution has been written in such a form that higher order derivatives for L_x, L_t are easily dealt with. In that case u_0 will involve more terms involving derivatives of u evaluated at the boundaries (analogous to our work for ordinary differential equations.) Other generalizations are readily apparent to higher dimensional equations, for example. In the Von Foerster system, if we suppose $f_1(u) = u^2$ we have then $A_0 = u_0^2$, $A_1 = 2u_0u_1$, $A_2 = u_1^2 + 2u_0u_2$, $A_3 = 2(u_1u_2 + u_0u_3)$, etc.

Consequently, using u_0 from (4) and

$$u_0 = \phi(x)$$

$$u_1 = -[L_x^{-1}L_t + L_t^{-1}L_x]\phi(x) - [L_x^{-1} + L_t^{-1}]b\phi^2(x) + \int_0^\infty \gamma(x, t)\phi(x)\,dx$$

$$u_2 = -[L_x^{-1}L_t + L_t^{-1}L_x]u_1 - [L_x^{-1} + L_t^{-1}]b(2u_0u_1) + \int_0^\infty \gamma(x, t)u_1\,dx$$

$$\vdots$$

involving simple differentiations and integrations. The solution $u = u_0 + u_1 + u_2 + \cdots$ and is given to some acceptable accuracy by the n term expression $u_0 + u_1 + \cdots u_{n-1}$.