Theoretical Computer Science 108 (1993) 151–171
Elsevier

Standard automata and semidirect products of transformation semigroups

Michael Kunze
U.S. National Park Service, Gatlinburg, TN, USA

Abstract

Kunze, M., Standard automata and semidirect products of transformation semigroups, Theoretical Computer Science 108 (1993) 151–171.

We propose a decomposition of transformation semigroups (X, S) on a finite set X that provides
(a) a composition of its elements out of idempotents/generators,
(b) a way by which S is obtained from semilattices/cyclic groups acting on X, namely by means of bilateral semidirect products and quotients.
The point is to provide both (a) and (b) simultaneously while still being accountable for the resources used in terms of cardinalities. This approach is applied to the semigroup $\mathrm{End}(X, \leqslant)$ of isotonic mappings of a linearly ordered set as well as the transition semigroups of automata that arise from certain varieties of formal languages. We discuss the semigroup varieties **D**, **R**, **J**, $\mathbf{LJ}_1$, and give a bilateral semidirect decomposition of the full transformation semigroup $\boldsymbol{T}(X)$ into $\mathrm{End}(X, \leqslant)$ and the symmetric group on X.

0. Introduction

Our topic is the decomposition of transformation semigroups by means of quotients and suitable products. Put another way, starting from actions of semilattices and cyclic groups as basic building blocks, we want to construct arbitrary finite transformation semigroups in a natural way. Simultaneously, for individual transformations $\alpha \in S$ of a given transformation semigroup (X, S), we want a decomposition of α into idempotent elements of S and/or permutations in S. The point is that this should be achieved uniformly for every $\alpha \in S$. (The problem of independent decompositions of transformations into idempotent mappings is investigated by Howie [5] and Saito [12].) So, the decomposition of elements of S should actually arise from

Correspondence to: M. Kunze, U.S. National Park Service, Great Smoky Mountains National Park, 1314 Cherokee Orchard Road, Gatlinburg, TN 37738, USA.

0304-3975/93/$06.00

a decomposition of the semigroup S into subsemilattices and subgroups of S. In contrast to ordinary (Schreier type) extension and decomposition theory of Rhodes and Weil [11], our approach is based exclusively on substructures rather than making use of homomorphic images also. Automata theoretic ideas will be encountered in motivation and definition of the product as well as proofs of algebraic theorems about semigroups, and applications.

1. The bilateral semidirect product

A fundamental question is: What kind of product to employ? For groups, the notion of semidirect product has led to an extensive theory with simple groups as basic building blocks, and examples of natural decompositions of transformation groups are abundant. Indeed, the semidirect product, or equivalently the wreath product, has been applied very successfully to the semigroup case, too, cf. e.g. [3]. However, semigroup theory is inherently different and we propose to study the bilateral semidirect product as a fundamental operation. This generalization comes in naturally from the point of view of automata theory, which we believe to be a typical realm of transformation semigroups.

Therefore, it may be appropriate to motivate the following Definition of bilateral semidirect products in terms of automata theory. Let $(S,\cdot)$, $(T,\cdot)$ be semigroups, $\varphi: S \to \boldsymbol{T}(T)$ be a homomorphism and $\delta: T \to \boldsymbol{T}(S)$ be an anti-homomorphism into the full transformation semigroup on T, and S, respectively.

Notation. For $s \in S$ and $t \in T$, denote the operation of $\varphi(s)$ on T by $t \mapsto t^{\varphi(s)}$ and the operation of $\delta(t)$ on S by $s \mapsto \delta_t(s)$.

Additionally, suppose that the following two conditions (Φ) and (Δ) hold true for every $s, s_1, s_2 \in S$ and $t, t_1, t_2 \in T$:

(Φ) $\quad (t_1 \cdot t_2)^{\varphi(s)} = t_1^{\varphi(\delta_{t_2}(s))} \cdot t_2^{\varphi(s)}$ [Sequential processing rule]

(Δ) $\quad \delta_t(s_1 \cdot s_2) = \delta_t(s_1) \cdot \delta_{t^{\varphi(s_1)}}(s_2)$ [Serial composition rule]

Under these assumptions $(S \times T, \circ)$ is a semigroup with respect to the multiplication (cf. [8]):

$$(s_1, t_1) \circ (s_2, t_2) = (s_1 \cdot \delta_{t_1}(s_2), t_1^{\varphi(s_2)} \cdot t_2).$$

Definition. This semigroup $(S\,{}_\delta\!\times_\varphi T, \circ)$ is called the bilateral semidirect product of S, T with respect to δ, φ. If $\varphi(S) = \{\mathrm{id}_T\}$ we have a semidirect product $S \times_\delta T$, if $\delta(T) = \{\mathrm{id}_S\}$ we have a reverse semidirect product $S\,{}_{®}\!\times_\varphi T$, and if both operations are trivial we get a direct product of S and T.

In order to understand conditions (Φ) and (Δ) more clearly, imagine a sequence $\mathscr{A}_1, \ldots, \mathscr{A}_k$ of finite automata acting on an input string $a_n \ldots a_1$ from right to left. The input of $a_1, \ldots, a_n$ to $\mathscr{A}_1$ produces a certain output string which is fed to $\mathscr{A}_2$. In general, each subsequent automaton $\mathscr{A}_i$ receives the output of the previous automation $\mathscr{A}_{i-1}$ as input. Denoting the current state of automaton $\mathscr{A}_i$ by A_i, the state transitions by δ, the output functions by φ (where the index of the automaton is understood from the state symbol), and using the formalism of semi-Thue systems we may write

$$a_1 A_1 \rightarrow \delta_{a_1}(A_1) a_1^{\varphi(A_1)}$$

$$\begin{aligned} a_n \ldots a_1 A_1 \ldots A_k &\rightarrow a_n \ldots a_2 \delta_{a_1}(A_1) a_1^{\varphi(A_1)} A_2 \ldots A_k \\ &\rightarrow a_n \ldots a_3 \delta_{a_2}(\delta_{a_1}(A_1)) a_2^{\varphi(\delta_{a_1}(A_1))} a_1^{\varphi(A_1)} A_2 \ldots A_k \\ &= a_n \ldots a_3 \delta_{a_2 a_1}(A_1)(a_2 a_1)^{\varphi(A_1)} A_2 \ldots A_k. \end{aligned}$$

In this automata theoretic setting, δ happens to be an anti-homomorphism and condition (Φ) holds true. This accounts for its mnemonic name: sequential processing rule. It is also evident that processors $\mathscr{A}_2, \ldots, \mathscr{A}_k$ may start operating as soon as the output of $\mathscr{A}_1$ appears. Concurrent operation of $\mathscr{A}_1, \ldots, \mathscr{A}_k$ in this fashion is called pipelining in VLSI circuits. Restricted to a single input symbol, we derive

$$\begin{aligned} a_1 A_1 A_2 &\rightarrow \delta_{a_1}(A_1) \delta_{a_1^{\varphi(A_1)}}(A_2)(a_1^{\varphi(A_1)})^{\varphi(A_2)} \\ &= \delta_{a_1}(A_1 A_2) a_1^{\varphi(A_1 A_2)}. \end{aligned}$$

In other words, φ is a homomorphism and (Δ) is aptly called "serial composition rule". It is worthwhile to note the dualism between input/output and state symbols which carries over to the properties (Φ) and (Δ).

The relationship between bilateral semidirect products and several important constructions in semigroup theory is discussed in [8]; for applications in automata theory see [7]. To get a feeling for this kind of product, we mention that a group G is a bilateral semidirect product of two subgroups U_1, U_2 iff $G = U_1 U_2$ and $U_1 \cap U_2 = \{1\}$. This product is semidirect iff one of the subgroups is normal. In the case of groups, however, the notions of semidirect and reverse semidirect products coincide. The symmetric group on X is a bilateral semidirect product of the stabilizer of a given element and any cyclic subgroup generated by a cyclic permutation of order $|X|$. This is an immediate consequence of the following lemma.

Lemma (On bilateral decompositions of transformation groups). *Let (X, G) be a transformation group and $C_G(z)$ the stabilizer of an element $z \in X$. If T is a subgroup of G that operates transitively on the orbit of z and $C_G(z) \cap T = \{\mathrm{id}\}$, then G is a bilateral semidirect product of $C_G(z)$ and T.*

Proof. For any $\alpha \in G$ there exists a $\tau \in T$ such that $(z\alpha)\tau = z$. Therefore, $\alpha\tau \in C_G(z)$ and $\alpha = \alpha\tau\tau^{-1} \in C_G(z) \cdot T$. This shows that $G = C_G(z) \cdot T$. $\square$

2. Semidirect decompositions

As in the case of groups, we would like to study decompositions of transformation semigroups (X,H) into transformation subsemigroups acting on the same set X. However, bilateral semidirect products of subsemigroups of H may happen to act on X and cover H in a natural way, but without that action being faithful. This technical complication is reflected in the following definition:

Definition. A transformation semigroup (X,H) is called a bilateral semidirect (semidirect, reverse semidirect) product of transformation subsemigroups (X,S) and (X,T), if

$$\mu: S\,{}_{\delta}\!\times_{\varphi} T \to \boldsymbol{T}(X), \quad x\mu(s,t)=xst$$

is a homomorphism and $H \cong \mu(S\,{}_{\delta}\!\times_{\varphi} T)$ ($\mu(S \times_{\delta} T)$, $\mu(S\,{}_{\circledR}\!\times_{\varphi} T)$).

This definition is a compromise that avoids talking about nonfaithful actions as a technical tool for decompositions of transformation semigroups.

Now let us have a look at the connection between semidirect product and wreath product as closure operations on classes of transformation semigroups. Although this should be considered folklore in general, we have to be a bit careful, because we chose not to manipulate the underlying set X when forming semidirect products. But two minor auxiliary operations will take care of that.

Definition. Given a transformation semigroup (X,S) and a set Y define transformation semigroups

$$(X \times Y, S), \quad \text{where } (x,y)\alpha=(x\alpha,y)$$

and

$$(X \cup Y, S), \quad \text{where } y\alpha=y,$$

for $x\in X$, $y\in Y$, $\alpha\in S$. $(X \times Y, S)$ is said to be obtained by splitting of states, and $(X\cup Y,S)$ is said to be obtained by adding fixed points.

Proposition 2.1. *Let* $\mathbf{X}$ *be a class of complete finite transformation semigroups containing* $(\{0,1\},\{\mathrm{id}\})$. *Then* $\mathbf{X}$ *is closed under wreath product and quotients if and only if it is closed under semidirect products, splitting of states, adding fixed points, and quotients.*

Proof. First suppose that $\mathbf{X}$ is closed under wreath product and quotients. This implies closure under direct products in the usual sense (e.g., [3]) and $(P,\{\mathrm{id}_P\})\in\mathbf{X}$ for any finite set P, because $(\{0,1\},\{\mathrm{id}\})\in\mathbf{X}$. Consequently, $(Q\times P, S\times\{\mathrm{id}_P\})\in\mathbf{X}$ whenever $(Q,S)\in\mathbf{X}$, i.e., $\mathbf{X}$ is closed under splitting of states. For adding fixed points we note that $(Q\cup P, S\times\{\mathrm{id}_P\})=(Q,S)+(P,\{\mathrm{id}_P\})$ and use the closure under sum, [3, III, Proposition 1.4]. Now let (Q,H) be a semidirect product of (Q,S) and (Q,T) with

$$
\begin{array}{ccc}
Q\times T & \xrightarrow{(\alpha_s,t)} & Q\times T \\
(q,t') & & (q\delta_{t'}(s),t't) \\
\downarrow & & \downarrow \\
qt' & & qt'st=q\mu(\mathrm{id},t')\mu(s,t)=q\delta_{t'}(s)t't \\
Q & \xrightarrow{\mu(s,t)} & Q
\end{array}
$$

Fig. 1.

respect to an anti-homomorphism $\delta: T\to \mathrm{End}(S)$ and suppose $(Q,S),(Q,T)\in \mathbf{X}$. Then $(T,T)\in\mathbf{X}$ by [3, I, Proposition 9.8]. We claim that

$$(Q,H)<(Q,S)\circ(T,T),$$

where $<$ denotes the covering relation. Since $(Q,S)\circ(T,T)\cong(Q\times T,\ S^T\times_\vartheta T)$, where $\vartheta: T\to\mathrm{End}(S^T)$, $x\vartheta_t(\alpha)=(xt)\alpha$, this is clear from Fig. 1. Here $\alpha_s: T\to S$, $x\alpha_s:=\delta_x(s)$ and the transformation $\mu(s,t)\in H$ (where $\mu: S\times_\varphi T\to H$ is the canonical homomorphism) is covered by (α_s,t).

For the converse, suppose that $\mathbf{X}$ is closed under semidirect products, splitting of states, adding fixed points, and quotients. Given any $(Q,S),(P,T)$ in $\mathbf{X}$ we have to show that the wreath product $(Q,S)\circ(P,T)$ belongs to $\mathbf{X}$. For each $p\in P$ define a transformation semigroup $(Q\times P,S_p)$ by

$$(q,p')s=\begin{cases}(qs,p') & \text{if } p=p',\\ (q,p') & \text{otherwise.}\end{cases}$$

By adding fixed points, $(Q\times P,S_p)$ belongs to $\mathbf{X}$. Say $P=\{p_1,\ldots,p_n\}$. Since the direct product is a special case of the semidirect product, $\mathbf{X}$ contains

$$(Q\times P,S_{p_1}\times\cdots\times S_{p_n}).$$

Observe that

$$(q,p)\cdot(s_1,\ldots,s_n)=(qs_i,p),\qquad \text{where } p=p_i.$$

Hence, $(Q\times P,S_{p_1}\times\cdots\times S_{p_n})\cong(Q\times P,S^P)$. Furthermore, by splitting of states, $(Q\times P,T)$, where $(q,p)t:=(q,pt)$ belongs to $\mathbf{X}$. Now we conclude by standard argument (e.g., [3, V, Proposition 4.1]) that the wreath product $(Q,S)\circ(P,T)$ is isomorphic to the semidirect product of $(Q\times P,S^P)$ and $(Q\times P,T)$ with respect to $\vartheta: T\to\mathrm{End}(S^P)$, $x\vartheta_t(\alpha)=(xt)\alpha$ and this semidirect product belongs to $\mathbf{X}$. $\square$

One of our goals is to find decompositions for classes of transformation semigroups that are characterized by algebraic or other properties. The following result is now more or less obvious from [3] but it nicely illustrates this idea.

Proposition 2.2. *For transformation semigroups* (X,S) *properties* (a)–(c) *are equivalent*:

(a) *The relation on* X *given by*

$$x\leqslant y \Leftrightarrow x=y \text{ or } y=x\alpha \text{ for some } \alpha\in S$$

is a partial order.

(b) S *is* $\mathfrak{R}$*-trivial where* $\mathfrak{R}$ *denotes Green's relation in the semigroup* S.

(c) (X,S) *is covered by a transformation semigroup which is an iterated semidirect product of semilattice actions on* X.

As usual, (X,S) is covered by (Z,H) iff (X,S) is a homomorphic image of a transformation subsemigroup of (Z,H). A characterization of semilattice actions by means of their transitivity order as defined in (a) is the topic of [2]. The flavor of that characterization can also be seen from the following example of a reverse semidirect product.

Example. Let $(X,\leqslant)$ be a finite distributive lattice. Define

$$S^-_{\leqslant}=\{\alpha_{\langle a\rangle}: x\mapsto \inf(x,a)\mid a\in X\}\subseteq \boldsymbol{T}(X),$$

$$S^+_{\leqslant}=\{\alpha_{[a]}: x\mapsto \sup(x,a)\mid a\in X\}\subseteq \boldsymbol{T}(X).$$

Because of the distributivity, the set theoretic product $S^-_{\leqslant}\cdot S^+_{\leqslant}$ is a subsemigroup of $\boldsymbol{T}(X)$. The transformation semigroup $(X,S^-_{\leqslant}\cdot S^+_{\leqslant})$ is a reverse semidirect product of the two semilattice actions $(X,S^-_{\leqslant})$ and $(X,S^+_{\leqslant})$ with respect to

$$\varphi: S^-_{\leqslant}\to \mathrm{End}(S^+_{\leqslant}),\quad \alpha_{[x]}^{\varphi(\alpha_{\langle a\rangle})}=\alpha_{[\inf(x,a)]}.$$

$S^-_{\leqslant}\cdot S^+_{\leqslant}$ is an $\boldsymbol{L}$-trivial band consisting of the projections

$$\pi_A: x\mapsto \sup(\inf(x,b),a)=x\alpha_{\langle b\rangle}\alpha_{[a]},$$

where $A=[a,b]$ is an interval of $(X,\leqslant)$. The transformation semigroup $(X,S^-_{\leqslant}\cdot S^+_{\leqslant})$ is transitive, because it contains the constant mappings. In case $(X,\leqslant)$ is a chain, i.e., a total order, $S^-_{\leqslant}\,{}_{\circledR}\!\times_{\varphi} S^+_{\leqslant}$ is a subsemigroup of the bilateral semidirect product $S^-\,{}_{\delta}\!\times_{\varphi} S^+$, which will be constructed later.

At a first glance, Propositions 2.1 and 2.2 may raise the question whether nested semidirect products offer any advantage over wreath products. This, of course, depends on the situation. Here is an application where the smaller size of semidirect products is impressive: Given an alphabet Σ, the set of states of a shift register of length k is Σ^k and its transition semigroup $D_{\Sigma,k}$ consists of the mappings induced by feeding input strings of lengths up to k.

Proposition 2.3. *The transformation semigroup* $(\Sigma^k, D_{\Sigma,k})$ *of a shift register of length* k *over some alphabet* Σ *is a transformation subsemigroup of* (Σ^k, S_k) *which is an iterated semidirect product of* k *right zero semigroup actions* $(\Sigma^k, R_1),\ldots,(\Sigma^k, R_k)$. *Every definite* *semigroup (i.e., a member of the pseudovariety* **D**, cf. [3, Example 3.7]) *is homomorphic image of a subsemigroup of S_k for suitable k.*

The semigroups R_k consist of certain projections and constant mappings, namely

$$\pi_k^i : \Sigma^k \to \Sigma^k, \quad (x_k, x_{k-1}, \ldots, x_1) \to (x_i, x_{k-1}, \ldots, x_1),$$

$$\kappa_k^a : \Sigma^k \to \Sigma^k, \quad (x_k, x_{k-1}, \ldots, x_1) \to (a, x_{k-1}, \ldots, x_1),$$

for $0 < i < k$ and $a \in \Sigma$. If $|\Sigma| = 2$ we have $|R_k| = k+1$ and $|\mathbf{D}_{\Sigma,k}| = 2^{k+1} - 2$, and $|S_k| = (k+1)!$ compares favorably with the cardinality of the wreath product:

$$|R_k \circ \cdots \circ R_1| = 2^{2^k - 1}.$$

Specifically, the 30-element transformation semigroup of a length-4 shift register over $\{a, b\}$ is covered by a semidirect product consisting of 120 mappings while the corresponding wreath product needs 2^{15} mappings.

3. An application to semigroup theory

First we consider the full transformation semigroup $\boldsymbol{T}(X)$ on a finite set X.

Proposition 3.1. *For any linear order $\leqslant$ on X whatsoever, $\boldsymbol{T}(X)$ is a canonical homomorphic image of a bilateral semidirect product*

$$S_X \,{}_\delta\!\times_\varphi \mathrm{End}(X, \leqslant)$$

of the symmetric group S_X on X and the semigroup $\mathrm{End}(X, \leqslant)$ *of isotonic mappings.*

Proof (*Outline*). For $\alpha \in \boldsymbol{T}(X)$ define a derived linear order $(X, \leqslant_\alpha)$ by

$$x \leqslant_\alpha y \iff \begin{cases} x\alpha \leqslant y\alpha & \text{if } x\alpha \neq y\alpha, \\ x \leqslant y & \text{otherwise.} \end{cases}$$

Let σ_α be the permutation of X that rearranges the order $(X, \leqslant)$ into $(X, \leqslant_\alpha)$ by sorting with respect to $\leqslant_\alpha$. This means that

$$x \leqslant_\alpha y \iff x\sigma_\alpha \leqslant y\sigma_\alpha.$$

As a consequence, $\sigma_\alpha^{-1}\alpha$ is isotonic with respect to the original order $(X, \leqslant)$:

$$x \leqslant y \iff (x\sigma_\alpha^{-1})\sigma_\alpha \leqslant (y\sigma_\alpha^{-1})\sigma_\alpha \iff x\sigma_\alpha^{-1} \leqslant_\alpha y\sigma_\alpha^{-1}$$
$$\Rightarrow x\sigma_\alpha^{-1}\alpha \leqslant y\sigma_\alpha^{-1}\alpha.$$

Therefore, $\alpha = \sigma_\alpha \cdot \sigma_\alpha^{-1}\alpha \in S_X \cdot \mathrm{End}(X, \leqslant)$. So, $\boldsymbol{T}(X) = S_X \cdot \mathrm{End}(X, \leqslant)$. Now define $\delta : \mathrm{End}(X, \leqslant) \to \boldsymbol{T}(S_X)$ and $\varphi : S_X \to \boldsymbol{T}(\mathrm{End}(X, \leqslant))$ by

$$\delta_\tau(\pi) = \sigma_{\tau\pi} \quad \text{and} \quad \tau^{\varphi(\pi)} = \sigma_{\tau\pi}^{-1}\tau\pi$$

for $\tau\in\mathrm{End}(X,\leqslant)$ and $\pi\in S_X$. One has to verify that δ,φ give rise to a bilateral semidirect product $S_X\,{}_\delta\!\times_\varphi \mathrm{End}(X,\leqslant)$. Obviously,

$$\mu: S_X\,{}_\delta\!\times_\varphi \mathrm{End}(X,\leqslant)\to \boldsymbol{T}(X),\quad \mu(\pi,\tau)=\pi\circ\tau,$$

where $\circ$ denotes the composition of mappings, is a homomorphism. □

The semigroup $\mathrm{End}(X,\leqslant)$ of isotonic mappings has been studied extensively in the context of semigroups acting on graphs [10]. Here our interest in $\mathrm{End}(X,\leqslant)$ originates from Proposition 3.1. Looking for decompositions of $\mathrm{End}(X,\leqslant)$, we first study its subsemigroup

$$\mathrm{End}(X,\leqslant)^- = \{\alpha\in\mathrm{End}(X,\leqslant)\mid x\alpha\leqslant x \text{ for every } x\in X\}$$

of monotone decreasing mappings. $\mathrm{End}(X,\leqslant)^+$ is defined dually.

For an interval A_i of $(X,\leqslant)$ with least element i we consider the transformation

$$A_i^-: X\to X,\quad x\mapsto\begin{cases} i & \text{if } x\in A_i,\\ x & \text{otherwise}\end{cases}$$

and dually we define B_j^+, where j is the greatest element in the interval B_j. For every $i\in X$ we have subsemilattices

$$S_i^- = \{A_i^- \mid A_i=[i,a]\subseteq X\}\subseteq \mathrm{End}(X,\leqslant)^-,$$

$$S_i^+ = \{B_i^+ \mid B_i=[b,i]\subseteq X\}\subseteq \mathrm{End}(X,\leqslant)^+.$$

For notational convenience let us assume $X=\{0,1,\ldots,n\}$ with the usual order $0<1<\cdots<n$. Then

$$\mathrm{End}(X,\leqslant)^- = S_0^-\circ S_1^-\circ\cdots\circ S_{n-1}^-,$$

$$\mathrm{End}(X,\leqslant)^+ = S_n^+\circ S_{n-1}^+\circ\cdots\circ S_1^+,$$

because any $\alpha\in\mathrm{End}(X,\leqslant)^-$ can be written as a composition of mappings

$$\alpha=A_0^-\circ\cdots\circ A_{n-1}^-,\quad \text{where } A_i=[i,\max([0,i]\alpha^{-1})],$$

and similarly for $\alpha\in\mathrm{End}(X,\leqslant)^+$. The unique minimal set of idempotent generators of $\mathrm{End}(X,\leqslant)\setminus\{\mathrm{id}\}$ given in [1] is $[0,1]^-,[1,2]^-,\ldots,[n-1,n]^-,[n-1,n]^+,\ldots,[0,1]^+$. For an n-tuple $A^-=(A_0^-,\ldots,A_{n-1}^-)$ and $a_i=\max(A_i)$, $i=0,\ldots,n-1$, we sometimes use the sequence notation $A^-=(a_0,\ldots,a_{n-1})^-$ or, equivalently, the function notation $A^-(i)=a_i$.

Proposition 3.2. *$\mathrm{End}(X,\leqslant)^-$ is covered by the following nested semidirect product of $n=|X|-1$ semilattices:*

$$S(X,\leqslant)^- = ((S_0^-\times_{\delta_1} S_1^-)\times_{\delta_2} S_2^-\cdots)\times_{\delta_{n-1}} S_{n-1}^-.$$

More precisely, $\mathrm{End}(X, \leqslant)^-$ *is the homomorphic image of* $S(X, \leqslant)^-$ *under the canonical homomorphism* $\mu:(A_0^-, \ldots, A_{n-1}^-)\mapsto A_0^- \circ \cdots \circ A_{n-1}^-$. *Alternatively,* $\mathrm{End}(X, \leqslant)^-$ *is also isomorphic to the subsemigroup*

$$S^- = \{A^- \in S(X, \leqslant)^- \mid A^-(0) \leqslant A^-(1) \leqslant \cdots \leqslant A^-(n-1)\}$$

of monotone sequences of $S(X, \leqslant)^-$.

Proof. For $i<j$ we form a semidirect product $S_i^- \times_{\delta_{ij}} S_j^-$ with respect to $\delta_{ji}: S_j^- \to \mathrm{End}(S_i^-)$, where

$$\delta_{ji}(B_j^-)[A_i^-] = \begin{cases} (A_i \cup B_j)_i^- & \text{if } A_i \cap B_j \neq \emptyset, \\ A_i^- & \text{otherwise.} \end{cases}$$

One observes $B_j^- \circ A_i^- = \delta_{ji}(B_j^-)[A_i^-] \circ B_j^-$ and verifies that δ_{ij} is a homomorphism and $\delta_{ji}(B_j^-) \in \mathrm{End}(S_i^-)$. In order to obtain an iterated semidirect product $S(X, \leqslant)^- = ((S_0^- \times_{\delta_1} S_1^-) \times_{\delta_2} S_2^- \cdots) \times_{\delta_{n-1}} S_{n-1}^-$, we check that

$$\delta_{ki}(C_k^-)[\delta_{ji}(B_j^-)[A_i^-]] = \delta_{ji}(\delta_{kj}(C_k^-)[B_j^-])[\delta_{ki}(C_k^-)[A_i^-]]$$

for $i<j<k$ and $A_i^- \in S_i^-$, $B_j^- \in S_j^-$, $C_k^- \in S_k^-$. Now the following lemma completes the proof.

Lemma (On nested semidirect products). *Let* $H_1, \ldots, H_n$ *be semigroups and* $\delta_{ji}: H_j \to \mathrm{End}(H_i)$ *for* $i<j$ *be anti-homomorphisms. Let* $\delta_{ji}(B)[A]$ *denote the image of* $A \in H_i$ *under the endomorphism* $\delta_{ji}(B) \in \mathrm{End}(H_i)$ *induced by* $B \in H_j$. *Suppose the following condition* $(\Delta\Delta)$ *holds for* $A \in H_i$, $B \in H_j$, $C \in H_k$, $i<j<k$:

$$(\Delta\Delta) \qquad \delta_{ki}(C)[\delta_{ji}(B)[A]] = \delta_{ji}(\delta_{kj}(C)[B])[\delta_{ki}(C)[A]].$$

Then $((H_1 \times_{\delta_2} H_2) \times_{\delta_3} H_3) \times_{\delta_4} \cdots \times_{\delta_n} H_n$ *is an iterated semidirect product with respect to*

$$\delta_j: H_j \to \mathrm{End}(((H_1 \times_{\delta_2} H_2) \times_{\delta_3} H_3) \times_{\delta_4} \cdots \times_{\delta_{j-1}} H_{j-1}),$$

where

$$\delta_j(B)[(A_1, \ldots, A_{j-1})] = (\delta_{j,1}(B)[A_1], \ldots, \delta_{j,j-1}(B)[A_{j-1}]).$$

Proof (*by induction*). For $n=2$ the statement holds vacuously. By induction hypothesis assume it holds for $n=k$ and $C \in H_{k+1}$. We have to show that

$$\delta_{k+1}(C) \in \mathrm{End}((H_1 \times_{\delta_2} H_2) \times \cdots \times_{\delta_k} H_k),$$

i.e.,

$$\delta_{k+1}(C)[A \cdot B] = \delta_{k+1}(C)[A] \cdot \delta_{k+1}(C)[B],$$

for $A=(A_1, \ldots, A_k)$, $B=(B_1, \ldots, B_k) \in (H_1 \times_{\delta_2} H_2) \times \cdots \times_{\delta_k} H_k$. The ith component of the left-hand side is

$$\begin{aligned} &\delta_{k+1,i}(C)[A_i \cdot \delta_{i+1,i}(A_{i+1})[\cdots \delta_{k,i}(A_k)[B_i]]] \\ &\quad = \delta_{k+1,i}(C)[A_i] \cdot \delta_{k+1,i}(C)[\delta_{i+1,i}(A_{i+1})[\cdots \delta_{k,i}(A_k)[B_i]]]. \end{aligned}$$

Evaluating the right-hand side, we obtain

$$\delta_{k+1}(C)[A]\cdot\delta_{k+1}(C)[B]$$
$$=(\delta_{k+1,1}(C)[A_1],\ldots,\delta_{k+1,k}(C)[A_k])\cdot(\delta_{k+1,1}(C)[B_1],\ldots,\delta_{k+1,k}(C)[B_k])$$

and the ith component of this is

$$\delta_{k+1,i}(C)[A_i]\cdot$$
$$\underbrace{\delta_{i+1,i}(\delta_{k+1,i+1}(C)[A_{i+1}])}_{\text{from }(i+1)\text{st component}}[\cdots\underbrace{\delta_{k,i}(\delta_{k+1,k}(C)[A_k])}_{\text{from }k\text{th component}}[\delta_{k+1,i}(C)[B_i]]].$$

Applying ($\Delta\Delta$) to the second factor yields the desired equality of the ith component on both sides:

$$\delta_{k+1,i}(C)[\delta_{i+1,i}(A_{i+1})[\cdots\delta_{k,i}(A_k)[B_i]]]$$
$$=\delta_{i+1,i}(\delta_{k+1,i+1}(C)[A_{i+1}])[\delta_{k+1,i}(C)[\cdots\delta_{k,i}(A_k)[B_i]]]$$
$$\vdots$$
$$=\delta_{i+1,i}(\delta_{k+1,i+1}(C)[A_{i+1}])[\cdots\delta_{k+1,i}(C)[\delta_{k,i}(A_k)[B_i]]]$$
$$=\delta_{i+1,i}(\delta_{k+1,i+1}(C)[A_{i+1}])[\cdots\delta_{k,i}(\delta_{k+1,k}(C)[A_k])[\delta_{k+1,i}(C)[B_i]]].$$

Finally, since $\delta_{k+1,1},\ldots,\delta_{k+1,k}$ are anti-homomorphisms, δ_{k+1} is obviously an anti-homomorphism. □

This completes the proof of Proposition 3.2.

Corollary. *Every $\mathfrak{J}$-trivial finite transformation semigroup is covered by a direct product of transformation semigroups of the form $S(X,\leqslant)^-$ which are nested semidirect products of semilattice actions on linear orders.*

Proof. The standard automata (cf. Section 4) given in [6] for piecewise testable languages (i.e., the languages corresponding to $\mathfrak{J}$-trivial monoids by Eilenberg/Schützenberger's 1–1 correspondence, [3]) are direct products of machines, each of which has a transition monoid contained in some $\mathrm{End}(X,\leqslant)^-$. Furthermore, every $\mathfrak{J}$-trivial monoid is homomorphic image of the transition monoid of such a standard automaton, cf. [3, VIII, 9]. □

Proposition 3.3. $\mathrm{End}(X,\leqslant)$ *is canonical homomorphic image of a bilateral semidirect product $S^-{}_\delta\times_\varphi S^+$, where S^-,S^+ are the subsemigroups of nested semidirect products of semilattices given in Proposition 3.2.*

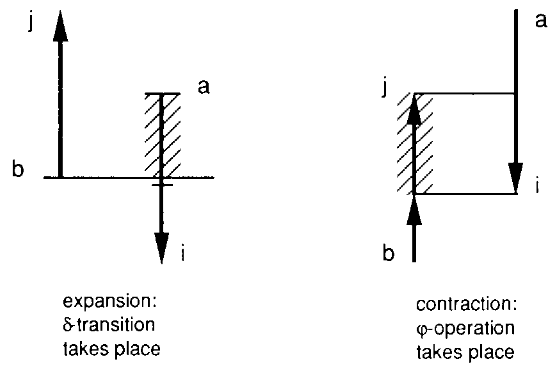

Fig. 2.

Proof. We are going to construct this bilateral semidirect product step by step. First we define δ and φ (see Fig. 2) for single components $[i,a]^- \in S_i^-$ and $[b,j]^+ \in S_j^+$:

$$\delta_{[b,j]^+}([i,a]^-) = \begin{cases} [i, \max(i, b-1)]^- & \text{if } b \leqslant a < j, \\ [i,a]^- & \text{otherwise,} \end{cases}$$

$$[b,j]^{+\varphi([i,a]^-)} = \begin{cases} [\min(i,b), i]^+ & \text{if } j \in [i,a], \\ [b,j]^+ & \text{otherwise.} \end{cases}$$

Observe that

$$[b,j]^+ \circ [i,a]^- = \delta_{[b,j]^+}([i,a]^-) \circ [b,j]^{+\varphi([i,a]^-)}.$$

The case where the δ-transition is nontrivial describes an expansion type mapping unless the downwards action is void (i.e., $b \leqslant i$), the case where the φ-operation is nontrivial describes a contraction type mapping unless the upwards action is void (i.e., $i \leqslant b$).

For $A^- = (a_0, \ldots, a_{n-1})^- \in S^-$ and $B^+ = (b_n, \ldots, b_1)^+ \in S^+$ define $\delta_{B^+}(A^-)$ by giving its ith component

$$\delta_{B^+}(A^-)[i] = \max(i, \min(a_i, b_{a_i+1} - 1)), \quad \text{where } b_{n+1} = n+1$$

is assumed for the case $a_i = n$. Since $(a_0, \ldots, a_{n-1})^-$ is monotone increasing, we have that

$$\delta_{B^+}(A^-)[i] = \delta_{[b_n,n]^+}(\cdots(\delta_{[b_1,1]^+}([i,a_i]^-))).$$

In particular, $\delta_{B^+}(A^-) \in S^-$. A good way of looking at it is to imagine the $\uparrow$-arrows and $\downarrow$-arrows as symbols of a semi-Thue system (Fig. 3) with alphabet $S_0^- \cup \cdots \cup S_{n-1}^- \cup S_n^+ \cup \cdots \cup S_1^+$ and derivation rules

$$[b,j]^+ \, [i,a]^- \to \delta_{[b,j]^+}([i,a]^-) \, [b,j]^{+\varphi([i,a]^-)}.$$

Since $(b_n, \ldots, b_1)$ is sorted in decreasing order and $(a_0, \ldots, a_{n-1})$ is sorted in increasing order, the $\varphi([i,a]^-)$-operation on $[b,j]^+$ takes place before $[i,a]^-$ is affected by

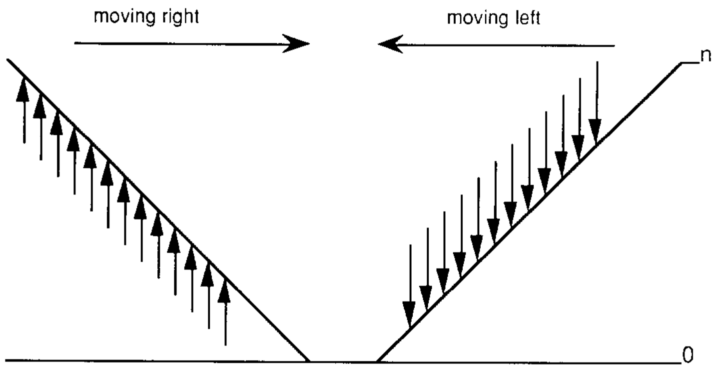

Fig. 3.

nontrivial $\delta_{[d,k]^+}$-transitions for $k \neq j$, and similarly for the $\delta_{[b,j]^+}$-transition of $[i,a]^-$. Therefore, the final result of the derivation is of the form

$$\delta_{B^+}(A^-)\,B^{+\varphi([0,a_0]^-)\cdots\varphi([n-1,a_{n-1}]^-)}.$$

The point is that the mapping induced on $(X, \leqslant)$ by B^+A^- is invariant under this derivation, so that we will get a homomorphism $\mu: S^-{}_\delta\times_\varphi S^+ \to \mathrm{End}(X, \leqslant)$:

$$\mu(A^-, B^+) = [0, a_0]^- \circ \cdots \circ [n-1, a_{n-1}]^- \circ [b_n, n]^+ \circ \cdots \circ [b_1, 1]^+.$$

To see that μ is surjective, pick any $\alpha \in \mathrm{End}(X, \leqslant)$ and set

$$a_i := \begin{cases} \max([0,i]\alpha^{-1}) & \text{if } [0,i]\alpha^{-1} \cap [i,n] \neq \emptyset, \\ i & \text{otherwise,} \end{cases}$$

$$b_j := \begin{cases} \min([j,n]\alpha^{-1}) & \text{if } [j,n]\alpha^{-1} \cap [0,j] \neq \emptyset, \\ j & \text{otherwise.} \end{cases}$$

This representation is almost precisely the canonical form given in [1]. According to the notation used in [1], we are working on the problem of finding the canonical form of the composition of two mappings.

A technical problem with the φ-operation is that the arrow heads are altered, so that the result $B^{+\varphi([0,a_0]^-)\cdots\varphi([n-1,a_{n-1}]^-)}$ of the derivation is not an element of S^+, even after applying additional rules to compute the products in $S_n^+, \ldots, S_1^+$, which amounts to merging arrows with identical heads. For a partial monotone function $f: \{1, \ldots, n\} \to \{1, \ldots, n\}$ satisfying $f(i) \leqslant i$ whenever $i \in \mathrm{Dom}(f)$, we define a monotone extension $\mathrm{Monex}(f): \{1, \ldots, n\} \to \{1, \ldots, n\}$ by

$$\mathrm{Monex}(f)[n] = \begin{cases} f(n) & \text{if } n \in \mathrm{Dom}(f), \\ n & \text{otherwise,} \end{cases}$$

$$\mathrm{Monex}(f)[i] = \begin{cases} f(i) & \text{if } i \in \mathrm{Dom}(f), \\ \min(i, \mathrm{Monex}(f)[i+1]) & \text{otherwise.} \end{cases}$$

With that concept in mind, we are ready to define

$$B^{+\varphi(A^-)}(n)=\begin{cases} b_n & \text{if } a_{n-1}=n-1, \\ n & \text{otherwise,}\end{cases}$$

$$B^{+\varphi(A^-)}(i)=\begin{cases} b_i & \text{if } a_{i-1}=i-1, \\ \min(i, b_{a_{i-1}+1}) & \text{if } a_i>a_{i-1}\geqslant i, \\ \min(i, B^{+\varphi(A^-)}(i+1)) & \text{if } a_i=a_{i-1}.\end{cases}$$

Lemma 3.4. $B^{+\varphi(A^-)}=\text{Monex}(\text{reduced}(B^{+\varphi([0,a_0]^-)\cdots\varphi([n-1,a_{n-1}]^-)}))$, *where the reduced sequence is obtained by computing the products in* $S_n^+,\ldots,S_1^+$, i.e., *applying rules* $[c,j]^+[d,j]^+\to[\min(c,d),j]^+$.

Proof. *Case* $a_{i-1}=i-1$: Then $a_k\leqslant i-1$ for every $k<i$. Hence, no $\varphi([k,a_k]^-)$ for $k<i$ can affect $[b_i,i]^+$ and $\varphi([k,a_k]^-)$ for $k\geqslant i$ will not affect it anyway. So $[b_i,i]^+$ remains as it is. Also note that $\uparrow$-arrows only can get shortened from the top (Fig. 4). Since $(b_n,\ldots,b_1)$ is monotone, no shortened $[b_j,j]^+$ for $j>i$ has a longer tail than $[b_i,i]^+$.

Otherwise we have $a_{i-1}\geqslant i$. Then $[b_i,i]^+$ gets its head cut by $\varphi([i-1,a_{i-1}]^-)$ and does not make a contribution to the ith component of $B^{+\varphi([0,a_0]^-)\cdots\varphi([n-1,a_{n-1}]^-)}$. The same is true for $[b_j,j]^+$ satisfying $j\leqslant a_{i-1}$. Hence, the first $[b_j,j]^+$ that may contribute is $[b_{a_{i-1}+1},a_{i-1}+1]^+$, and $[b_j,j]^+$ for $j>a_{i-1}+1$ will not contribute any more.

Case $a_i>a_{i-1}$: Then $a_i\geqslant a_{i-1}+1$ and we actually get $[b_{a_{i-1}+1},a_{i-1}+1]^{+\varphi([i,a_i]^-)}=[b_{a_{i-1}+1},i]^+$ as long as $b_{a_{i-1}+1}\leqslant i$, and we obtain $[i,i]^+$ otherwise.

If finally $a_i=a_{i-1}$, then $B^{+\varphi([0,a_0]^-)\cdots\varphi([n-1,a_{n-1}]^-)}$ does not contain an arrow with head i, and some value at i is provided from values at $i+1$, $i+2,\ldots,n$ by monotone extension. $\square$

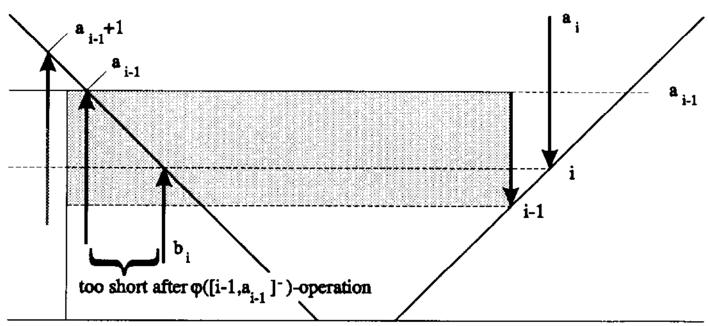

Fig. 4.

The following two lemmata will complete the proof of Proposition 3.3.

Lemma 3.5. *For* $A^- = (a_0, \ldots, a_{n-1})^- \in S^-$, $C^- = (c_0, \ldots, c_{n-1})^- \in S^-$, $B^+ = (b_n, \ldots, b_1)^+ \in S^+$,

(a) $B^{+\varphi(A^- \circ C^-)} = (B^{+\varphi(A^-)})^{\varphi(C^-)}$,

(b) $\delta_{B^+}(A^- \circ C^-) = \delta_{B^+}(A^-) \circ \delta_{B^{+\varphi(A^-)}}(C^-)$.

Proof. First note that $A^- \circ C^- = (a_{c_0}, \ldots, a_{c_{n-1}})^- \in S^-$.

(a): $\varphi([0, a_0]^-), \ldots, \varphi([n-1, a_{n-1}]^-)$ act on the heads of B^+ like $\mu(A^-)$. Thereafter, $\varphi([0, c_0]^-), \ldots, \varphi([n-1, c_{n-1}]^-)$ act on the heads of the result, joined by the monotone extension arrows, like $\mu(C^-)$. Finally, monotone extension arrows are added again (Fig. 5). The same effect on the heads of B^+ is achieved by applying the composition $\mu(A^-) \circ \mu(C^-) = \mu(A^- \circ C^-)$ and filling the result with monotone extension arrows afterwards: Arrows resulting from the intermediate monotone extension in $B^{+\varphi(A^-)}$ do have identical base as some original arrow to the left. After applying $\varphi([0, c_0]^-), \ldots, \varphi([n-1, c_{n-1}]^-)$ this original arrow causes monotone extension arrows in the final step to be added that supersede those arrows from the intermediate extension in $B^{+\varphi(A^-)}$.

(b): We want to show that the ith component of the right-hand side equals that of the left-hand side, namely $\delta_{B^+}(A^- \circ C^-)[i] = \delta_{[b_n, n]^+}(\cdots(\delta_{[b_1, 1]^+}([i, a_{c_i}]^-)))$. Those $[b_j, j]^+$ with $j \leqslant c_i$ do not make any contribution to either side by the definition of δ, because the $\varphi(A^-)$-operation can only shorten the $\uparrow$-arrows by cutting their heads. Those $[b_j, j]^+$ with j in the range $c_i < j \leqslant a_{c_i}$ do not affect the left-hand side, nor the c_ith component of A^-. But they would act on $[i, c_i]^-$ in case $b_j \leqslant c_i$, if that would not be taken care of by the φ-operation: $[b_j, j]^{+\varphi([c_i, a_{c_i}]^-)} = [b_j, c_i]^+$ which is too short to be able to act on $[i, c_i]^-$. Also, $\uparrow$-arrows added by monotone extension in $B^{+\varphi(A^-)}$ may be disregarded for further δ-transition anyway. It remains to consider $[b_j, j]^+$

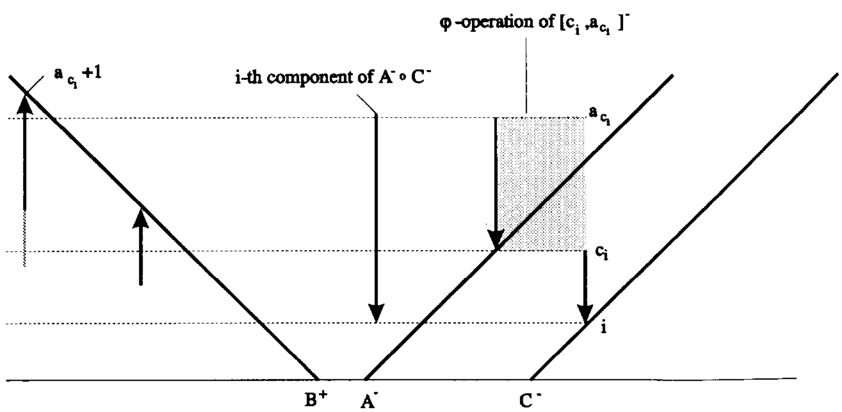

Fig. 5.

with $j > a_{c_i}$ and, in order to have any influence, $b_j \leqslant a_{c_i}$. These $[b_j, j]^+$ are not altered by the φ-operations $\varphi([0, a_0]^-), \ldots, \varphi([c_i, a_{c_i}]^-)$, because the corresponding $\downarrow$-arrows are too short. Although these $[b_j, j]^+$ may possibly have cut their heads by $\varphi([c_i + 1, a_{c_i+1}]^-), \ldots, \varphi([n-1, a_{n-1}]^-)$, they still remain tall enough (i.e., with head $\geqslant c_i$) to be able to act on $[0, c_0]^-, \ldots, [i, c_i]^-$ the same way as $[b_j, j]^+$ would. Because of the monotone order in B^+ the final result is determined by $[b_j, j]^+$ with $j = a_{c_i} + 1$. So, we may assume that $j = a_{c_i} + 1$. Cf. Fig. 5.

Case $b_j \leqslant i$: Then $\delta_{B^+}(A^- \circ C^-), \delta_{B^+}(A^-), \delta_{B^{+\varphi(A^-)}}(C^-)$ all have trivial ith component.

Case $i < b_j \leqslant c_i$: Then $\delta_{B^+}(A^- \circ C^-)[i] = \delta_{B^{+\varphi(A^-)}}(C^-)[i] = b_j - 1$ and $\delta_{B^+}(A^-)[b_j - 1] = b_j - 1$, because either $b_j - 1 = a_{b_j - 1}$ to start with or $b_j \leqslant a_{b_j - 1} \leqslant a_{c_i} < j$ and δ-transition takes place.

Case $c_i < b_j \leqslant a_{c_i}$: Then $\delta_{B^+}(A^- \circ C^-)[i] = b_j - 1 = \delta_{B^+}(A^-)[c_i]$ and $\delta_{B^{+\varphi(A^-)}}(C^-)[i] = c_i$. $\square$

Lemma 3.6. *For* $A^- = (a_0, \ldots, a_{n-1})^- \in S^-$, $B^+ = (b_n, \ldots, b_1)^+ \in S^+$, $D^+ = (d_n, \ldots, d_1)^+ \in S^+$,

(a) $\delta_{B^+ \circ D^+}(A^-) = \delta_{B^+}(\delta_{D^+}(A^-))$,

(b) $(B^+ \circ D^+)^{\varphi(A^-)} = B^{+\varphi(\delta_{D^+}(A^-))} \circ D^{+\varphi(A^-)}$.

Proof. First note $B^+ \circ D^+ = (b_{d_n}, \ldots, b_{d_1})^+ \in S^+$.

(a) is obvious from the definition of δ.

(b): We apply Lemma 3.4 and study the operation of $\varphi([0, a_0]^-), \ldots, \varphi([n-1, a_{n-1}]^-)$ on $B^+ \circ D^+$. Consider the jth component $[b_{d_j}, j]^+$ of $B^+ \circ D^+$ (Fig. 6). Since δ_{D^+} shortens the $\downarrow$-arrows of A^- at their tails, only those $\varphi([i, a_i]^-)$ satisfying $b_{d_j} \leqslant i < j$ may act nontrivially on $[b_{d_j}, j]^+, [b_{d_j}, d_j]^+$, or $[d_j, j]^+$.

Case $a_i < j$: Then the φ-operation of $[i, a_i]^-$ on $[b_{d_j}, j]^+$ and $[d_j, j]^+$ is trivial. For the φ-operation of $[i, a_i]^-$ on $[b_{d_j}, d_j]^+$ to be nontrivial, we must have $i \leqslant d_j \leqslant a_i$. But in that case $\delta_{D^+}(A^-)[i] \leqslant \delta_{[d_j, j]^+}(A^-)[i] < d_j$, so that the d_jth component of B^+ is

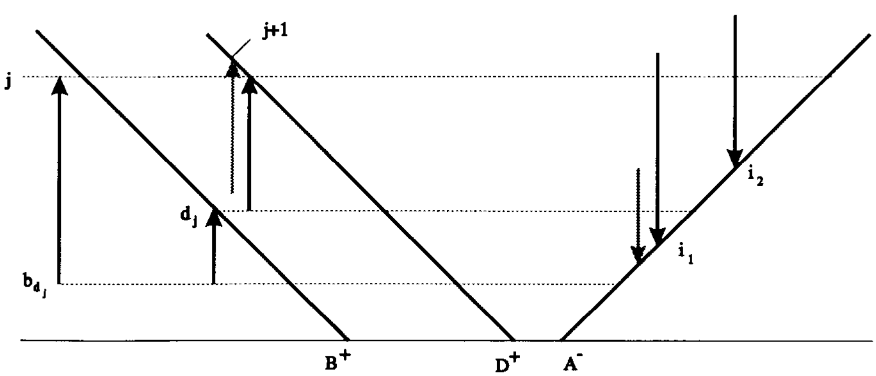

Fig. 6.

not affected either by the φ-operation of the $\downarrow$-arrow of $\delta_{D^+}(A^-)$ that results from $[i, a_i]^-$. Thus, only the following case is of interest:

Case $j \leqslant a_i$: Define $I_1 = \{i \mid b_{d_j} \leqslant i < d_j \text{ and } j \leqslant a_i\}$, $I_2 = \{i \mid d_j \leqslant i < j \text{ and } j \leqslant a_i\}$.

Subcase $I_1 = \emptyset$: If $I_2 = \emptyset$, too, then no action takes place. Otherwise let $i_2 = \min(I_2)$. Then $[b_{d_j}, j]^{+\varphi(A^-)} = [b_{d_j}, j]^{+\varphi([i_2, a_{i_2}]^-)} = [b_{d_j}, i_2]^+$, $[d_j, j]^{+\varphi(A^-)} = [d_j, j]^{+\varphi([i_2, a_{i_2}]^-)} = [d_j, i_2]^+$, and $[b_{d_j}, d_j]^{+\varphi(\delta_{D^+}(A^-))} = [b_{d_j}, d_j]^+$ because of the discussion above.

Subcase $I_1 \neq \emptyset$: Let $i_1 = \min(I_1)$. Then $[b_{d_j}, j]^{+\varphi(A^-)} = [b_{d_j}, j]^{+\varphi([i_1, a_{i_1}]^-)} = [b_{d_j}, i_1]^+$. It may happen that $b_{d_j} = b_{d_{j-1}}$. However, in that case we may drop $[b_{d_j}, j]^+$ from its consideration as a component of $B^+ \circ D^+$ altogether, because its contribution to $(B^+ \circ D^+)^{\varphi(A^-)}$ will be superseded by monotone extension arrows anyway. Now assuming $b_{d_j} \neq b_{d_{j+1}}$, $d_{j+1} > d_j$ follows. Since for $k \leqslant j$ the δ-transition of $[i_1, a_{i_1}]^-$ induced by $[d_k, k]^+$ is trivial and $d_{j+1} > d_j$, $\delta_{D^+}([i_1, a_{i_1}]^-)$ has a tail at least as long as d_j. Therefore, the φ-operation of $\delta_{D^+}([i_1, a_{i_1}]^-)$ on $[b_{d_j}, d_j]^+$ yields $[b_{d_j}, i_1]^+$.

Regarding $D^{+\varphi(A^-)}$, we note in this case: For $k \leqslant j$ the $\uparrow$-arrows of $D^{+\varphi(A^-)}$ which are based at i_1 or below cannot extend their heads beyond i_1 (even after monotone extension). For $k > j$ the $\uparrow$-arrows of $D^{+\varphi(A^-)}$ are already based at values $> d_j > i_1$. Thus, forming the product on the right-hand side of (b) still gives i_1st component $[b_{d_j}, i_1]^+$. Monotone extension does not make a difference for the value of the product. $\square$

This completes the proof of Proposition 3.3.

For efficiency considerations it is interesting to note that

$$|\mathrm{End}(X, \leqslant)| / |\mathrm{End}(X, \leqslant)^+| = (|X| + 1)/2.$$

To this end one determines $|\mathrm{End}(X, \leqslant)^+|$ to be the $|X|$th Catalan number, which is conveniently done by a counting technique discussed in [4].

When investigating the structure of a finite semigroup S, a usual approach is to study the idempotent elements of S. Our idea is to elaborate that approach by involving part of the semigroup structure and to study subsemilattices and their products. While a complete survey of all subsemilattices and their products is most likely an unrealistic goal (e.g., just think of the full transformation semigroup $\boldsymbol{T}(X)$), it is probably sensible to single out certain subsemilattices that provide some insight, such as a bilateral semidirect decomposition. This is what we did for $\mathrm{End}(X, \leqslant)$ in Fig. 7, the upper part illustrating Propositions 3.2 and 3.3. The part below $S_{\leqslant}^- S_{\leqslant}^+$ is merely included to exhibit the relationship to the Example in Section 2. Since the general bilateral semidirect product is a very powerful operation, it may be interesting to observe that both Propositions 3.1 and 3.3 require only one application of that product, independently of the size of X. This is in contrast to the bilateral semidirect decomposition of the symmetric group on X where nested application is necessary, indicating once more that a complex semigroup structure is often related to nontrivial subgroups.

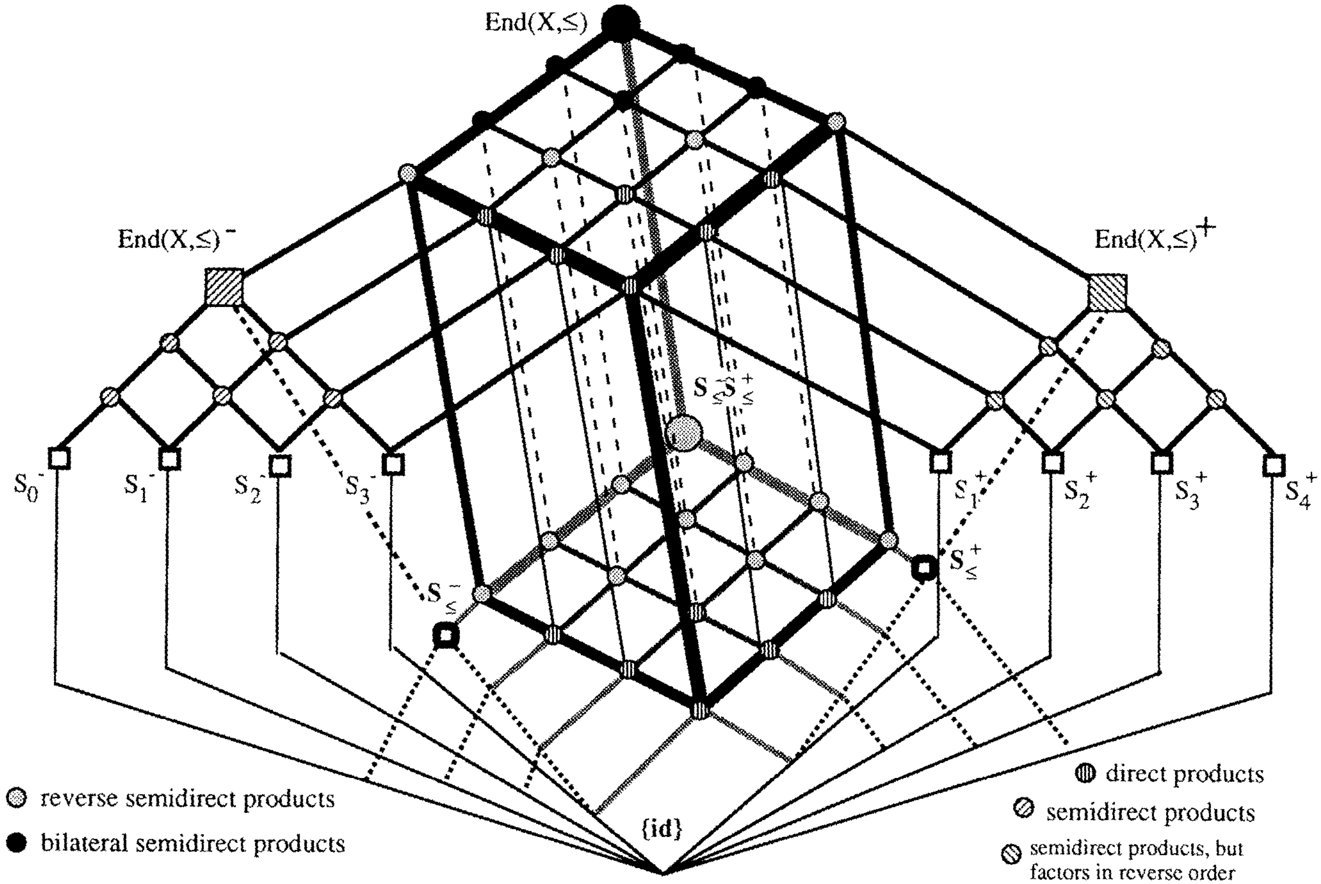

Fig. 7. Products of some transformation subsemigroups of $(X, \mathrm{End}(X, \leqslant))$ where $X = \{0, 1, 2, 3, 4\}$.

4. Applications to formal languages

Since automata theory forms a link between semigroups on one side and formal languages on the other side, and studying distinguished classes of objects is one of the most fruitful concepts for both of them, let us try an analogous approach to automata. We start with a definition of Jang [6] explaining what it means that a class of automata recognizes a class of languages:

Definition. A class $\mathcal{M}$ of initialized finite automata (i.e., with initial state fixed, but the set of final states left open) is a class of standard automata for a class $\mathcal{L}$ of formal languages, if

(1) for given alphabet Σ, the automata in $\mathcal{M}$ with input alphabet Σ form a chain with respect to the covering relation of automata,

(2) every language in $\mathcal{L}$ is recognized by some automaton in $\mathcal{M}$, and

(3) every automaton in $\mathcal{M}$ accepts only languages in $\mathcal{L}$ whatever subset of states is chosen to be final.

If the class $\mathcal{M}$ happens to be a one- or two-parameter family of machines, then this concept enables us to study the entire corresponding class $\mathcal{L}$ of languages by investigating the properties of so to speak a single machine, with those parameters as variables. This concept is a generalization of McNaughton and Papert's [9] approach to locally testable languages via counter-free automata. It turns out to be useful in formal language theory, even if one is not interested in automata in their own right.

So, let us have a look at the variety of locally testable languages. The corresponding semigroup variety is $\mathbf{LJ}_1 = \mathbf{J}_1 * \mathbf{D}$, the pseudo-variety generated by semidirect products of semilattices and definite semigroups. The original proof of that was not easy (cf. Eilenberg [3], Straubing [13], and further development by Tilson [14]) and relies on Simon's theorem on graphs. Indeed, Simon's theorem carries essentially the main workload of the proof, as Jang [6] showed by comparing two classes of standard automata for locally testable languages. Both of them are built around a shift register scanning the length-k segments of an input string: one is a straightforward formalization of McNaughton and Papert's machine motivating the concept of k-testability, while the other is closer to the graph theoretic setting of Simon's theorem.

As an exercise in semidirect decompositions, one may try to find a natural decomposition of the transition semigroup of these machines, in particular because their existence is already guaranteed by membership in $\mathbf{LJ}_1 = \mathbf{J}_1 * \mathbf{D}$. This can be done, but may require an embedding of the transition semigroup into a larger semigroup. A natural solution is the following semigroup which actually gives rise to a third class of standard automata for locally testable languages. This class of automata is closer to the semigroup theoretic point of view, and the equivalence of all three classes of machines can be seen by adjustment of the length parameter k.

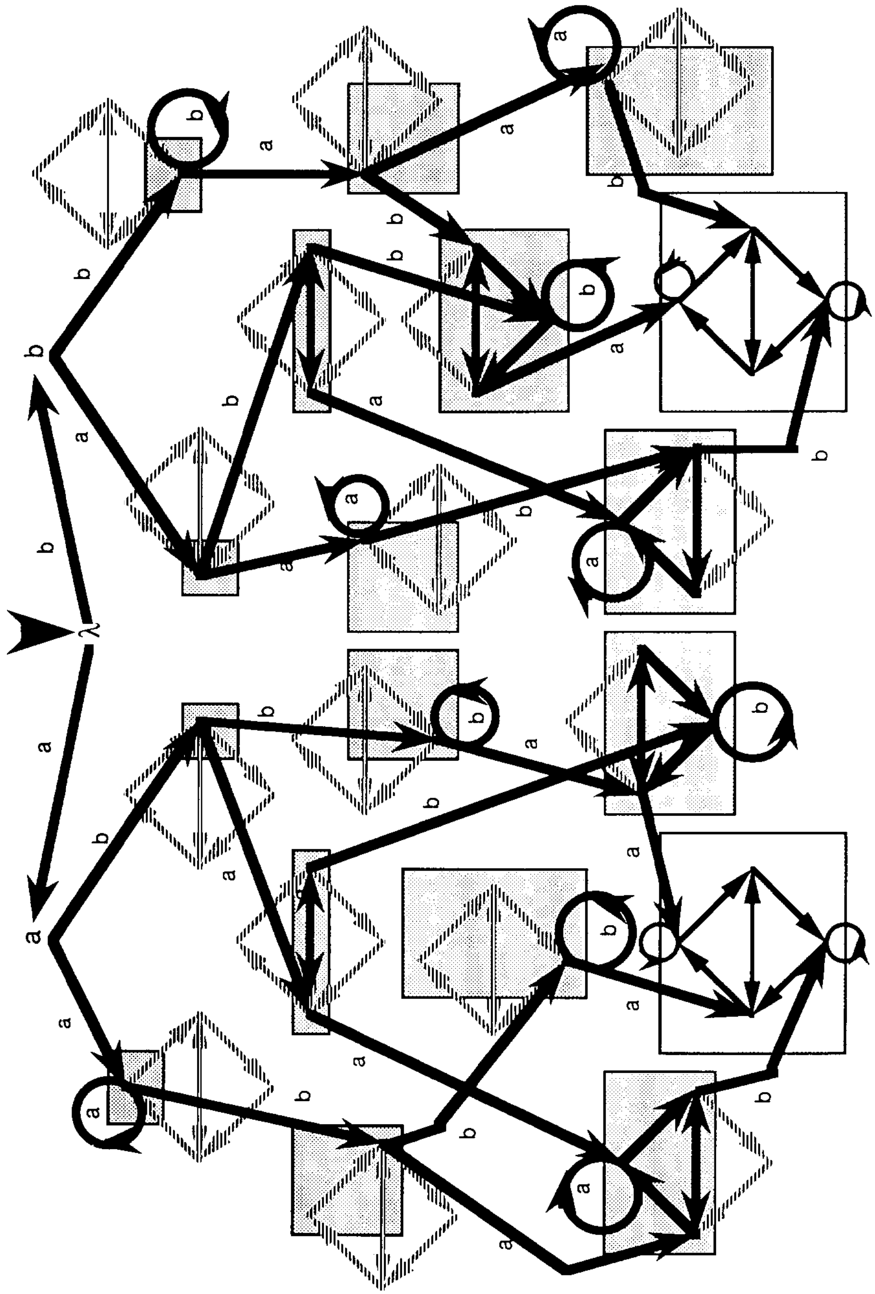

Fig. 8. State diagram of $\mathrm{DLT}_{3,2}$ where $\Sigma = \{a, b\}$.

Using the shift-register semigroup of Section 2, define the semidirect product

$$S_{\Sigma,k} = 2^{D_{\Sigma,k}} \times_\delta D_{\Sigma,k},$$

where $2^{D_{\Sigma,k}}$ denotes the power set of $D_{\Sigma,k}$ together with the union of sets as semigroup multiplication and

$$\delta_u(A) = \{u \cdot w \mid w \in A\}, \quad \text{for } u \in D_{\Sigma,k} \text{ and } A \subseteq D_{\Sigma,k}.$$

Here, elements of $D_{\Sigma,k}$ are conveniently denoted by the string fed to the shift register. To obtain an automaton $\mathrm{DLT}_{\Sigma,k}$, just take $S_{\Sigma,k}$ as the set of states with $(\{\lambda\},\lambda)$ being the initial state (λ denotes the empty string), and the multiplication as transition function: For input $a \in \Sigma$ multiply the current state with $(\{a\},a)$ from the right. The transition semigroup of $\mathrm{DLT}_{\Sigma,k}$, obviously, is isomorphic to a subsemigroup of $S_{\Sigma,k}$, and the class of all $\mathrm{DLT}_{\Sigma,k}$'s is a class of standard automata for locally testable languages. It may be interesting to see how that semidirect decomposition can be visualized in the state diagram of $\mathrm{DLT}_{\Sigma,k}$. Fig. 8 is for $k=2$ and $\Sigma = \{a,b\}$. The shaded boxes indicate the past history of length-k segments of the input string. These segments are represented by their location in a copy of the shift register state diagram.

For the pseudo-variety $\mathbf{LJ}_1$ this approach yields a set of generators without making use of the direct product as a variety operation: we just need homomorphic images of subsemigroups of $S_{\Sigma,k}$.

Proposition 4.1. *Every semigroup in* $\mathbf{LJ}_1$ *is covered by a semidirect product of the form* $2^{D_{\Sigma,k}} \times_\delta D_{\Sigma,k}$.

Proof. Given a semigroup S in $\mathbf{LJ}_1$, find a syntactic semigroup S' of some language L such that $S \subseteq S' \in \mathbf{LJ}_1$. L is locally testable. Thus, S' is a homomorphic image of the transition semigroup of some $\mathrm{DLT}_{\Sigma,k}$ for suitable Σ and k. $\square$

References

[1] A. Aizenstat, The defining relations of the endomorphism semigroup of a finite linearly ordered set, *Sibirsk. Mat. Zh.* **3** (1962) 161–169; Russian, Math. Reviews **26** (1963) #6287.

[2] S. Crvenković and M. Kunze, Actions of semilattices, *Semigroup Forum* **34** (1986) 139–156.

[3] S. Eilenberg, *Automata, Languages, and Machines, Vol. B* (Academic Press, New York, 1976).

[4] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics* (Addison-Wesley, Reading, MA, 1988) 344–346.

[5] J.M. Howie, Products of idempotents in finite full transformation semigroups, *Proc. Roy. Soc. Edinburgh* **98A** (1984) 25–35.

[6] W. Jang, Standard automata for varieties of formal languages, Dissertation, Fayetteville, AR, 1989.

[7] M. Kunze, Lineare Parallelrechner, *EIK J. Inform. Process. and Cybernetics* **20** (1984) 9–39 and 111–147.

[8] M. Kunze, Zappa products, *Acta Math Hungar.* **41** (1983) 225–239.

[9] R. McNaughton and S. Papert, *Counter-Free Automata* (MIT Press, Cambridge, MA, 1971).

[10] V.A. Molchanov, Semigroups of mappings on graphs, *Semigroup Forum* **27** (1983) 155–199.

[11] J. Rhodes and P. Weil, Decomposition techniques for finite semigroups, using categories, *J. Pure Appl. Algebra* **62** (1989) 269–312.
[12] T. Saito, Products of idempotents in finite full transformation semigroups, *Semigroup Forum* **39** (1989) 295–309.
[13] H. Straubing, Finite semigroup varieties of the form $\mathbf{V} * \mathbf{D}$, *J. Pure Appl. Algebra* **36** (1985) 53–94.
[14] B. Tilson, Categories as algebras: an essential ingredient in the theory of monoids, *J. Pure Appl. Algebra* **48** (1987) 83–198.