Vanishing theorem for cohomology groups of $c_2$-self-dual bundles on quaternionic Kähler manifolds

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Communicated by S.M. Salamon
Received 3 August 1993

Abstract: It is known that certain self-dual connections are defined on quaternionic Kähler manifolds. A.D.H.M-vanishing theorem $\tilde{H}^1(\mathbb{P}^3, F(-2)) = 0$ is generalized to anti-self-dual bundles on an arbitrary quaternionic Kähler manifolds with positive scalar curvature. The cohomology group is related to the solution space of quaternionic operators.

Keywords: Yang–Mills, quaternionic Kähler manifold, twistor space.

MSC classification: 53C07.

1. Introduction

In [1] Atiyah, Hitchin and Singer gave an account of ideas of R. Penrose connecting the Riemannian geometry of certain four-dimensional spaces with complex three-dimensional geometry. The Penrose twistor fibration is a fundamental and important tool in algebraic geometry, if we want to find solutions to the anti-self-dual Yang–Mills equations. Especially, Atiyah–Drinfeld–Hitchin–Manin [2] showed how a complete solution to the problem of finding all anti-self-dual Yang–Mills fields on $S^4$ was constructed via the twistor space $\mathbb{P}^3$. The crucial step in this work was the vanishing of the sheaf cohomology group

$$H^1(\mathbb{P}^3, \tilde{F}(-2)) = 0,$$

where $\tilde{F}$ is the pull-back bundle on $\mathbb{P}^3$ of anti-self-dual bundle $F$ on $S^4$. First, the above vanishing theorem was proved by Drinfeld–Manin [5] and Rawnsley [13] in case of $S^4$. Hitchin [7] gave a generalization to a half-conformally-flat manifold with positive scalar curvature.

The purpose of the present paper is to show that a vanishing theorem of this type still holds even in a generalization to a higher-dimensional case. The object in this
paper is a quaternionic Kähler manifold which is a $4n$-dimensional oriented Riemannian manifold whose holonomy group is contained in the subgroup $\text{Sp}(n) \cdot \text{Sp}(1)$, $n \geq 1$. For example, we take $\mathbb{H}P^n$ instead of $S^4 \cong \mathbb{H}P^1$. In fact, Mamone Capria-Salamon [4] conjectured that

$$H^1(\mathbb{P}^{2n+1}, \tilde{F}(-2)) = 0,$$

where $\mathbb{P}^{2n+1}$ is a twistor space of $\mathbb{H}P^n$ and $F$ is a $c_2$-self-dual bundle on $\mathbb{H}P^n$, which are generalizations of the notion of 4-dimensional case.

First of all, Salamon [14] showed that there is a twistor fibering over an arbitrary quaternionic Kähler manifold $M$. The total space $Z$ with compact fibres $\mathbb{P}^1$ has naturally a complex structure in the same way as in the 4-dimensional case. Moreover, the twistor space $Z$ is actually a Kähler manifold when $M$ has positive scalar curvature. We are interested in such a case and we use this Kähler structure.

Next, as for Yang-Mills connection, Nitta [12] and Mamone Capria-Salamon [4] have developed independently higher dimensional analogues of the notion of self-dual or anti-self-dual connections on a quaternionic Kähler manifold $M$. Those connections are called $c_1$, $c_2$ and $c_3$-self-dual connections in Galicki and Poon [6]. We use this terminology in this paper and focus attention on $c_2$-self-dual connections, because $c_2$-self-duality is related with holomorphy on the twistor space $Z$. Alternatively, if we pull back $c_2$-self-dual form on $M$ to $Z$, we get $(1,1)$ form on $Z$. Therefore every $c_2$-self-dual bundle on $M$ is pulled back to a holomorphic bundle on $Z$. In Section 2 we recall the twistor space and the $c$-self-dual connection briefly.

In the third section, we prove the main theorem. The main idea is to establish a correspondence between the elements of $H^1(Z, \tilde{F}(-2))$ and solutions of a second-order operator on $M$, where $\tilde{F}$ is the pull-back bundle on $Z$ of $c_2$-self-dual bundle $F$ on $M$. A difference to be emphasized in higher-dimensional case from 4-dimensional case is that such a correspondence is not surjective. We identify $H^1(Z, \tilde{F}(-2))$ with the corresponding Dolbeault cohomology group. The elements of this Dolbeault cohomology group are uniquely represented by fiber-harmonic forms on $Z$. The notion of fiber-harmonicity is introduced by Rawnsley [13] in case of $S^4$ and based on the twistor fibering $\mathbb{P}^3 \to S^4$. We extend this notion to the twistor space of an arbitrary quaternionic Kähler manifold with positive scalar curvature. The essential point is that the horizontal part of the element of the above Dolbeault cohomology group is completely determined in terms of vertical part. Hence, we turn our attention to the vertical part. If the vertical part restricted to every fibre is required to be harmonic, we can explicitly apply Serre duality for it on fibres $\mathbb{P}^1$ and get a section of $F$, which satisfies a second-order operator on $M$. The positivity of this second-order operator implies our result.

In Section 4, we attempt to find a background of the appearance of this operator. To do so, we introduce a quaternionic Dirac operator and a twistor operator on a quaternionic Kähler manifold. Then, the solution spaces to these operators are related to the cohomology groups on the twistor space. But we do not pursue the details of this argument. For example, see [3].

Finally, I would like to make a grateful acknowledgement to Professors K. Ogiue,
Y. Ohnita and S. Udagawa for many suggestions and kindly encouragement.

2. Preliminaries

Let $M$ be a connected quaternionic Kähler manifold. Using the reduction theorem (see Kobayashi and Nomizu [9]), we see that the orthonormal frame bundle of the tangent bundle $TM$ can be reduced to a principal $\text{Sp}(n) \cdot \text{Sp}(1)$-bundle $P$. Since the action of $\text{Ad}(g)$ ($g \in \text{Sp}(n)$) on $\mathfrak{sp}(1)$ is trivial, we take the vector bundle $E = P \times_{\text{Ad}} \mathfrak{sp}(1)$ associated with the adjoint representation. Then the vector bundle $E$ has the following properties.

1. $E$ is a rank 3 subbundle of $\text{End}(TM)$.
2. $E$ has a local basis $I, J, K$ satisfying that
   \begin{itemize}
   \item[(i)] the Riemannian metric $g$ is hermitian for $I, J, K$, in the sense that
     \[ g_x(I X, I Y) = g_x(J X, J Y) = g_x(K X, K Y) = g_x(X, Y), \]
     for all $X, Y \in T_xM$, \( \forall x \in M \).
   \item[(ii)] $I^2 = J^2 = -1$, $IJ = -JI = K$.
   \end{itemize}
3. The connection induced by the Riemannian connection preserves $E$. In other words, if $D$ is the induced connection, then
   \[ D(E) \subset E \otimes T^*M. \]
Conversely, the existence of such a vector bundle turns a Riemannian manifold into a quaternionic Kähler manifold. Therefore the vector bundle $E$ is called the quaternionic Kähler structure bundle of $M$.

2.1. $c$-self-dual connection

The vector bundle $\wedge^2 T^*M$ has the following holonomy invariant decomposition:

\[ \wedge^2 T^*M = S^2 \mathbb{H} \oplus S^2 \mathbb{E} \oplus (S^2 \mathbb{H} \oplus S^2 \mathbb{E})^\perp, \]

where $\mathbb{H}$ and $\mathbb{E}$ are vector bundles associated with the standard representations of $\text{Sp}(1)$ and $\text{Sp}(n)$, respectively. For example, $\mathbb{H}$ is a tautological quaternionic line bundle when the base space is a quaternionic projective space $\mathbb{H}P^n$. This decomposition can also be explained in terms of the Hodge $*$-operator as in the 4-dimensional case. To show this, we note that $E$ is isomorphic to $S^2 \mathbb{H}$ via the metric $g$. Explicitly, an element $A \in E_x$ is mapped to $\omega_A$ by

\[ \omega_A(X, Y) = g(A X, Y) \quad \text{for} \ X, Y \in T_xM. \]

Then, making use of $\omega_I, \omega_J, \omega_K$ which are locally defined 2-forms, we define a global 4-form $\Omega$ by

\[ \Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K. \]

It is known that $\Omega$ is non-degenerate and parallel on $M$. This form $\Omega$ is called the fundamental 4-form on $M$ [10].
Definition 2.1.1. [6] An \( \omega \in \bigwedge^2 T^*M \) is called a c-self-dual form if

\[ *\omega = c\omega \wedge \Omega^{n-1}, \]

where "*" is the Hodge *-operator.

We notice that the above equation can be viewed as the self-dual or anti-self-dual equation on a 4-dimensional oriented Riemannian manifold in case of \( n = 1 \).

Theorem 2.1.2. [6] An \( \omega \in \bigwedge^2 T^*M \) is a non-zero c-self-dual form if and only if

\begin{align*}
(1) & \quad \omega \in S^2\mathbb{H} \quad \text{and} \quad c = c_1 = \frac{6n}{(2n+1)!}, \\
(2) & \quad \omega \in S^2\mathbb{E} \quad \text{and} \quad c = c_2 = -\frac{1}{(2n-1)!}, \\
(3) & \quad \omega \in (S^2\mathbb{H} \oplus S^2\mathbb{E})^\perp \quad \text{and} \quad c = c_3 = \frac{3}{(2n-1)!}. 
\end{align*}

We shall investigate metric connections on a complex vector bundle \( F \) equipped with a hermitian metric \( h \).

Definition 2.1.3. [6] A connection \( \nabla \) is called c-self-dual if its curvature 2-form \( R^\nabla \) is a c-self-dual form.

Remark. As we mentioned in the introduction, Nitta and Mamone Capria-Salamon have found these connections independently [4, 12]. But they used different terminology. \( A_2, B_2 \) and \( A_2' \) connections used in Nitta's paper correspond to \( c_1, c_2 \) and \( c_3 \)-self-dual connections, respectively. On the other hand, by Mamone Capria-Salamon, a \( c_2 \)-self-dual connection is called a self-dual connection.

Theorem 2.1.4. [4, 6, 12] Every c-self-dual connection is a Yang–Mills connection.

Remark. Moreover, if \( M \) is compact, then for \( i = 1 \) or 2 an arbitrary \( c_i \)-self-dual connection minimizes the Yang–Mills functional [4, 6]. It is known that we have essentially unique non-flat \( c_1 \)-self-dual connection over a simply connected quaternionic Kähler manifold whose dimension is greater than or equal to 8 [11].

In this paper, we are concerned with \( c_2 \)-self-dual connections. Therefore, we give a property of \( c_2 \)-self-dual form. The decomposition (2.1) is based on the adjoint representation of \( \text{Sp}(n) \cdot \text{Sp}(1) \). When \( \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \) is regarded as one of Lie subalgebras of \( \mathfrak{so}(4n) \), the subspace invariant by the adjoint action of \( \text{Sp}(1) \) is just \( \mathfrak{sp}(n) \). This observation implies the following.

Lemma 2.1.5. (see also [4]) An \( \omega \in \bigwedge^2 T^*M \) is a \( c_2 \)-self-dual form if and only if

\[ \omega_x(IX, IY) = \omega_x(JX, JY) = \omega_x(KX, KY), \quad \forall \ x \in M, \quad \forall \ X, Y \in T_x M, \]

where \( I, J, K \) is a local basis which satisfies the property (2) of the quaternionic Kähler structure bundle \( E \).
2.2 Twistor space

We give a quick review of the twistor space $Z$ of an arbitrary quaternionic Kähler manifold $M$, mostly in order to fix our notation (for details, see [14]).

Let $E, H$ denote the standard complex representations of $Sp(n), Sp(1)$ on $\mathbb{C}^{2n}, \mathbb{C}^2$ respectively. We identify the complex numbers $\mathbb{C}$ with the subfield of the quaternions $\mathbb{H}$ generated by 1 and $i$. Similarly $\mathbb{H}^m$ gets identified with $\mathbb{C}^{2m}$. Then the complex vector spaces $E, H$ possess a quaternionic structure $j$ induced by the right multiplication by a unit quaternion $j$. In addition there are invariant skew forms $\omega_H \in \Lambda^2 H^*, \omega_E \in \Lambda^2 E^*$ satisfying $\omega_H(hj,kj) = \omega_H(h,k)$, $\omega_H(h,hj) > 0$ if $h \neq 0$, and similarly for $\omega_E$. These give identifications $H \cong H^*, E \cong E^*$. From here, although we refer mainly to $H$, there is no problem of replacing $H$ by $E$. By a standard basis of $H$ we mean a unitary basis of the form $\{h_1 = h, h_2 = hj\}$, so that $\omega_H(h, hj) = 1$. All these structures carry over to the fibres of the associated vector bundles, for example, $\mathbb{H}$, $\mathbb{E}$. We denote by $\omega_H$ a canonical section induced by $\omega_H$. By a standard basis of $\mathbb{H}$ we shall simply mean a unitary pair of sections $\{h_1 = h, h_2 = hj\}$ defined over some open set of $M$.

It is known that any irreducible $Sp(n) \times Sp(1)$-module can be realized as a subspace of some $(\otimes^p E) \otimes (\otimes^q H)$, $p, q \geq 0$. The representations which factor through $Sp(n) \cdot Sp(1)$ are precisely those corresponding to $p + q$ even. An arbitrary such representation has a real structure induced from the quaternionic structures of $E, H$, and we adopt the convention that it is to be thought of as a real vector space even if expressed in terms of $E$ and $H$.

Since $M$ is a quaternionic Kähler manifold, $T^*M$ is expressed by the representation $E \otimes H$ as

$$T^*M = P \times_{Sp(n),Sp(1)} (E \otimes H) = E \otimes \mathbb{H}.$$ 

Then, the Riemannian metric is given by

$$g_M = \omega_E \otimes \omega_H.$$ 

We define an orthonormal basis of $T^*M$ over some open set $U$ in $M$ by

$$\theta^{1p} = \frac{1}{\sqrt{2}}(e_{2p-1} \otimes h_1 + e_{2p} \otimes h_2),$$

$$\theta^{2p} = \frac{\sqrt{-1}}{\sqrt{2}}(e_{2p-1} \otimes h_1 - e_{2p} \otimes h_2),$$

$$\theta^{3p} = \frac{1}{\sqrt{2}}(e_{2p} \otimes h_1 - e_{2p-1} \otimes h_2),$$

$$\theta^{4p} = \frac{\sqrt{-1}}{\sqrt{2}}(e_{2p} \otimes h_1 + e_{2p-1} \otimes h_2),$$

for $p = 1, 2, \ldots, n$,

where $\{h_1, h_2\}$ is a standard basis of $\mathbb{H}$ and $\{e_1, e_2, \ldots, e_{2n}\}$ is a standard basis of $E$. 

The dual basis of $T^nM$ to $\{\theta^{11}, \theta^{21}, \ldots, \theta^{3n}, \theta^{4n}\}$ is denoted by

$$E_{1p} = \frac{1}{\sqrt{2}}(e^{\theta}_{2p-1} \otimes h_1^{p} + e^{\theta}_{2p} \otimes h_2^{p}),$$

$$E_{2p} = \frac{\sqrt{-1}}{\sqrt{2}}(e^{\theta}_{2p-1} \otimes h_1^{p} - e^{\theta}_{2p} \otimes h_2^{p}),$$

$$E_{3p} = \frac{1}{\sqrt{2}}(e^{\theta}_{2p} \otimes h_1^{p} - e^{\theta}_{2p-1} \otimes h_2^{p}),$$

$$E_{4p} = \frac{\sqrt{-1}}{\sqrt{2}}(e^{\theta}_{2p} \otimes h_1^{p} + e^{\theta}_{2p-1} \otimes h_2^{p}),$$

for $p = 1, 2, \ldots, n$,

where $\{h_1^{p}, h_2^{p}\}$ is a standard basis of $\mathbb{H}^*$ which corresponds to $\{h_1, h_2\}$ under $\omega_{\mathbb{H}}$ and so on. If we choose a suitable basis $\{I, J, K\}$ of the quaternionic Kähler structure bundle $\mathcal{E}$, the above $\{E_{11}, E_{21}, \ldots, E_{3n}, E_{4n}\}$ satisfies that

$$E_{2p} = I E_{1p}, \quad E_{3p} = J E_{1p}, \quad E_{4p} = K E_{1p}, \quad \text{for } p = 1, 2, \ldots, n.$$ 

In general, $\mathbb{H}$ and $\mathbb{E}$ may not be well-defined over the whole of $M$. So, we fix $x \in M$ and take a neighborhood $U$ of $x$ on which the vector bundles $\mathbb{H}$ and $\mathbb{E}$ may be defined. The tangent space $T_xM$ has an almost complex structure parametrized by $\mathbb{H}_x \setminus 0$ in the following way:

$$A_x^{1,0} = \mathbb{E}_x \otimes C \subset T^*_x M_{\mathbb{C}} \quad \text{at } h \in \mathbb{H}_x \setminus 0. \tag{2.2}$$

The structure is clearly unchanged under non-zero scalar multiplication and hence, the complex projective bundle $P(\mathbb{H})$ is considered to parametrize those almost complex structure. Moreover, $P(\mathbb{H})$ always exists globally over $M$. The projective bundle $P(\mathbb{H})$ is called the twistor space of $M$ and denoted by $Z$. Hereafter, we use these notations without notice, where no confusion can arise. In order to show that $Z$ has a natural complex structure, we need the following theorem due to Salamon, which is one of the essential points used in this paper.

**Theorem 2.2.1.** [14] *Let $\Omega_{\mathbb{H}}$ be the curvature of the induced connection on $\mathbb{H}$. For an arbitrary standard basis $\{h_1 = h, h_2 = \bar{h}\}$, the curvature 2-forms are given by

$$\Omega_i^j = -c th_i \vee \bar{h}_j \in \Gamma(S^2 \mathbb{H}),$$

where

$$\Omega_{\mathbb{H}}(h_i) = \sum_{j=1}^{2} h_j \otimes \Omega_i^j,$$

$c > 0$ is a constant depending only on $n$, and $t$ is the scalar curvature.*
Theorem 2.2.2. [14] The twistor space $Z$ of $M$ has a natural complex structure.

In order to use notations in the proof of this theorem due to Salamon, we review the outline of it briefly. Let $\pi_Z : Z \to M$ and $\Pi : \mathbb{H}\setminus 0 \to U$ be the projections. Pick $h \in \mathbb{H}_x \setminus 0 = \Pi^{-1}(x)$. The connection on $\mathbb{H}$ induced by the Riemannian connection on $M$ gives a decomposition

$$T^*_h(\mathbb{H}\setminus 0) \cong T^*_h(\mathbb{H}_x \setminus 0) \oplus \Pi^*T^*_xM.$$  

The cotangent space $T^*_h(\mathbb{H}_x \setminus 0)$ to the fibre inherits a natural almost complex structure from the complex structure of $\mathbb{H}_x \cong \mathbb{C}^2$. On the other hand, the cotangent space $\Pi^*T^*_xM \cong T^*_xM$ to $M$ has an almost complex structure determined by $[h] \in P(\mathbb{H})$ as in (2.2). Thus we get an almost complex structure on $\mathbb{H}\setminus 0$ and shall show that the integrability condition for this structure is satisfied.

Let $\sigma^q_i$, $\omega^j_i$ denote the 1-forms of the induced connections on $\mathbb{E}$, $\mathbb{H}$ with respect to standard bases;

$$\nabla e_p = \sum_{q=1}^{2n} e_q \otimes \sigma^q_p, \quad \nabla h_i = \sum_{j=1}^2 h_j \otimes \omega^j_i.$$  

If coordinates are given by $z^1, z^2$ on the total space of $\mathbb{H}$, an arbitrary $k \in \mathbb{H}$ may be written $\sum_{i=1}^2 k = z^i h_i$. Define the 1-forms by

$$\theta^i = dz^i + \sum_{j=1}^2 z^j \omega^j_i, \quad i = 1, 2.$$  

Since $z^i$ are holomorphic functions on each fibre $\mathbb{H}_x$, $\{\theta^i\}$ spans the distribution of $(1,0)$-forms corresponding to the space $T^*_h(\mathbb{H}_x \setminus 0)$. Put

$$\eta_p = \sum_{i=1}^2 z^i \Pi^*(e_p \otimes h_i).$$  

We see that $\{\eta_p\}$ spans the distribution of $(1,0)$-forms corresponding to the space $\Pi^*T^*_xM$. By a direct computation and Theorem 2.2.1, the distribution $D = \text{span}\{\theta^i, \eta_p\}$ is involutive. The structure is invariant under any non-zero scalar multiplication and passes to one on $P(\mathbb{H}) = Z$. In the future, we shall omit $\pi^*, \Pi^*$, where no confusion can arise.

From the definition of the complex structure of $Z$, each fibre $\pi^{-1}(x) \cong \mathbb{P}^1$ is a complex submanifold. We may define the tautological line bundle $O(-1)$ locally over $\pi^{-1}(U)$, which is holomorphic. However, $O(-1)^{\otimes 2} = O(-2)$ is a well-defined holomorphic line bundle over $Z$. We denote by $\Omega^1$ the bundle of holomorphic 1-forms on $Z$. Then, we get the following exact sequence of holomorphic vector bundles on $Z$ [14]:

$$0 \to O(-2) \to \Omega^1 \to O(-1) \otimes \pi^*E \to 0.$$
By Sp(1)-invariance, the mapping
\[ k \mapsto \sum_{i=1}^{2} \omega_{\mathbb{H}}(h_{i}, k) \otimes \theta^{i}, \quad k \in \mathbb{H} \setminus 0, \]
defines a canonical \((1,0)\)-form \(\alpha\) on \(\mathbb{H} \setminus 0\). In terms of the coordinates \(z^{i}\),
\[ \alpha = z^{2} \theta^{1} - z^{1} \theta^{2}. \]
It is known that \(\alpha\) is a holomorphic 1-form on \(\mathbb{H} \setminus 0\) and pushed down to \(Z\) which is denoted by \(\beta\). Because \(\alpha\) is quadratic in \(z^{1}, z^{2}\), \(\beta\) is a nowhere-vanishing holomorphic section of \(\Omega^{1} \otimes O(2)\). Then, \(\beta\) determines an embedding \(O(-2) \hookrightarrow \Omega^{1}\).

From now on, we shall confine ourselves exclusively to a quaternionic Kähler manifold \(M\) with positive scalar curvature which is complete and connected. Then its twistor space \(Z\) admits a Kähler–Einstein metric of positive scalar curvature \([14]\). It is important that the induced metric on each fibre \(\pi^{-1}(x)\) (which is denoted by \(\mathbb{P}_{x}\)) is the same as the standard Fubini–Study metric on \(\mathbb{P}^{1}\) up to a constant scalar multiple. Consequently, \(\mathbb{P}_{x}\) is a Kähler submanifold of \(Z\). On the other hand, \(\pi : Z \rightarrow M\) is a Riemannian submersion.

### 2.3 Main result

We mention the main theorem after preparing definitions and notations for our proof. Let \(F\) be a \(c_{2}\)-self-dual bundle with a unitary structure on \(M\). The pull-back bundle of \(F\) on \(Z\) is denoted by \(\tilde{F}\). \(\tilde{F} \otimes O(-2)\) is abbreviated to \(\tilde{F}(-2)\) as usual. Since \(O(-1)\) has the standard metric inherited from \(\mathbb{H}\), \(\tilde{F}(-2)\) is given a Hermitian structure.

**Definition 2.3.1.** An element \(\omega\) of \(\Gamma(\Lambda^{0,1} \otimes \tilde{F}(-2))\) is called **fibre-harmonic** if the restriction \(\omega_{x}\) of \(\omega\) to an arbitrary fibre \(\mathbb{P}_{x}\) is harmonic for \(x \in M\). We denote by \(\Gamma_{h}(\Lambda^{0,1} \otimes \tilde{F}(-2))\) the space of fibre-harmonic elements of \(\Gamma(\Lambda^{0,1} \otimes \tilde{F}(-2))\).

**Remark.** This definition is a generalization of the original one due to Rawnsley \([13]\), but a fibre-harmonic form also applies to (an arbitrary vector bundle valued) \((1,0)\) or \((0,1)\)-form on \(Z\).

Using this notion, we define \(O(-2)\)-valued forms which are often treated in this paper. Recall that \(\beta\) is a nowhere-vanishing holomorphic \(O(2)\)-valued 1-form. If we restrict \(\beta\) to each fibre \(\mathbb{P}_{x}\) and apply Serre duality on \(\mathbb{P}_{x}\), then we have an \(O(-2)\)-valued \((0,1)\)-form, which is, after changing a constant factor, denoted by \(\sigma\). Since \(\beta\) is a fibre-harmonic form, \(\sigma\) is written explicitly, for which we need only a local expression. So, we pull back \(\sigma\) to \(\mathbb{H} \setminus 0\) and denote it by \(\tilde{\sigma}\). By a direct computation, \(\tilde{\sigma}\) is described in the following way:
\[ \tilde{\sigma} = \frac{1}{(|z^{1}|^{2} + |z^{2}|^{2})^{2}}(z^{2} \theta^{1} - z^{1} \theta^{2}). \]
We summarize properties of \(\sigma\) obtained easily from the definition.
Lemma 2.3.2. \( \sigma \) is a vertical, nowhere-vanishing fibre-harmonic form.

Next, we compute \( \bar{\partial} \sigma \) which is an \( \mathcal{O}(-2) \)-valued horizontal \((0,2)\)-form. Using \( \text{Sp}(1) \)-invariance, we get the pull-back form \( \bar{\partial} \sigma \) on \( \mathbb{H} \setminus 0 \):

\[
\bar{\partial} \sigma = \frac{2ct}{(|z_1|^2 + |z_2|^2)^2} \sum_{p=1}^{n} \eta_{2p} \wedge \eta_{2p-1}.
\] (2.3)

To avoid using terms such as the horizontal or vertical part of a form on \( Z \), we adopt the following notation.

\[
H^{(0,1)} = \{ \omega \in \Lambda^{(0,1)}_Z | \omega(X) = 0 \text{ for all } X \in T\mathbb{P}_x, \text{ and for all } x \in M \},
\]

and we denote by \( V^{(0,1)} \) the orthogonal complement of \( H^{(0,1)} \) with respect to the Kähler metric of \( Z \).

Here, we state the main theorem.

**Main Theorem.** Let \( M \) be a quaternionic Kähler manifold with positive scalar curvature and \( F \) be a \( c_2 \)-self-dual bundle on \( M \) with a unitary structure. If the twistor space of \( M \) is denoted by \( Z \) and the pull-back bundle of \( F \) on \( Z \) is denoted by \( \tilde{F} \), then we have

\[
H^1(\tilde{F}(-2)) = 0.
\]

3. Proof of Main Theorem

**Proposition 3.1.** For an arbitrary \( \omega \in \Gamma(\Lambda^{(0,1)}_Z \otimes \tilde{F}(-2)) \), we have the following decomposition:

\[
\omega = \psi + \bar{\partial}t,
\]

where

\[
\psi \in \Gamma_{\mathbb{H}}(\Lambda^{(0,1)}_Z \otimes \tilde{F}(-2)), \quad t \in \Gamma(\tilde{F}(-2)).
\]

**Proof.** If \( \omega \) is restricted to each fibre \( \mathbb{P}_x, \), \( \omega_x \) is \( \bar{\partial} \)-closed. So, the Hodge–Kodaira theory implies that there are a unique harmonic \( \tilde{F}(-2)|_{\mathbb{P}_x} \)-valued \((0,1)\)-form \( \omega_1 \) and a section \( t_x \) of \( \Gamma(\tilde{F}(-2)|_{\mathbb{P}_x}) \) satisfying that

\[
\omega_x = \omega_1 + \bar{\partial}t_x.
\]

Since \( \tilde{F}|_{\mathbb{P}_x} \) is holomorphically trivial, Bott formula gives

\[
H^0(\mathbb{P}_x, \tilde{F}(-2)) \cong H^0(\mathbb{P}_x, \mathcal{O}(-2)) \otimes F_x = 0.
\]

Consequently, \( t_x \) is also unique and by elliptic regularity, we obtain a smooth section \( t \) of \( \tilde{F}(-2) \) which, restricted to each fibre, corresponds to \( t_x \). It follows from \( (\bar{\partial}t)_x = 0 \) that

\[
\Gamma(\Lambda^{(0,1)}_Z \otimes \tilde{F}(-2)) = \Gamma_{\mathbb{H}}(\Lambda^{(0,1)}_Z \otimes \tilde{F}(-2)) + \partial \Gamma(\tilde{F}(-2)).
\]
Next, we show the uniqueness of this decomposition and so assume that

$$0 = \psi + \bar{\partial}t,$$

where $\psi \in \Gamma_{\text{fr}}(\Lambda^r_2(0,1) \otimes \tilde{F}(-2))$ and $t \in \Gamma(\tilde{F}(-2))$. Restricting this to each fibre, we get

$$\psi_x = -\bar{\partial}_x t_x.$$

Thus $\psi_x$ is a harmonic and exact form and hence it must vanish. Then $t_x$ is a holomorphic section of $O(-2) \otimes F_x$, but we use Bott formula again to have $t_x = 0$ and $t = 0$. By the above assumption, $\psi = 0$. □

We give more detailed decomposition theorem suitable for our purpose.

**Proposition 3.2.** The fibre-harmonic form $\omega$ in $\Gamma_{\text{fr}}(\Lambda^r_2(0,1) \otimes \tilde{F}(-2))$ can be decomposed into the following:

$$\omega = \omega_h + \pi^* s \otimes \sigma,$$

where $\omega_h \in \Gamma(H^{(0,1)} \otimes \tilde{F}(-2))$ and $s \in \Gamma(F)$.

**Proof.** First, we divide $\omega$ into the horizontal part $\omega_h \in \Gamma(H^{(0,1)} \otimes \tilde{F}(-2))$ and the vertical part $\omega_v \in \Gamma(V^{(0,1)} \otimes \tilde{F}(-2))$:

$$\omega = \omega_h + \omega_v.$$

The existence of non-vanishing section $\sigma$ on the line bundle $V^{(0,1)} \otimes O(-2)$ implies that $\omega_v$ must be written

$$\omega_v = t \otimes \sigma,$$

with $t$ a section of $\tilde{F}$. If we pull back $\omega$ to an arbitrary fibre, we have

$$\omega_x = t_x \otimes \sigma_x.$$

Since $\omega$ and $\sigma$ are fibre-harmonic and $\tilde{F}|_{F_x}$ is holomorphically trivial, $t_x$ is a constant section of $\tilde{F}|_{F_x}$. Therefore, $t$ can be pushed down to $M$, or alternatively there exists a section $s$ of $F$ such that $t = \pi^* s$. □

Combined this with Proposition 3.1, for an arbitrary $\tilde{F}(-2)$-valued $(0,1)$-form $\omega$, we have the unique decomposition as follows:

$$\omega = \omega_h + \pi^* s \otimes \sigma + \bar{\partial}t.$$  (3.1)

We thus have a transform

$$T : H^1(Z, \tilde{F}(-2)) \rightarrow \Gamma(F),$$

under the Dolbeault isomorphism $H^1(Z, \tilde{F}(-2)) \cong H^{0,1}(Z, \tilde{F}(-2))$,

$$T : \omega \rightarrow s.$$
Proposition 3.3. This transformation $T : H^1(Z, \bar{F}(-2)) \rightarrow \Gamma(F)$ is injective.

Proof. If we suppose that

$$T(\omega) = 0, \quad \omega \in H^{0,1}(Z, \bar{F}(-2)),$$

then, from the decomposition (3.1), $\omega$ is written in the form:

$$\omega = \omega_h + \bar{\partial}t.$$

Since $\omega$ is $\bar{\partial}$-closed, so is $\omega_h$. Proposition 3.3 follows from the next lemma.

Lemma 3.4. If $\omega_h \in \Gamma(H^{(0,1)} \otimes \bar{F}(-2))$ satisfies $\bar{\partial}\omega_h = 0$, $\omega_h$ itself is zero as a 1-form.

Proof. We pull back this form to $\hat{\omega}_h$ on $\mathbb{H}\{0$ and express it locally in terms of a basis of $(0,1)$-forms. Since $\hat{\omega}_h$ is restricted to zero on each fibre,

$$\hat{\omega}_h = \sum_{p=1}^{2n} s_p \otimes \{e_p \otimes (z^1 h_2 - z^2 h_1)\}, \quad s_p \in \Gamma(\bar{F} \otimes \mathcal{O}(-2)).$$

On the other hand $\bar{\partial}\hat{\omega}_h = 0$, so

$$0 = \sum_p \bar{\partial}s_p \wedge \{e_p \otimes (z^1 h_2 - z^2 h_1)\} + s_p \otimes \bar{\partial}\{e_p \otimes (z^1 h_2 - z^2 h_1)\}.$$

We are free to assume that $\nabla e_{p|x} = 0, \ x \in M$, then by the $\text{Sp}(1)$-invariance,

$$d\{e_p \otimes (z^1 h_2 - z^2 h_1)\}$$

$$= (dz^1 + \bar{z}^1 \omega_2^1 - \bar{z}^2 \omega_2^1) \wedge e_p \otimes h_2 - (dz^2 + \bar{z}^2 \omega_1^1 + z^2 \omega_1^1) \wedge e_p \otimes h_1$$

$$= \bar{\theta}^1 \wedge e_p \otimes h_2 - \bar{\theta}^2 \wedge e_p \otimes h_1.$$  \hspace{1cm} (3.2)

To find $(0,1)$-part of this form, recall the definition of $(1,0)$-form $\eta_p$ and the real structure of $\mathbb{E} \otimes \mathbb{H}$. Explicitly,

$$\eta_{2p-1} = e_{2p} \otimes (z^1 h_2 - \bar{z}^2 h_1), \quad \eta_{2p} = -e_{2p-1} \otimes (z^1 h_2 - \bar{z}^2 h_1).$$

Thus we have

$$\bar{\partial}\eta_p = \frac{\bar{\partial}\eta_{2p-1} \wedge \eta_{2p}}{1 + |z^1|^2}.$$  \hspace{1cm} (3.2)

Consequently, from (3.2)

$$0 = \sum_p \bar{\partial}s_p \wedge \eta_p + s_p \otimes \frac{1}{1 + |z^1|^2} \{z^1 \bar{\theta}^1 + z^2 \bar{\theta}^2\} \wedge \eta_p,$$

and, for each $p$,

$$\bar{\partial}s_p + s_p \otimes \frac{1}{1 + |z^1|^2} (z^1 \bar{\theta}^1 + z^2 \bar{\theta}^2) = 0,$$
on an arbitrary fibre, hence

\[ \bar{\partial}\{(|z^1|^2 + |z^2|^2)s_p\} = 0, \quad \text{for each } p. \]

Since \( \tilde{\omega}_h \) is pulled back from \( Z \cong P(\mathbb{H}) \) and \( \eta_p \) is linear in \( z \) and furthermore \( \bar{F} \) is holomorphically trivial on each fibre, each \( s_p \) may be regarded as a vector valued function of \( k = z^1h_1 + z^2h_2 \in (\mathbb{H}\setminus 0)_z \) satisfying the homogeneity equation

\[ s_p(\lambda z^1, \lambda z^2) = \lambda^{-2} s_p(z^1, z^2). \]

Thus \( (|z^1|^2 + |z^2|^2)s_p \) is homogeneous of degree \(-1\) and holomorphic, so that it must vanish. Hence \( \tilde{\omega}_h = 0 \) and \( \omega_h = 0. \quad \square \)

We will show that \( T(\omega) \) satisfies some differential equations. To do so, we will find a constraint on \( s \) imposed by

\[ \bar{\partial}\omega = \bar{\partial}(\omega_h + \pi^*s \otimes \sigma) = 0. \]

We now pull back \( \omega \) to \( \mathbb{H}\setminus 0 \) and express it locally in terms of a basis of \((0, 1)\)-forms. Since \( \omega_h \) is a horizontal form,

\[ \omega = \sum s_p \otimes \bar{\eta}_p + \Pi^*s \otimes \bar{\sigma}, \]

where, \( s_p \in \Gamma(\mathfrak{F} \otimes \mathcal{O}(-2)) \). We may choose \( \{e_p\} \) such that, at a fixed point \( x \in M \), \( \nabla e_p = 0 \) and so, from now on, we fix such a point and such a basis of \( \mathfrak{E} \). Applying \( \bar{\partial} \) to both sides, we obtain

\[ \bar{\partial}\omega = \sum \left( \bar{\partial}s_p \wedge \eta_p + s_p \otimes \frac{1}{|z^1|^2 + |z^2|^2}(z^1d\bar{\theta}^1 + z^2d\bar{\theta}^2) \wedge \bar{\eta}_p \right) \]

\[ + \bar{\partial}(\Pi^*s) \wedge \bar{\sigma} + \Pi^*s \otimes \bar{\partial}\bar{\sigma}. \quad (3.3) \]

To compute \( \nabla(\Pi^*s) \), we pull back \( \theta^p \), \( i = 1, \ldots, 4, p = 1, \ldots, n \) and get

\[ \Pi^*\theta^{1p} = \frac{1}{\sqrt{2}} \frac{1}{|z^1|^2 + |z^2|^2}(z^1\eta_{2p-1} + z^2\bar{\eta}_{2p} + z^2\eta_{2p} + z^1\bar{\eta}_{2p-1}), \]

\[ \Pi^*\theta^{2p} = \frac{\sqrt{-1}}{\sqrt{2}} \frac{1}{|z^1|^2 + |z^2|^2}(z^1\eta_{2p-1} + z^2\bar{\eta}_{2p} - z^2\eta_{2p} - z^1\bar{\eta}_{2p-1}), \]

\[ \Pi^*\theta^{3p} = \frac{1}{\sqrt{2}} \frac{1}{|z^1|^2 + |z^2|^2}(-z^2\eta_{2p-1} + z^1\bar{\eta}_{2p} + z^1\eta_{2p} - z^2\bar{\eta}_{2p-1}), \]

\[ \Pi^*\theta^{4p} = \frac{\sqrt{-1}}{\sqrt{2}} \frac{1}{|z^1|^2 + |z^2|^2}(z^2\eta_{2p-1} - z^1\bar{\eta}_{2p} + z^1\eta_{2p} - z^2\bar{\eta}_{2p-1}). \]

From the definition of the pull-back connection, we have

\[ \bar{\partial}(\Pi^*s) = \frac{1}{\sqrt{2}} \frac{1}{|z^1|^2 + |z^2|^2} \sum \left( (z^1I_p - z^2J_p) \otimes \bar{\eta}_{2p-1} + (z^2K_p + z^1L_p) \otimes \bar{\eta}_{2p} \right) \]
Cohomology groups of $c_2$-self-dual bundles

where, for an abbreviation, we adopt

\[ I_p = \Pi^* \nabla_{E_1} s - \sqrt{-1} \Pi^* \nabla_{E_2} s, \quad J_p = \Pi^* \nabla_{E_3} s + \sqrt{-1} \Pi^* \nabla_{E_4} s, \]

\[ K_p = \Pi^* \nabla_{E_1} s + \sqrt{-1} \Pi^* \nabla_{E_2} s, \quad L_p = \Pi^* \nabla_{E_3} s - \sqrt{-1} \Pi^* \nabla_{E_4} s. \]

Taking account of Proposition 3.5, we focus on the component of $\tilde{\omega}$ which has its value in $\Gamma(H^{0,1} \otimes V^{0,1} \otimes \tilde{F}(-2))$:

\[
\sum (\partial s_p \wedge \eta_p + s_p \otimes \frac{1}{|z|^2 + |z^2|^2} (z^1 \bar{\theta}^1 + z^2 \bar{\theta}^2) \wedge \eta_p) + \bar{\partial}(\Pi^* s) \wedge \bar{\sigma}
\]

\[
= \frac{1}{(|z|^2 + |z^2|^2)^3} \times \sum \left[ \left( \frac{z^2}{\sqrt{2}} (z^1 I_p - z^2 J_p) + s_{2p-1} z^1 (|z|^2 + |z^2|^2)^2 \theta^1 \wedge \eta_{2p-1} \right)
+ \left( \frac{\bar{z}^2}{\sqrt{2}} (z^1 I_p + z^2 J_p) + s_{2p-1} z^2 (|z|^2 + |z^2|^2)^2 \theta^2 \wedge \eta_{2p-1} \right)
+ \left( \frac{z^2}{\sqrt{2}} (z^2 K_p + z^1 L_p) + s_{2p} z^1 (|z|^2 + |z^2|^2)^2 \theta^1 \wedge \eta_{2p} \right)
+ \left( \frac{\bar{z}^2}{\sqrt{2}} (z^2 K_p + z^1 L_p) + s_{2p} z^2 (|z|^2 + |z^2|^2)^2 \theta^2 \wedge \eta_{2p} \right)
+ (\partial s_{2p-1} \wedge \eta_{2p-1} + \bar{\partial} s_{2p} \wedge \eta_{2p}) \right].
\]

Since $\tilde{\omega}$ is pulled back from $Z$ and $\tilde{F}$ is holomorphically trivial on each fibre, we may regard $s_p$ as a vector valued function of $k = z^1 h_1 + z^2 h_2 \in (H \setminus 0)|_x$. The hypothesis on $\omega$ yields that the component of $\partial \omega$ in $\Gamma(H^{0,1} \otimes V^{0,1} \otimes \tilde{F}(-2))$ must vanish, so $s_p$ satisfies the following equations:

\[
\frac{\partial s_{2p-1}}{\partial z^1} = \frac{z^2}{\sqrt{2} (|z|^2 + |z^2|^2)^3} (z^1 I_p - z^2 J_p) - \frac{z^1}{|z|^2 + |z^2|^2} s_{2p-1}, \quad (3.4)
\]

\[
\frac{\partial s_{2p-1}}{\partial z^2} = \frac{-\bar{z}^2}{\sqrt{2} (|z|^2 + |z^2|^2)^3} (z^1 I_p - z^2 J_p) - \frac{z^2}{|z|^2 + |z^2|^2} s_{2p-1}, \quad (3.5)
\]

\[
\frac{\partial s_{2p}}{\partial z^1} = \frac{\bar{z}^2}{\sqrt{2} (|z|^2 + |z^2|^2)^3} (z^2 K_p + z^1 L_p) - \frac{z^2}{|z|^2 + |z^2|^2} s_{2p}, \quad (3.6)
\]

\[
\frac{\partial s_{2p}}{\partial z^2} = \frac{-\bar{z}^2}{\sqrt{2} (|z|^2 + |z^2|^2)^3} (z^2 K_p + z^1 L_p) - \frac{z^2}{|z|^2 + |z^2|^2} s_{2p}. \quad (3.7)
\]

From (3.4) and (3.5), we have

\[
z^2 \frac{\partial s_{2p-1}}{\partial z^1} - \bar{z}^1 \frac{\partial s_{2p-1}}{\partial z^2} = \frac{1}{\sqrt{2} (|z|^2 + |z^2|^2)^2} (z^1 I_p - z^2 J_p),
\]

\[
z^1 \frac{\partial s_{2p-1}}{\partial z^1} + \bar{z}^2 \frac{\partial s_{2p-1}}{\partial z^2} = -s_{2p-1}.
\]
Proposition 3.3 assures the existence and the uniqueness of the solution to these equations. In fact, considering that $s_p$ also satisfies the homogeneity equation as Lemma 3.4, we can solve these explicitly:

$$s_{2p-1} = \frac{1}{\sqrt{2}|z^1|^2 + |z^2|^2} \left(-\bar{z}^1 J_p - \bar{z}^2 L_p \right).$$

Applying the same computation to (3.6) and (3.7), we get

$$s_{2p} = \frac{1}{\sqrt{2}|z^1|^2 + |z^2|^2} \left(\bar{z}^1 K_p - \bar{z}^2 I_p \right).$$

Before we go to the next step, we may suppose without loss of generality that $\nabla h_1 = \nabla h_2 = 0$, at the fixed point $x \in M$. Then this, combined with the previous hypothesis that $\nabla e_p = 0$, at $x \in M$, implies that

$$\nabla_{E_p} E_{jq} = 0, \text{ for } i, j = 1, \ldots, 4 \text{ and } p, q = 1, \ldots, n, \text{ at } x \in M.$$ 

Since we no longer have necessity of taking the $H^{(0,1)} \otimes V^{(0,1)} \otimes \bar{F}(-2)$-component of $\bar{\omega}$ into account, substituting $s_p$ into (3.3), we obtain

$$\bar{\omega} = \Pi^* s \otimes \frac{2ct}{|z^1|^2 + |z^2|^2} \sum \eta_{2p} \wedge \bar{\eta}_{2p-1}$$

$$+ \frac{1}{\sqrt{2}|z^1|^2 + |z^2|^2} \left(-\bar{z}^1 \partial J_p \wedge \bar{\eta}_{2p-1} - \bar{z}^2 \partial I_p \wedge \bar{\eta}_{2p-1} \right. (3.8)$$

$$\left. + \bar{z}^1 \partial K_p \wedge \bar{\eta}_{2p} - \bar{z}^2 \partial L_p \wedge \bar{\eta}_{2p} \right),$$

where we used the equation (2.3) in Section 2, to get the first part of the right-hand-side of the above equation. Recalling the definition of $I_p, J_p, K_p, L_p$ and considering that these are the pull-back sections to $\mathbb{H}\setminus 0$, we can apply the same way to compute $\bar{\partial}(\Pi^* s)$ to the second part of the right-hand-side of (3.8).

We pick up $\eta_{2p} \wedge \bar{\eta}_{2p-1}$-terms in $\bar{\omega}$ and ignore the other terms, which do not appear in 4-dimensional case. From $c_2$-self-duality and Lemma 2.1.5, the curvature form $R^\nabla$ of $\nabla$ satisfies that

$$R^\nabla(E_{1p}, E_{2q}) = -R^\nabla(E_{3p}, E_{4q}),$$

$$R^\nabla(E_{1p}, E_{3q}) = R^\nabla(E_{2p}, E_{4q}),$$

$$R^\nabla(E_{1p}, E_{4q}) = -R^\nabla(E_{2p}, E_{3q}),$$

for $p, q = 1, \ldots, n$, and

$$R^\nabla(E_{1p}, E_{1q}) = R^\nabla(E_{2p}, E_{2q}) = R^\nabla(E_{3p}, E_{3q}) = R^\nabla(E_{4p}, E_{4q}).$$
Making use of these equations and Ricci identity, we have

\[
\frac{1}{2(|z_1|^2 + |z_2|^2)^2} \sum_{p=1}^{n} \left[ -|z|^2 \Pi^* \left( \sum_{i=1}^{4} \nabla_E_{ip} \nabla_E_{ip} s \right) \right. \\
- \left. |z|^2 \Pi^* \left( \sum_{i=1}^{4} \nabla_E_{ip} \nabla_E_{ip} s \right) \right] \otimes \bar{\eta}_{2p} \wedge \bar{\eta}_{2p-1} \\
+ \Pi^* s \otimes \frac{2ct}{(|z_1|^2 + |z_2|^2)^2} \sum_{p=1}^{n} \bar{\eta}_{2p} \wedge \bar{\eta}_{2p-1} = 0,
\]

and so

\[
\frac{1}{2(|z_1|^2 + |z_2|^2)^2} \sum_{p=1}^{n} \Pi^* \left( - \sum_{i=1}^{4} \nabla_E_{ip} \nabla_E_{ip} s \right) + 4ct s \otimes \bar{\eta}_{2p} \wedge \bar{\eta}_{2p-1} = 0.
\]

Consequently, we obtain

\[
\sum_{i} (-\nabla_E_{ip} \nabla_E_{ip} s) + 4ct s = 0, \quad \text{for} \quad p = 1, \ldots, n.
\]

Summing up the both-sides with respect to \( p = 1, \ldots, n \), we get

\[
\sum_{p,i} (-\nabla_E_{ip} \nabla_E_{ip} s) + 4nct s = 0.
\]

The assumption that \( \nabla_E_{ip} E_{jq} = 0 \) at a fixed point \( x \) in \( M \) implies

\[
(\nabla^* \nabla + 4nct) s = 0,
\]

where \( \nabla^* \nabla \) is the Laplace operator.

Since \( \nabla^* \nabla \geq 0 \) as an operator and \( ct \) is positive from Theorem 2.2.1, the equation \( (\nabla^* \nabla + 4nct) s = 0 \) has only a trivial solution. The Main Theorem is thereby proved.

4. Linear field equations in a Yang–Mills background

We have given the above proof in a computational and coordinate-based fashion. However, the operator \( \nabla^* \nabla + 4nct \) has its origin in linear field equations. As mentioned in the introduction, we give the outline briefly. For the details of quaternionic invariant operators, see Baston [3].

Using the isomorphism \( TM_C \cong \mathbb{H}^* \otimes \mathbb{E}^* \cong \mathbb{H}^* \otimes \mathbb{E} \cong \text{Hom}(\mathbb{H}, \mathbb{E}) \), we define

\[
\gamma : TM_C \rightarrow \text{Hom}(\mathbb{H}, \mathbb{E}).
\]

Moreover, we denote by \( \gamma^*(E_{ip}) \) the adjoint operator of \( \gamma(E_{ip}) \in \text{Hom}(\mathbb{H}, \mathbb{E}) \) with respect to \( h_H \) and \( h_E \), i.e.

\[
h_H(\gamma^*(E_{ip}) e, h) = h_E(e, \gamma(E_{ip}) h),
\]

where \( \{E_{ip}\} \) is a basis of \( TM \) in the Section 2.2. Then, we have the following.
Lemma 4.1. It holds
(1) $\gamma^*(E_{ip})\gamma(E_{ip}) = \frac{1}{2} \text{Id}_\mathbb{H}$,
(2) $\gamma^*(E_{ip})\gamma(E_{jp}) + \gamma^*(E_{jp})\gamma(E_{ip}) = 0 \ (i \neq j)$,
(3) $\gamma^*(E_{ip})\gamma(E_{jq}) = 0 \ (p \neq q)$.

Making use of $\gamma$, we define

$$D : \Gamma(\mathbb{H}) \rightarrow \Gamma(\mathbb{E}),$$

$$D\phi = \sum_{p,i} \gamma(E_{ip})\nabla_{E_{ip}}\phi, \quad \phi \in \Gamma(\mathbb{H}).$$

This operator $D$ is called a quaternionic Dirac operator, because this corresponds to a Dirac operator up to a constant multiple in 4-dimensional case. The formal adjoint operator $D^* : \Gamma(\mathbb{E}) \rightarrow \Gamma(\mathbb{H})$ is expressed as

$$D^*\psi = -\sum_{p,i} \left(\nabla_{E_{ip}}\gamma^*\psi\right)(E_{ip}).$$

Lemma 4.1 implies that the Weitzenböck formula of these operators is

$$D^* D\phi = \frac{1}{2} \nabla^*\nabla \phi - \sum_p \sum_{i<j} \gamma^*(E_{ip})\gamma(E_{jp}) R_\mathbb{H}(E_{ip}, E_{jp}) \phi,$$

where $\phi \in \Gamma(\mathbb{H})$ and $R_\mathbb{H}$ is the curvature of $\mathbb{H}$. This combined with Theorem 2.2.1 implies that

$$D^* D\phi = \frac{1}{2} \nabla^*\nabla \phi + 3nct \phi.$$

Next, we define the twistor operator $\overline{D} : \Gamma(\mathbb{H}) \rightarrow \Gamma(\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{E})$ as follows:

$$\overline{D}\phi = \nabla \phi - \frac{1}{2} \sum \gamma^*(E_{ip}) D\phi \otimes \theta^{ip}.$$

The next proposition explains the reason why the operator $\overline{D}$ is called the twistor operator.

Proposition 4.2. $H^0(Z, \mathcal{O}(1))$ is isomorphic to the space of solutions of $\overline{D}\phi = 0$.

Proof. For an arbitrary $x \in M$, the restriction to $\mathbb{P}_x$ implies that

$$H^0(\mathbb{P}_x, \mathcal{O}(1)) \cong \mathbb{H}^* \cong \mathbb{H}_{0\mathbb{H}},$$

and thus gives a transformation

$$S : H^0(Z, \mathcal{O}(1)) \rightarrow \Gamma(\mathbb{H}).$$

Let $\{h_1, h_2\}$ be a standard basis of $\mathbb{H}$ with corresponding coordinate functions $z^1, z^2$ and suppose that $\nabla h_i|_x = 0$ for some fixed $x \in M$. For any $t \in H^0(Z, \mathcal{O}(1))$, $t$ is written as

$$t = \alpha z^1 + \beta z^2,$$
where \( \alpha, \beta \) are smooth functions on \( M \). By the definition of \( S \), we get
\[
S(t) = \alpha h_2 - \beta h_1.
\]
Assume that
\[
da(x) = e_1 \otimes h_1 + e_2 \otimes h_2, \quad \dbeta(x) = e_3 \otimes h_1 + e_4 \otimes h_2.
\]
As usual, we regard \( t \) as a function of \( \mathbb{H} \setminus 0 \). Then,
\[
dt = z^1 e_1 \otimes h_1 + z^1 e_2 \otimes h_2 + z^2 e_3 \otimes h_1 + z^2 e_4 \otimes h_2 + \alpha dz^1 + \beta dz^2.
\]
For an arbitrary \( k \in (\mathbb{H} \setminus 0)_x \), the complex structure of \( \mathbb{H} \setminus 0 \) yields that \( \theta^i = dz^i, \ i = 1, 2 \) at a fixed point \( x \in M \). Therefore, \( \bar{\partial} t_x = 0 \) is equivalent to
\[
e_1 = e_4, \quad e_2 = e_3 = 0.
\]
On the other hand, after a long computation, \( \bar{\partial} S(t) = 0 \) implies that
\[
\dbeta(E_{1p}) = -\sqrt{-1} \alpha(E_{2p}) = \dbeta(E_{3p}) = \sqrt{-1} \alpha(E_{4p}),
\]
\[
-\dbeta(E_{3p}) = \sqrt{-1} \alpha(E_{4p}) = \dbeta(E_{1p}) = \sqrt{-1} \alpha(E_{2p}).
\]
It is easy to see that these are equivalent to equations (4.1). □

The adjoint operator \( \bar{D}^* : \Gamma(\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{E}) \to \Gamma(\mathbb{H}) \) of the twistor operator \( D \) is expressed as
\[
\bar{D}^* \psi = \nabla^* \psi + \frac{1}{2} \sum \nabla_{E_{ip}} \left( \gamma^* \sum \gamma(E_{jq}) \psi(E_{jq}) \right)(E_{ip}),
\]
where \( \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{E} \cong \mathbb{H} \otimes T^* M \) is used. Then, the Weitzenböck formula is written as follows:
\[
\bar{D}^* \bar{D} \phi = \frac{3}{4} \nabla^* \nabla \phi + \frac{1}{2} \sum \sum \gamma^*(E_{ip}) \gamma(E_{jp}) R_{\mathbb{H}}(E_{ip}, E_{jp}) \phi
\]
\[
= \frac{3}{4} \nabla^* \nabla \phi - \frac{3}{2} \nabla^2 \phi.
\]
Suppose that \( \bar{D} \phi = 0 \) and \( D \psi = 0 \), then we can contract \( \phi \) and \( \psi \) using the metric \( h_{\mathbb{H}} \) and obtain a scalar function \( f = h_{\mathbb{H}}(\phi, \psi) \). Note that
\[
\sum h_{\mathbb{H}}(\nabla_{E_{ip}} \phi, \nabla_{E_{ip}} \psi) = \sum \frac{1}{2} h_{\mathbb{H}}(\gamma^*(E_{ip}) D \phi, \nabla_{E_{ip}} \psi)
\]
\[
= \frac{1}{2} \sum h_{\mathbb{E}}(D \phi, \gamma(E_{ip}) \nabla_{E_{ip}} \psi) = 0.
\]
Consequently, we have
\[
\nabla^2 f = h_{\mathbb{H}}(\nabla^2 \phi, \psi) + h_{\mathbb{H}}(\phi, \nabla^2 \psi),
\]
and hence
\[
\nabla^* \nabla f = h_{\mathbb{H}}(\nabla^* \nabla \phi, \psi) + h_{\mathbb{H}}(\phi, \nabla^* \nabla \psi).
\]
The Weitzenböck formulae yield that
\[(\nabla^* \nabla + 4nct)f = 0.\]
This is just the operator which is used in the proof of the Main Theorem.

By replacing the covariant derivative of \(\mathbb{H}\) by general ones, we extend a quaternionic Dirac operator to an operator in a Yang–Mills background:
\[D: \Gamma(\mathbb{H} \otimes F) \rightarrow \Gamma(\mathbb{E} \otimes F).\]
In particular, if \(F\) is a \(c_2\)-self-dual bundle, the Weitzenböck formula still holds, since it follows that
\[\sum_p \sum_{i<j} \gamma^*(E_{ip})\gamma(E_{jp})R_F(E_{ip}, E_{jp})\phi = 0, \quad \text{for any } \phi \in \Gamma(F).\]

Imitating a correspondence between \(H^1(Z, \tilde{F}(-2))\) and \(\Gamma(F)\), we consider
\[H^{0,1}(Z, \tilde{F}(-3)) \rightarrow H^{0,1}(\mathbb{P}_x, \mathcal{O}(-3)) \otimes F_x \quad \text{(restriction)}
\sim H^{1,0}(\mathbb{P}_x, \mathcal{O}(3))^* \otimes F_x \quad \text{(Serre duality)}
\cong H^0(\mathbb{P}_x, \mathcal{O}(1))^* \otimes F_x
\cong \mathbb{H}_x \otimes F_x,
\]
and thus we have a transformation
\[T_1 : H^{0,1}(Z, \tilde{F}(-3)) \rightarrow \Gamma(\mathbb{H} \otimes F).\]
Along the almost same line as the proof of the Main Theorem, we may show that
\[D(T_1(\omega)) = 0 \quad \text{for an arbitrary } \omega \in H^{0,1}(Z, \tilde{F}(-3)).\]

At the end, the natural product map
\[H^0(Z, \mathcal{O}(1)) \otimes H^{0,1}(Z, \tilde{F}(-3)) \rightarrow H^{0,1}(Z, \tilde{F}(-2))\]
gives the operator \(\nabla^* \nabla + 4nct\).

**Remark.** When the base space is \(\mathbb{H} P^n\), we have another proof of this vanishing theorem, using an algebraic geometrical technique and induction with respect to the dimension [8].

**References**


Cohomology groups of $c_2$-self-dual bundles