Hyperbolic polynomial diffeomorphisms of $\mathbb{C}^2$. II: Hubbard trees

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Abstract

This paper is a sequel to Part I [Y. Ishii, Hyperbolic polynomial diffeomorphisms of $\mathbb{C}^2$. I: A non-planar map, Adv. Math. 218 (2) (2008) 417–464]. In the current article we construct an object analogous to a Hubbard tree consisting of a pair of trees decorated with loops and a pair of maps between them for a hyperbolic polynomial diffeomorphism $f$ of $\mathbb{C}^2$. Key notions in the construction are the pinching disks and the pinching locus which determine how local dynamical pieces are glued together to obtain a global picture. It is proved that the shift map on the orbit space of a Hubbard tree is topologically conjugate to $f$ on its Julia set. Several examples of Hubbard trees are also given.
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1. Introduction and statements of the main results

As the name indicates, Hubbard trees for polynomial maps in one complex variable have been first introduced by J.H. Hubbard in the well-known “Orsay Notes” [4]. A Hubbard tree describes how the orbits of the critical points of a polynomial are sitting inside its Julia set. In this sense, it can be interpreted as a complex extension of the kneading sequence for maps of the interval [13]. In fact, in Exposé VI of [4] the Hubbard tree has been used to establish the rigidity of the real quadratic maps $p_c(x) = x^2 + c$ with superattractive cycles, which finally implies the monotonicity of the topological entropy of $p_c$ on the real line with a help of the kneading theory [13].

This article is a sequel to Part I [8]. The purpose of the current paper Part II is to construct an object analogous to a Hubbard tree consisting of a pair of trees decorated with loops and a pair of maps between them for certain complex Hénon map or, more generally, a polynomial diffeomorphism $f$ of $\mathbb{C}^2$. We also show that a Hubbard tree gives a topological model for the dynamics of $f$ on its Julia set in terms of its finite data. Recall that a polynomial diffeomorphism of $\mathbb{C}^2$ with non-trivial dynamics is, up to conjugacy, expressed as the composition of finitely many generalized Hénon maps $f_{p,b} : (x, y) \mapsto (p(x) - by, x)$, where $p$ is a polynomial in one variable with degree at least two and $b$ is a non-vanishing complex constant [6].

Let us state the main results of this article. Let $\{A_\varepsilon\}_{\varepsilon \in \Sigma}$ be a family of finitely many Poincaré boxes in $\mathbb{C}^2$. A Poincaré box $A_\varepsilon$ is an open subset in $\mathbb{C}^2$ which is biholomorphic to a product set of the form $A_x,\varepsilon \times A_y,\varepsilon$ where $A_x,\varepsilon$ and $A_y,\varepsilon$ are bounded open subsets of $\mathbb{C}$, and at each point of $A_\varepsilon$ two kinds of cone fields, the horizontal Poincaré cone field and the vertical Poincaré cone field, are equipped (see Definition 2.5). Hereafter we always assume that $A_x,\varepsilon$ is connected and $A_y,\varepsilon$ is connected and simply connected. We put $A \equiv \bigcup_{\varepsilon \in \Sigma} A_\varepsilon$ and let an injective holomorphic map $f : A \cap f^{-1}(A) \to A$ be a hyperbolic system over $\Gamma \subset \Sigma \times \Sigma$ (see Definition 3.1). This is an adaptation to our setting of hyperbolicity for a polynomial diffeomorphism of $\mathbb{C}^2$ with respect to the two kinds of Poincaré cone fields. We will introduce five assumptions, Assumption 1 to Assumption 5, on a hyperbolic system in the sequel.
The first main result of this article is

**Theorem A.** Let \( \{A_\epsilon\}_{\epsilon \in \Sigma} \) be a family of finitely many Poincaré boxes and let \( f : A \cap f^{-1}(A) \to A \) be a hyperbolic system over \( \Gamma \subset \Sigma \times \Sigma \) satisfying Assumptions 2 to 5. Then, one can construct a Hubbard tree \( \iota_T, \tau : T^1 \to T^0 \) from the hyperbolic system.

See Definition 4.5 for more details on the definition of a Hubbard tree. Fig. 1 above describes the Hubbard tree of a cubic Hénon map which has been shown in [8] to be hyperbolic, i.e. its Julia set is a hyperbolic set, but non-planar, i.e. the map is not topologically conjugate on the Julia set to a small perturbation of any expanding polynomial in one variable. The dotted arrows in the figure mean degree one transitions and the solid arrow is a degree three transition between the loops by \( \tau \). The other map \( \iota_T \) smashes the six small loops and the associated six short edges in \( T^1 \) into the unique dot in \( T^0 \), and two right-bottom edges in \( T^1 \) to points in the right-bottom loop in \( T^0 \). The dots in \( T^0 \) and \( T^1 \) represent the points in the pinching loci (see Definition 3.5). The procedure to construct a Hubbard tree as well as the precise statements of the assumptions are presented in Sections 3 and 4.

The pair of maps \( \iota_T, \tau : T^1 \to T^0 \) induces the space of bi-infinite orbits:

\[
T^\infty = \{(t_i)_{i \in \mathbb{Z}} \in (T^1)^\mathbb{Z} : \tau(t_i) = \iota_T(t_{i+1})\}
\]

as well as the shift map \( \tau : T^\infty \to T^\infty \) on it. Let \( A^\infty \equiv \bigcap_{n \in \mathbb{Z}} f^n(A) \) and consider the restriction \( f : A^\infty \to A^\infty \). Our second main result (Theorem 5.17) is

**Theorem B.** Let \( \{A_\epsilon\}_{\epsilon \in \Sigma} \) be a family of finitely many Poincaré boxes and let \( f : A \cap f^{-1}(A) \to A \) be a hyperbolic system over \( \Gamma \subset \Sigma \times \Sigma \) satisfying Assumptions 1 to 5. Then, the shift map \( \tau : T^\infty \to T^\infty \) is topologically conjugate to \( f : A^\infty \to A^\infty \).
Now, let $f$ be a polynomial diffeomorphism of $\mathbb{C}^2$ and denote by $J_f$ the Julia set of $f$. As an immediate consequence of Theorem B above, we are able to obtain the following result (Corollary 5.18).

**Corollary C.** Let $\{A_\varepsilon\}_{\varepsilon \in \Sigma}$ be a family of finitely many Poincaré boxes and let $f$ be a polynomial diffeomorphism of $\mathbb{C}^2$ so that $f : A \cap f^{-1}(A) \rightarrow A$ is a hyperbolic system over $\Gamma \subset \Sigma \times \Sigma$ satisfying Assumptions 1 to 5. Assume that $A^\infty$ is hyperbolic for $f$ and $J_f \subset A$. Then the shift map $\tau : T^\infty \rightarrow T^\infty$ is topologically conjugate to $f : J_f \rightarrow J_f$.

Thus, a Hubbard tree represents the combinatorial, dynamical and topological information of the Julia set of $f$ in terms of its finite data. See Section 6, where several examples of the Hubbard trees for complex Hénon maps are presented.

One of the main ingredients of the proof of Theorem B is the homotopy shadowing theorem developed in [10] which roughly states that a “homotopy equivalence” between two expanding/hyperbolic dynamical systems induces a topological conjugacy between the shift maps on their orbit spaces. By using this result, in the first step we construct a conjugacy between the Hénon map on its Julia set and the shift map on the orbit space of the branched surface model, and in the second step we construct a conjugacy between the shift maps on the orbit spaces of the branched surface model and of the Hubbard tree. In order to prove this second part, we define a metric in the Hubbard tree so that it becomes an expanding dynamical system by using the Perron–Frobenius theory (see Proposition 4.4), and apply the homotopy shadowing theorem. We note that this method provides a new proof of a well-known result for complex one-dimensional polynomials as in Theorem 2.2.

The structure of this article is as follows. In Section 2, we recall the construction of a Hubbard tree in the complex one-dimensional case, generalize some definitions and basic facts on hyperbolic systems established in [8,10] to adapt them to the setting of the current paper, and outline the proof of Theorem A. In Section 3, we first construct the branched surface model starting from a hyperbolic system. Key notions in the construction are the pinching disks and the pinching locus which determine how local dynamical pieces are glued together to obtain a global picture. In Section 4, we define a Hubbard tree starting from the branched surface model, which concludes the proof of Theorem A. Section 5 is dedicated to the proof of Theorem B. In Section 6, we present three types of examples of Hubbard trees. The first one is the hyperbolic cubic Hénon map which cannot be obtained as a small perturbation [8] described in Fig. 1. The second example consists of small perturbations of expanding polynomials in one variable. The third one is a crossed mapping model for a Hénon map with connected Julia set. At the end of this article we present a problem on the canonical construction of a Hubbard tree. We believe this will be crucial for further study of the parameter space for the complex Hénon family from a combinatorial point of view. In particular, a solution to the problem together with the Hubbard trees presented in this article may enables us to define the concept of “limbs” in the parameter space of the Hénon family.

In a forthcoming paper [9] we plan to compare our combinatorial description for the Julia sets of Hénon maps in terms of the Hubbard trees with other methods such as quotients of solenoids [2,14].
2. Background material and sketch of construction

In this section we summarize some background material which will be used to prove Theorems A and B in Section 1 and sketch how to construct Hubbard trees for some polynomial diffeomorphisms of $\mathbb{C}^2$.

2.1. Hubbard trees in dimension one

First we recall a recipe to construct a Hubbard tree for the quadratic map:

$$p_c(z) = z^2 + c \quad (c \in \mathbb{C})$$

defined on $\mathbb{C}$ based on an excellent survey [3] (see also Appendix of [12]). The method of the construction explained here will be employed on the way to construct Hubbard trees for polynomial diffeomorphisms of $\mathbb{C}^2$.

Let $K_c \equiv \{ z \in \mathbb{C} : \{ p_c^n(z) \}_{n \geq 0} \text{ is bounded in } \mathbb{C} \}$ be the filled Julia set of $p_c$ and let $J_c \equiv \partial K_c$ be its Julia set. We here consider the case where $J_c$ is connected and expanding for $p_c$. On may then assume that $p_c$ has a unique superattractive cycle denoted by $C$.

For each connected component $U$ of $\text{Int } K_c$, we fix a point $p \in U$ and choose a homeomorphism $\phi_U : U \to \Delta$ so that $\phi_U^{-1}(0) = p$, where $\Delta$ is an open unit disk in the complex plane. The point $p = \phi_U^{-1}(0)$ is called the center of $U$. We may choose $p \in U$ so that any point in $\mathcal{C}$ is a center. An arc of the form $\phi_U^{-1}((re^{i\theta} : 0 \leq r \leq 1))$ is called a ray of $U$. An arc $\gamma$ in $K_c$ is called a legal arc if, for any connected component $U$ of $\text{Int } K_c$, $\gamma \cap U$ is contained in the union of two rays of $U$. Note that for any two points $z_1$ and $z_2$ in $K_c$, there exists a unique legal arc having $z_1$ and $z_2$ as its extremities. We denote the unique legal arc by $[z_1, z_2]$. Given a finite set of points $\{z_1, \ldots, z_N\}$ in $K_c$, the union $\bigcup_{1 \leq i, j \leq N} [z_i, z_j]$ is called the legal hull of $\{z_1, \ldots, z_N\}$.

Let $H_0$ be the legal hull of $\mathcal{C}$ and $H_1$ be the legal hull of $p_c^{-1}(\mathcal{C})$. We replace each point of $\mathcal{C}$ in $H_0$ by a loop to obtain $T_0$ and each point of $p_c^{-1}(\mathcal{C})$ in $H_1$ by a loop to obtain $T_1$. Then, the quadratic polynomial $p_c$ naturally induces a covering map $\tau : T_1 \to T_0$. Since $T_0$ can be seen as a subset of $T_1$, we can also define a “smashing” map $\iota_T : T_1 \to T_0$ by letting $\iota_T$ be the identity map on $T_0(\subset T_1)$ and shrink the rest $T_1 \setminus T_0$ to appropriate points in $T_0$ so that $\iota_T$ becomes continuous. Then, we obtain a pair of spaces together with a pair of maps between them:

$$\iota_T, \tau : T_1 \longrightarrow T_0,$$

which we call the (decorated) Hubbard tree for $p_c$.

Remark 2.1. The definition of Hubbard trees has been first appeared in [4, Exposé IV] for post-critically finite polynomials in one variable. As a combinatorial model for the filled Julia set of an expanding polynomial, a similar object called “un arbre décoré” in [4, Exposé VI] as well as
A Hubbard tree $\iota, \tau : T^1 \to T^0$ induces the one-sided orbit space:

$$T^+ \equiv \{(t_i)_{i \geq 0} \in (T^1)^\mathbb{N} : \tau(t_i) = \iota(t_{i+1})\}$$

and the shift map $\tau : T^+ \to T^+$ on it. The following result is well-known.

**Theorem 2.2.** Assume that the Julia set $J_c$ of $p_c$ is connected and expanding. Then, the restriction $p_c : J_c \to J_c$ is topologically conjugate to the shift map $\tau : T^+ \to T^+$.

See [3] for a purely topological proof of this fact (the author thanks Peter Haissinsky for communicating the details of the proof). As a by-product of our discussion we give its metrical new proof in Proposition 5.15.

**Remark 2.3.** There is another construction of a Hubbard tree à la Thurston’s lamination theory [15]. This is called the “pinched disk model” in [3].

### 2.2. Multivalued dynamical systems

In this subsection we recall as well as slightly generalize the setting and the results in [8,10] for our purpose.

As in [10, Definition 2.1] we first introduce the following generalization of the notion of a dynamical system.

**Definition 2.4.** A pair of spaces $X^0$ and $X^1$ with a pair of maps $\iota, \varphi : X^1 \to X^0$ is called a multivalued dynamical system. It is also denoted as $\mathcal{X} = (X^0, X^1 ; \iota, \varphi)$.

A multivalued dynamical system $\iota, \varphi : X^1 \to X^0$ induces pull-backs $\iota, \varphi : X^{m+1} \to X^m$ $(m \geq 1)$ as well as the space of bi-infinite orbits:

$$X^\infty \equiv \{(x_i)_{i \in \mathbb{Z}} \in (X^1)^\mathbb{Z} : \varphi(x_i) = \iota(x_{i+1})\}$$

with the shift map $\varphi : X^\infty \to X^\infty$ on it. An element of $X^\infty$ is called an orbit of the multivalued dynamical system $\iota, \varphi : X^1 \to X^0$.

There are two important classes of multivalued dynamical systems. One is a hyperbolic system and another is an expanding system. First we recall the notion of a hyperbolic system in the special case of complex dimension two.

Let $A_x$ and $A_y$ be bounded open sets in $\mathbb{C}$ and let $|\cdot|_{A_x}$ and $|\cdot|_{A_y}$ be Poincaré metrics in $A_x$ and $A_y$ respectively. Let us define a cone field in $\mathcal{A} = A_x \times A_y$ in terms of the “slope” with respect to the Poincaré metrics in $A_x$ and $A_y$ as

$$C^h_p \equiv \{v = (v_x, v_y) \in T_p \mathcal{A} : |v_x|_{A_x} \geq |v_y|_{A_y}\}$$
and a metric in this cone by $\|v\|_h = |D\pi_x(v)|_{A_x}$. Similarly we define

$$C^v_p \equiv \{ v = (v_x, v_y) \in T_p\mathcal{A}: |v_x|_{A_x} \leq |v_y|_{A_y} \}$$

and a metric in this cone by $\|v\|_v = |D\pi_x(v)|_{A_x}$. We call $\{(C^v_p)_{p \in A}, \|\cdot\|_h \}$ the horizontal Poincaré cone field in $\mathcal{A}$ and $\{(C^v_p)_{p \in A}, \|\cdot\|_v \}$ the vertical Poincaré cone field in $\mathcal{A}$.

**Definition 2.5.** An open subset $\mathcal{A}$ in $\mathbb{C}^2$ which is biholomorphic to a product set of the form $A_x \times A_y$ equipped with the induced horizontal and the vertical Poincaré cone fields is called a Poincaré box.

As in [8, Corollaries 2.17 and 2.18] Poincaré boxes have been used to give criteria for hyperbolicity of polynomial diffeomorphisms of $\mathbb{C}^2$.

Let $\mathcal{A}^D$ and $\mathcal{A}^R$ be two Poincaré boxes in $\mathbb{C}^2$ which are biholomorphic to product sets $A_x^D \times A_y^D$ and $A_x^R \times A_y^R$ respectively. Let $\iota_A : \iota_A^{-1}(\mathcal{A}^D) \cap f^{-1}(\mathcal{A}^R) \to \mathcal{A}^D$ be the inclusion and $f : \iota_A^{-1}(\mathcal{A}^D) \cap f^{-1}(\mathcal{A}^R) \to \mathcal{A}^R$ be a holomorphic injection defined on $\iota_A^{-1}(\mathcal{A}^D) \cap f^{-1}(\mathcal{A}^R) \subset \mathbb{C}^2$. Below, the pair of these maps $\iota_A : \iota_A^{-1}(\mathcal{A}^D) \cap f^{-1}(\mathcal{A}^R) \to \mathcal{A}^D$ and $f : \iota_A^{-1}(\mathcal{A}^D) \cap f^{-1}(\mathcal{A}^R) \to \mathcal{A}^R$ is denoted as

$$\mathcal{A}^D \overset{\iota_A^{-1}}{\leftarrow} \iota_A^{-1}(\mathcal{A}^D) \cap f^{-1}(\mathcal{A}^R) \overset{f}{\to} \mathcal{A}^R$$

and is again called a multivalued dynamical system in an extended sense. When $\mathcal{A}^D = \mathcal{A}^R = \mathcal{A}$, we write

$$\iota_A, f : \iota_A^{-1}(\mathcal{A}) \cap f^{-1}(\mathcal{A}) \to \mathcal{A}.$$

Let $\pi_x^D : \mathcal{A}^D \to A_x^D$ (resp. $\pi_x^R : \mathcal{A}^R \to A_x^R$) and $\pi_y^D : \mathcal{A}^D \to A_y^D$ (resp. $\pi_y^R : \mathcal{A}^R \to A_y^R$) be the maps induced from the projections to each coordinate.

**Definition 2.6.** A multivalued dynamical system $\mathcal{A}^D \overset{\iota_A^{-1}}{\leftarrow} \iota_A^{-1}(\mathcal{A}^D) \cap f^{-1}(\mathcal{A}^R) \overset{f}{\to} \mathcal{A}^R$ is called a crossed mapping if

$$\rho_f \equiv (\pi_x^R \circ f, \pi_y^D \circ \iota_A) : \iota_A^{-1}(\mathcal{A}^D) \cap f^{-1}(\mathcal{A}^R) \to A_x^R \times A_y^D$$

is proper of degree $d \geq 1$ (see [10, Definition 5.1]).

In particular, it follows that $\iota_A^{-1}(\mathcal{A}^D) \cap f^{-1}(\mathcal{A}^R)$ is non-empty for a crossed mapping $\mathcal{A}^D \overset{\iota_A^{-1}}{\leftarrow} \iota_A^{-1}(\mathcal{A}^D) \cap f^{-1}(\mathcal{A}^R) \overset{f}{\to} \mathcal{A}^R$ since the degree of $\rho_f$ is at least one.

We denote by $\{(C^D_z)_{z \in \mathcal{A}^D}, \|\cdot\|_D \}$ the horizontal Poincaré cone field in $\mathcal{A}^D$ and denote by $\{(C^R_z)_{z \in \mathcal{A}^R}, \|\cdot\|_R \}$ the horizontal Poincaré cone field in $\mathcal{A}^R$. A multivalued dynamical system $\mathcal{A}^D \overset{\iota_A^{-1}}{\leftarrow} \iota_A^{-1}(\mathcal{A}^D) \cap f^{-1}(\mathcal{A}^R) \overset{f}{\to} \mathcal{A}^R$ is said to expand the pair of the horizontal Poincaré cone fields if there exists $\lambda > 1$ so that for any $z \in \iota_A^{-1}(\mathcal{A}^D) \cap f^{-1}(\mathcal{A}^R)$, we have $Df^{-1}(C^D_{\iota_A(z)}) \subset Df^{-1}(C^R_{\iota_A(z)})$ and $\lambda \|D\iota_A(v)\|_D \leq \|Df(v)\|_R$ for any $v \in T_z\mathbb{C}^2$ with $D\iota_A(v) \subset C^D_{\iota_A(z)}$. Similarly, we say that $\mathcal{A}^D \overset{\iota_A^{-1}}{\leftarrow} \iota_A^{-1}(\mathcal{A}^D) \cap f^{-1}(\mathcal{A}^R) \overset{f}{\to} \mathcal{A}^R$ contracts the pair of the vertical
Poincaré cone fields if \( \mathcal{A}^\mathbb{R} \overset{f}{\leftarrow} f^{-1}(\mathcal{A}^\mathbb{R}) \cap \iota^{-1}_A(\mathcal{A}^\mathbb{D}) \overset{\iota_A}{\rightarrow} \mathcal{A}^\mathbb{D} \) expands the pair of the vertical Poincaré cone fields.

**Definition 2.7.** A crossed mapping \( \mathcal{A}^\mathbb{D} \overset{\iota_A}{\leftarrow} \iota^{-1}_A(\mathcal{A}^\mathbb{D}) \cap f^{-1}(\mathcal{A}^\mathbb{R}) \overset{f}{\rightarrow} \mathcal{A}^\mathbb{R} \) is called a hyperbolic system if it expands the pair of the horizontal Poincaré cone fields and contracts the pair of the vertical Poincaré cone fields (see [10, Definition 5.3]).

Let \( \mathcal{F}_h^\mathbb{D} = \{ A^\mathbb{D}_x(y) \}_{y \in A^\mathbb{D}_x} \) be the horizontal foliation of \( \mathcal{A}^\mathbb{D} \) where the leaves \( A^\mathbb{D}_x(y) \) are the biholomorphic images of \( A^\mathbb{D}_x \times \{ y \} \), and let \( \mathcal{F}_v^\mathbb{R} = \{ A^\mathbb{R}_x(x) \}_{x \in A^\mathbb{R}_x} \) be the vertical foliation of \( \mathcal{A}^\mathbb{R} \) where the leaves \( A^\mathbb{R}_x(x) \) are the biholomorphic images of \( \{ x \} \times A^\mathbb{R}_x \).

**Definition 2.8.** A crossed mapping \( \mathcal{A}^\mathbb{D} \overset{\iota_A}{\leftarrow} \iota^{-1}_A(\mathcal{A}^\mathbb{D}) \cap f^{-1}(\mathcal{A}^\mathbb{R}) \overset{f}{\rightarrow} \mathcal{A}^\mathbb{R} \) is said to satisfy the no-tangency condition (NTC) if \( \iota^{-1}_A(\mathcal{F}_h^\mathbb{D}) \) and \( f^{-1}(\mathcal{F}_v^\mathbb{R}) \) have no tangencies in \( \iota^{-1}_A(\mathcal{A}^\mathbb{D}) \cap f^{-1}(\mathcal{A}^\mathbb{R}) \) (see [10, Definition 5.16]).

The proof of the following statement is identical to that of [8, Theorem 2.14], hence we omit it here.

**Theorem 2.9.** A crossed mapping \( \mathcal{A}^\mathbb{D} \overset{\iota_A}{\leftarrow} \iota^{-1}_A(\mathcal{A}^\mathbb{D}) \cap f^{-1}(\mathcal{A}^\mathbb{R}) \overset{f}{\rightarrow} \mathcal{A}^\mathbb{R} \) satisfies the (NTC) if and only if it is a hyperbolic system.

Next we recall the notion of an expanding system [10, Definition 4.1]. Let \( X^0 \) and \( X^1 \) be two length spaces with metrics \( d_{X^0} \) and \( d_{X^1} \) respectively.

**Definition 2.10.** A multivalued dynamical system \( \iota, g : X^1 \rightarrow X^0 \) is called an expanding system if (i) \( g : X^1 \rightarrow X^0 \) is a covering map, and (ii) there are \( \delta > 0 \) and \( \lambda > 1 \) so that \( d_{X^0}(g(x), g(y)) \geq \lambda d_{X^0}(\iota(x), \iota(y)) \) whenever \( d_{X^1}(x, y) < \delta \).

To a hyperbolic system one can naturally associate an expanding system as follows. Let \( \mathcal{A} \) be a Poincaré box which is biholomorphic to \( A^0_x \times A^0_y \) with \( A^0_x \) being simply connected. Assume that \( \iota, \mathcal{A} \), \( f : \iota^{-1}_A(\mathcal{A}) \cap f^{-1}(\mathcal{A}) \rightarrow \mathcal{A} \) is a hyperbolic system. Then, by [10, Proposition 5.12] we know that \( \iota^{-1}_A(\mathcal{A}) \cap f^{-1}(\mathcal{A}) \) is biholomorphic to a product set of the form \( A^1_x \times A^1_y \). Take \( y_0 \in A^0_y \) and let \( \sigma_{y_0} \equiv \pi \circ f \circ \chi_{y_0} : A^1_x \rightarrow A^0_x \), where \( \pi : \mathcal{A} \rightarrow A^0_x \) is the projection and \( \chi_{y_0} : A^1_x \rightarrow \mathcal{A} \) is given by \( \chi_{y_0}(x) \equiv (x, y_0) \). Let \( \iota_{y_0} \equiv \pi \circ \iota_A \circ \chi_{y_0} : A^1_x \rightarrow A^0_x \). Then, one can easily see that \( \iota_{y_0}, \sigma_{y_0} : A^1_x \rightarrow A^0_x \) is an expanding system with respect to the Poincaré metrics in \( A^0_x \) and \( A^1_x \).

**Definition 2.11.** The expanding system \( \iota_{y_0}, \sigma_{y_0} : A^1_x \rightarrow A^0_x \) is called an associated expanding system of \( \iota, \mathcal{A}, f : \iota^{-1}_A(\mathcal{A}) \cap f^{-1}(\mathcal{A}) \rightarrow \mathcal{A} \) at \( y_0 \in A^0_y \).

See [10, Definition 9.2]. One also knows that the topological conjugacy class of the shift map on the orbit space of the associated expanding system at \( y_0 \in A^0_y \) does not depend on the choice of \( y_0 \) (see Corollary 5.10), thus we will drop \( y_0 \) from the notation and simply write \( \iota, \sigma : A^1_x \rightarrow A^0_x \) for the associated expanding system.
2.3. Sketch of the proof of Theorem A

Here we briefly describe how to construct a Hubbard tree in Theorem A. Let $\{A_\varepsilon\}_{\varepsilon \in \Sigma}$ be a family of finitely many Poincaré boxes in $\mathbb{C}^2$ and consider a multivalued dynamical system $\iota_A, f : \tilde{i}_A^{-1}(A) \cap \tilde{f}^{-1}(A) \to A$, where $A \equiv \bigcup_{\varepsilon \in \Sigma} A_\varepsilon$. We lift the maps $\iota_A$ and $f$ to the “abstract disjoint union” $\tilde{A} \equiv \bigsqcup_{\varepsilon \in \Sigma} A_\varepsilon$ to get a hyperbolic system $\tilde{\iota}_A, \tilde{f} : \tilde{i}_A^{-1}(\tilde{A}) \cap \tilde{f}^{-1}(\tilde{A}) \to \tilde{A}$. Since we may assume that the stable direction of $\tilde{f}$ is straight vertical in each Poincaré box $A_\varepsilon$ which is biholomorphic to $A_{x,\varepsilon} \times A_{y,\varepsilon}$, we can “squeeze” $\tilde{\iota}_A, \tilde{f}$ and $\tilde{A}$ along the vertical direction to get an associated expanding system $\tilde{\iota}_S, \tilde{\sigma} : \tilde{i}_S^{-1}(\tilde{S}) \cap \tilde{\sigma}^{-1}(\tilde{S}) \to \tilde{S}$, where $\tilde{S} \equiv \bigsqcup_{\varepsilon \in \Sigma} A_{x,\varepsilon}$.

Now the key notions in the construction are the pinching disks which are some special degree one vertical disks in $\tilde{A}$ and the pinching locus $L^0 \subset \tilde{S}$ which is the squeezed image of the pinching disks (see Subsection 3.2 for their definitions). Similarly we define $L^1 \subset \tilde{i}_S^{-1}(\tilde{S}) \cap \tilde{\sigma}^{-1}(\tilde{S})$. These loci define equivalence relations $\sim_{\varepsilon^0}$ in $\tilde{S}$ and $\sim_{\varepsilon^1}$ in $\tilde{i}_S^{-1}(\tilde{S}) \cap \tilde{\sigma}^{-1}(\tilde{S})$ respectively so that we obtain two branched surfaces $S^0 \equiv \tilde{S}/\sim_{\varepsilon^0}$ and $S^1 \equiv (\tilde{i}_S^{-1}(\tilde{S}) \cap \tilde{\sigma}^{-1}(\tilde{S}))/\sim_{\varepsilon^1}$. Then, the maps $\tilde{i}_S$ and $\tilde{\sigma}$ factor through to $\iota_S, \sigma : S^1 \to S^0$. We call the multivalued dynamical system $\iota_S, \sigma : S^1 \to S^0$ the branched surface model of $\iota_A, f : \tilde{i}_A^{-1}(A) \cap \tilde{f}^{-1}(A) \to A$. See Sections 3 for more details on the construction of the branched surface model starting from a family of Poincaré boxes.

Next we proceed as in Subsection 2.1 with taking the pinching locus $L^0$ into account with the set $\mathcal{C}^0$ of centers in the holes of $\tilde{S}$. More precisely, we take the legal hull $\mathcal{H}^0$ of $\mathcal{C}^0 \cup L^0$ in $\tilde{S}$ and replace each point of $\mathcal{C}^0$ in $\mathcal{H}^0$ by a loop to get $\tilde{\mathcal{H}}^0$. Similarly let $\mathcal{C}^1$ be the set of centers in the holes of $\tilde{i}_S^{-1}(\tilde{S}) \cap \tilde{\sigma}^{-1}(\tilde{S})$. We take the legal hull $\mathcal{H}^1$ of $\mathcal{C}^1 \cup L^1$ in $\tilde{i}_S^{-1}(\tilde{S}) \cap \tilde{\sigma}^{-1}(\tilde{S})$ and replace each point of $\mathcal{C}^1$ in $\mathcal{H}^1$ by a loop to get $\tilde{\mathcal{H}}^1$. Then, the expanding system $\tilde{i}_S, \tilde{\sigma} : \tilde{i}_S^{-1}(\tilde{S}) \cap \tilde{\sigma}^{-1}(\tilde{S}) \to \tilde{S}$ naturally induces a new multivalued dynamical system $\tilde{\iota}_T, \tilde{\tau} : \tilde{\mathcal{H}}^1 \to \tilde{\mathcal{H}}^0$. Since $L^0$ can be seen as a subset of $\tilde{\mathcal{H}}^0$ and $L^1$ can be seen as a subset of $\tilde{\mathcal{H}}^1$, the equivalence relations $\sim_{\varepsilon^0}$ in $\tilde{S}$ and $\sim_{\varepsilon^1}$ in $\tilde{i}_S^{-1}(\tilde{S}) \cap \tilde{\sigma}^{-1}(\tilde{S})$ are transferred to equivalence relations $\sim_{\varepsilon^0}$ in $\tilde{\mathcal{H}}^0$ and $\sim_{\varepsilon^1}$ in $\tilde{\mathcal{H}}^1$ respectively so that we get two “trees decorated with loops” $T^0 \equiv \tilde{\mathcal{H}}^0/\sim_{\varepsilon^0}$ and $T^1 \equiv \tilde{\mathcal{H}}^1/\sim_{\varepsilon^1}$. Finally the maps $\tilde{\iota}_T$ and $\tilde{\tau}$ factor through to $\iota_T, \tau : T^1 \to T^0$. This multivalued dynamical system $\iota_T, \tau : T^1 \to T^0$ is the Hubbard tree of $\iota_A, f : \tilde{i}_A^{-1}(A) \cap \tilde{f}^{-1}(A) \to A$ as in Theorem A. See Section 4 for more details on the construction of a Hubbard tree starting from the branched surface model.

3. From Poincaré boxes to a branched surface model

3.1. A family of Poincaré boxes

Let $\Sigma$ be a finite set and choose $\Gamma \subset \Sigma \times \Sigma$. We also write $\Sigma^0 \equiv \Sigma$ and $\Sigma^1 \equiv \Gamma$.

For $\varepsilon \in \Sigma$, we let $B_{x,\varepsilon}$ and $B_{y,\varepsilon}$ be bounded open topological disks in $\mathbb{C}$. Let us put $A_{x,\varepsilon} \equiv B_{x,\varepsilon} \setminus \bigcup_{l \in L_\varepsilon} H_{l,\varepsilon}$ and $A_{y,\varepsilon} \equiv B_{y,\varepsilon}$, where $\{H_{l,\varepsilon}\}_{l \in L_\varepsilon}$ is a family of finitely many mutually disjoint closed topological disks in $B_{x,\varepsilon}$. We consider a family of Poincaré boxes $\{A_\varepsilon\}_{\varepsilon \in \Sigma}$, where each $A_\varepsilon$ is biholomorphic to the product set $A_{x,\varepsilon} \times A_{y,\varepsilon}$. Note that $A_\varepsilon$ may overlap with each other and their horizontal/vertical directions induced from the product structure of $A_{x,\varepsilon} \times A_{y,\varepsilon}$ are not necessarily the same on the overlap.
Let $\mathcal{A} \equiv \bigcup_{\varepsilon \in \Sigma} \mathcal{A}_{\varepsilon}$ and consider the multivalued dynamical system:

$$\iota_{\mathcal{A}}, f : \iota_{\mathcal{A}}^{-1}(\mathcal{A}) \cap f^{-1}(\mathcal{A}) \rightarrow \mathcal{A},$$

where $\iota_{\mathcal{A}}$ is the inclusion and $f$ is a holomorphic injection defined on $f^{-1}(\mathcal{A}) \subset \mathbb{C}^2$. We write $\mathcal{A}^0 \equiv \mathcal{A}$ and $\mathcal{A}^1 \equiv \iota_{\mathcal{A}}^{-1}(\mathcal{A}) \cap f^{-1}(\mathcal{A})$. Then, the above multivalued dynamical system induces the space of all bi-infinite orbits:

$$\mathcal{A}^{\infty} \equiv \{(z_i)_{i \in \mathbb{Z}} \in (\mathcal{A}^1)^{\mathbb{Z}} : f(z_i) = \iota_{\mathcal{A}}(z_{i+1})\}$$

as well as the shift map $f : \mathcal{A}^{\infty} \rightarrow \mathcal{A}^{\infty}$ on it. Note that $\mathcal{A}^{\infty}$ can be identified with $\bigcap_{n \in \mathbb{Z}} f^n(\mathcal{A})$ and the shift map becomes the restriction of $f$ to it.

For $m = 0, 1$, we set

$$\widetilde{\mathcal{A}}^m \equiv \bigcup_{\varepsilon \in \Sigma^m} \mathcal{A}_{\varepsilon},$$

where $\mathcal{A}_{\varepsilon} \equiv \mathcal{A}_{\varepsilon_0}$ for $\varepsilon = \varepsilon_0 \in \Sigma^0$ and $\mathcal{A}_{\varepsilon} \equiv \iota_{\mathcal{A}}^{-1}(\mathcal{A}_{\varepsilon_0}) \cap f^{-1}(\mathcal{A}_{\varepsilon_1})$ for $\varepsilon = \varepsilon_0 \varepsilon_1 \in \Sigma^1$. Then, the multivalued dynamical system $\iota_{\mathcal{A}}, f : \iota_{\mathcal{A}}^{-1}(\mathcal{A}) \cap f^{-1}(\mathcal{A}) \rightarrow \mathcal{A}$ together with $\Gamma = \Sigma^1$ determines a family of multivalued dynamical systems $\{\mathcal{A}_{\varepsilon_0} \ni \iota_{\mathcal{A}}^{-1}(\mathcal{A}_{\varepsilon_0}) \rightarrow \mathcal{A}_{\varepsilon_1}\}_{\varepsilon_0 \varepsilon_1 \in \Sigma^1}$ which will be denoted by $\tilde{\iota}_{\mathcal{A}}, \tilde{f} : \tilde{\mathcal{A}}^1 \rightarrow \tilde{\mathcal{A}}^0$. As before, this induces the space of bi-infinite orbits:

$$\tilde{\mathcal{A}}^{\infty} \equiv \{(\tilde{z}_i)_{i \in \mathbb{Z}} \in (\tilde{\mathcal{A}}^1)^{\mathbb{Z}} : \tilde{f}(\tilde{z}_i) = \tilde{\iota}_{\mathcal{A}}(\tilde{z}_{i+1})\}$$

as well as the shift map $\tilde{f} : \tilde{\mathcal{A}}^{\infty} \rightarrow \tilde{\mathcal{A}}^{\infty}$ on it.

**Definition 3.1.** A family of multivalued dynamical systems $\tilde{\iota}_{\mathcal{A}}, \tilde{f} : \tilde{\mathcal{A}}^1 \rightarrow \tilde{\mathcal{A}}^0$ is said to be hyperbolic if $\mathcal{A}_{\varepsilon_0} \ni \iota_{\mathcal{A}}^{-1}(\mathcal{A}_{\varepsilon_0}) \rightarrow \mathcal{A}_{\varepsilon_1}$ is hyperbolic in the sense of Definition 2.7 for all $\varepsilon_0 \varepsilon_1 \in \Sigma^1$.

A multivalued dynamical system $\iota_{\mathcal{A}}, f : \mathcal{A}^1 \rightarrow \mathcal{A}^0$ (or, simply $f : \mathcal{A} \cap f^{-1}(\mathcal{A}) \rightarrow \mathcal{A}$) is said to be hyperbolic over $\Gamma = \Sigma^1$ if $\tilde{\iota}_{\mathcal{A}}, \tilde{f} : \tilde{\mathcal{A}}^1 \rightarrow \tilde{\mathcal{A}}^0$ is hyperbolic.

**Hereafter, we always assume that the multivalued dynamical system $\iota_{\mathcal{A}}, f : \iota_{\mathcal{A}}^{-1}(\mathcal{A}) \cap f^{-1}(\mathcal{A}) \rightarrow \mathcal{A}$ is a hyperbolic system over $\Gamma = \Sigma^1$.**

In order to construct the Hubbard tree from $\iota_{\mathcal{A}}, f : \mathcal{A}^1 \rightarrow \mathcal{A}^0$, some assumptions are needed. The first assumption requires that the set $\Sigma^1 = \Gamma \subset \Sigma \times \Sigma$ is in some sense “abundant enough” to cover $\mathcal{A}^{\infty}$. To state this, let us put

$$J^m(\tilde{f}) \equiv \bigcap_{k \in \mathbb{Z}} (\tilde{f} \circ \tilde{\iota}_{\mathcal{A}}^{-1})^{-k}(\tilde{\mathcal{A}}^m)$$

as a subset of $\tilde{\mathcal{A}}^m$ for $m = 0, 1$ (replace $\tilde{f} \circ \tilde{\iota}_{\mathcal{A}}^{-1}$ by $\tilde{f} \circ \tilde{\iota}_{\mathcal{A}}^{-1} \circ \tilde{f}$ for the case $m = 1$). Since $\tilde{f}$ is injective and $\tilde{\iota}_{\mathcal{A}}$ is the inclusion, $J^m(\tilde{f})$ can be identified with $\tilde{\mathcal{A}}^{\infty}$. We also define

$$J^m(f) \equiv \bigcap_{k \in \mathbb{Z}} (f \circ \iota_{\mathcal{A}}^{-1})^{-k}(A^m)$$
as a subset of $\mathcal{A}^m$ for $m = 0, 1$. Since $f$ is injective and $\iota_A$ is the inclusion, $J^m(f)$ can be identified with $\mathcal{A}^\infty$. Let $\text{pr}_A: \tilde{\mathcal{A}}^m \to \mathcal{A}^m$ be the projection induced from the inclusion $\mathcal{A}_\xi \to \mathcal{A}^m$ for $\xi \in \Sigma^m$.

**Assumption 1 (Admissibility).** The map $\text{pr}_A: J^1(\tilde{f}) \to J^1(f)$ is surjective.

We note that this is an assumption on $\Gamma = \Sigma^1$. The inclusion $\text{pr}_A(J^1(\tilde{f})) \subset J^1(f)$ always holds. In fact, Assumption 1 is equivalent to the following condition: for any $\xi \in J^0(f)$ there exists a bi-infinite sequence $\ldots \varepsilon_{-1} \varepsilon_0 \varepsilon_1 \ldots \in (\Sigma^0)^\mathbb{Z}$ with $\varepsilon_n \varepsilon_{n+1} \in \Sigma^1$ so that $(f \circ \iota_A^{-1})^n(\varepsilon) \in \mathcal{A}_{\varepsilon_n}$ for all $n \in \mathbb{Z}$. An element in $\Sigma^1$ is called an admissible transition of the multivalued dynamical system $\iota_A, f : \mathcal{A}^1 \to \mathcal{A}^0$.

### 3.2. Pinching disks and pinching locus

We define the forward Julia set of $\tilde{f}$ by

$$J_+^m(\tilde{f}) \equiv \bigcap_{k \geq 0} (\tilde{f} \circ \iota_A^{-1})^{-k}(\tilde{\mathcal{A}}^m),$$

as a subset of $\tilde{\mathcal{A}}^m$ for $m = 0, 1$. Since $\iota_A, f : \mathcal{A}^1 \to \mathcal{A}^0$ is hyperbolic over $\Sigma^1$ and since $A_{y, \varepsilon}$ is simply connected, $J_+^m(\tilde{f}) \cap \mathcal{A}_\varepsilon$ is laminated by local stable manifolds which are holomorphic vertical-like disks of degree one in $\mathcal{A}_\varepsilon$ for each $\varepsilon \in \Sigma^m$. Moreover, we may assume that these disks are straight vertical thanks to the comment following [10, Lemma 5.5]. This defines a lamination $\mathcal{W}^m(\tilde{f})$ of $J_+^m(\tilde{f})$ by straight vertical disks which we call the stable lamination of $J_+^m(\tilde{f})$.

The first central concept in this section is

**Definition 3.2.** A leaf $D$ of the stable lamination $\mathcal{W}^m(\tilde{f})$ is called a pinching disk in $\tilde{\mathcal{A}}^m$ if there exists a leaf $D' \neq D$ of $\mathcal{W}^m(\tilde{f})$ with $\text{pr}_A(D) \cap \text{pr}_A(D') \neq \emptyset$. Such a pair $\{D, D'\}$ is called an intersecting pair of pinching disks in $\tilde{\mathcal{A}}^m$.

Note that if $\{D, D'\}$ is an intersecting pair of pinching disks in $\tilde{\mathcal{A}}^m$ with $\mathcal{D} \subset J_+^m(\tilde{f}) \cap \mathcal{A}_\varepsilon$ and $D' \subset J_+^m(\tilde{f}) \cap \mathcal{A}_{\varepsilon'}$, then it follows that $\varepsilon \neq \varepsilon'$ since $\text{pr}_A$ is injective on each $\mathcal{A}_\varepsilon$.

Let $\Delta^m$ be the totality of the pinching disks in $\tilde{\mathcal{A}}^m$ for $m = 0, 1$. We have the following invariance property of $\Delta^m$.

**Lemma 3.3.** For any $D \in \Delta^1$ there exists $D' \in \Delta^0$ such that $\tilde{f}(D) \subset D'$.

**Proof.** Take $D_1 \in \Delta^1$ and let $\{D_1, D_2\}$ be a pair of pinching disks in $\tilde{\mathcal{A}}^1$. Then, there exist leaves $D'_1$ and $D'_2$ of $\mathcal{W}^0(\tilde{f})$ so that $D'_1 \supset \tilde{f}(D_1)$ and $D'_2 \supset \tilde{f}(D_2)$. Assume that these two leaves coincide and write $D' \equiv D'_1 = D'_2$. Since $D_1 \neq D_2$ and $\tilde{f}$ is injective, we have $\tilde{f}(D_1) \cap \tilde{f}(D_2) = \emptyset$. Since $\text{pr}_A$ is injective on $D'$, we see $f(\text{pr}_A(D_1) \cap \text{pr}_A(D_2)) = f(\text{pr}_A(D_1) \cap f(\text{pr}_A(D_2))) = \text{pr}_A(f(\tilde{f}(D_1)) \cap \text{pr}_A(\tilde{f}(D_2))) = \text{pr}_A(\tilde{f}(D_1) \cap \tilde{f}(D_2)) = \emptyset$. It follows that $\text{pr}_A(D_1) \cap \text{pr}_A(D_2) = \emptyset$, which contradicts the fact that $\{D_1, D_2\}$ forms a pair of pinching disks in $\tilde{\mathcal{A}}^1$. Hence, $D'_1 \neq D'_2$. Since $\tilde{f}$ is injective, we also see that $\text{pr}_A(D'_1) \cap \text{pr}_A(D'_2) \supset \text{pr}_A(\tilde{f}(D_1)) \cap \text{pr}_A(\tilde{f}(D_2)) = \emptyset$. Thus, there exists $D' \in \Delta^0$ such that $\tilde{f}(D_1) \subset D'$. This completes the proof.
Fig. 2. The landing point $q$ and the sets $A_{x,i}$.

$f(\text{pr}_{\mathcal{A}}(D_1)) \cap f(\text{pr}_{\mathcal{A}}(D_2)) = f(\text{pr}_{\mathcal{A}}(D_1) \cap \text{pr}_{\mathcal{A}}(D_2)) \neq \emptyset$. This shows that $\{D'_1, D'_2\}$ forms a pair of pinching disks in $\tilde{\mathcal{A}}^0$ and hence $D'_1 \in \Delta^0$. This finishes the proof. \hfill \Box

**Remark 3.4.** For any $D \in \Delta^0$ it is easy to see that $\tilde{\iota}_{\mathcal{A}}^{-1}(D) \in \Delta^1$. In this way, a pinching disk in $\tilde{\mathcal{A}}^0$ can be regarded as a pinching disk in $\tilde{\mathcal{A}}^1$.

The next assumption may look very restrictive, but actually not (see Example below as well as Section 6).

**Assumption 2 (Finiteness).** $\Delta^0$ is a finite set.

Since the system of crossed mappings $\tilde{\iota}_{\mathcal{A}}$, $\tilde{f} : \tilde{\mathcal{A}}^1 \to \tilde{\mathcal{A}}^0$ has bounded degree, it follows from Lemma 3.3 and Assumption 2 that $\Delta^1$ is also a finite set.

Let $\pi^m : \mathcal{A}_\varepsilon \to A_{x,\varepsilon}$ be the projection to the first coordinate. The second central concept in this section is

**Definition 3.5.** We call $\mathcal{L}^m = \{\pi^m(D)\}_{D \in \Delta^m}$ the pinching locus for $m = 0, 1$.

Since the stable direction is straight vertical in $\mathcal{A}_\varepsilon$, the set $\pi^m(D)$ consists of one point for each $D \in \Delta^m$. Thus, by Assumption 2, $\mathcal{L}^m$ is a finite set for $m = 0, 1$.

In order to grasp Definitions 3.2 and 3.5, we here present the following example.

**Example.** Let $p_c(x) = x^2 + c$ be the quadratic polynomial whose Julia set is the so-called basilica, i.e. below we put $c \equiv -1$. We investigate a small perturbation $f = f_{-1,b}$ of this polynomial map.

Consider the external rays $\mathcal{R}_{\frac{1}{3}}$ and $\mathcal{R}_{\frac{2}{3}}$ of angles $1/3$ and $2/3$ respectively for $p_c$ together with their common landing point denoted by $q$. Then, the union $\{q\} \cup \mathcal{R}_{\frac{1}{3}} \cup \mathcal{R}_{\frac{2}{3}}$ divides the complex
plane into two pieces. Let \( G_c \) be the Green function for \( p_c \). For \( r > 0 \) large enough, we define \( U_i \) \((i = 0, 1)\) to be the connected component of \( \{ x \in \mathbb{C}: G_c(x) < r \} \setminus (\{ q \} \cup \mathcal{R}_1 \cup \mathcal{R}_2) \)

containing \( p_c^i(0) \). For \( \varepsilon > 0 \) small, let \( A_{x,i} \) be the \( \varepsilon \)-neighborhood of \( U_i \) with a small neighborhood of \( p_c^i(0) \) removed (see Fig. 2). Then, \( p_c: A_{x,i} \cap p_c^{-1}(A_{x,j}) \rightarrow A_{x,j} \) is a polynomial-like map of degree two for \((i, j) = (0, 1)\) and degree one for \((i, j) = (1, 0), (0, 0)\). We note that the sets \( U_i \) decompose the Julia set \( J_c \) of \( p_c \) into two pieces. More precisely, we let \( (J_c)_i \equiv (J_c \cap U_i) \cup \{ q \} \), then we have \( (J_c)_0 \cap (J_c)_1 = \{ q \} \) and \( (J_c)_0 \cup (J_c)_1 = J_c \).

Let \( \Sigma = \{ 0, 1 \} \) and \( \Gamma \equiv \{(0, 1), (1, 0), (0, 0)\} \subset \Sigma \times \Sigma \). We put \( A_i \equiv A_{x,i} \times \Delta(0; R) \) for \( R > 0 \) large, where \( \Delta(0; R) = \{ y \in \mathbb{C}: |y| < R \} \). Then, it is easy to see that \( A_i \xleftarrow{\iota_A} A_j \) is a crossed mapping satisfying the (NTC) for \((i, j) \in \Gamma \). Thus, \( \iota_A, f: \iota_A^{-1}(A) \cap f^{-1}(A) \rightarrow A \) is a hyperbolic system over \( \Gamma \), where \( A = A_0 \cup A_1 \). Let \( J^m_+(\tilde{f}) \) be the forward Julia set of the corresponding disjoint system \( \tilde{\iota}_A, \tilde{f}: \tilde{A}_1 \rightarrow \tilde{A}_0 \). Recall that the leaves of the stable lamination of \( J^m_+(\tilde{f}) \cap A_2 \) are straight vertical.

The unique landing point \( q \) satisfies \( q \in A_{x,i} \) for any \( i = 0, 1 \). We then let \( D_i \equiv \{ q \} \times \Delta(0; R) \subset A_i \). Since we have \( (J_c)_0 \cap (J_c)_1 = \{ q \} \) and \( p_c: A_{x,i} \cap p_c^{-1}(A_{x,j}) \rightarrow A_{x,j} \) is a polynomial-like map for \((i, j) \in \Gamma \), one obtains \( J^m_+(\tilde{f}) \cap A_i = (J_c)_i \times \Delta(0; R) \). This immediately implies that \( \{ D_0, D_1 \} \) is the only pair of pinching disks for the map \( f = f_{-1,b} \).

See Fig. 3. More examples of pinching disks are presented in Section 6.

**Remark 3.6.** Note that the projected image by \( \text{pr}_A \) of the left-hand side \( J^m_+(\tilde{f}) \cap A_i \) of the above equality is in general different from \( J^m_+(f) \cap A_i \), where

\[
J^m_+(f) \equiv \bigcap_{k \geq 0} (f \circ \iota_A^{-1})^{-k}(A^m).
\]
We can only prove \( \text{pr}_A(J^m_+ (\tilde{f}) \cap A_i) \subset J^m_+ (f) \cap A_i \). In fact, in Example above, one has \( \text{pr}_A(J^m_+ (\tilde{f}) \cap A_i) = \text{pr}_A((J_{c_i} \Delta c) \cap A_i) \subset (J_{c_i} \Delta c) \times \Delta (0; R) = J^m_+ (f) \cap A_i \) but the equality never holds. This is because, for the map \( \tilde{f} \), we only consider the admissible transitions \( \Gamma \); there is a \( (i, j) \in \Sigma \times \Sigma \) which is not admissible but \( f(A_i) \cap A_j \neq \emptyset \).

### 3.3 Branched surface models

Write \( S_{x} \equiv Ax_{x_{\xi}} \) for \( x \in \Sigma^{m} \) and put

\[
\tilde{S}^{m} = \bigsqcup_{x \in \Sigma^{m}} S_{x}.
\]

Recall that \( \pi^{m} : A_{x} \rightarrow S_{x} \) is the projection to the first coordinate. Hereafter, we fix \( y_{x} \in A_{y_{x} \xi} \) and define \( \chi^{m} : S_{x} \rightarrow A_{x} \) by \( \chi^{m}(x) = (x, y_{x}) \) for \( x \in \Sigma^{m} \). We will see that the homotopy equivalence class (see Definition 5.4) of the construction below does not depend on the choice of \( y_{x} \).

The two maps \( \tilde{\sigma} = \pi^{1} \circ \tilde{f} \circ \chi^{1} : S_{x_{i} \xi_{1}} \rightarrow S_{x_{j} \xi_{1}} \) and \( \tilde{\iota}_{S} \equiv \text{pr}_{0} \circ \tilde{\iota}_{A} \circ \chi^{1} : S_{x_{i} \xi_{1}} \rightarrow S_{x_{0}} \) induce a multivalued dynamical system \( \tilde{\iota}_{S}, \tilde{\sigma} : \tilde{S}^{1} \rightarrow \tilde{S}^{0} \) (see Diagram 1). Consider Poincaré metrics in \( \tilde{S}^{0} \) and \( \tilde{S}^{1} \). Then, \( \tilde{\iota}_{S}, \tilde{\sigma} : \tilde{S}^{1} \rightarrow \tilde{S}^{0} \) becomes a family of associated expanding systems of \( \tilde{\iota}_{A}, \tilde{f} : \tilde{A}^{1} \rightarrow \tilde{A}^{0} \). Namely,

**Lemma 3.7.** The multivalued dynamical system \( \tilde{\iota}_{S}, \tilde{\sigma} : \tilde{S}^{1} \rightarrow \tilde{S}^{0} \) is expanding.

**Proof.** Since \( \tilde{\iota}_{A}, \tilde{f} : \tilde{A}^{1} \rightarrow \tilde{A}^{0} \) is a crossed mapping, it follows that \( \tilde{\sigma} : \tilde{S}^{1} \rightarrow \tilde{S}^{0} \) is a polynomial-like map. Thus, \( \tilde{\iota}_{S} \) is a contraction with respect to the Poincaré metrics in \( \tilde{S}^{m} \). Since \( \tilde{\iota}_{A}, \tilde{f} : \tilde{A}^{1} \rightarrow \tilde{A}^{0} \) satisfies the (NTC), it follows that \( \tilde{\sigma} : \tilde{S}^{1} \rightarrow \tilde{S}^{0} \) is a non-branched covering. Hence it is an isometry with respect to the Poincaré metrics in \( \tilde{S}^{m} \). This finishes the proof. \( \square \)

We will also consider the space of bi-infinite orbits:

\[
\tilde{S}^{\infty} = \{ (\tilde{\iota}_{S} \tilde{\sigma})_{\xi \in \mathbb{Z}} \in (\tilde{S}^{1})^{\mathbb{Z}} : \tilde{\sigma}(\tilde{\iota}_{S}) = \tilde{\iota}_{S}(\tilde{\iota}_{S} + 1) \}
\]

for the multivalued dynamical system \( \tilde{\iota}_{S}, \tilde{\sigma} : \tilde{S}^{1} \rightarrow \tilde{S}^{0} \) as well as the shift map \( \tilde{\sigma} : \tilde{S}^{\infty} \rightarrow \tilde{S}^{\infty} \) on it.

Now we construct branched surfaces \( S^{0} \) and \( S^{1} \) and a pair of maps \( \iota_{S}, \sigma : S^{1} \rightarrow S^{0} \). For \( \tilde{s}, \tilde{s}' \in \Sigma^{m} \) with \( \tilde{s} = \pi^{m}(D) \) and \( \tilde{s}' = \pi^{m}(D') \), we first say that \( \tilde{s} \equiv \Sigma^{m} \tilde{s}' \) iff either \( D = D' \) holds.
or \( \{D, D'\} \) forms an intersecting pair of pinching disks in \( \tilde{A}^m \). We then say that \( \tilde{s} \sim \sim \tilde{s}' \) for \( \tilde{s}, \tilde{s}' \in \tilde{L}^m \) iff either \( \tilde{s} = \tilde{s}' \) holds or there exists a chain of points \( \tilde{r}_0, \tilde{r}_1, \ldots, \tilde{r}_k \in \tilde{L}^m \) so that \( \tilde{s} = \tilde{r}_0 \approx \sim \tilde{r}_1 \approx \sim \cdots \approx \sim \tilde{r}_k = \tilde{s}' \). Write
\[
\mathcal{S}^m = \tilde{\mathcal{S}}^m / \sim \mathbb{Z} = \left( \bigsqcup_{\xi \in \tilde{\mathcal{S}}^m} \mathcal{S}_\xi \right) / \sim \mathbb{Z}.
\]

Let \( \text{pr}_S : \tilde{\mathcal{S}}^m \to \mathcal{S}^m \) be the natural projection with respect to the equivalence relation \( \sim \mathbb{Z} \) defined above. Put \( \sigma = \text{pr}_S \circ \tilde{\sigma} \circ \text{pr}_S^{-1} \) and \( \iota_S = \text{pr}_S \circ \tilde{\iota}_S \circ \text{pr}_S^{-1} \). We first need

**Lemma 3.8.** The maps \( \iota_S, \sigma : S^1 \to S^0 \) are well-defined.

**Proof.** We only show the well-definedness of \( \iota_S \) here. The argument for \( \sigma \) is similar by using Lemma 3.3.

Let \( s \in S^1 \) and take distinct points \( \tilde{s}, \tilde{s}' \in \text{pr}_S^{-1}(s) \subset \tilde{S}^1 \). Without loss of generality one may assume that \( \{D_1, D'_1\} \) forms an intersecting pair of pinching disks, where \( \tilde{s} = \pi^1(D_1) \) and \( \tilde{s}' = \pi^1(D'_1) \). Then, there are straight vertical disks \( D_0 \) and \( D'_0 \) in \( \tilde{A}^0 \) so that \( \tilde{i}_A(D_1) \subset D_0 \) and \( \tilde{i}_A(D'_1) \subset D'_0 \). If these disks are distinct, then \( \{D_0, D'_0\} \) forms an intersecting pair of pinching disks in \( \tilde{A}^0 \). It follows that \( \text{pr}_S \circ \pi^0 \circ \tilde{i}_A(D_1) = \text{pr}_S \circ \pi^0 \circ \tilde{i}_A(D'_1) \), which means that \( \text{pr}_S \circ \tilde{i}_S(\tilde{s}) = \text{pr}_S \circ \tilde{i}_S(\tilde{s}') \). Hence \( \iota_S \) is well-defined in this case. If not, \( \tilde{i}_A(D_1) \) and \( \tilde{i}_A(D'_1) \) are contained in one straight vertical disk in \( \tilde{A}^0 \). We then have \( \pi^0 \circ \tilde{i}_A(D_1) = \pi^0 \circ \tilde{i}_A(D'_1) \), which again implies that \( \text{pr}_S \circ \tilde{i}_S(\tilde{s}) = \text{pr}_S \circ \tilde{i}_S(\tilde{s}') \). Hence \( \iota_S \) is well-defined in this case as well. This finishes the proof. \( \square \)

Thus, the pair of these maps defines the multivalued dynamical system \( \iota_S, \sigma : S^1 \to S^0 \). Note that \( \sigma : S^1 \to S^0 \) is not necessarily a covering map.

**Definition 3.9.** The multivalued dynamical system
\[
\iota_S, \sigma : S^1 \to S^0
\]
equipped with the induced Poincaré metrics in \( S^0 \) and \( S^1 \) is called the branched surface model of the hyperbolic system \( \iota_A, f : A^1 \to A^0 \) over \( \Gamma \).

One can define the space of bi-infinite orbits:
\[
S^\infty = \{ (s_i)_{i \in \mathbb{Z}} \in (S^1)^\mathbb{Z} : \sigma(s_i) = \iota_S(s_{i+1}) \}
\]
for the multivalued dynamical system \( \iota_S, \sigma : S^1 \to S^0 \) as well as the shift map \( \sigma : S^\infty \to S^\infty \) on it.

It is in fact possible to equip the structure of a non-singular branched surface in the sense of Williams [17] to \( \mathcal{S}^m \). See [17, Section 1] for more details.
4. From a branched surface model to a Hubbard tree

4.1. Legal arcs and legal hulls

We define the Julia set of the multivalued dynamical system $\tilde{i}_S : \tilde{S}^1 \to \tilde{S}^0$ as

$$J^m(\tilde{\sigma}) \equiv \bigcap_{k \geq 0} (\tilde{\sigma} \circ \tilde{i}_S)^{-k}(\tilde{S}^m)$$

and put $J(\tilde{\sigma})_\varepsilon \equiv J^m(\tilde{\sigma}) \cap S_\varepsilon$ for $\varepsilon \in \Sigma^m$. Each $J(\tilde{\sigma})_\varepsilon$ is a subset of $A_{x,\varepsilon} = S_\varepsilon$, thus of $B_{x,\varepsilon}$. We say that a connected component of $B_{x,\varepsilon} \setminus J(\tilde{\sigma})_\varepsilon$ is bounded if its closure in $\mathbb{C}$ does not intersect with $\partial B_{x,\varepsilon}$. Let $K(\tilde{\sigma})_\varepsilon$ be the union of $J(\tilde{\sigma})_\varepsilon$ and the bounded connected components of $B_{x,\varepsilon} \setminus J(\tilde{\sigma})_\varepsilon$. Each $K(\tilde{\sigma})_\varepsilon$ is simply connected. Moreover, each connected component of $\text{Int} K(\tilde{\sigma})_\varepsilon$ is also simply connected. When $U_1$ and $U_2$ are distinct connected components of $\text{Int} K(\tilde{\sigma})_\varepsilon$, we see that $\overline{U_1} \cap \overline{U_2}$ consists of at most one point, since $K(\tilde{\sigma})_\varepsilon$ is simply connected.

Recall that $A_{x,\varepsilon} \equiv B_{x,\varepsilon} \setminus \bigcup_{l \in L_\varepsilon} H_l$. For each $\varepsilon \in \Sigma^m$ and each $l \in L_\varepsilon$, we choose a center $p \in H_l$ as in Subsection 2.1. Let $\tilde{\varphi}^m$ be the totality of such finitely many points. We may take $\tilde{\varphi}^m$ so that $\tilde{\sigma}(\mathcal{C}^1) \subset \mathcal{C}^0$. Write $\mathcal{C}_\varepsilon \equiv \tilde{\varphi}^m \cap K(\tilde{\sigma})_\varepsilon$ and $\mathcal{L}_\varepsilon \equiv \mathcal{L}^m \cap K(\tilde{\sigma})_\varepsilon$ for $\varepsilon \in \Sigma^m$. Let $\mathcal{H}_\varepsilon$ be the legal hull of $\mathcal{C}_\varepsilon \cup \mathcal{L}_\varepsilon$ in $K(\tilde{\sigma})_\varepsilon$ and set

$$\tilde{\mathcal{H}}^m \equiv \bigcup_{\varepsilon \in \Sigma^m} \mathcal{H}_\varepsilon.$$

The legal hull $\mathcal{H}_\varepsilon$ is a subset of $K(\tilde{\sigma})_\varepsilon$, so one may restrict $\tilde{\sigma}$ to $\tilde{\mathcal{H}}^1$. Since $\tilde{\sigma}(\mathcal{C}^1 \cup \mathcal{L}^1) \subset \mathcal{C}^0 \cup \mathcal{L}^0$, we obtain $\tilde{\sigma} : \tilde{\mathcal{H}}^1 \to \tilde{\mathcal{H}}^0$ (by modifying $\varphi_U$, if necessary). Here we need

**Assumption 3 (Non-triviality).** For each $\varepsilon_0 \in \Sigma^0$, we have $\mathcal{C}_{\varepsilon_0} \cup \mathcal{L}_{\varepsilon_0} \neq \emptyset$.

**Remark 4.1.** The horseshoe case is excluded from our framework because of Assumption 3.

Since $\tilde{\mathcal{H}}^0$ can be viewed as a subset of $\tilde{\mathcal{H}}^1$, one may define the “smashing map” $\tilde{i}_T : \tilde{\mathcal{H}}^1 \to \tilde{\mathcal{H}}^0$ as follows. We have $\mathcal{L}_{\varepsilon_0} \subset \tilde{i}_S(\bigcup_{l \in L_{\varepsilon_1}} \mathcal{L}_{\varepsilon_0 l})$ from Remark 3.4 and we can take $\tilde{\varphi}^m$ so that $\mathcal{C}_{\varepsilon_0} \subset \tilde{i}_S(\bigcup_{l \in L_{\varepsilon_1}} \mathcal{C}_{\varepsilon_0 l})$. Thus, one has $\mathcal{H}_{\varepsilon_0} \subset \tilde{i}_S(\bigcup_{l \in L_{\varepsilon_1}} \mathcal{H}_{\varepsilon_0 l})$ by an appropriate choice of $\varphi_U$. We first define $\tilde{i}_T$ on $\tilde{i}_S^{-1}(\mathcal{H}_{\varepsilon_0})$ to be the restriction of $\tilde{i}_S$. Next, let us define $\tilde{i}_T$ on $(\bigcup_{l \in L_{\varepsilon_1}} \mathcal{H}_{\varepsilon_0 l}) \setminus \tilde{i}_S^{-1}(\mathcal{H}_{\varepsilon_0})$. Since $\mathcal{H}_{\varepsilon_0}$ is connected and $\mathcal{H}_{\varepsilon_0 l}$ is simply connected, the intersection of the closure of a connected component of $(\bigcup_{l \in L_{\varepsilon_1}} \mathcal{H}_{\varepsilon_0 l}) \setminus \tilde{i}_S^{-1}(\mathcal{H}_{\varepsilon_0})$ with $\tilde{i}_S^{-1}(\mathcal{H}_{\varepsilon_0})$ consists of at most one point. If the closure of a connected component does not intersect with $\tilde{i}_S^{-1}(\mathcal{H}_{\varepsilon_0})$, then its image by $\tilde{i}_T$ is defined to be any point in $\mathcal{H}_{\varepsilon_0}$ (which is non-empty by Assumption 3). If the closure intersects with $\tilde{i}_S^{-1}(\mathcal{H}_{\varepsilon_0})$ at one point, then its image by $\tilde{i}_T$ is defined to be that point. This defines a continuous map $\tilde{i}_T : \bigcup_{l \in L_{\varepsilon_1}} \mathcal{H}_{\varepsilon_0 l} \to \mathcal{H}_{\varepsilon_0}$, thus $\tilde{i}_T : \tilde{\mathcal{H}}^1 \to \tilde{\mathcal{H}}^0$.

Next, we replace each $p \in \mathcal{C}_\varepsilon$ by a loop (a circle) $\mathcal{T}_p$ to obtain $\mathcal{T}_\varepsilon$. Formally, we put

$$\mathcal{T}_\varepsilon \equiv (\mathcal{H}_\varepsilon \setminus \mathcal{C}_\varepsilon) \cup \bigcup_{p \in \mathcal{C}_\varepsilon} \mathcal{T}_p.$$
which is a “decorated” tree consisting of loops and edges. Let

\[ \tilde{T}^m = \bigsqcup_{\varepsilon \in \Sigma^m} T_{\varepsilon}. \]

One can still regard \( T_\varepsilon \) as a subset of \( S_\varepsilon \). To do this, we shrink some connected components of \( \mathcal{H}_\varepsilon \setminus \mathcal{C}_\varepsilon \) and insert \( \mathbb{T}_p \) so that \( \mathbb{T}_p \) surrounds the hole of \( S_\varepsilon \) containing \( p \in \mathcal{C}_\varepsilon \). Then, a continuous injective map \( \chi^m : \tilde{T}^m \to \tilde{S}^m \) is induced from the inclusion map from \( T_{\varepsilon} \) into \( S_\varepsilon \). Let \( \pi^m : \tilde{S}^m \to \tilde{T}^m \) be a homotopy equivalence so that \( \pi^m \circ \chi^m \) is homotopic to the identity map of \( \tilde{T}^m \) and \( \chi^m \circ \pi^m \) is homotopic to the identity map of \( \tilde{S}^m \).

Denote by \( U_p \) the connected component of \( \text{Int} \ K(\tilde{\sigma})_\varepsilon \) containing \( p \in \mathcal{C}_\varepsilon \). For \( \varepsilon \in \Sigma^1 \) and \( p \in \mathcal{C}_\varepsilon \), let \( \tilde{r} : \mathbb{T}_p \to \mathbb{T}_{\tilde{\sigma}(p)} \) be a degree \( d_p \) covering map, where \( d_p \) is the degree of the branched covering \( \tilde{\sigma} : U_p \to U_{\tilde{\sigma}(p)} \). By choosing the covering map \( \tilde{r} \) appropriately, \( \tilde{r} : \mathbb{T}_1 \to \mathbb{T}_0 \) extends to a continuous map \( \tilde{r} : \tilde{T}^1 \to \tilde{T}^0 \). Next we extend \( \tilde{r}_T \). Take \( p \in \mathcal{C}_{e_0e_1} \). If \( \tilde{i}_S(p) \in \mathcal{C}_0 \), then we define \( \tilde{i}_T : \mathbb{T}_p \to \mathbb{T}_{\tilde{i}_S(p)} \) as a homeomorphism. If not, we put \( \tilde{i}_T : \mathbb{T}_p \to \{ \tilde{i}_S(p) \} \) to be a constant map. By choosing the homeomorphism appropriately for each \( p \in \mathcal{C}^1 \), \( \tilde{i}_T : \mathbb{T}_1 \to \mathbb{T}_0 \) extends to a continuous map \( \tilde{i}_T : \tilde{T}^1 \to \tilde{T}^0 \). Thus, we have obtained a multivalued dynamical system \( \tilde{i}_T, \tilde{r} : \tilde{T}^1 \to \tilde{T}^0 \) (see Diagram 2).

We also consider the space of bi-infinite orbits:

\[ \tilde{T}^\infty = \{(\tilde{r}_i)_{i \in \mathbb{Z}} \in (\tilde{T}^1)^\mathbb{Z} : \tilde{r}(\tilde{r}_i) = \tilde{i}_T(\tilde{r}_{i+1}) \}\]

for the multivalued dynamical system \( \tilde{i}_T, \tilde{r} : \tilde{T}^1 \to \tilde{T}^0 \) as well as the shift map \( \tilde{\tau} : \tilde{T}^\infty \to \tilde{T}^\infty \) on it.

Recall that \( \tilde{\sigma} |_{S_{e_0e_1}} : S_{e_0e_1} \to S_{e_1} \) is a covering map. Let \( d_{e_0e_1} \) be its degree. We assume that the pinching locus inherits this property.

**Assumption 4 (Covering).** For each \( e_0e_1 \in \Sigma^1 \), the restriction:

\[ \tilde{\sigma} |_{S_{e_0e_1}} : S_{e_0e_1} \to S_{e_1} \]

is a covering map of degree \( d_{e_0e_1} \).

**Proposition 4.2.** Assumption 4 implies that the map \( \tilde{\tau} : \tilde{T}^1 \to \tilde{T}^0 \) is a covering.
Proof. We show that the restriction $\bar{\tau}|_{\mathcal{T}_{\hat{e}_{\hat{1}}}^1} : \mathcal{T}_{\hat{e}_{\hat{1}}} \to \mathcal{T}_{\hat{e}_{\hat{1}}}^1$ is a covering for $\varepsilon_0 \varepsilon_1 \in \Sigma^1$.

The covering map $\bar{\sigma}|_{\mathcal{S}_{\hat{e}_{\hat{1}}}^1} : \mathcal{S}_{\hat{e}_{\hat{1}}} \to \mathcal{S}_{\hat{e}_{\hat{1}}}^1$ induces a covering map $\bar{\sigma}|_{\partial \mathcal{S}_{\hat{e}_{\hat{1}}}^1} : \partial \mathcal{S}_{\hat{e}_{\hat{1}}} \to \partial \mathcal{S}_{\hat{e}_{\hat{1}}}^1$, and this is true for the inner boundaries of $\partial \mathcal{S}_{\hat{e}_{\hat{1}}}^1$ and $\partial \mathcal{S}_{\hat{e}_{\hat{1}}}$. Thus, for any loop $\gamma$ in $\mathcal{T}_{\hat{e}_{\hat{1}}}$, the restriction $\bar{\tau}|_{\bar{\tau}^{-1}(\gamma)} : \bar{\tau}^{-1}(\gamma) \to \gamma$ is a covering.

Take a point $t \in \mathcal{T}_{\hat{e}_{\hat{1}}}$. If $t$ is contained only in a single loop but not contained in edges, then $\bar{\tau}|_{\bar{\tau}^{-1}(U)} : \bar{\tau}^{-1}(U) \to U$ becomes a covering for a sufficiently small neighborhood $U$ of $t$ in $\mathcal{T}_{\hat{e}_{\hat{1}}}$ by the previous discussion.

Next assume that $t$ is contained in more than one loop but not contained in closed edges of $\mathcal{T}_{\hat{e}_{\hat{1}}}$. Then, there are corresponding Jordan curves in $J(\hat{\sigma})_{\hat{e}_{\hat{1}}}$ which are identified at $\chi^0(t)$. Since $\hat{\sigma}$ is a local conformal map near $J(\hat{\sigma})_{\hat{e}_{\hat{1}}}$, the inverse image $\hat{\sigma}^{-1}(U \cap K(\hat{\sigma})_{\hat{e}_{\hat{1}}})$ consists of biholomorphic copies of $U \cap K(\hat{\sigma})_{\hat{e}_{\hat{1}}}$, where $U$ is a neighborhood of $\chi^0(t)$. This implies that $\bar{\tau}$ is a covering on a neighborhood of $t$.

Finally, assume that $t$ is contained in a closed edge $e$ of $\mathcal{T}_{\hat{e}_{\hat{1}}}$. We recall that each endpoint of $e$ is contained in either a loop or the pinching locus. Suppose first the case where one endpoint is contained in a loop $\gamma$ and the other endpoint is $p \in \mathcal{L}_{\hat{e}_{\hat{1}}}$. Note that $\bar{\tau}$ is a covering over $\gamma$ and over $p$ of the same degree $d$ by Assumption 4. Since $\hat{\sigma}$ is a covering over $\chi^0(e)$, $\hat{\sigma}^{-1}(\chi^0(e))$ consists of $d$ distinct curves in $K(\hat{\sigma})_{\hat{e}_{\hat{1}}}$ by the path-lifting property of $\hat{\sigma}$. Since $K(\hat{\sigma})_{\hat{e}_{\hat{1}}}$ is simply connected, we see that each of the $d$ curves is homotopic to a unique legal arc in $K(\hat{\sigma})_{\hat{e}_{\hat{1}}}$ which connects $\chi^0(\hat{\gamma})$ and $\chi^0(\hat{\rho})$, where $\hat{\gamma}$ is a loop in $\bar{\tau}^{-1}(\gamma)$ and $\hat{\rho}$ is a point in $\bar{\tau}^{-1}(p)$. Moreover, each curve cannot pass through either the connected components of $\text{int} K(\hat{\sigma})_{\hat{e}_{\hat{1}}}$ containing the holes or the pinching locus, since the interior of $e$ is assumed not to intersect either with a loop or the pinching locus. It follows that $\bar{\tau}^{-1}(e)$ consists of $d$ distinct edges in $\mathcal{T}_{\hat{e}_{\hat{1}}}$ each of which connects a loop $\hat{\gamma}$ in $\bar{\tau}^{-1}(\gamma)$ and a point $\hat{\rho}$ in $\bar{\tau}^{-1}(p)$. Thus, $\bar{\tau}$ is a covering over a small neighborhood of $e$. The case where the both endpoints of $e$ are contained in loops and the case where the both endpoints of $e$ are contained in the pinching locus can be argued similarly. This completes the proof. $\square$

4.2. Construction of metrics

Next we define metrics in $\hat{\mathcal{T}}^0$ and $\hat{\mathcal{T}}^1$ so that $\hat{\iota}_{\hat{T}} : \hat{\mathcal{T}}^1 \to \hat{\mathcal{T}}^0$ becomes an expanding system.

We first recall some basic terminologies. Let $A$ be an $n \times n$ matrix with non-negative integer entries. This defines a directed graph $\mathcal{G}$ which consists of arrows with non-zero weights (we define the weight of an arrow from $i$ to $j$ to be $a_{ij}$) and vertices. An irreducible component $\mathcal{G} \subset \mathcal{G}$ is called a sink if there is no out-going edges from $\mathcal{G}$. Similarly, a source is an irreducible component without in-coming edges. We use the same notations for irreducible components of $A$. By $\lambda_{PF}(A)$ we mean the Perron–Frobenius eigenvalue of $A$. For two vectors $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we write $x > y$ iff $x_k > y_k$ for all $1 \leq k \leq n$ and $x \geq y$ iff $x_k \geq y_k$ for all $1 \leq k \leq n$. We use similar notations for matrices as well. We also write $1 = 1^t(1, \ldots, 1)$ and $0 = 1^t(0, \ldots, 0)$.

Let $\{l_1, \ldots, l_M\}$ be the finite collection of loops in $\hat{\mathcal{T}}^1$. For $1 \leq i, j \leq M$ with $\bar{\tau}(l_i) = \bar{\tau}(l_j)$ we let $a_{ij}$ be the degree of the map $\bar{\tau} : l_i \to \bar{\tau}(l_j)$ (otherwise we put $a_{ij} = 0$). This defines a $M \times M$ matrix $A_L = (a_{ij})$ and a directed graph $\mathcal{G}_L$ with positive weights on the arrows of the graph. A cycle (a closed path) $l_i \to \cdots \to l_k \to l_i$ in $\mathcal{G}_L$ is called prime if $i_k \neq i_l$ for all $1 \leq k < l \leq K$. By the degree of a cycle, we mean the product of the weights on the arrows in the cycle. Similarly we let $\{e_1, \ldots, e_N\}$ be the collection of edges in $\hat{\mathcal{T}}^1$. Note that we can take $N$ to be finite by Assumption 2. We let $b_{ij} = 1$ iff $\bar{\tau}(e_i) \supset \bar{\tau}(e_j)$ (otherwise we put $b_{ij} = 0$). This defines a $N \times N$ matrix $A_E = (b_{ij})$ and a directed graph $\mathcal{G}_E$ (we do not consider weights for $\mathcal{G}_E$).
Assumption 5 (Expansion). We assume that

(i) any prime cycle in $G_L$ has degree greater than one, and
(ii) for any sink component $\tilde{G}_E \subset G_E$ there is a vertex of $\tilde{G}_E$ which corresponds to an edge in $\tilde{T}^1$

so that the edge touches a loop in $\tilde{T}^1$.

Lemma 4.3. Let $\tilde{G}_E \subset G_E$ be an irreducible component. Then, there is a vertex of $\tilde{G}_E$ so that it touches a loop in $\tilde{T}^1$ if and only if all vertices of $\tilde{G}_E$ touch loops in $\tilde{T}^1$.

Proof. Let $e_{i_0}$ be a vertex of $\tilde{G}_E$ which touches a loop in $\tilde{T}^1$. Let $e_{i_0} \to e_{i_1} \to \cdots \to e_{i_n} \to e_{i_0}$ be any cycle in $\tilde{G}_E$ and assume that $e_{i_k}$ does not touch loops in $\tilde{T}^1$ for some $1 \leq k \leq n$. Since $\tilde{\tau}(\mathcal{L}^1) \subset \mathcal{L}^0$, it follows that $\tilde{\tau}(e_{i_k})$ does not touch loops in $\tilde{T}^0$. Since $e_{i_k} \to e_{i_{k+1}}$, we have $\tilde{\tau}(e_{i_k}) \supset \tilde{\iota}(e_{i_{k+1}})$. This implies that $e_{i_{k+1}}$ does not touch loops in $\tilde{T}^1$. By repeating this argument, we conclude that $e_{i_0}$ does not touch loops in $\tilde{T}^1$, a contradiction. $\square$

Under Assumption 5, we are able to show

Proposition 4.4. We can define metrics in $\tilde{T}^m$ so that $\tilde{\iota}_T, \tilde{\tau}: \tilde{T}^1 \to \tilde{T}^0$ becomes an expanding system (by modifying $\tilde{\iota}_T$ if necessarily). These also induces metrics in $\tilde{T}^m$.

Proof. The proof is divided into several steps.

Step 1. We first define a metric in each loop in $\tilde{T}^1$. For each vertex of $G_L$ there is only one arrow going out from it. Thus, in each connected component of $G_L$ there is only one irreducible component which forms a prime cycle, and the arrows outside the cycle are all directed to the cycle.

Let $l_1 \to l_2 \to \cdots \to l_{i_K} \to l_{i_1}$ be a prime cycle. By Assumption 5(i), the degree $d$ of the cycle which is the product of the weights appeared in the cycle above is at least two. We fix a constant $C > 0$ and take $\lambda \equiv d {\tilde{\tau}} > 1$. To start with, we define the total length of $l_{i_1}$ to be $C$. We next define the total length of $l_{i_2}$ to be $C \cdot \frac{d_{1,2}}{\lambda}$, where $d_{1,2}$ is the degree of $\tilde{\tau}: l_{i_1} \to \tilde{\tau}(l_{i_2})$. We inductively define the total length of $l_{i_{k+1}}$ by multiplying $\frac{d_{k,K}}{\lambda}$ to the length of $l_{i_k}$ for $1 \leq k < K$. This definition is consistent, since the length of $l_{i_1}$ determined by the last arrow $l_{i_K} \to l_{i_1}$ is

$$C \cdot \frac{d_{1,2}}{\lambda} \cdots \frac{d_{K-1,K}}{\lambda} \cdot \frac{d_{K,1}}{\lambda} = C \cdot \frac{d}{\lambda^K},$$

which is equal to $C$. Note that the length of $l_{i_{k+1}}$ (resp. $l_{i_1}$) is smaller than the length of $l_{i_k}$ (resp. $l_{i_1}$) multiplied by $d_{k,K+1}$ (resp. $d_{K,1}$).

It is then not difficult to define a metric in each loop outside of this cycle in a similar manner. More precisely, when the total length of $l_i$ is determined and there is an arrow $l_j \to l_i$, then the total length of $l_j$ is defined so that the length of $l_i$ is slightly smaller than the length of $l_j$ multiplied by the degree $d_{i,j}$. This defines the structure of a length space in each loop in $\tilde{T}^1$.

Step 2. Next we define a metric in each loop in $\tilde{T}^0$. Take a loop $l^1$ in $\tilde{T}^0$. Then there is a unique loop $l^1$ in $\tilde{T}^1$ which is mapped homeomorphically to $l^0$ by $\tilde{\iota}_T$. We define a metric in $l^0$ so that $\tilde{\iota}_T: l^1 \to l^0$ becomes an isometry. Note that some loops in $\tilde{T}^1$ may shrink to centers in $\tilde{T}^0$ by $\tilde{\iota}_T$, so $\tilde{\iota}_T$ is not a global isometry.
This construction implies that the map $\tilde{\tau} : \tilde{T}^1 \to \tilde{T}^0$ restricted to the loops does not increase the metric. Recall that in the previous paragraph we saw that, when there is an arrow $l_j \to l_i$, the length of $l_i$ is slightly smaller than the length of $l_j$ multiplied by the degree $d_{l,j}$. This implies that the map $\tilde{\tau} : \tilde{T}^1 \to \tilde{T}^0$ restricted to the loops strictly increase the metric. Combining these two facts, one can conclude that $\tilde{\tau} : \tilde{T}^1 \to \tilde{T}^0$ restricted to the loops becomes an expanding system.

**Step 3.** We first define a metric in each edge in $\tilde{T}^1$. Recall that $\tilde{T}^0$ can be viewed as a subset of $\tilde{H}^1$. Thus, once we define metrics in the edges in $\tilde{T}^1$, they are transferred to metrics in the edges in $\tilde{T}^0$. The structure of $G_E$ is more complicated than $G_L$, since there may be multiple arrows going out from one vertex of $G_E$. This implies that there may be several irreducible components in $G_E$.

We take an irreducible component $\tilde{G}_E \subset G_E$ and define metrics in the edges $e_{i_1}, \ldots, e_{i_n}$ of $\tilde{T}^1$ which appear as the vertices of $\tilde{G}_E$. Let $A_E = (b_{ij})$ be the corresponding $n \times n$ matrix. Since for each $i$ there is $j$ so that $b_{ij} = 1$, we see that $A_E \mathbf{1} \geq \mathbf{1}$. It then follows from the monotonicity of $\lambda_{PF}(\cdot)$ that $\lambda_{PF}(A_E) \geq 1$. Let $p = (p_1, \ldots, p_n)$ be a Perron–Frobenius eigenvector of $A_E$. Since $A_E$ is irreducible, we may take $p > 0$ (see [11, Theorem 4.2.3]). Thus, to each edge $e_{i_k}$ we may define its total length to be $p_k$. This gives metrics in edges which appear as vertices of every irreducible component of $G_E$ with the property that $\tilde{\tau} : e_i \cap \tilde{\tau}^{-1}(e_j) \to e_j$ does not decrease the metric.

Now we consider the total graph $G_E$. When an irreducible component $\tilde{G}_E \subset G_E$ is a sink, then we associate metrics to the vertices in $\tilde{G}_E$ (i.e. the corresponding edges in $\tilde{T}^1$) in the way described as above. Let $G_E^1$ be the union of such components. Let $\tilde{G}_E \subset G_E$ be an irreducible component so that all of the out-going edges from it go into $G_E^1$ only. Let $p$ be the Perron–Frobenius eigenvector associated to $\tilde{G}_E'$ as we found in the discussion above. If we choose $\epsilon > 0$ sufficiently small and equip metrics to the vertices in $\tilde{G}_E'$ (i.e. the corresponding edges in $\tilde{T}^1$) by $\epsilon \cdot p$, then $\tilde{\tau}$ strictly increases the metric for any transition from a vertex of $\tilde{G}_E'$ to a vertex of $G_E^1$. Let $G_E^2$ be the union of $G_E^1$ and such components. We repeat this argument finitely many times to define metrics in all vertices in $G_E$ (i.e. all edges in $\tilde{T}^1$) so that (i) $\tilde{\tau}$ does not decrease the metric for any transition inside an irreducible component, and (ii) $\tilde{\tau}$ strictly increases the metric any transition from $G_E^{n-1}$ to $G_E^n$.

**Step 4.** Let $\tilde{G}_E \subset G_E$ be a sink component. Then, by Assumption 5(ii), there is an edge $e^1$ which appears as a vertex of $\tilde{G}_E$ and touches a loop $l^1$ in $\tilde{T}^1$ (in fact, all edges in the component satisfy this property by Lemma 4.3). Let $[\alpha^1, \beta^1]$ are the endpoints of $e^1$ and $e^1$ touches a loop at $\beta^1$. Write $e^0 = \tilde{\tau}(e^1)$ and let $\alpha_0 = \tilde{\tau}(\alpha^1)$ and $\beta_0 = \tilde{\tau}(\beta^1)$ be the endpoints of $e^0$. Now, we modify the definition of the length of $e^0$ to be $1 + \epsilon$ times the length of $e^1$ defined in Step 3, while we keep the condition $\alpha_0 = \tilde{\tau}(\alpha^1)$ and some portion of $l^1$ is mapped into $e^0$ as in Fig. 4 so that $\tilde{\tau}$ stays continuous. By this construction, $\tilde{\tau}$ keeps to be an isometry on $e^1$ and slightly (locally) increase the metric on $l^1$ by the factor of $1 + 2\epsilon$. Thus, if we choose $\epsilon > 0$ to be sufficiently small, then the expansion rate of $\tilde{\tau}$ becomes smaller than that of $\tilde{\tau}$ on the loop $l^1$. Note that by this modification of the length of $e^0$, the map $\tilde{\tau}$ strictly increase the metric on $e^1$ by the factor of $1 + \epsilon$.

When an irreducible component is not a sink, then we may arrange the metrics in the vertices in the component so that $\tilde{\tau}$ strictly increase the metric for all transitions in the component as follows. Let $\tilde{G}_E$ be an irreducible component which is not a sink with vertices $\{e_{i_1}, \ldots, e_{i_n}\}$ and let $\tilde{A}_E$ be the corresponding matrix. Then, there is at least one out-going edge from the corre-
Corresponding component. Recall that $\lambda_{PF}(A_E) \geq 1$ as in Step 3. For any out-going arrow from $\tilde{G}_E$, we modify its head to any vertex in $\tilde{G}_E$ so that it becomes a new arrow inside the component. The matrix $A'_E$ for this new component satisfies $\tilde{A}_E \leq A'_E$ and $(\tilde{A}_E)_{ij} < (A'_E)_{ij}$ for some $(i, j)$. It then follows that $\lambda_{PF}(A'_E) > 1$ by [11, Theorem 4.4.7]. Let $p = (p_1, \ldots, p_n) > 0$ be a Perron–Frobenius eigenvector of $A'_E$ which is also irreducible. We define the total length of $e_k$ to be $\varepsilon p_k$, where $\varepsilon > 0$ is so small that $\tilde{\tau}$ strictly increases the metric for the transition corresponding to any out-going edge from $\tilde{G}_E$. Also, $\tilde{\tau}$ strictly increases the metric for the transitions inside $\tilde{G}_E$, since $A'_E p = \lambda_{PF}(A'_E) p > p$.

We can summarize the above discussion in Step 4 as follows. For any transition in $G_E$, $\tilde{\tau}$ strictly increase the metric, and $\tilde{\iota}_T$ does not increases the metric. Thus, one can conclude that $\tilde{\iota}_T, \tilde{\tau}: \tilde{T}^1 \to \tilde{T}^0$ restricted to the edges becomes an expanding system.

Step 5. The above Steps 2 and 4 with Proposition 4.2 complete the proof. □

4.3. Definition of Hubbard trees

Since $\tilde{T}^m$ can be regarded as a subset of $\tilde{S}^m$ by the injection $\chi^m: \tilde{T}^m \to \tilde{S}^m$ and since the image $\chi^m(\tilde{T}^m)$ contains the pinching locus in $\tilde{S}^m$, we are allowed to define the notion of the pinching locus in $\tilde{T}^m$ to be $(\chi^m)^{-1}(L^m)$. Then, the equivalence relation $\sim_{L^m}$ in $\tilde{S}^m$ is transferred to an equivalence relation in $\tilde{T}^m$ again denoted by $\sim_{L^m}$ to obtain

$$T^m \equiv \tilde{T}^m / \sim_{L^m} = \left( \bigcup_{\varepsilon \in \Sigma^m} T^m_\varepsilon \right) / \sim_{L^m}.$$ 

Let $pr_T: \tilde{T}^m \to T^m$ be a natural projection with respect to the equivalence relation $\sim_{L^m}$. As in Proposition 4.4, one can equip a metric in $T^m$ induced from the one in $\tilde{T}^m$. Put $i_T \equiv pr_T \circ \tilde{i}_T \circ pr_T^{-1}$ and $\tau \equiv pr_T \circ \tilde{\tau} \circ pr_T^{-1}$. We then see that these maps are well-defined thanks to Lemma 3.3 as in Lemma 3.8. Note that $\tau: T^1 \to T^0$ is not necessarily a covering map.

The main object of this article is
Definition 4.5. The multivalued dynamical system

\[ \iota_T, \tau : T^1 \rightarrow T^0 \]

equipped with the metrics in \( T^0 \) and \( T^1 \) as in Proposition 4.4 is called a (decorated) Hubbard tree of a hyperbolic system \( \iota_A, f : A^1 \rightarrow A^0 \) over \( \Gamma \).

Note that only Assumption 3 is used to construct the multivalued dynamical system \( \iota_T, \tau : T^1 \rightarrow T^0 \) from \( \iota_A, f : A^1 \rightarrow A^0 \) together with \( \Gamma \), but Assumptions 2, 4 and 5 are required to define the structure of an expanding system in \( \iota_T, \hat{\tau} : \tilde{T}^1 \rightarrow \tilde{T}^0 \).

We also consider the space of all orbits:

\[ T^\infty = \{(t_i)_{i \in \mathbb{Z}} \in (T^1)^\mathbb{Z} : \tau(t_i) = \iota_T(t_{i+1})\} \]

for the Hubbard tree \( \iota_T, \tau : T^1 \rightarrow T^0 \) as well as the shift map \( \tau : T^\infty \rightarrow T^\infty \) on it.

Remark 4.6. Since \( A_{y,\xi} \) is assumed to be simply connected, the choice of \( y_\xi \in A_{y,\xi} \) in the definition of \( \chi^m : S_{\xi} \rightarrow A_{\xi} \) does not change the homotopy equivalence class (see Definition 5.4) of the Hubbard tree \( \iota_T, \tau : T^1 \rightarrow T^0 \) thanks to [10, Corollary 9.3]. Hence, the topological conjugacy class of \( \tau : T^\infty \rightarrow T^\infty \) does not depend on the choice of \( y_\xi \in A_{y,\xi} \) (see Corollary 5.10).

5. Homotopy shadowing theorems and conjugacies

5.1. Homotopy semi-conjugacies

First we recall some notions from [10]. Let \( \mathcal{X} = (X^0, X^1; \iota, g) \) and \( \mathcal{Y} = (Y^0, Y^1; \iota', g') \) be two multivalued dynamical systems. The following is the notion of a “semi-conjugacy up to homotopy” adopted to our setting.

Definition 5.1. \( \mathcal{X} \) is said to be homotopy semi-conjugate to \( \mathcal{Y} \) if there exist \( h^0 : X^0 \rightarrow Y^0 \) and \( h^1 : X^1 \rightarrow Y^1 \) so that \( h^0 g \) is homotopic to \( g' h^1 \) by \( G = G_t \) (\( G_0 = h^0 g \) and \( G_1 = g' h^1 \)) and \( h^0 t = \text{homotopic to } \iota' h^1 \) by \( H = H_t \) (\( H_0 = h^0 t \) and \( H_1 = \iota' h^1 \)). We call the quadruple \( h = (h^0, h^1; G, H) \) a homotopy semi-conjugacy from \( \mathcal{X} \) to \( \mathcal{Y} \) (see [10, Definition 3.6]).

The pair of identity maps \( \text{id}_{X^0} : X^0 \rightarrow X^0 \) and \( \text{id}_{X^1} : X^1 \rightarrow X^1 \) together with the pair of constant homotopies \( g \) and \( \iota \) defines a homotopy semi-conjugacy from \( \mathcal{X} \) to itself denoted by \( \text{id}_{\mathcal{X}} = (\text{id}_{X^0}, \text{id}_{X^1} ; g, \iota) \).

Definition 5.2. We call \( \text{id}_{\mathcal{X}} = (\text{id}_{X^0}, \text{id}_{X^1} ; g, \iota) \) the identity semi-conjugacy of \( \mathcal{X} \) (see [10, Definition 3.7]).

Let \( h = (h^0, h^1; G, H) \) and \( k = (k^0, k^1; G', H') \) be two homotopy semi-conjugacies from \( \mathcal{X} = (X^0, X^1; \iota, g) \) to \( \mathcal{Y} = (Y^0, Y^1; \iota, g') \).

Definition 5.3. \( h \) is said to be homotopic to \( k \) if there exist \( \mathcal{S} = \mathcal{S}_t : X^1 \rightarrow Y^1 \) with \( S_0 = h^1 \) and \( S_1 = k^1 \) and \( T = T_t : X^0 \rightarrow Y^0 \) with \( T_0 = h^0 \) and \( T_1 = k^0 \) so that (i) \( g' S \cdot (G')^{-1} \) is homotopic to \( G^{-1} \cdot T g \) and (ii) \( H \cdot \iota' S \) is homotopic to \( T \iota \cdot H' \). The pair \( (T, S) \) is called a homotopy from \( h \) to \( k \) (see [10, Definition 3.8]).
Let \( h = (h^0, h^1; G, H) \) be a homotopy semi-conjugacy from \( \mathcal{X} \) to \( \mathcal{Y} \) and let \( k = (k^0, k^1; G', H') \) be a homotopy semi-conjugacy from \( \mathcal{Y} \) to \( \mathcal{Z} \). We define these composition \( kh : \mathcal{X} \to \mathcal{Z} \) as

\[
kh = (k^0 h^0, k^1 h^1; k^0 G \cdot G'h^1, k^0 H \cdot H'h^1).
\]

**Definition 5.4.** \( \mathcal{X} \) and \( \mathcal{Y} \) are said to be **homotopy equivalent** if there exist homotopy semi-conjugacies \( h \) from \( \mathcal{X} \) to \( \mathcal{Y} \) and \( k \) from \( \mathcal{Y} \) to \( \mathcal{X} \) so that \( kh \) is homotopic to the identity semi-conjugacy \( \text{id}_X \) of \( \mathcal{X} \) and \( hk \) is homotopic to the identity semi-conjugacy \( \text{id}_Y \) of \( \mathcal{Y} \) (see [10, Definition 3.9]).

5.2. Homotopy shadowing theorems

In this subsection we slightly extend the homotopy shadowing theorems established in [10] for our purpose.

Let \( (X_i)_{i \in \mathbb{Z}} \) be a sequence of metric spaces and let

\[
\mathcal{X} = (X_i \leftarrow \iota^{-1}(X_i) \cap g^{-1}(X_{i+1}) \xrightarrow{g} X_{i+1})_{i \in \mathbb{Z}}
\]

be a sequence of multivalued dynamical systems. In the proof of the homotopy shadowing theorem, the following notion is important (compare with [10, Definition 6.1]).

**Definition 5.5.** A **homotopy pseudo-orbit** \((x, \alpha)\) of a sequence of multivalued dynamical systems \( \mathcal{X} \) is a sequence \( x = (x_i)_{i \in \mathbb{Z}} \) of points \( x_i \in \iota^{-1}(X_i) \cap g^{-1}(X_{i+1}) \) together with a sequence \( \alpha = (\alpha_i)_{i \in \mathbb{Z}} \) of paths \( \alpha_i : [0, 1] \to X_i \) so that \( \alpha_i(0) = g(x_{i-1}) \) and \( \alpha_i(1) = \iota(x_i) \) hold and the length of \( \alpha_i \) is uniformly bounded with respect to \( i \in \mathbb{Z} \).

When \( \alpha \) consists of constant homotopies, the homotopy pseudo-orbit \((x, \alpha)\) is called an **orbit**.

In this case, the homotopies \( \alpha \) may be omitted from \((x, \alpha)\) and simply write \( x \) if there will be no confusion. This definition of an orbit is consistent with the previous one presented in Subsection 2.2 (compare with [10, Definition 6.4]). Below, \( \cdot \) denotes the concatenation of paths.

**Definition 5.6.** Two homotopy pseudo-orbits \((x, \alpha)\) and \((x', \alpha')\) of a sequence of multivalued dynamical systems \( \mathcal{X} \) are said to be **homotopic** if there is a sequence \( \beta = (\beta_i)_{i \in \mathbb{Z}} \) of paths \( \beta_i : [0, 1] \to \iota^{-1}(X_i) \cap g^{-1}(X_{i+1}) \) of bounded length with \( \beta_i(0) = x_i \) and \( \beta_i(1) = x'_i \) so that \( \alpha_i \cdot \iota(\beta_i) \) is homotopic to \( g(\beta_{i-1}) \cdot \alpha'_i \).

Now, let \( \{A_e\}_{e \in \Sigma} \) be a finite collection of Poincaré boxes where \( A_e = A_{x,e} \times A_{y,e} \) and \( A_{y,e} \) is simply connected. Let

\[
\mathcal{A} = (A_{e_k} \leftarrow \iota^{-1}(A_{e_k}) \cap f^{-1}(A_{e_{k+1}}) \xrightarrow{f} A_{e_{k+1}})_{k \in \mathbb{Z}}
\]

be a sequence of hyperbolic systems, where \( e_k \in \Sigma \). The next theorem generalizes [10, Theorem 8.1] whose proof is identical to it and hence omitted.

**Theorem 5.7.** Every homotopy pseudo-orbit for a sequence of hyperbolic systems \( \mathcal{A} \) is homotopic to an orbit.
The following is a generalization of [10, Proposition 8.8] which implies the uniqueness of the shadowing orbit appeared in Theorem 5.7 above.

**Proposition 5.8.** If two orbits for a sequence of hyperbolic systems \( A \) are homotopic, then they are equal.

Next we explain about the homotopy shadowing theorem for expanding systems. The following theorem is a restatement of [10, Theorem 4.2].

**Theorem 5.9.** A homotopy equivalence between two expanding systems \( \iota, g : X^1 \to X^0 \) and \( \iota, g' : Y^1 \to Y^0 \) induces a topological conjugacy between the shift maps on the orbit spaces \( g : X^\infty \to X^\infty \) and \( g' : Y^\infty \to Y^\infty \).

It can be shown that the homotopy equivalence class of an associated expanding system does not depend on the choice of \( y_0 \in A^1_y \) (see [10, Corollary 9.3]). This fact together with the previous theorem implies

**Corollary 5.10.** The topological conjugacy class of an associated expanding system does not depend on the choice of \( y_0 \in A^1_y \).

In what follows, we thus drop \( y_0 \) in the notation of an associated expanding system.

It can be also shown that a hyperbolic system \( X = (A^0_x \times A^0_y, A^1_x \times A^1_y; \iota_A, f) \) with \( A^0_y \) being contractible and its associated expanding system \( Y = (A^0_x, A^1_x; \iota_S, \sigma) \) are homotopy equivalent [10, Theorem 9.4]. Hence we have the following theorem which makes an connection between the dynamics of a hyperbolic system and its associated expanding system [10, Corollary 9.5].

**Theorem 5.11.** Let \( X = (A^0_x \times A^0_y, A^1_x \times A^1_y; \iota_A, f) \) be a hyperbolic system with \( A^0_y \) being contractible and \( Y = (A^0_x, A^1_x; \iota_S, \sigma) \) be its associated expanding system. Then, the shift maps \( f \) and \( \sigma \) on the corresponding orbit spaces are topologically conjugate.

### 5.3. End of the proof of Theorem B

In this subsection we prove Theorem B and Corollary C in Section 1. The construction of the conjugacy between \( f \) and \( \tau \) splits into two parts. First we construct a conjugacy between \( f \) and \( \sigma \).

**Proposition 5.12.** There exists a homeomorphism

\[ \Phi : A^\infty \to S^\infty \]

so that \( \Phi \circ f = \sigma \circ \Phi \).

**Proof.** Choose an orbit \( (z_i)_{i \in \mathbb{Z}} \in A^\infty \). This is equivalent to the fact that \( z_0 \) belongs to \( J^1(f) \) (by letting \( z_i \equiv (\iota^{-1}_A \circ f)^i(z_0) \) for \( i \in \mathbb{Z} \)). By Assumption 1, there exists \( \tilde{z}_i \in J^1(\tilde{f}) \) so that \( \text{pr}_A(\tilde{z}_i) = z_i \). Note that such a point \( \tilde{z}_i \) is not necessarily unique in general. One then sees that \( \tilde{f}(\tilde{z}_i) = \iota_A(\tilde{z}_{i+1}) \) for \( i \in \mathbb{Z} \). Since
Proof. Take another sequence \((\tilde{\chi}_i)_{i \in \mathbb{Z}}\) \(\sim \iota\). Then, the sequence \((\chi_i)_{i \in \mathbb{Z}}\) becomes an orbit of the branched surface model \(\iota_S, \sigma : S^1 \to S^0\). Here, we need

Lemma 5.13. The orbit \((\text{pr}_S \circ \pi^1(\tilde{\chi}_i))_{i \in \mathbb{Z}}\) does not depend on the choice of \((\tilde{\chi}_i)_{i \in \mathbb{Z}} \in \tilde{A}^\infty\) with \(\text{pr}_A(\tilde{\chi}_i) = z_i\).

Proof. Take another \((\tilde{\chi}'_i)_{i \in \mathbb{Z}} \in \tilde{A}^\infty\) so that \(\text{pr}_A(\tilde{\chi}'_i) = z_i\). For \(i \in \mathbb{Z}\), let \(D_i\) (resp. \(D'_i\)) be the unique leaf of the stable foliation in \(\tilde{A}^1\) through \(z_i\) (resp. \(\tilde{z}'_i\)). Since \(\text{pr}_A(\tilde{\chi}_i) = \text{pr}_A(\tilde{\chi}'_i)\), either \(D_i, D'_i\) forms an intersecting pair of pinching disks or \(D = D'\). This implies that \(\pi^1(D_i)\) and \(\pi^1(D'_i)\) in \(\tilde{S}^1\) are identified by \(\sim_{\Sigma^1}\). Hence we have \(\text{pr}_S \circ \pi^1(D_i) = \text{pr}_S \circ \pi^1(D'_i)\) and \(\text{pr}_S \circ \pi^1(\tilde{\chi}_i) = \text{pr}_S \circ \pi^1(\tilde{\chi}'_i)\). This finishes the proof. \(\square\)

We can thus define \(\Phi : A^\infty \to S^\infty\) by

\[
\Phi : A^\infty \ni (z_i)_{i \in \mathbb{Z}} \longmapsto (\text{pr}_S \circ \pi^1(\tilde{\chi}_i))_{i \in \mathbb{Z}} \in S^\infty.
\]

Then, \(\Phi\) satisfies \(\Phi \circ f = \sigma \circ \Phi\).

Conversely, from an orbit of \(\iota_S, \sigma : S^1 \to S^0\), one can define a homotopy pseudo-orbit of \(\tilde{\iota}_A, \tilde{f} : \tilde{A}^1 \to \tilde{A}^0\) as follows. Given an orbit \((\tilde{s}_i)_{i \in \mathbb{Z}} \in S^\infty\), we choose \((\tilde{z}_i)_{i \in \mathbb{Z}} \in \tilde{S}^\infty\) so that \(\text{pr}_S(\tilde{z}_i) = s_i\) for \(i \in \mathbb{Z}\). Again, note that the choice of \(\tilde{z}_i\) is not necessarily unique. Then, the sequence \((\chi^1(\tilde{z}_i))_{i \in \mathbb{Z}}\) has the property that \(\tilde{f}(\chi^1(\tilde{z}_{i-1}))\) and \(\tilde{\iota}_A(\chi^1(\tilde{z}_i))\) are in a straight vertical disk of some Poincaré box \(A_i\). These points can be joined by a homotopy \(\alpha_i\) whose support is in the vertical disk. Note that since \(A_{Y,x_i}\) is simply connected, the homotopy class of \(\alpha_i\) is unique. Then, \((\chi^1(\tilde{z}_i))_{i \in \mathbb{Z}}, (\alpha_i)_{i \in \mathbb{Z}}\) is a homotopy pseudo-orbit of the hyperbolic system \(\tilde{\iota}_A, \tilde{f} : \tilde{A}^1 \to \tilde{A}^0\). Now, we do the shadowing. We can find an orbit \((\tilde{z}_i)_{i \in \mathbb{Z}} \in \tilde{A}^\infty\) homotopic to \((\chi^1(\tilde{z}_i))_{i \in \mathbb{Z}}, (\alpha_i)_{i \in \mathbb{Z}}\) thanks to Theorem 5.7. It is easy to check that \((\text{pr}_A(\tilde{z}_i))_{i \in \mathbb{Z}}\) is an orbit of \(\iota_A, \tilde{f} : A^1 \to A^0\).

The most crucial step is to prove the following claim.

Lemma 5.14. The orbit \((\text{pr}_A(\tilde{\chi}_i))_{i \in \mathbb{Z}} \in A^\infty\) does not depend on the choice of \((\tilde{s}_i)_{i \in \mathbb{Z}} \in \tilde{S}^\infty\) with \(\text{pr}_S(\tilde{s}_i) = s_i\).

Proof. Let us take another orbit \((\tilde{s}'_i) \in \tilde{S}^\infty\) so that \(\tilde{s}_i \sim_{\Sigma^1} \tilde{s}'_i\) for all \(i \in \mathbb{Z}\). By the definition of \(\sim_{\Sigma^1}\) and \(\approx_{\Sigma^1}\), for each \(i \in \mathbb{Z}\) there exists a chain of points \([\tilde{a}_i,0, \ldots, \tilde{a}_{i,L}] \subset \tilde{A}^1\) so that \(\text{pr}_A(\tilde{a}_{i,l})\) and \(\text{pr}_A(\tilde{a}_{i,l+1})\) belong to \(\text{pr}_A(D^1_{L+1})\) for some pinching disk \(D^1_{L+1} (l = 0, \ldots, L - 1)\) in \(\tilde{A}^1\), and \(\tilde{a}_i,0 = \chi^1(\tilde{s}_i)\) and \(\tilde{a}_{i,L} = \chi^1(\tilde{s}'_i)\). Note that we can take \(L\) to be independent of \(i \in \mathbb{Z}\) thanks to Assumption 2. Let \(\alpha_{i,l}\) be a homotopy in \(\text{pr}_A(D^1_{L+1})\) connecting \(\text{pr}_A(\tilde{a}_{i,l})\) to \(\text{pr}_A(\tilde{a}_{i,l+1})\). Then, the
pair \((pr_A(\tilde{\alpha}_i,l))_i\in\mathbb{Z}, (\alpha_i,l)_i\in\mathbb{Z}\) becomes a homotopy pseudo-orbit for a sequence of hyperbolic systems:

\[
(A_{\varepsilon_k} \leftarrow t^{-1}_A(A_{\varepsilon_k}) \cap f^{-1}(A_{\varepsilon_{k+1}}) \xrightarrow{f} A_{\varepsilon_{k+1}})_{k\in\mathbb{Z}}
\]

in the sense of Definition 2.7, where \(\varepsilon_k \in \Sigma\). By Theorem 5.7, there exists an orbit which is homotopic to the homotopy pseudo-orbit \((pr_A(\tilde{\alpha}_i,l))_i\in\mathbb{Z}, (\alpha_i,l)_i\in\mathbb{Z}\) for every \(l = 0, \ldots, L\). Notice that the two homotopy pseudo-orbits \((pr_A(\tilde{\alpha}_i,l))_i\in\mathbb{Z}, (\alpha_i,l)_i\in\mathbb{Z}\) and \((pr_A(\tilde{\alpha}_i,l+1))_i\in\mathbb{Z}, (\alpha_i,l+1)_i\in\mathbb{Z}\) are homotopic by a sequence of homotopies \((\beta_i)_i\in\mathbb{Z}\) whose support is contained in the disk \(pr_A(D^1_{i,l})\). In fact, there is a vertical disk \(D^0_{i+1,l} \subset \mathcal{A}^0\) so that \(f(D^1_{i,l}) \subset D^0_{i+1,l}\), and thus the two homotopy pseudo-orbits above coincide for every \(l = 0, \ldots, L - 1\). It follows that the orbit \((pr_A(\tilde{\alpha}_i)_i\in\mathbb{Z})\) is independent of the choice of \((\tilde{s}_i)_i\in\mathbb{Z}\).

We can thus define \(\Psi : S^\infty \rightarrow A^\infty\) by

\[
\Psi : S^\infty \ni (s_i)_{i\in\mathbb{Z}} \mapsto (pr_A(\tilde{\alpha}_i))_{i\in\mathbb{Z}} \in A^\infty.
\]

Then, \(\Psi\) satisfies \(f \circ \Psi = \Psi \circ \sigma\).

**End of the proof of Proposition 5.12.** To finish the proof we will show that \(\Phi \circ \Psi = \Psi \circ \Phi = \text{id}\). Define \(\tilde{\Phi} : \tilde{A}^\infty \rightarrow \tilde{S}^\infty\) by \(\tilde{\Phi}((\tilde{s}_i)_i\in\mathbb{Z}) \equiv (\tilde{s}_i)_i\in\mathbb{Z}\) and \(\tilde{\Psi} : \tilde{S}^\infty \rightarrow \tilde{A}^\infty\) by \(\tilde{\Psi}((\tilde{s}_i)_i\in\mathbb{Z}) \equiv (\tilde{s}_i)_i\in\mathbb{Z}\), where \((\tilde{s}_i)_i\in\mathbb{Z} \in \tilde{A}^\infty\) and \((\tilde{s}_i)_i\in\mathbb{Z} \in \tilde{S}^\infty\) are appeared in the above construction. Then, as in the last part of the proof of Theorem 5.11 (see [10, Section 10]), we see that \(\tilde{\Phi} \circ \tilde{\Psi} = \tilde{\Psi} \circ \tilde{\Phi} = \text{id}\).

Recall that we can write \(\Phi \circ \Psi : S^\infty \rightarrow S^\infty\) as

\[
\Phi \circ \Psi = pr_S \circ \tilde{\Phi} \circ pr_A^{-1} \circ pr_A \circ \tilde{\Psi} \circ pr^{-1}_S.
\]

Let \((s_i)_i\in\mathbb{Z} \in S^\infty\) and take any \((\tilde{s}_i)_i\in\mathbb{Z} \in \tilde{S}^\infty\) so that \(pr_S((\tilde{s}_i)_i\in\mathbb{Z}) = (s_i)_i\in\mathbb{Z}\). Let \((\tilde{z}_i)_i\in\mathbb{Z} \equiv \tilde{\Psi}((\tilde{s}_i)_i\in\mathbb{Z})\) and take any \((\tilde{z}_i)_i\in\mathbb{Z} \in pr_A^{-1} \circ pr_A((\tilde{z}_i)_i\in\mathbb{Z})\). This implies that \(pr_A((\tilde{z}_i)_i\in\mathbb{Z}) = pr_A((\tilde{z}_i)_i\in\mathbb{Z})\). Thanks to Lemma 5.13, we obtain \(pr_S \circ \tilde{\Phi}((\tilde{z}_i)_i\in\mathbb{Z}) = pr_S \circ \tilde{\Psi}((\tilde{z}_i)_i\in\mathbb{Z}) = pr_S \circ \Phi \circ \Psi((s_i)_i\in\mathbb{Z}) = pr_S((s_i)_i\in\mathbb{Z}) = (s_i)_i\in\mathbb{Z}\). So, \(\Phi \circ \Psi((s_i)_i\in\mathbb{Z}) = pr_S \circ \tilde{\Phi}((\tilde{z}_i)_i\in\mathbb{Z}) = (s_i)_i\in\mathbb{Z}\) and hence \(\Phi \circ \Psi = \text{id}\). The other equality \(\Psi \circ \Phi = \text{id}\) can be obtained similarly by using Lemma 5.14. Hence the proof of Proposition 5.12 is done.

Next we construct a conjugacy between \(\sigma\) and \(\tau\).

**Proposition 5.15.** There exists a homeomorphism

\[
\Phi : S^\infty \rightarrow T^\infty
\]

so that \(\Phi \circ \sigma = \tau \circ \Phi\).

**Proof.** Lemma 3.7 tells that \(\tilde{\iota}_S, \tilde{\sigma} : \tilde{S}^1 \rightarrow \tilde{S}^0\) is an expanding system. By Proposition 4.4, one can define metrics in \(\tilde{T}^m\) so that \(\tilde{\iota}_T, \tilde{\tau} : \tilde{T}^1 \rightarrow \tilde{T}^0\) becomes an expanding system. Since we
choose \( \chi^m : \tilde{T}^m \to \tilde{S}^m \) and \( \pi^m : \tilde{S}^m \to \tilde{T}^m \) so that \( \pi^m \circ \chi^m \) is homotopic to the identity map of \( \tilde{T}^m \) and \( \chi^m \circ \pi^m \) is homotopic to the identity map of \( \tilde{S}^m \), it follows that \( \tilde{t}_S, \tilde{\sigma} : \tilde{S}^1 \to \tilde{S}^0 \) and \( \tilde{t}_T, \tilde{\tau} : \tilde{T}^1 \to \tilde{T}^0 \) are homotopy equivalent. We then know from Theorem 5.9 that \( \tilde{\sigma} : \tilde{S}^\infty \to \tilde{S}^\infty \) and \( \tilde{\tau} : \tilde{T}^\infty \to \tilde{T}^\infty \) are topologically conjugate. Then, this induces a topological conjugacy between \( \sigma : S^\infty \to S^\infty \) and \( \tau : T^\infty \to T^\infty \) because the equivalence relations \( \sim_{\Sigma^m} \) in \( \tilde{S}^m \) and \( \sim_{\Sigma^m} \) in \( \tilde{T}^m \) are essentially the same. This finishes the proof. \( \Box \)

**Remark 5.16.** Compare the above proof of Proposition 5.15 with the one by Douady [3] for polynomial maps (see Theorem 2.2) which is purely topological and need not to define expanding metrics. Our method of proof using expanding metrics seems to be new even for the polynomial map case and can apply to some disconnected Julia sets.

Now we arrive at the precise statement of Theorem B in Section 1.

**Theorem 5.17.** Let \( \{A_x\}_{x \in \Sigma} \) be a finite collection of Poincaré boxes with \( A_{Y_x} \) being simply connected and let \( t_A, f : t_A^{-1}(A) \cap f^{-1}(A) \to A \) be a hyperbolic system over \( \Gamma \subset \Sigma \times \Sigma \), where \( \mathcal{A} \equiv \bigcup_{x \in \Sigma} A_x \). Suppose that it satisfies Assumptions 1 to 5. Then, the shift map \( f : \mathcal{A}^\infty \to \mathcal{A}^\infty \) is topologically conjugate to \( \tau : \mathcal{T}^\infty \to \mathcal{T}^\infty \).

**Proof.** Combine Propositions 5.12 and 5.15. \( \Box \)

Let \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) be a polynomial diffeomorphism of \( \mathbb{C}^2 \). We define

\[
K^\pm \equiv \{ z \in \mathbb{C}^2 : \{ f^{\pm n}(z) \}_{n \geq 0} \text{ is bounded in } \mathbb{C}^2 \}.
\]

Then, one can easily see that

\[
J^\pm \equiv \partial K^\pm = \{ z \in K^\pm : \{ f^{\pm n}(z) \}_{n \geq 0} \text{ is not normal at } z \}.
\]

Recall that \( f \) is said to be hyperbolic if its Julia set \( J_f \equiv J^+ \cap J^- \) is a hyperbolic set for \( f \). Corollary C in Section 1 is precisely stated as

**Corollary 5.18.** Under the circumstance of Theorem 5.17, we let \( f \) be a polynomial diffeomorphism of \( \mathbb{C}^2 \). Suppose that \( \mathcal{A}^\infty \) is hyperbolic for \( f \) and \( J_f \subset \mathcal{A} \). Then, the restriction \( f : J_f \to J_f \) is topologically conjugate to \( \tau : \mathcal{T}^\infty \to \mathcal{T}^\infty \).

**Proof.** Recall that we may identify \( \mathcal{A}^\infty \) with \( \bigcap_{n \in \mathbb{Z}} f^n(A) \). Take a point \( z \in \bigcap_{n \in \mathbb{Z}} f^n(A) \). Since \( A \) is bounded, we have \( z \in K^\pm \). Since there is a hyperbolic splitting of \( T_z \mathbb{C}^2 \) at \( z \), one sees that \( \{ f^{\pm n}(z) \}_{n \geq 0} \) are not normal at \( z \). Thus, we have \( z \in J^\pm \) and hence \( z \in J_f \). This shows \( \bigcap_{n \in \mathbb{Z}} f^n(A) \subset J_f \). By the assumption \( J_f \subset \mathcal{A} \), it follows that \( \mathcal{A}^\infty = \bigcap_{n \in \mathbb{Z}} f^n(A) = J_f \). The conclusion then follows from Theorem 5.17. \( \Box \)

**Remark 5.19.** Hyperbolicity of \( t_A, f : t_A^{-1}(A) \cap f^{-1}(A) \to A \) over \( \Gamma \) satisfying Assumption 1 implies the existence of a hyperbolic splitting at many points in \( \mathcal{A}^\infty \) by employing a new cone field \((\mathbb{C}^n)_{p \in \mathcal{A}}, \| \cdot \|_n \) as in [8, Definition 4.1], showing that the cones are non-empty [8, Lemmas 4.18 and 4.19] and applying a criterion in [8, Proposition 4.3]. To prove hyperbolicity over all points in \( \mathcal{A}^\infty \), however, we often need some additional argument (see [8, Numerical Check 6]...
6. Some examples of Hubbard trees for Hénon maps

This section discusses three classes of examples of Hubbard trees for complex Hénon maps. In Subsection 6.1 we consider a hyperbolic cubic Hénon map which exhibits essentially two-dimensional dynamics. Subsection 6.2 treats small perturbations of expanding quadratic polynomials in one variable. In the last subsection we discuss a crossed mapping model for a Hénon map of degree two with connected Julia set.

6.1. A non-planar cubic Hénon map

By [1,5,7] we know that, for any expanding polynomial \( p \) in one variable, the generalized Hénon map \( f = f_{p,b} \) is hyperbolic for \( b \in \mathbb{C}^\times \) sufficiently close to zero. Moreover, it has been shown in [7] that the restriction \( f: J_f \to J_f \) is topologically conjugate to the shift map on the projective limit space \( p: \lim\langle p, J_p \rangle \to \lim\langle p, J_p \rangle \) where \( J_p \) denotes the Julia set of \( p \).

Consider the cubic Hénon map:

\[
    f = f_{a,b} : (x, y) \mapsto (-x^3 + a - by, x),
\]

where \((a, b) = (-1.35, 0.2)\). In [8, Theorem A] we have shown that the cubic Hénon map above is hyperbolic but is not topologically conjugate on its Julia set to a small perturbation of any expanding polynomial in one variable. Hence this is the first example of a hyperbolic Hénon map which exhibits essentially two-dimensional dynamics.

Fig. 5 represents the Poincaré boxes \( \{A_i\}_{i=0}^3 \) and their images by the cubic map \( f = f_{a,b} \) which have been used to show the hyperbolicity of \( f \) on its Julia set in [8]. We put \( \Sigma \equiv \{0, 1, 2, 3\} \) and let

\[
    \Gamma \equiv \{(0, 3), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)\}
\]

be the set of admissible transitions for \( f \). Then, it follows from [8, Proposition 4.16] that \( f: \mathcal{A} \cap f^{-1}(\mathcal{A}) \to \mathcal{A} \) is a hyperbolic system over \( \Gamma \), where \( \mathcal{A} = \bigcup_{i=0}^3 A_i \).
Hereafter we write $$W^s_{loc}(q; X) \equiv \{ z \in W^s(q): f^n(z) \in X \text{ for all } n \geq 0 \}$$ for a saddle fixed point $$q$$ of $$f$$. We use the following fact from [8, Numerical Check 6]. There, $$D_0$$ is the projective polydisk obtained by filling up the hole of $$\mathcal{A}_0$$.

**Lemma 6.1.** There exists a polydisk $$\mathcal{V} \supseteq \bigcap_{|p| \leq 2} f^n(D_0)$$ so that $$f: \mathcal{V} \cap f^{-1}(\mathcal{V}) \rightarrow \mathcal{V}$$ is a crossed mapping of degree one. In particular, there exists a unique saddle fixed point $$p \in \mathcal{V}$$ and $$\bigcap_{n \geq 0} f^{-n}(\mathcal{V}) = W^s_{loc}(p; \mathcal{V})$$.

Let us define $$D_i \equiv W^s_{loc}(p; \mathcal{A}_i) \subset \mathcal{A}_i$$ for $$i = 0, 3$$. Note that $$D_i$$ is the connected component of $$W^s(p) \cap \mathcal{A}_i$$ containing the saddle fixed point $$p$$. Moreover, $$D_i$$ becomes a vertical disk of degree one in $$\mathcal{A}_i$$ for $$i = 0, 3$$. Similarly we let $$D_i \subset \mathcal{A}_i$$ be the connected component of $$f^{-1}(D_0 \cap f(\mathcal{A}_i))$$ intersecting with $$D_0$$ for $$i = 1, 2$$. Since $$f: \mathcal{A}_i \cap f^{-1}(\mathcal{A}_0) \rightarrow \mathcal{A}_0$$ is a crossed mapping satisfying the (NTC), $$D_i$$ becomes a vertical disk of degree one in $$\mathcal{A}_i$$ for $$i = 1, 2$$.

**Lemma 6.2.** We have $$\bigcap_{n \geq 0} f^{-n}(\mathcal{A}_0 \cap \mathcal{A}_3) = D_0 \cap D_3$$.

**Proof.** Since $$z \in W^s_{loc}(p; \mathcal{A}_0)$$ implies that $$f^n(z) \in \mathcal{A}_0$$ and since $$z \in W^s_{loc}(p; \mathcal{A}_3)$$ implies that $$f^n(z) \in \mathcal{A}_3$$ for all $$n \geq 0$$, the inclusion $$\bigcap_{n \geq 0} f^{-n}(\mathcal{A}_0 \cap \mathcal{A}_3) \supseteq D_0 \cap D_3$$ is trivial.

Conversely, let us assume that $$f^n(z) \in \mathcal{A}_0 \cap \mathcal{A}_3$$ for all $$n \geq 0$$. Then, $$f^n(z) \in \bigcap_{-2 \leq k \leq 2} f^k(\mathcal{A}_0 \cap \mathcal{A}_3)$$. If $$f^{n_i}(z) \not\in \bigcap_{0 \leq k \leq 2} f^k(\mathcal{A}_0 \cap \mathcal{A}_3)$$ for more than two positive integers $$n_1 < n_2 < n_3 < \ldots$$, then $$n_3 \geq 2$$ and hence either $$f^{n_3}(z) \not\in \mathcal{A}_0 \cap \mathcal{A}_3$$, $$f^{n_3-1}(z) \not\in \mathcal{A}_0 \cap \mathcal{A}_3$$ or $$f^{n_3-2}(z) \not\in \mathcal{A}_0 \cap \mathcal{A}_3$$ holds, a contradiction. It follows that there exists $$N \geq 0$$ with $$f^n(z) \in \bigcap_{|k| \leq 2} f^k(\mathcal{A}_0 \cap \mathcal{A}_3)$$ for all $$n \geq N$$. Thus, $$f^n(z) \in \mathcal{V}$$ for all $$n \geq N$$ and hence $$f^n(z) \rightarrow p$$ as $$n \rightarrow +\infty$$ by Lemma 6.1. Since we have assumed that $$f^n(z) \in \mathcal{A}_0$$ and $$f^n(z) \in \mathcal{A}_3$$ for all $$n \geq 0$$, this implies $$z \in W^s_{loc}(p; \mathcal{A}_0)$$ and $$z \in W^s_{loc}(p; \mathcal{A}_3)$$. This finishes the proof. \(\square\)

Next we define some disks in $$\tilde{\mathcal{A}}^1$$. For $$(i, j) \in \Gamma$$ with $$j \neq 0$$, let $$D_{ij} \equiv f^{-1}(D_j) \cap \mathcal{A}_i$$. For $$(i, j) \in \Gamma$$ with $$j = 0$$, let $$\{D^k_{10}\}_{k=1}^3$$ be the connected components of $$f^{-1}(D_0) \cap \mathcal{A}_i$$ such that $$f(D^k_{10}) \cap D_k \neq \emptyset$$. Now we determine the intersecting pairs of pinching disks in $$\tilde{\mathcal{A}}^{m}$$.

**Proposition 6.3.**

(i) The intersecting pairs of pinching disks in $$\tilde{\mathcal{A}}^0$$ for $$f$$ are $$\{D_0, D_1\}$$, $$\{D_0, D_2\}$$ and $$\{D_0, D_3\}$$.

(ii) The intersecting pairs of pinching disks in $$\tilde{\mathcal{A}}^1$$ for $$f$$ are $$\{D^1_{10}, D_{11}\}$$, $$\{D^2_{10}, D_{12}\}$$, $$\{D^1_{20}, D_{21}\}$$, $$\{D^2_{20}, D_{22}\}$$, $$\{D^1_{30}, D_{31}\}$$, $$\{D^2_{30}, D_{32}\}$$, $$\{D^3_{10}, D_{33}\}$$, $$\{D^3_{20}, D_{33}\}$$ and $$\{D^3_{30}, D_{33}\}$$.

**Proof.** Since the proofs for (i) and (ii) are similar, we only examine the case (i).

For $$\varepsilon = \varepsilon_0 \varepsilon_1 \ldots \in \Sigma^\mathbb{N}$$ with $$\varepsilon_i, \varepsilon_{i+1} \in \Gamma$$, we set

$$J^{+}(f)_{\varepsilon} \equiv \bigcap_{n \geq 0} f^{-n}(\mathcal{A}_{\varepsilon_n}).$$

Then, the proof of [8, Theorem 4.23(iv)] implies that $$J^{+}(f)_{\varepsilon} \cap J^{+}(f)_{\varepsilon'} \neq \emptyset$$ and $$\varepsilon_0 \neq \varepsilon'_0$$ hold only if either $$\{\varepsilon, \varepsilon'\} = \{0303, \ldots, 1030, \ldots\}$$, $$\{\varepsilon, \varepsilon'\} = \{0303, \ldots, 2030, \ldots\}$$ or $$\{\varepsilon, \varepsilon'\} = \{0303, \ldots, 3030, \ldots\}$$ is satisfied. Thus, it follows that any pinching disk must intersect with
Lemma 6.2 implies that \((A_0 \cap A_i) \cap (\bigcap_{n \geq 1} f^{-n}(A_0 \cap A_3))\) for some \(1 \leq i \leq 3\). By the definition of \(D_0\) and \(D_3\), one obtains \(f(A_0) \cap D_3 = f(D_0)\). By the definition of \(D_i\), one gets \((f^{-1}(D_0) \cap A_i) \cap A_0 = D_i\) for \(i = 1, 2\). These imply that \((A_0 \cap A_i) \cap f^{-1}(D_0 \cap D_3) = D_0 \cap D_i\) for \(1 \leq i \leq 3\). Thus, the totality of the pinching disks is \(\{D_i\}_{i=0}^3\).

We next determine the intersecting pairs of pinching disks. We know that \(D_0 \cap D_3 \neq \emptyset\), since \(p \in D_0 \cap D_3\). Thus, \(\{D_0, D_3\}\) forms an intersecting pair of pinching disk. Next, one has \(D_0 \cap D_i \neq \emptyset\) by the definition of \(D_i\) for \(i = 1, 2\). Thus, \(\{D_0, D_1\}\) and \(\{D_0, D_2\}\) form intersecting pairs of pinching disk. Since \(A_i \cap A_j = \emptyset\) for all \(1 \leq i < j \leq 3\), we have \(D_i \cap D_j = \emptyset\) for all \(1 \leq i < j \leq 3\). Thus, \(\{D_0, D_1\}\), \(\{D_0, D_2\}\) and \(\{D_0, D_3\}\) are the only intersecting pairs of pinching disks. This finishes the proof of (i). □

Now we construct a Hubbard tree for the cubic Hénon map \(f\). Assumption 2 is trivially satisfied thanks to Proposition 6.3. It is not difficult to see that Assumptions 3 and 4 hold. One can thus define a multivalued dynamical system \(\iota, \tau : T^1 \rightarrow T^0\). We then label the edges and loops in \(T^1\) (thus, in \(\widetilde{T}^1\)) as shown in Fig. 6. Note that the dots in \(T^m\) represent the points in \(\text{pr}_T(L^m)\).

Let us check Assumption 5. The matrix \(A_L\) for the loops in \(\widetilde{T}^m\) is expressed as

\[
A_L = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
Then, its unique irreducible component is
\[
\tilde{A}_L = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}
\]
over \{l_1, l_2\} and its degree is 3. The matrix \(A_E\) for edges is given by
\[
A_E = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]
Then, its unique sink irreducible component is
\[
\tilde{A}_E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
over \{e_1, e_2\} and the edges \(e_1\) and \(e_2\) touch loops. Thus, Assumption 5 is satisfied and the Hubbard tree can be defined by Theorem A in Section 1.

Assumption 1 holds thanks to [8, Theorem 4.23(i)]. Since we already know that \(\bigcup_{i=0}^3 A_i \supset J_f\) as stated at the end of Subsection 4.4 of [8] and since \(A^\infty\) is hyperbolic for \(f\) by [8, Corollaries 4.17 and 4.21], it follows from Corollary C that the cubic Hénon map \(f : J_f \to J_f\) is topologically conjugate to the shift map \(\tau : T^\infty \to T^\infty\).

### 6.2. Basilica, rabbit and airplane

In this subsection, we construct Hubbard trees for small perturbations of expanding quadratic polynomials \(p_c(x) = x^2 + c\) inside the complex Hénon family:
\[
f = f_{c,b} : (x, y) \mapsto (x^2 + c - by, x),
\]
where \((c, b) \in \mathbb{C} \times \mathbb{C}\) and \(b \neq 0\). Recall that, for any expanding quadratic polynomial \(p_c\), there exists a small \(b_\ast > 0\) so that the complex Hénon map \(f = f_{c,b}\) is hyperbolic for \(0 < |b| < b_\ast\). See [8, Theorem C] for explicit bounds on such \(b_\ast\) when \(c = 0, c = -1\) and some horseshoe parameters.

(i) **Basilica.** The first example we discuss is a small perturbation of \(p_{-1}(x) = x^2 - 1\) which has a superattractive cycle of period two whose Julia set is called the **basilica**.

Let \(\Sigma \equiv \{0, 1\}\) and \(\Gamma \equiv \{(0, 0), (0, 1), (1, 0)\}\). Let \(f = f_{-1,b}\) where \(b \neq 0\) is sufficiently close to zero. In Example of Subsection 3.2, we have constructed a family of two Poincaré boxes \(\{A_0, A_1\}\) and shown that \(f : A \cap f^{-1}(A) \to A\) is a hyperbolic system over \(\Gamma\), where \(A = A_0 \cup A_1\). We have also seen that the disks \(\{D_0, D_1\}\) defined in Example form the unique pair of pinching disks in \(A^0 = A\) for \(f\).
Fig. 7. Hubbard tree for a perturbation of the basilica.

One can easily show that Assumptions 2 to 4 are satisfied. We then get a multivalued dynamical system $\iota_{\mathcal{T}}: \mathcal{T}^1 \rightarrow \mathcal{T}^0$. We label the edges and loops in $\mathcal{T}^1$ (thus, in $\tilde{T}^1$) as in Fig. 7. The unique irreducible component of the matrix $A_L$ for loops is

$$
\tilde{A}_L = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}
$$

over $\{l_1, l_2\}$ and the degree is 2. The unique sink irreducible component of the matrix $A_E$ for edges is

$$
\tilde{A}_E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

over $\{e_1, e_2\}$ and the edges $e_1$ and $e_2$ touch loops. Thus, Assumption 5 is satisfied and we can construct a Hubbard tree $\iota_{\mathcal{T}}, \tau: \mathcal{T}^1 \rightarrow \mathcal{T}^0$. Assumption 1 can be easily verified. Since one has chosen $A_i$ so that $A_0 \cup A_1 \supset J_f$ and since $A^\infty$ is hyperbolic for $f$ as shown in [8, Corollary 4.4], it follows from Corollary C that $f: J_f \rightarrow J_f$ is topologically conjugate to $\tau: \mathcal{T}^\infty \rightarrow \mathcal{T}^\infty$.

(ii) *Rabbit.* Next we describe a small perturbation of a quadratic map with a superattractive cycle of period three whose Julia set is called the rabbit.

Let $c_0 \approx -0.122561 + 0.744862i$ be the rabbit parameter and fix a large $r > 0$. It is known that the external angles in the parameter space for the hyperbolic component of the Mandelbrot set containing $c_0$ are $1/7$ and $2/7$. Consider the orbits $\{1/7, 2/7, 4/7\}$ of these angles by the angle doubling map. Then, the three external rays of the angles $1/7, 2/7$ and $4/7$ together with their unique landing point $z_0$ (which is a repelling fixed point of $p_{c_0}$) divide the region $\{x \in \mathbb{C}: G_{c_0}(x) < r\}$ into three pieces $\{U_i\}_{i=0}^2$, where $G_{c_0}$ is the Green function for the filled Julia set of $p_{c_0}$ and $p_{c_0}^i(0) \in U_i$. We fatten $U_i$ slightly and remove a sufficiently small neighborhood of the superattractive cycle of period three to get three annuli which we denote by $A_i$ ($i = 0, 1, 2$). Define the three Poincaré boxes as $A_i \equiv A_i \times \Delta(0; R)$ and write $A = \bigcup_{i=0}^2 A_i$. Let $\Sigma \equiv \{0, 1, 2\}$ and $\Gamma = \{(0, 0), (0, 1), (1, 2), (2, 0)\}$. Then, $f: A \cap f^{-1}(\mathcal{A}) \rightarrow \mathcal{A}$ becomes a hyperbolic system over $\Gamma$, where $f$ is a small perturbation of $p(x) = x^2 + c_0$ inside the complex Hénon family.

One can then show that Assumptions 2 to 4 are satisfied. In fact, the pinching disks in $\mathcal{A}^0$ are the local stable manifolds of the fixed saddle point which can be obtained as the continuation of $z_0$ through a small perturbation. There are three pinching disks $D_i \subset A_i$ ($i = 0, 1, 2$) and any
two of the three disks form an intersecting pair of pinching disks. We then get a multivalued dynamical system \( \iota, \tau : T^1 \to T^0 \). We label the edges and loops in \( T^1 \) (thus, in \( \tilde{T}^1 \)) as shown in Fig. 8. The unique irreducible component of the matrix \( A_L \) for loops is

\[
\tilde{A}_L = \begin{pmatrix}
0 & 2 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]

over \( \{l_1, l_2, l_3\} \) and the degree is 2. The unique sink irreducible component of the matrix \( A_E \) for edges is

\[
\tilde{A}_E = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]

over \( \{e_1, e_2, e_3\} \) and the edges \( e_1, e_2 \) and \( e_3 \) touch loops. Thus, Assumption 5 is satisfied and we can construct a Hubbard tree \( \iota, \tau : T^1 \to T^0 \). Assumption 1 can be easily verified. Since one can check that \( \bigcup_{i=0}^2 A_i \supset J_f \) and since \( A^\infty \) is hyperbolic for \( f \) as in the case of basilica, it follows from Corollary C that \( f : J_f \to J_f \) is topologically conjugate to \( \tau : T^\infty \to T^\infty \).

(iii) Airplane. Finally we describe a small perturbation of a quadratic map with a superattracting cycle of period three on the real axis whose Julia set is called the airplane.

Let \( c_1 \approx -1.75488 \) be the airplane parameter and fix a large \( r > 0 \). Then, the two external rays of angles 3/7 and 4/7 together with their unique landing point \( z_0 \) (which is a repelling periodic point of period three), the two external rays of angles 2/7 and 5/7 together with their unique landing point \( z_1 \) (which is again a repelling periodic point of period three) and the two external
rays of angles $1/7$ and $6/7$ together with their unique landing point $z_2$ (which is again a repelling periodic point of period three) divide the region $\{x \in \mathbb{C}: G_{c_1}(x) < r\}$ into four pieces $\{U_i\}_{i=0}^3$, where $p_i^c(0) \in U_i$ ($i = 0, 1, 2$) and $U_3$ does not intersect with the attractive cycle of period three.

We fatten the pieces slightly and remove a small neighborhood of the attractive cycle to get four regions denoted by $\{A_i\}_{i=0}^3$. Define the four Poincaré boxes as $A_i \equiv A_i \times \Delta(0; R)$ and write $A = \bigcup_{i=0}^3 A_i$. Let $\Sigma \equiv\{0, 1, 2, 3\}$ and $\Gamma \equiv\{(0, 1), (0, 3), (1, 2), (2, 0), (2, 2), (3, 0), (3, 3)\}$. Then, $f: A \cap f^{-1}(A) \to A$ becomes a hyperbolic system over $\Gamma$, where $f$ is a small perturbation of $p(x) = x^2 + c_1$ inside the complex Hénon family.

One can then show that Assumptions 2 to 4 are satisfied. In fact, there are six pinching disks in $\tilde{A}^0$ and they are the local stable manifolds of the saddle periodic points of period three which can be obtained as the continuation of $\{z_0, z_1, z_2\}$. We then get a multivalued dynamical system $\iota_T, \tau: T^1 \to T^0$. We label the edges and loops in $T^1$ (thus, in $\tilde{T}^1$) as shown in Fig. 9. The unique irreducible component of the matrix $A_L$ for loops is

$$\tilde{A}_L = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

over $\{l_1, l_2, l_3\}$ and the degree is 2. The unique sink irreducible component of the matrix $A_E$ for edges is

$$\tilde{A}_E = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

over $\{e_1, e_5, e_8\}$ and the edges $e_1, e_5$ and $e_8$ touch loops. Thus, Assumption 5 is satisfied and we can construct a Hubbard tree $\iota_T, \tau: T^1 \to T^0$. As before, it follows from Corollary C that $f: J_f \to J_f$ is topologically conjugate to $\tau: T^\infty \to T^\infty$.

6.3. A model with connected Julia set

The purpose of this subsection is to present a combinatorial model of a hyperbolic system for a quadratic Hénon map exhibiting a connected Julia set which has been proposed in [8, Fig. 12].
Let $\Sigma \equiv \{0, 1, 2, 3, 4\}$ and

$$\Gamma \equiv \{(0, 4), (1, 3), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (4, 2)\}.$$ 

We consider the hyperbolic system $f : A \cap f^{-1}(A) \to A$ over $\Gamma$ as described in Fig. 10. Here, $f : A_i \cap f^{-1}(A_j) \to A_j$ is (i) a degree two map of horseshoe type for $(i, j) = (2, 1), (3, 0)$, (ii) a degree two map of solenoidal type for $(i, j) = (2, 0), (3, 1)$, and (iii) a degree one map for the other admissible transitions $(i, j) \in \Gamma$ (see [8, Definition 3.8] for the notions of the solenoidal type and the horseshoe type).

The Hubbard tree of this map is given in Fig. 11. Here, again the dotted transitions mean the transitions of degree one and the others are of degree two by $\tau$. The other map $\iota_T$ can be defined as before. Note that, by $\iota_T$, (i) the part consisting of $e_{10}, e_{11}, l_7, e_{12}, e_{13}$ and $l_8$ shrinks to a point in $\iota_T(l_5)$, (ii) the part $e_{14}$, $e_{15}$ and $l_9$ (resp. $e_{16}$, $e_{17}$ and $l_{10}$) shrinks to a point (resp. another point) in $\iota_T(l_3)$, and (iii) $l_6$ shrinks to a point between $\iota_T(e_7)$ and $\iota_T(l_5)$.

**Remark 6.4.** This combinatorial model appears to arise for the Hénon map with $(c, b) = (-1.325, 0.2)$. For instance, the model $\iota_T, \tau : T^1 \to T^0$ as well as its pull-backs $T^m$ ($m \geq 2$) seem to approximate the shape of the Julia set as it appears in the 3-D computer picture drawn by Ushiki [16]. Further, as evidence that this Hénon map may be hyperbolic, the computer pictures appear to be consistent with the characteristics of hyperbolic maps presented in [2].

Imagine that there is a saddle fixed point $p \in A_1 \cap A_3$ and let $D_1 \equiv W_{\text{loc}}^s(p; A_1)$. Let $D_3$ be the connected component of $f^{-1}(D_1 \cap f(A_3))$ containing $p$ and $D'_3$ be the other connected component of $f^{-1}(D_1 \cap f(A_3))$. Let $D_2$ be the connected component of $f^{-1}(D_1 \cap f(A_2))$ intersecting with $D_1$. Let $D_4 \equiv f^{-1}(D_2 \cap f(A_4)), D_0 \equiv f^{-1}(D_4 \cap f(A_0))$ and $D'_4 \equiv f^{-1}(D'_3 \cap f(A_1))$. We assume that the intersecting pairs of pinching disks in $T^0$ are $\{D_1, D'_1\}$, $\{D_1, D_2\}$, $\{D_1, D_3\}$ and $\{D'_3, D_4\}$. The intersecting pairs of pinching disks in $T^1$ can be defined as the inverse images of these pairs (see Fig. 11 again). More details are left to the reader.
Assumption 5 is also easily checked. In fact, the irreducible components of the matrix $A_L$ are

$$
\tilde{A}_L = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
2 & 0 & 0
\end{pmatrix}
$$

over $\{l_1, l_4, l_5\}$ and

$$
\tilde{A}'_L = \begin{pmatrix}
0 & 1 \\
2 & 0
\end{pmatrix}
$$

over $\{l_2, l_3\}$ respectively and their degrees are both 2.

For the edges, the unique sink irreducible component of the matrix $A_E$ is

$$
\tilde{A}_E = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
$$

over $\{e_3, e_4\}$ and $e_3$ and $e_4$ touch loops. Thus, Assumption 5 is satisfied. It follows that $f : A^\infty \to A^\infty$ is topologically conjugate to $\tau : T^\infty \to T^\infty$ by Theorem B. Note that we do not know if $A^\infty$ is hyperbolic for $f$ and $A \supset J_f$ for the Hénon map $f$ in Remark 6.4, hence we do not know if one can apply Corollary C.

For this example, it would be interesting to ask the following two questions.
Question 1. Can we understand this crossed mapping model through the Hubbard trees for airplane and basilica?

In [2] it has been shown that a connected and hyperbolic Julia set of a polynomial diffeomorphism of \( \mathbb{C}^2 \) can be represented as a quotient space of a solenoid.

Question 2. How is our Hubbard tree description of this example related to the one by a solenoid?

This issue will be discussed in detail in a forthcoming paper [9].

To conclude this article, we address the following problem which will be a crucial step for combinatorial study of the parameter space for the complex Hénon family.

Problem. Give a canonical construction of a Hubbard tree as well as its pinching locus, i.e. construct a Hubbard tree independently of the initial choice of Poincaré boxes.

A solution to this problem would give us a hope that the projected pinching locus \( \text{pr}_T (\mathcal{L}^m) \) for a hyperbolic Hénon map \( f \) would play the role of “external angles” of the hyperbolic component containing \( f \) in the parameter space of the complex Hénon family. This may enable us to define the concept of “limbs” for the Hénon family.

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