Partial Shape Preserving Approximation by Bivariate Hermite-Fejér Polynomials

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Abstract—Extending the results from the univariate case in a paper by Gal and Szabados, in this paper, we prove that the bivariate interpolation operators of Hermite-Fejér preserve some kinds of monotonicity and convexity of bivariate functions, in the neighborhoods of some points. Also, quantitative results are proved, i.e., estimates of the magnitudes for these neighborhoods are obtained.

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1. INTRODUCTION

It is evident that because of the interpolation conditions, the interpolating operators do not preserve the shape of a univariate function \( f \), on the whole interval that contains the points of interpolation. Many years ago, Popoviciu [1–3] proved the following result of qualitative type: for the classical univariate Hermite-Fejér polynomials based on the Jacobi-type knots, there still exist some points (independent of the function \( f \)) around whom the monotonicity of \( f \) is preserved. Containing these ideas, in the very recent papers [4,5], quantitative versions of Popoviciu's results (i.e., estimates for the lengths of these neighborhoods of preservation) were first obtained, then, in addition, the case of convexity of \( f \) was considered, and finally, all these problems were treated for Shepard operators, for Grünwald interpolation polynomials, and for Kryloff-Stayermann interpolation polynomials.

A key result used in the above-mentioned papers for the proofs of qualitative-type results is the following simple one.

**Lemma 1.1.** (See [3].) Let \( f : [a, b] \to \mathbb{R} \), \( a \leq x_1 < x_2 < \cdots < x_n \leq b \), and \( F_n(f)(x) = \sum_{i=1}^{n} h_i(x)f(x_i) \), where \( h_i \in C^1[a, b] \) and \( \sum_{i=1}^{n} h_i(x) = 1, \forall x \in [a, b] \).

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(i) We have
\[ F_n'(f)(x) = \sum_{i=1}^{n-1} \left( - \sum_{j=1}^{i} h_j'(x) \right) [f(x_{i+1}) - f(x_i)]. \]

(ii) If there exists \( x_0 \in (a,b) \) such that \( h_i'(x_0) < 0, h_n'(x_0) > 0 \), and the sequence \( h_1'(x_0), h_2'(x_0), \ldots, h_n'(x_0) \) has a unique variation of sign, then
\[ -\sum_{j=0}^{i} h_j'(x_0) < 0, \quad \text{for all} \ i = 1, n - 1, \]
and consequently, by (i), there exists a neighborhood \( V(x_0) \) of \( x_0 \), where the monotonicity of \( f \) assumed on the whole \([a,b]\) is preserved.

In this paper, qualitative and quantitative results for bivariate Hermite-Fejér polynomials are obtained.

2. BIVARIATE HERMITE-FEJÉR POLYNOMIALS

If \( g: [-1, 1] \to \mathbb{R} \) and \(-1 < x_{n,n} < x_{n-1,n} < \ldots < x_{1,n} < 1\) are the roots of Jacobi polynomials \( J_n(x) \), then it is well known (see, e.g., [3] or [5]) that the (univariate) Hermite-Fejér polynomials based on the roots above are given by \( F_n(g)(x) = \sum_{i=1}^{n} h_i(x)g(x_i) \), where
\[ h_i(x) = \ell_i(x) \left[ 1 - \frac{\ell_i'(x_i)}{\ell_n'(x_i)} (x - x_i) \right], \]
\[ \ell_i(x) = \prod_{i=1}^{n} (x - x_i). \]

We have \( \sum_{i=1}^{n} h_i(x) = 1 \), for all \( x \in [-1, 1] \).

Now, if \( f: [-1, 1] \times [-1, 1] \to \mathbb{R} \), then according to, e.g., [6], the bivariate Hermite-Fejér polynomial is defined by
\[ F_{n_1,n_2}(f)(x, y) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} h_i^{(1)}(x)h_j^{(2)}(y)f(x^{(1)}_{i,n_1}, x^{(2)}_{j,n_2}), \]
where \( h_i^{(1)}(x), x^{(1)}_{i,n_1}, \ i = 1, n_1, \) and \( h_j^{(2)}(y), x^{(2)}_{j,n_2}, \ j = 1, n_2 \) are defined as in the univariate case above, \( n_1, n_2 \in \mathbb{N} \).

We easily see (see, e.g., [6]) that
\[ F_{n_1,n_2}(f)(x^{(1)}_{i,n_1}, x^{(2)}_{j,n_2}) = f(x^{(1)}_{i,n_1}, x^{(2)}_{j,n_2}), \quad \forall i = 1, n_1, \quad j = 1, n_2. \]

The key result of this section is the following.

**Theorem 2.1.** With the notations above, we have
\[ \frac{\partial^2 F_{n_1,n_2}(f)(x,y)}{\partial x \partial y} = \sum_{i=1}^{n_1-1} \left[ \sum_{p=1}^{i} h_p^{(1)}(x) \right] \left[ \sum_{j=1}^{n_2-1} \sum_{q=1}^{j} h_q^{(2)}(y) \right] \times \left( f(x^{(1)}_{i,n_1}, x^{(2)}_{j,n_2}) - f(x^{(1)}_{i+1,n_1}, x^{(2)}_{j+1,n_2}) \right) \]
\[ -f(x^{(1)}_{i,n_1}, x^{(2)}_{j,n_2}) + f(x^{(1)}_{i+1,n_1}, x^{(2)}_{j+1,n_2}) \right). \]
PROOF. We observe

\[
\frac{\partial F_{n_1,n_2}(f)(x,y)}{\partial x} = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} h_{i,n_1}^{(1)'}(x) h_{j,n_2}^{(2)}(y) f \left( x_{i,n_1}, x_{j,n_2} \right)
\]

\[
= \sum_{j=1}^{n_2} \left( \sum_{i=1}^{n_1} h_{i,n_1}^{(1)'}(x) f \left( x_{i,n_1}, x_{j,n_2} \right) \right) h_{j,n_2}^{(2)}(y) \quad \text{(by Lemma 1.1(i))}
\]

\[
= \sum_{j=1}^{n_2} \left[ \sum_{i=1}^{n_1} \left( \sum_{p=1}^{i} h_{p,n_1}^{(1)'}(x) \right) \left( \sum_{j=1}^{n_2} h_{j,n_2}^{(2)}(y) \right) \right]
\]

\[
\times \left( f \left( x_{i,n_1}, x_{j,n_2} \right) - f \left( x_{i+1,n_1}, x_{j,n_2} \right) \right) \quad \text{(by Lemma 1.1(i))}
\]

It follows

\[
\frac{\partial^2 F_{n_1,n_2}(f)(x,y)}{\partial x \partial y} = \sum_{i=1}^{n_1-1} \left[ \left( \sum_{p=1}^{i} h_{p,n_1}^{(1)'}(x) \right) \left( \sum_{j=1}^{n_2-1} h_{j,n_2}^{(2)}(y) \right) \right]
\]

\[
\times \left( f \left( x_{i,n_1}, x_{j,n_2} \right) - f \left( x_{i+1,n_1}, x_{j,n_2} \right) \right) \quad \text{(by Lemma 1.1(i))}
\]

which proves the theorem.

Also, we need the following.

DEFINITION 2.2. (See, e.g., [7, p. 33].) We say that \( f: [a, b] \times [c, d] \to \mathbb{R} \) is bidimensional or hyperbolical upper (lower) monotone on \([a, b] \times [c, d]\) if

\[
\Delta_2(f)(x,y; \alpha, \beta) = f(x+\alpha, y+\beta) - f(x, y+\beta) - f(x+\alpha, y) + f(x, y) \geq 0,
\]

\((\leq 0, \text{respectively}), \text{for all } \alpha, \beta \geq 0 \text{ and } (x, y) \in [a, b] \times [c, d] \text{ such that } (x+\alpha, y+\beta) \in [a, b] \times [c, d].\)

REMARK. If \( f \in C^2([a, b] \times [c, d]) \) and \( \frac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0 \), for all \((x,y) \in [a, b] \times [c, d],\) then \( f \) is bidimensional upper monotone on \([a, b] \times [c, d]\) (see, e.g., [8]).

COROLLARY 2.3. Let \( n_1 = 2p_1, n_2 = 2p_2 \) be even numbers; and let us consider the bivariate Hermite-Fejér polynomials \( F_{n_1,n_2}(f)(x,y) \) given by (1), based on the roots \( x_{i,n_1}^{(1)} \) \( i = 1, n_1 \) of \( \lambda_1 \)-ultra-spherical polynomials of degree \( n_1 \) with \( \lambda_1 > -1 \) (i.e., the Jacobi polynomials \( J_{n_1}^{(\alpha_1, \beta_1)} \) with \( \alpha_1 = \beta_1, \lambda_1 = \alpha_1 + \beta_1 + 1, -1 < \alpha_1 , \beta_1 \leq 1 \), and on the roots \( x_{j,n_2}^{(2)} \) \( j = 1, n_2 \) of
\( \lambda_2 \)-ultraspherical polynomials of degree \( n_2 \), \( J_n^{(\alpha_2, \beta_2)}_2 \), \( \lambda_2 > -1 \) (i.e., \( \alpha_2 = \beta_2, \lambda_2 = \alpha_2 + \beta_2 + 1, -1 < \alpha_2, \beta_2 \leq 1 \)). There exists a constant \( c > 0 \) (independent of \( f \) and \( n_1, n_2 \)) such that if \( f: [-1,1] \times [-1,1] \to \mathbb{R} \) is bidimensional monotone on \([-1,1] \times [-1,1] \), then \( F_{n_1, n_2}(f)(x, y) \) is bidimensional monotone (of the same monotonicity) on \((l(-c/n_1^2, c/n_1^2) \times (-c/n_2^2, c/n_2^2)) \).

**Proof.** By the proof of Theorem 2.1 in \([5]\) (see relation (2) and the last relation there), we have

\[
\sum_{p=1}^{i} h_p^{(1)}(x) > 0, \quad \sum_{q=1}^{j} h_q^{(2)}(y) > 0, \quad \forall i = 1, n_1 - 1, \quad j = 1, n_2 - 1,
\]

\( \forall x \in \left(-\frac{c}{n_1^2}, \frac{c}{n_1^2}\right), \quad \forall y \in \left(-\frac{c}{n_2^2}, \frac{c}{n_2^2}\right) \).

Taking into account Theorem 2.1, we obtain

\[
\frac{\partial^2 F_{n_1, n_2}(f)(x, y)}{\partial x \partial y} \geq 0, \quad \forall (x, y) \in \left(-\frac{c}{n_1^2}, \frac{c}{n_1^2}\right) \times \left(-\frac{c}{n_2^2}, \frac{c}{n_2^2}\right),
\]

which by the remark after Definition 2.2 proves the theorem.

**Corollary 2.4.** Let us consider \( F_{n_1, n_2}(f)(x, y) \) given by (1), based on the roots of Jacobi polynomials \( J_{nr}^{(\alpha_i, \beta_i)}, J_{n_2}^{(\alpha_2, \beta_2)} \), of degree \( n_1 \) and \( n_2 \), respectively, with \( \alpha_i, \beta_i \in (-1,0], i = 1, 2 \).

If \( \xi \) is any root of the polynomial \( \ell_{n_1}^{(1)}(x) \) and \( \eta \) is any root of the polynomial \( \ell_{n_2}^{(2)}(y) \) (here \( \ell_{n_1}^{(1)}(x) = \prod_{i=1}^{n_1}(x - x_{i, n_1}^{(1)}), \ell_{n_2}^{(2)}(y) = \prod_{j=1}^{n_2}(y - x_{j, n_2}^{(2)})) \), then there exists a constant \( c > 0 \) (independent of \( n_1, n_2 \) and \( f \)) such that if \( f \) is bidimensional monotone on \([-1,1] \times [-1,1] \), then \( F_{n_1, n_2}(f)(x, y) \) is bidimensional monotone (of the same monotonicity) on

\[
\left( \xi - \frac{c \xi}{n_1^{\gamma_2}}, \xi + \frac{c \xi}{n_1^{\gamma_2}} \right) \times \left( \eta - \frac{c \eta}{n_2^{\gamma_2}}, \eta + \frac{c \eta}{n_2^{\gamma_2}} \right) \subset (-1,1) \times (-1,1),
\]

where

\[
c\xi = \frac{c}{(1 - \xi^2)^{5/2 + \delta_1}}, \quad c\eta = \frac{c}{(1 - \eta^2)^{5/2 + \delta_2}}, \quad \gamma_i = \max\{\alpha_i, \beta_i\}, \quad i = 1, 2,
\]

and

\[
\delta_1 = \left\{ \begin{array}{l}
\alpha_1, \quad \text{if} \ 0 \leq \xi < 1, \\
\beta_1, \quad \text{if} \ -1 < \xi \leq 0,
\end{array} \right.
\]

\[
\delta_2 = \left\{ \begin{array}{l}
\alpha_2, \quad \text{if} \ 0 \leq \eta < 1, \\
\beta_2, \quad \text{if} \ -1 < \eta \leq 0.
\end{array} \right.
\]

**Proof.** An immediate consequence of Theorem 2.1 above and of Theorem 2.2 in \([5]\).

**Remarks.**

(1) Because \( \ell_{n_1}^{(1)}(x) \) and \( \ell_{n_2}^{(2)}(y) \) have exactly \( n_1 - 1 \) and \( n_2 - 1 \) roots in \((-1,1)\), respectively, it follows that in \((-1,1) \times (-1,1) \), there exists a grid of \((n_1 - 1)(n_2 - 1)\) points \((\xi, \eta)\) from Corollary 2.4.

(2) From Remark 1, after Theorem 2.2 in \([5]\), it follows that if \( \xi \) and \( \eta \) are near the endpoints in the ultraspherical case, for example, \( \xi, \beta_i \in (-1,0), i = 1, 2 \), then the best possible bidimensional interval of preservation of bidimensional monotonicity is \((-1,1) \times (-1,1) \) for \( \xi = c/n_1^2, \eta = c/n_2^2 \).
In what follows, we will extend the convexity problem from the univariate case of [5]. In this sense, we need the following.

**Definition 2.5.** (See, e.g., [9].) We say that \( f: [-1, 1] \times [-1, 1] \rightarrow \mathbb{R} \) is strictly double convex on \([-1, 1] \times [-1, 1]\), if \( \Delta_{h_1}^{2, x} f(a, b) > 0 \), for all \( h_1, h_2 > 0 \), \( (a, b) \in [-1, 1] \times [-1, 1] \), with \( a \pm h_2, b \pm h_1 \in [-1, 1] \), where

\[
\Delta_{h_2}^{2, x} f(a, \beta) = f(a + h_2, \beta) - 2f(a, \beta) + f(a - h_2, \beta)
\]

and

\[
\Delta_{h_1}^{2, y} f(\alpha, \beta) = f(\alpha, \beta + h_1) - 2f(\alpha, \beta) + f(\alpha, \beta - h_1).
\]

**Remark.** By the mean value theorem, it is easy to see that if \( \frac{\partial^2 f(x, y)}{\partial x^2 \partial y^2}(x, y) > 0 \), for all \( (x, y) \in [-1, 1]^2 \), then \( f \) is strictly double convex on \([-1, 1]^2\).

Now, let \( n_1, n_2 \geq 3 \) be odd and let us consider as \( F_{n_1, n_2}(f)(x, y) \) the Hermite-Fejér interpolation polynomials given by (1), based on the roots \( x_{i, n_1}^{(1)} \), \( i = 1, n_1 \), and \( x_{j, n_2}^{(2)} \), \( j = 1, n_2 \) of the \( \lambda_1 \)-ultraspherical polynomials \( P_n^{(\lambda_1)} \) of degree \( n_1 \) and \( \lambda_2 \)-ultraspherical polynomials \( P_n^{(\lambda_2)} \) of degree \( n_2 \), respectively, \( \lambda_1, \lambda_2 \in [0, 1] \), and the Cotes-Christoffel numbers of the Gauss-Jacobi quadrature.

**Theorem 2.6.** If \( f \in C([-1, 1] \times [-1, 1]) \) satisfies

\[
\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \lambda_{i, n_1}^{(1)} \lambda_{j, n_2}^{(2)} \Delta_{x, x}^{2, y} f(0, 0) \left[ \frac{\Delta_{x, x}^{2, y} f(0, 0)}{\left( x_{i, n_1}^{(1)} x_{j, n_2}^{(2)} \right)^2} \right] > 0,
\]

then \( F_{n_1, n_2}(f)(x, y) \) is strictly double convex in \( V(0, 0) = \{(x, y); x^2 + y^2 < d_{n_1, n_2}^2\} \), with

\[
|d_{n_1, n_2}| \geq c_f, \lambda_1, \lambda_2 \left[ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \lambda_{i, n_1}^{(1)} \lambda_{j, n_2}^{(2)} \Delta_{x, x}^{2, y} f(0, 0) \right] \left[ \left( x_{i, n_1}^{(1)} x_{j, n_2}^{(2)} \right)^2 \right] \]

where \( c_f, \lambda_1, \lambda_2 > 0 \) is independent of \( n_1 \) and \( n_2 \).

**Proof.** We observe

\[
\frac{\partial^4 F_{n_1, n_2}(f)(x, y)}{\partial x^2 \partial y^2} = \sum_{j=1}^{n_2} h_{j, n_2}^{(2)} \left( \sum_{i=1}^{n_1} h_{i, n_1}^{(1)} (x) f (x_{i, n_1}^{(1)} x_{j, n_2}^{(2)}) \right)
\]

and reasoning as in the proof of Theorem 2.3 in [5] (see relation (7) there), we obtain

\[
\frac{\partial^4 F_{n_1, n_2}(f)(0, 0)}{\partial x^2 \partial y^2} = \sum_{j=1}^{n_2} h_{j, n_2}^{(2)} (0) \left[ \sum_{i=1}^{n_1} h_{i, n_1}^{(1)} (0) \Delta_{x, x}^{2, y} f (0, x_{j, n_2}^{(2)}) \right].
\]
Denoting $G(y) = \sum_{i=1}^{(n-1)/2} h_{i,n_i}^{(1)}(0) \cdot \Delta^2_{x_{i,n_i}} f(0,y)$, we get

$$\frac{\partial^4 F_{n_1,n_2}(f)(0,0)}{\partial x^2 \partial y^2} = \sum_{j=1}^{n_2} h_{j,n_2}^{(2)}(0) G_x^{(2)}(x_{j,n_2}) = \sum_{j=1}^{(n_2-1)/2} h_{j,n_2}^{(2)}(0) \Delta^2_Y x_{j,n_2} G(0) = \sum_{j=1}^{(n_2-1)/2} \sum_{i=1}^{(n_2-1)/2} h_{j,n_2}^{(2)}(0) h_{i,n_1}^{(1)}(0) \Delta^2_Y x_{j,n_2} \left[ \Delta^2_{x_{i,n_1}} f(0,0) \right].$$

Therefore, again by relation (7) of [5] and by hypothesis, we obtain

$$\frac{\partial^4 F_{n_1,n_2}(f)(0,0)}{\partial x^2 \partial y^2} > c_3 \lambda_1 \lambda_2 n_1 n_2 \sum_{i=1}^{n_2} \sum_{j=1}^{n_2} \frac{\Delta^2_Y x_{j,n_2}}{G(x_{j,n_2})} > 0. \quad (3)$$

So it follows that $F_{n_1,n_2}(f)(x,y)$ is strictly double convex in a neighborhood of $(0,0)$. Let $(\alpha_{n_1,n_2}, \beta_{n_1,n_2})$ be the nearest root of the nearest roots of $\frac{\partial^4 F_{n_1,n_2}(f)}{\partial x^2 \partial y^2}$ to $(0,0)$, in the sense that the distance $d_{n_1,n_2} = \sqrt{\alpha_{n_1,n_2}^2 + \beta_{n_1,n_2}^2}$ is minimum for all the roots of $\frac{\partial^4 F_{n_1,n_2}(f)}{\partial x^2 \partial y^2}$. Then, for all $(x,y) \in V(0,0) = \{(x,y) \in \mathbb{R}^2; \sqrt{x^2 + y^2} < d_{n_1,n_2}\}$, we necessarily have $\frac{\partial^4 F_{n_1,n_2}(f)(x,y)}{\partial x^2 \partial y^2} > 0$. By the mean value theorem for bivariate functions, we get

$$\frac{\partial^4 F_{n_1,n_2}(f)(0,0)}{\partial x^2 \partial y^2} = \left| \frac{\partial F_{n_1,n_2}(f)(0,0)}{\partial x^2 \partial y^2} - \frac{\partial F_{n_1,n_2}(f)(\alpha_{n_1,n_2}, \beta_{n_1,n_2})}{\partial x^2 \partial y^2} \right| \leq |\alpha_{n_1,n_2}| \left| \frac{\partial^5 F_{n_1,n_2}(f)(\xi,\eta)}{\partial x^3 \partial y^2} \right| + |\beta_{n_1,n_2}| \left| \frac{\partial^5 F_{n_1,n_2}(f)(\xi,\eta)}{\partial x^2 \partial y^3} \right| \leq |d_{n_1,n_2}| \left[ \left| \frac{\partial^5 F_{n_1,n_2}(f)(\xi,\eta)}{\partial x^3 \partial y^2} \right| + \left| \frac{\partial^5 F_{n_1,n_2}(f)(\xi,\eta)}{\partial x^2 \partial y^3} \right| \right].$$

Because degree $(F_{n_1,n_2}(f)) \leq 2n_1 - 1 + 2n_2 - 1 = 2n_1 + 2n_2 - 2$, we have degree $(\frac{\partial^4 F_{n_1,n_2}(f)}{\partial x^2 \partial y^2}) \leq 2n_1 + 2n_2 - 7$, degree $(\frac{\partial^5 F_{n_1,n_2}(f)}{\partial x^3 \partial y^2}) \leq 2n_1 + 2n_2 - 7$. As in the proof of Theorem 2.3 in [5], we can assume that the interval of convexity cannot be larger than $[-c_1/n_1, c_1/n_1] \times [-c_2/n_2, c_2/n_2]$. Consequently, we may assume that $|d_{n_1,n_2}| \leq c/\min\{n_1, n_2\}$.

Now, by the Bernstein theorem in [10, p. 136, relation (8)], we obtain

$$\left| \frac{\partial^5 F_{n_1,n_2}(f)(\xi,\eta)}{\partial x^3 \partial y^2} \right| \leq c(2n_1 + 2n_2 - 7) (2n_1 + 2n_2 - 6) (2n_1 + 2n_2 - 5) \times (2n_1 + 2n_2 - 4) (2n_1 + 2n_2 - 3) \cdot \|F_{n_1,n_2}\|_{C([-1,1] \times [-1,1])} \leq c(n_1 + n_2)^5 \cdot \|F_{n_1,n_2}\|_{C([-1,1] \times [-1,1])}.$$

But because by [11], the fundamental interpolation polynomials $h_{i,n_i}(x)$ and $h_{j,n_2}(y)$ are $\geq 0$, $\forall i = 1, n_1, \forall j = 1, n_2, \forall (x,y) \in [-1,1] \times [-1,1]$, denoting $M_f = \|f\|_{C([-1,1] \times [-1,1])}$, it follows

$$\left| \frac{\partial^5 F_{n_1,n_2}(f)(\xi,\eta)}{\partial x^3 \partial y^2} \right| \leq c(n_1 + n_2)^5 M_f, \quad \left| \frac{\partial^5 F_{n_1,n_2}(f)(\xi,\eta)}{\partial x^2 \partial y^3} \right| \leq c(n_1 + n_2)^5 M_f.$$
and consequently,
\[ \frac{\partial^4 F_{n_1,n_2}(f)(0,0)}{\partial x^2 \partial y^2} \leq c_f (n_1 + n_2)^5 d_{n_1,n_2}, \]

where \( c_f > 0 \) is independent of \( n_1 \) and \( n_2 \) (but dependent on \( f \)).

Combining this estimate with (3), we easily get the lower estimate for \( |d_{n_1,n_2}| \) in the statement of Theorem 2.6.

**REMARKS.**

1. As in the univariate case, the neighborhood \( V(0,0) \) of preservation of strict convexity depends on \( f \) also.

2. The estimate of \( |d_{n_1,n_2}| \) in the bivariate case seems to be weaker, in a sense, than that of the univariate case, because it was not yet proved to be a Stechkin-type inequality for bivariate polynomials. That would be useful for a better estimate.

3. If \( f : [-1, 1] \times [-1, 1] \to \mathbb{R} \) is strictly double convex on \([-1, 1] \times [-1, 1] \), then condition (2) is obviously satisfied, and consequently, \( F_{n_1,n_2}(f)(x,y) \) preserve the strictly double convexity in a disc centered at \((0,0)\), having for its ray \( |d_{n_1,n_2}| \) the lower estimate of Theorem 2.6.

4. Let \( F_{n_1,n_2}(f)(x,y) \) be given by (1), based on the roots \( x_{i,n_1}^{(1)}, \ i = 1,n_1 \), and \( x_{j,n_2}^{(2)}, \ j = 1,n_2 \) of the \( \lambda_1 \)-ultraspherical polynomials \( P_{n_1}^{(\lambda_1)} \) and \( \lambda_2 \)-ultraspherical polynomials \( P_{n_2}^{(\lambda_2)} \), respectively, where \( \lambda_1, \lambda_2 \in [0,1] \). Because by [11], the polynomials \( h_{i,n_1}^{(1)}(x), h_{j,n_2}^{(2)}(y) \geq 0, \forall i = 1,n_1, \forall j = 1,n_2, \forall (x,y) \in [-1,1] \times [-1,1] \), by the formulas

\[
\frac{\partial^p F_{n_1,n_2}(f)(x,y)}{\partial y^p} = \sum_{i=1}^{n_1} h_{i,n_1}^{(1)}(x) \left[ \sum_{j=1}^{n_2} \frac{\partial^p h_{j,n_2}^{(2)}(y)}{\partial y^p} f \left( x_{i,n_1}^{(1)}, x_{j,n_2}^{(2)} \right) \right],
\]

\( p = 1,2 \), from the univariate case in [5], the following results are immediate.

If \( f(x,y) \) is nondecreasing with respect to \( y \in [-1,1] \) (for all fixed \( x \in [-1,1] \)), then for \( n_1, n_2 \in \mathbb{N} \) and \( \eta \) root of \( P_{n_2}^{(\lambda_2)}(\eta) \), \( F_{n_1,n_2}(f)(x,y) \) is nondecreasing with respect to

\[
y \in \left( \eta - \frac{c_\eta}{n_2^{\lambda_2+2}}, \eta + \frac{c_\eta}{n_2^{\lambda_2+2}} \right),
\]

for all fixed \( x \in [-1,1] \) (here \( c_\eta \) and \( \gamma_2 \) are given by Corollary 2.4).

If \( f(x,y) \) is strictly convex with respect to \( y \in [-1,1] \) (for all fixed \( x \in [-1,1] \)), then for all \( n_1 \in \mathbb{N} \), arbitrary and \( n_2 \in \mathbb{N} \), \( n_2 \geq 3, n_2 \) odd number, there exists a neighborhood \( V(0) \) of 0, such that for all fixed \( x \in [-1,1] \), \( F_{n_1,n_2}(f)(x,y) \) is strictly convex with respect to \( y \in V(0) \).

Similar results hold if we consider \( \frac{\partial^p F_{n_1,n_2}(f)(x,y)}{\partial x^p}, \ p = 1,2 \).

5. All the results above can easily be extended for \( n \) variables, \( n > 2 \).

**REFERENCES**