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On the structure of fixed-point sets of asymptotically regular semigroups

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ABSTRACT

We show that the set of fixed points of an asymptotically regular mapping acting on a convex and weakly compact subset of a Banach space is, in some cases, a Hölder continuous retract of its domain. Our results qualitatively complement the corresponding fixed point existence theorems and extend a few recent results of Górnicki [15–17]. We also characterize Bynum's coefficients and the Opial modulus in terms of nets.

1. Introduction

The notion of asymptotic regularity, introduced by Browder and Petryshyn in [1], has become a standing assumption in many results concerning fixed points of nonexpansive and more general mappings. Recall that a mapping $T : M \to M$ acting on a metric space (M, d) is said to be asymptotically regular if

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$$

for all $x \in M$. Ishikawa [2] proved that if C is a bounded closed convex subset of a Banach space X and $T : C \to C$ is nonexpansive, then the mapping $T_\lambda = (1-\lambda)I + \lambda T$ is asymptotically regular for each $\lambda \in (0, 1)$. Edelstein and O'Brien [3] showed independently that T_λ is uniformly asymptotically regular over $x \in C$, and Goebel and Kirk [4] proved that the convergence is even uniform with respect to all nonexpansive mappings from C into C. Other examples of asymptotically regular mappings are given by the result of Anzai and Ishikawa [5] (see also [6]): if T is an affine mapping acting on a bounded closed convex subset of a locally convex space X, then $T_\lambda = (1-\lambda)I + \lambda T$ is uniformly asymptotically regular.

In 1987, Lin [7] constructed a uniformly asymptotically regular Lipschitz mapping in ℓ_2 without fixed points which extended an earlier construction of Tingley [8]. Subsequently, Maluta et al. [9] proved that there exists a continuous fixed-point free asymptotically regular mapping defined on any bounded convex subset of a normed space which is not totally bounded (see also [10]). For the fixed-point existence theorems for asymptotically regular mappings we refer the reader to [11–13].

It was shown in [14] that the set of fixed points of a k-uniformly Lipschitzian mapping in a uniformly convex space is a retract of its domain if k is close to 1. In recent papers [15–17], Górnicki proved several results concerning the structure of fixed-point sets of asymptotically regular mappings in uniformly convex spaces. In this paper we continue this work and extend a few results of Górnicki in two aspects: we consider a more general class of spaces and prove that in some cases, the fixed-point set Fix T is not only a (continuous) retract but even a Hölder continuous retract of the domain. We present our results in a more general case of a one-parameter nonlinear semigroup. We also characterize Bynum's coefficients and the Opial modulus in terms of nets.

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2. Preliminaries

Let G be an unbounded subset of $[0, \infty)$ such that $t+s, t-s \in G$ for all $t, s \in G$ with $t > s$ (e.g., $G = [0, \infty)$ or $G = \mathbb{N}$). By a nonlinear semigroup on C we shall mean a one-parameter family of mappings $\mathcal{T} = \{T_t : t \in G\}$ from C into C such that $T_{t+s}x = T_t T_s x$ for all $t, s \in G$ and $x \in C$. In particular, we do not assume in this paper that $\{T_t : t \in G\}$ is strongly continuous. We use a symbol $|T|$ to denote the exact Lipschitz constant of a mapping $T : C \to C$, i.e.,

$$|T| = \inf\{k > 0 : \|Tx - Ty\| \le k\|x - y\| \text{ for all } x, y \in C\}.$$

If T is not Lipschitzian we define $|T| = \infty$.

A semigroup $\mathcal{T} = \{T_t : t \in G\}$ from C into C is said to be asymptotically regular if $\lim_t \|T_{t+h}x - T_t x\| = 0$ for every $x \in C$ and $h \in G$.

Assume now that C is convex and weakly compact and $\mathcal{T} = \{T_t : t \in G\}$ is a nonlinear semigroup on C such that $s(\mathcal{T}) = \liminf_t |T_t| < \infty$. Choose a sequence (t_n) of elements in G such that $\lim_{n\to\infty} t_n = \infty$ and $s(\mathcal{T}) = \lim_{n\to\infty} \left|T_{t_n}\right|$. By Tikhonov's theorem, there exists a pointwise weakly convergent subnet $(T_{t_{n_\alpha}})_{\alpha\in A}$ of (T_{t_n}). We denote it briefly by $(T_{t_\alpha})_{\alpha\in A}$. For every $x \in C$, define

$$Lx = w\text{-}\lim_\alpha T_{t_\alpha} x, \tag{1}$$

i.e., Lx is the weak limit of the net $(T_{t_\alpha}x)_{\alpha\in A}$. Notice that Lx belongs to C since C is convex and weakly compact. The weak lower semicontinuity of the norm implies

$$\|Lx - Ly\| \le \liminf_\alpha \|T_{t_\alpha}x - T_{t_\alpha}y\| \le \limsup_{n\to\infty} \|T_{t_n}x - T_{t_n}y\| \le s(\mathcal{T})\|x - y\|.$$

We formulate the above observation as a separate lemma.

Lemma 2.1. *Let C be a convex weakly compact subset of a Banach space X and let $\mathcal{T} = \{T_t : t \in G\}$ be a semigroup on C such that $s(\mathcal{T}) = \liminf_t |T_t| < \infty$. Then the mapping $L : C \to C$ defined by (1) is $s(\mathcal{T})$-Lipschitz.*

We end this section with the following variant of a well known result which is crucial for our work (see, e.g., [18, Proposition 1.10]).

Lemma 2.2. *Let (X, d) be a complete bounded metric space and let $L : X \to X$ be a k-Lipschitz mapping. Suppose there exist $0 < \gamma < 1$ and $c > 0$ such that $d(L^{n+1}x, L^n x) \le c\gamma^n$ for every $x \in X$. Then $Rx = \lim_{n\to\infty} L^n x$ is a Hölder continuous mapping.*

Proof. We may assume that $\operatorname{diam} X = 1$. Fix $x \ne y$ in X and notice that for any $n \in \mathbb{N}$,

$$d(Rx, Ry) \le d(Rx, L^n x) + d(L^n x, L^n y) + d(L^n y, Ry) \le 2c\frac{\gamma^n}{1-\gamma} + k^n d(x, y).$$

Take $\alpha < 1$ such that $k \le \gamma^{1-\alpha^{-1}}$ and put $\gamma^{n-r} = d(x, y)^\alpha$ for some $n \in \mathbb{N}$ and $0 < r \le 1$. Then $k^{n-1} \le (\gamma^{1-\alpha^{-1}})^{n-r}$ and hence

$$d(Rx, Ry) \le 2c\frac{\gamma^{n-r}}{1-\gamma} + k(\gamma^{n-r})^{1-\alpha^{-1}} d(x, y) = \left(\frac{2c}{1-\gamma} + k\right) d(x, y)^\alpha. \quad \square$$

3. Bynum's coefficients and the Opial modulus in terms of nets

From now on, C denotes a nonempty convex weakly compact subset of a Banach space X. Let $\mathcal{A}$ be a directed set, $(x_\alpha)_{\alpha\in\mathcal{A}}$ a bounded net in X, $y \in X$ and write

$$r(y, (x_\alpha)) = \limsup_\alpha \|x_\alpha - y\|,$$
$$r(C, (x_\alpha)) = \inf\{r(y, (x_\alpha)) : y \in C\},$$
$$A(C, (x_\alpha)) = \{y \in C : r(y, (x_\alpha)) = r(C, (x_\alpha))\}.$$

The number $r(C, (x_\alpha))$ and the set $A(C, (x_\alpha))$ are called, respectively, the asymptotic radius and the asymptotic center of $(x_\alpha)_{\alpha\in\mathcal{A}}$ relative to C. Notice that $A(C, (x_\alpha))$ is nonempty convex and weakly compact. Write

$$r_a(x_\alpha) = \inf\{\limsup_\alpha \|x_\alpha - y\| : y \in \overline{\operatorname{conv}}(\{x_\alpha : \alpha \in \mathcal{A}\})\}$$

and let

$$\operatorname{diam}_a(x_\alpha) = \inf_\alpha \sup_{\beta,\gamma\ge\alpha} \|x_\beta - x_\gamma\|$$

denote the asymptotic diameter of (x_α).

The normal structure coefficient $\mathrm{N}(X)$ of a Banach space X is defined by

$$\mathrm{N}(X) = \sup \{k : k\, r(K) \le \operatorname{diam} K \text{ for each bounded convex set } K \subset X\},$$

where $r(K) = \inf_{y\in K} \sup_{x\in K} \|x - y\|$ is the Chebyshev radius of K relative to itself. Assuming that X does not have the Schur property, the weakly convergent sequence coefficient (or Bynum's coefficient) is given by

$$\mathrm{WCS}(X) = \sup \left\{ k : k\, r_a(x_n) \le \operatorname{diam}_a(x_n) \text{ for each sequence } x_n \xrightarrow{w} 0 \right\},$$

where $x_n \xrightarrow{w} 0$ means that (x_n) is weakly null in X (see [19]). For Schur spaces, we define $\mathrm{WCS}(X) = 2$.

It was proved independently in [20–22] that

$$\mathrm{WCS}(X) = \sup \left\{ k : k \limsup_n \|x_n\| \le \operatorname{diam}_a(x_n) \text{ for each sequence } x_n \xrightarrow{w} 0 \right\} \tag{2}$$

and, in [23], that

$$\mathrm{WCS}(X) = \sup \left\{ k : k \limsup_n \|x_n\| \le D[(x_n)] \text{ for each sequence } x_n \xrightarrow{w} 0 \right\},$$

where $D[(x_n)] = \limsup_m \limsup_n \|x_n - x_m\|$.

Kaczor and Prus [24] initiated a systematic study of assumptions under which one can replace sequences by nets in a given condition. We follow the arguments from that paper and use the well known method of constructing basic sequences attributed to Mazur (see [25]). Let us first recall a variant of a classical lemma which can be proved in the same way as for sequences (see, e.g., [25, Lemma]).

Lemma 3.1. *Let $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ be a bounded net in X weakly converging to 0 such that $\inf_\alpha \|x_\alpha\| > 0$. Then for every $\varepsilon > 0$, $\alpha' \in \mathcal{A}$ and for every finite dimensional subspace E of X, there is $\alpha > \alpha'$ such that*

$$\|e + t x_\alpha\| \ge (1 - \varepsilon)\|e\|$$

for any $e \in E$ and every scalar t.

Recall that a sequence (x_n) is basic if and only if there exists a number $c > 0$ such that $\|\sum_{i=1}^q t_i x_i\| \le c\|\sum_{i=1}^p t_i x_i\|$ for any integers $p > q \ge 1$ and any sequence of scalars (t_i). In the proof of the next lemma, based on Mazur's technique, we follow in part the reasoning given in [24, Corollary 2.6]. Set $D[(x_\alpha)] = \limsup_\alpha \limsup_\beta \left\|x_\alpha - x_\beta\right\|$.

Lemma 3.2. *Let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be a bounded net in X which converges to 0 weakly but not in norm. Then there exists an increasing sequence (α_n) of elements of $\mathcal{A}$ such that $\lim_n \|x_{\alpha_n}\| = \limsup_\alpha \|x_\alpha\|$, $\operatorname{diam}_a(x_{\alpha_n}) \le D[(x_\alpha)]$ and (x_{α_n}) is a basic sequence.*

Proof. Since $(x_\alpha)_{\alpha\in\mathcal{A}}$ does not converge strongly to 0 and $D[(x_{\alpha_s})] \le D[(x_\alpha)]$ for any subnet $(x_{\alpha_s})_{s\in\mathcal{B}}$ of $(x_\alpha)_{\alpha\in\mathcal{A}}$, we can assume, passing to a subnet, that $\inf_\alpha \|x_\alpha\| > 0$ and the limit $c = \lim_\alpha \|x_\alpha\|$ exists. Write $d = D[(x_\alpha)]$. Let (ε_n) be a sequence of reals from the interval $(0, 1)$ such that $\Pi_{n=1}^\infty (1 - \varepsilon_n) > 0$. We shall define the following sequences (α_n) and (β_n) by induction.

Let us put $\alpha_1 < \beta_1 \in \mathcal{A}$ such that $\left|\|x_{\alpha_1}\| - c\right| < 1$ and $\sup_{\beta\ge\beta_1} \|x_{\alpha_1} - x_\beta\| < d + 1$. By the definitions of c and d, there exists $\alpha' > \beta_1$ such that $|\|x_\alpha\| - c| < \frac{1}{2}$ and $\inf_{\beta'} \sup_{\beta\ge\beta'} \|x_\alpha - x_\beta\| < d + \frac{1}{2}$ for every $\alpha \ge \alpha'$. It follows from Lemma 3.1 that there exists $\alpha_2 > \alpha'$ such that

$$\|t_1 x_{\alpha_1} + t_2 x_{\alpha_2}\| \ge (1 - \varepsilon_2)\|t_1 x_{\alpha_1}\|$$

for any scalars t_1, t_2. Furthermore, $\left|\|x_{\alpha_2}\| - c\right| < \frac{1}{2}$, and we can find $\beta_2 > \alpha_2$ such that $\sup_{\beta\ge\beta_2} \|x_{\alpha_2} - x_\beta\| < d + \frac{1}{2}$.

Suppose now that we have chosen $\alpha_1 < \beta_1 < \cdots < \alpha_n < \beta_n$ $(n > 1)$ in such a way that $\left|\|x_{\alpha_k}\| - c\right| < \frac{1}{k}$, $\sup_{\beta\ge\beta_k} \|x_{\alpha_k} - x_\beta\| < d + \frac{1}{k}$ and

$$(1 - \varepsilon_k)\|t_1 x_{\alpha_1} + \cdots + t_{k-1} x_{\alpha_{k-1}}\| \le \|t_1 x_{\alpha_1} + \cdots + t_k x_{\alpha_k}\|$$

for any scalars $t_1, \ldots, t_k$, $k = 2, \ldots, n$. From the definitions of c and d, and by Lemma 3.1, we can find $\beta_{n+1} > \alpha_{n+1} > \beta_n$ such that $\left|\|x_{\alpha_{n+1}}\| - c\right| < \frac{1}{n+1}$, $\sup_{\beta\ge\beta_{n+1}} \|x_{\alpha_{n+1}} - x_\beta\| < d + \frac{1}{n+1}$ and (considering a subspace E spanned by the elements $x_{\alpha_1}, \ldots, x_{\alpha_n}$ and putting $e = t_1 x_{\alpha_1} + \cdots + t_n x_{\alpha_n}$),

$$(1 - \varepsilon_{n+1})\|t_1 x_{\alpha_1} + \cdots + t_n x_{\alpha_n}\| \le \|t_1 x_{\alpha_1} + \cdots + t_{n+1} x_{\alpha_{n+1}}\|$$

for any scalars $t_1, \ldots, t_{n+1}$.

Notice that the sequence (x_{α_n}) defined in this way satisfies $\lim_{n\to\infty} \|x_{\alpha_n}\| = c$ and $\operatorname{diam}_a(x_{\alpha_n}) \le d$. Furthermore,

$$\|t_1 x_{\alpha_1} + \cdots + t_p x_{\alpha_p}\| \ge \Pi_{n=q+1}^p (1 - \varepsilon_n)\|t_1 x_{\alpha_1} + \cdots + t_q x_{\alpha_q}\|$$

for any integers $p > q \ge 1$ and any sequence of scalars (t_i). Hence (x_{α_n}) is a basic sequence. $\square$

We are now in a position to give a characterization of the coefficient $\mathrm{WCS}(X)$ in terms of nets. The abbreviation "$\{x_\alpha\}$ is r.w.c." means that the set $\{x_\alpha : \alpha \in \mathcal{A}\}$ is relatively weakly compact.

Theorem 3.3. *Let X be a Banach space without the Schur property and write*

$$w_1 = \sup\left\{k : k\, r_a(x_\alpha) \le \mathrm{diam}_a(x_\alpha) \text{ for each net } x_\alpha \xrightarrow{w} 0,\ \{x_\alpha\} \text{ is r.w.c.}\right\},$$

$$w_2 = \sup\left\{k : k \limsup_\alpha \|x_\alpha\| \le \mathrm{diam}_a(x_\alpha) \text{ for each net } x_\alpha \xrightarrow{w} 0,\ \{x_\alpha\} \text{ is r.w.c.}\right\},$$

$$w_3 = \sup\left\{k : k \limsup_\alpha \|x_\alpha\| \le D[(x_\alpha)] \text{ for each net } x_\alpha \xrightarrow{w} 0,\ \{x_\alpha\} \text{ is r.w.c.}\right\}.$$

Then

$$\mathrm{WCS}(X) = w_1 = w_2 = w_3.$$

Proof. Fix $k > w_3$ and choose a weakly null net (x_α) such that the set $\{x_\alpha : \alpha \in \mathcal{A}\}$ is relatively weakly compact and $k \limsup_\alpha \|x_\alpha\| > D[(x_\alpha)]$. Then, by Lemma 3.2, there exists an increasing sequence (α_n) such that

$$k \lim_n \|x_{\alpha_n}\| > D[(x_\alpha)] \ge \mathrm{diam}_a(x_{\alpha_n})$$

and (x_{α_n}) is a basic sequence. Since the set $\{x_\alpha : \alpha \in \mathcal{A}\}$ is relatively weakly compact, we can assume (passing to a subsequence) that (x_{α_n}) is weakly convergent. Since it is a basic sequence, its weak limit equals zero. It follows from (2) that $\mathrm{WCS}(X) \le k$ and letting k go to w_3 we have

$$\mathrm{WCS}(X) \le w_3 \le w_2 \le w_1 \le \mathrm{WCS}(X). \quad \square$$

Notice that a similar characterization holds for the normal structure coefficient.

Theorem 3.4. *For a Banach space X,*

$$\mathrm{N}(X) = \sup\{k : k\, r_a(x_\alpha) \le \mathrm{diam}_a(x_\alpha) \text{ for each bounded net } (x_\alpha) \text{ in } X\}.$$

Proof. Let

$$N_1 = \sup\{k : k\, r_a(x_\alpha) \le \mathrm{diam}_a(x_\alpha) \text{ for each bounded net } (x_\alpha) \text{ in } X\}.$$

Set $k > N_1$ and choose a bounded net (x_α) such that $k\, r_a(x_\alpha) > \mathrm{diam}_a(x_\alpha)$. Fix $y \in \overline{\mathrm{conv}}(\{x_\alpha : \alpha \in \mathcal{A}\})$ and notice that $k \limsup_\alpha \|x_\alpha - y\| > \mathrm{diam}_a(x_\alpha)$. In a straightforward way, we can choose a sequence (α_n) such that

$$k \lim_n \|x_{\alpha_n} - y\| = k \limsup_\alpha \|x_\alpha - y\| > \mathrm{diam}_a(x_\alpha) \ge \mathrm{diam}_a(x_{\alpha_n}).$$

It follows from [19, Theorem 1] that $\mathrm{N}(X) \le k$ and letting k go to N_1 we have $\mathrm{N}(X) \le N_1$. By [26, Theorem 1], $\mathrm{N}(X) \ge N_1$ and the proof is complete. $\square$

In the next section we shall need a similar characterization for the Opial modulus of a Banach space X, defined for each $c \ge 0$ by

$$r_X(c) = \inf\left\{\liminf_{n\to\infty} \|x_n + x\| - 1\right\},$$

where the infimum is taken over all $x \in X$ with $\|x\| \ge c$ and all weakly null sequences (x_n) in X such that $\liminf_{n\to\infty} \|x_n\| \ge 1$ (see [27]). We first prove the following counterpart of Lemma 3.2.

Lemma 3.5. *Let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be a bounded net in X which converges to 0 weakly but not in norm and $x \in X$. Then there exists an increasing sequence (α_n) of elements of $\mathcal{A}$ such that $\lim_n \|x_{\alpha_n} + x\| = \liminf_\alpha \|x_\alpha + x\|$, $\lim_n \|x_{\alpha_n}\| \ge \liminf_\alpha \|x_\alpha\|$ and (x_{α_n}) is a basic sequence.*

Proof. Since $(x_\alpha)_{\alpha\in\mathcal{A}}$ does not converge strongly to 0 and

$$\liminf_s \|x_{\alpha_s}\| \ge \liminf_\alpha \|x_\alpha\|$$

for any subnet $(x_{\alpha_s})_{s\in\mathcal{B}}$ of $(x_\alpha)_{\alpha\in\mathcal{A}}$, it is sufficient (passing to a subnet) to consider only the case that $\inf_\alpha \|x_\alpha\| > 0$ and the limits $c_1 = \liminf_\alpha \|x_\alpha + x\|$, $c_2 = \liminf_\alpha \|x_\alpha\|$ exist. Let (ε_n) be a sequence of reals from the interval $(0, 1)$ such that $\Pi_{n=1}^{\infty}(1 - \varepsilon_n) > 0$. We shall define the sequence (α_n) by induction.

Let us put $\alpha_1 \in \mathcal{A}$ such that $\left|\|x_{\alpha_1} + x\| - c_1\right| < 1$ and $\left|\|x_{\alpha_1}\| - c_2\right| < 1$. By the definitions of c_1 and c_2, there exists $\alpha' > \alpha_1$ such that $|\|x_\alpha + x\| - c_1| < \frac{1}{2}$ and $|\|x_\alpha\| - c_2| < \frac{1}{2}$ for every $\alpha \geq \alpha'$. It follows from Lemma 3.1 that there exists $\alpha_2 > \alpha'$ such that

$$\|t_1 x_{\alpha_1} + t_2 x_{\alpha_2}\| \geq (1 - \varepsilon_2)\|t_1 x_{\alpha_1}\|$$

for any scalars t_1, t_2. We can now proceed analogously to the proof of Lemma 3.2 to obtain a basic sequence (x_{α_n}) with the desired properties. □

Theorem 3.6. *For a Banach space X without the Schur property and for $c \geq 0$,*

$$r_X(c) = \inf\left\{\liminf_{\alpha} \|x_\alpha + x\| - 1\right\},$$

where the infimum is taken over all $x \in X$ with $\|x\| \geq c$ and all weakly null nets (x_α) in X such that $\liminf_\alpha \|x_\alpha\| \geq 1$ and the set $\{x_\alpha : \alpha \in \mathcal{A}\}$ is relatively weakly compact.

Proof. Let $r_1(c) = \inf\{\liminf_\alpha \|x_\alpha + x\| - 1\}$, where the infimum is taken as above. Fix $c \geq 0$ and take $k > r_1(c)$. Then there exist $x \in X$ with $\|x\| \geq c$ and a weakly null net $(x_\alpha)_{\alpha\in\mathcal{A}}$ such that $\liminf_\alpha \|x_\alpha\| \geq 1,\ \{x_\alpha : \alpha \in \mathcal{A}\}$ is relatively weakly compact and

$$\liminf_{\alpha} \|x_\alpha + x\| - 1 < k.$$

By Lemma 3.5, there exists an increasing sequence (α_n) of elements of $\mathcal{A}$ such that $\lim_n \|x_{\alpha_n}\| \geq 1$, $\lim_n \|x_{\alpha_n} + x\| - 1 < k$ and (x_{α_n}) is a basic sequence. Since $\{x_\alpha : \alpha \in \mathcal{A}\}$ is relatively weakly compact, we can assume (passing to a subsequence) that (x_{α_n}) is weakly null. Hence $r_X(c) < k$ and since k is an arbitrary number greater than $r_1(c)$, it follows that $r_X(c) \leq r_1(c)$. The reverse inequality is obvious. □

4. Fixed-point sets as Hölder continuous retracts

The following lemma may be proved in a similar way to [28, Theorem 7.2].

Lemma 4.1. *Let C be a nonempty convex weakly compact subset of a Banach space X and $\mathcal{T} = \{T_t : t \in G\}$ an asymptotically regular semigroup on C such that $s(\mathcal{T}) = \lim_\alpha \left|T_{t_\alpha}\right|$ for a pointwise weakly convergent subnet $(T_{t_\alpha})_{\alpha\in A}$ of $(T_t)_{t\in G}$. Let $x_0 \in C$, $x_{m+1} = w\text{-}\lim_\alpha T_{t_\alpha} x_m,\ m = 0, 1, \ldots,$ and*

$$B_m = \limsup_{\alpha} \left\|T_{t_\alpha} x_m - x_{m+1}\right\|.$$

Assume that

(a) $s(\mathcal{T}) < \sqrt{\mathrm{WCS}(X)}$ *or,*
(b) $s(\mathcal{T}) < 1 + r_X(1)$.

Then, there exists $\gamma < 1$ such that $B_m \leq \gamma B_{m-1}$ for any $m = 1, 2, \ldots.$

Proof. It follows from the asymptotic regularity of $\{T_t : t \in G\}$ that

$$\limsup_{\alpha} \left\|T_{t_\alpha - l} x - y\right\| = \limsup_{\alpha} \left\|T_{t_\alpha} x - y\right\|$$

for any $l \in G$ and $x, y \in C$. Thus

$$\begin{aligned} D[(T_{t_\alpha} x_m)] &= \limsup_{\beta} \limsup_{\alpha} \left\|T_{t_\alpha} x_m - T_{t_\beta} x_m\right\| \\ &\leq \limsup_{\beta} \left|T_{t_\beta}\right| \limsup_{\alpha} \left\|T_{t_\alpha - t_\beta} x_m - x_m\right\| = s(\mathcal{T}) \limsup_{\alpha} \left\|T_{t_\alpha} x_m - x_m\right\|. \end{aligned}$$

Hence, from Theorem 3.3 and from the weak lower semicontinuity of the norm,

$$\begin{aligned} B_m &\leq \frac{D[(T_{t_\alpha} x_m)]}{\mathrm{WCS}(X)} \leq \frac{s(\mathcal{T})}{\mathrm{WCS}(X)} \limsup_{\alpha} \left\|T_{t_\alpha} x_m - x_m\right\| \\ &\leq \frac{s(\mathcal{T})}{\mathrm{WCS}(X)} \limsup_{\alpha} \liminf_{\beta} \left\|T_{t_\alpha} x_m - T_{t_\beta} x_{m-1}\right\| \\ &\leq \frac{s(\mathcal{T})}{\mathrm{WCS}(X)} \limsup_{\alpha} \left|T_{t_\alpha}\right| \limsup_{\beta} \left\|x_m - T_{t_\beta - t_\alpha} x_{m-1}\right\| = \frac{(s(\mathcal{T}))^2}{\mathrm{WCS}(X)} B_{m-1}. \end{aligned}$$

This gives (a). For (b), we can use Theorem 3.6 and proceed analogously to the proof of [28, Theorem 7.2] (see also [17, Theorem 5]). □

We are now in a position to prove a qualitative semigroup version of [28, Theorem 7.2 (a) (b)] which is in turn based on the results given in [11,12] (see also [29]). It also extends, in a few directions, [17, Theorem 5].

Theorem 4.2. *Let C be a nonempty convex weakly compact subset of a Banach space X and $\mathcal{T} = \{T_t : t \in G\}$ an asymptotically regular semigroup on C. Assume that*

(a) $s(\mathcal{T}) < \sqrt{\mathrm{WCS}(X)}$ *or,*
(b) $s(\mathcal{T}) < 1 + r_X(1)$.

Then $\mathcal{T}$ has a fixed point in C and $\mathrm{Fix}\,\mathcal{T} = \{x \in C : T_t x = x,\ t \in G\}$ is a Hölder continuous retract of C.

Proof. Choose a sequence (t_n) of elements in G such that $\lim_{n\to\infty} t_n = \infty$ and $s(\mathcal{T}) = \lim_{n\to\infty} |T_{t_n}|$. Let $(T_{t_{n_\alpha}})_{\alpha\in A}$ (denoted briefly by $(T_{t_\alpha})_{\alpha\in A}$) be a pointwise weakly convergent subnet of (T_{t_n}). Define, for every $x \in C$,

$$Lx = w\text{-}\lim_{\alpha} T_{t_\alpha} x.$$

Fix $x_0 \in C$ and put $x_{m+1} = Lx_m$, $m = 0, 1, \ldots$. Let

$$B_m = \limsup_{\alpha} \left\| T_{t_\alpha} x_m - x_{m+1} \right\|.$$

By Lemma 4.1, there exists $\gamma < 1$ such that $B_m \le \gamma B_{m-1}$ for any $m \ge 1$. Since the norm is weak lower semicontinuous and the semigroup is asymptotically regular,

$$\begin{aligned}\|L^{m+1}x_0 - L^m x_0\| &= \|x_{m+1} - x_m\| \le \liminf_{\alpha} \left\| T_{t_\alpha} x_m - x_m \right\| \\ &\le \liminf_{\alpha} \liminf_{\beta} \left\| T_{t_\alpha} x_m - T_{t_\beta} x_{m-1} \right\| \le \limsup_{\alpha} \left| T_{t_\alpha} \right| \limsup_{\beta} \left\| x_m - T_{t_\beta - t_\alpha} x_{m-1} \right\| \\ &= s(\mathcal{T}) B_{m-1} \le s(\mathcal{T}) \gamma^{m-1} \operatorname{diam} C\end{aligned}$$

for every $x_0 \in C$ and $m \ge 1$. Furthermore, by Lemma 2.1, the mapping $L : C \to C$ is $s(\mathcal{T})$-Lipschitz. It follows from Lemma 2.2 that $Rx = \lim_{n\to\infty} L^n x$ is a Hölder continuous mapping on C. We show that R is a retraction onto $\mathrm{Fix}\,\mathcal{T}$. It is clear that if $x \in \mathrm{Fix}\,\mathcal{T}$, then $Rx = x$. Furthermore, for every $x \in C$, $m \ge 1$ and $\alpha \in A$,

$$\|T_{t_\alpha} Rx - Rx\| \le \left\| T_{t_\alpha} Rx - T_{t_\alpha} L^m x \right\| + \left\| T_{t_\alpha} L^m x - L^{m+1} x \right\| + \left\| L^{m+1} x - Rx \right\|$$

and hence

$$\lim_{\alpha} \|T_{t_\alpha} Rx - Rx\| \le s(\mathcal{T}) \left\| Rx - L^m x \right\| + B_m + \left\| L^{m+1} x - Rx \right\|.$$

Letting m go to infinity, $\limsup_\alpha \|T_{t_\alpha} Rx - Rx\| = 0$. Since $s(\mathcal{T}) = \lim_\beta \left| T_{t_\beta} \right| < \infty$, there exists $\beta_0 \in A$ such that $\left| T_{t_\beta} \right| < \infty$ for every $\beta \ge \beta_0$. Then, the asymptotic regularity of $\mathcal{T}$ implies

$$\|T_{t_\beta} Rx - Rx\| \le \left| T_{t_\beta} \right| \limsup_{\alpha} \|Rx - T_{t_\alpha} Rx\| + \lim_{\alpha} \|T_{t_\beta + t_\alpha} Rx - T_{t_\alpha} Rx\| + \limsup_{\alpha} \|T_{t_\alpha} Rx - Rx\| = 0.$$

Hence $T_{t_\beta} Rx = Rx$ for every $\beta \ge \beta_0$ and, from the asymptotic regularity again,

$$\|T_t Rx - Rx\| = \lim_{\beta} \left\| T_{t + t_\beta} Rx - T_{t_\beta} Rx \right\| = 0$$

for each $t \in G$. Thus $Rx \in \mathrm{Fix}\,\mathcal{T}$ for every $x \in C$ and the proof is complete. □

It is well known that the Opial modulus of a Hilbert space H,

$$r_H(c) = \sqrt{1 + c^2} - 1,$$

and the Opial modulus of $\ell_p,\ p > 1$,

$$r_{\ell_p}(c) = (1 + c^p)^{1/p} - 1$$

for all $c \ge 0$ (see [27]). The following corollaries are sharpened versions of [15, Theorem 2.2] and [17, Corollary 8].

Corollary 4.3. *Let C be a nonempty bounded closed convex subset of a Hilbert space H. If $\mathcal{T} = \{T_t : t \in G\}$ is an asymptotically regular semigroup on C such that*

$$\liminf_{t} |T_t| < \sqrt{2},$$

then $\mathrm{Fix}\,\mathcal{T}$ *is a Hölder continuous retract of C.*

Corollary 4.4. *Let C be a nonempty bounded closed convex subset of ℓ_p, $1 < p < \infty$. If $\mathcal{T} = \{T_t : t \in G\}$ is an asymptotically regular semigroup on C such that*

$$\liminf_t |T_t| < 2^{1/p},$$

then $\operatorname{Fix}\mathcal{T}$ *is a Hölder continuous retract of C.*

Let $1 \leq p, q < \infty$. Recall that the Bynum space $\ell_{p,q}$ is the space ℓ_p endowed with the equivalent norm $\|x\|_{p,q} = (\|x^+\|_p^q + \|x^-\|_p^q)^{1/q}$, where x^+, x^- denote, respectively, the positive and the negative part of x. If $p > 1$, then

$$r_{\ell_{p,q}}(c) = \min\{(1+c^p)^{1/p} - 1, (1+c^q)^{1/q} - 1\}$$

for all $c \geq 0$ (see, e.g., [30]). The following corollary extends [17, Corollary 10].

Corollary 4.5. *Let C be a nonempty convex weakly compact subset of $\ell_{p,q}$, $1 < p < \infty$, $1 \leq q < \infty$. If $\mathcal{T} = \{T_t : t \in G\}$ is an asymptotically regular semigroup on C such that*

$$\liminf_t |T_t| < \min\{2^{1/p}, 2^{1/q}\},$$

then $\operatorname{Fix}\mathcal{T}$ *is a Hölder continuous retract of C.*

Let us now examine the case of p-uniformly convex spaces. Recall that a Banach space X is p-uniformly convex if $\inf_{\varepsilon>0} \delta(\varepsilon)\varepsilon^{-p} > 0$, where δ denotes the modulus of uniform convexity of X. If X is p-uniformly convex, then (see [31])

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda \|x\|^p + (1-\lambda)\|y\|^p - c_p W_p(\lambda)\|x-y\|^p \tag{3}$$

for some $c_p > 0$ and every $x, y \in X$, $0 \leq \lambda \leq 1$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$. A Banach space X satisfies the Opial property if

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for every sequence $x_n \xrightarrow{w} x$ and $y \neq x$.

The following theorem is an extension of [17, Theorem 7], and a partial extension of [16, Theorem 9].

Theorem 4.6. *Let C be a nonempty bounded closed convex subset of a p-uniformly convex Banach space X with the Opial property and $\mathcal{T} = \{T_t : t \in G\}$ an asymptotically regular semigroup on C such that*

$$\liminf_t |T_t| < \max\left\{(1+c_p)^{1/p}, \left(\frac{1}{2}\left(1 + (1 + 4c_p \operatorname{WCS}(X)^p)^{1/2}\right)\right)^{1/p}\right\}.$$

Then $\mathcal{T}$ has a fixed point in C and $\operatorname{Fix}\mathcal{T}$ *is a Hölder continuous retract of C.*

Proof. Choose a sequence (t_n) of elements in G, $\lim_{n\to\infty} t_n = \infty$, such that $s(\mathcal{T}) = \lim_{n\to\infty} |T_{t_n}|$ and let $(T_{t_\alpha})_{\alpha\in A}$ denotes a pointwise weakly convergent subnet of (T_{t_n}). Define, for every $x \in C$,

$$Lx = w\text{-}\lim_\alpha T_{t_\alpha} x.$$

Fix $x_0 \in C$ and put $x_{m+1} = Lx_m$, $m \geq 0$. Let $B_m = \limsup_\alpha \|T_{t_\alpha} x_m - x_{m+1}\|$. Since X satisfies the Opial property, it follows from [24, Proposition 2.9] that

$$\limsup_\alpha \|T_{t_\alpha} x_m - x_{m+1}\| < \limsup_\alpha \|T_{t_\alpha} x_m - y\|$$

for every $y \neq x_{m+1}$, i.e., x_{m+1} is the unique point in the asymptotic center $A(C, (T_{t_\alpha} x_m))$, $m \geq 0$. Applying (3) yields

$$\begin{aligned} &c_p W_p(\lambda)\|x_m - T_{t_\alpha} x_m\|^p + \|\lambda x_m + (1-\lambda)T_{t_\alpha} x_m - T_{t_\beta} x_{m-1}\|^p \\ &\quad \leq \lambda \|x_m - T_{t_\beta} x_{m-1}\|^p + (1-\lambda)\|T_{t_\alpha} x_m - T_{t_\beta} x_{m-1}\|^p \end{aligned}$$

for every $\alpha, \beta \in A$, $0 < \lambda < 1$, $m > 0$. Following [16, Theorem 9] (see also [32]) and using the asymptotic regularity of $\mathcal{T}$, we obtain

$$\limsup_\alpha \|T_{t_\alpha} x_m - x_m\|^p \leq \frac{s(\mathcal{T})^p - 1}{c_p}(B_{m-1})^p \tag{4}$$

for any $m > 0$. By Theorem 3.3 and the weak lower semicontinuity of the norm, we have

$$B_m \leq \frac{D[(T_{t_\alpha} x_m)]}{\operatorname{WCS}(X)} \leq \frac{s(\mathcal{T})}{\operatorname{WCS}(X)} \limsup_\alpha \|T_{t_\alpha} x_m - x_m\|. \tag{5}$$

Furthermore, by the Opial property,

$$B_m \leq \limsup_{\alpha} \left\| T_{t_\alpha} x_m - x_m \right\| . \tag{6}$$

Combining (4) with (5) and (6) we see that

$$(B_m)^p = \limsup_{\alpha} \left\| T_{t_\alpha} x_m - x_{m+1} \right\|^p \leq \gamma^p (B_{m-1})^p,$$

where

$$\gamma^p = \max\left\{ \frac{s(\mathcal{T})^p - 1}{c_p}, \frac{s(\mathcal{T})^p - 1}{c_p} \left(\frac{s(\mathcal{T})}{\mathrm{WCS}(X)} \right)^p \right\} < 1,$$

by assumption. Hence $B_m \leq \gamma B_{m-1}$ for every $m \geq 1$ and, proceeding in the same way as in the proof of Theorem 4.2, we conclude that Fix $\mathcal{T}$ is a nonempty Hölder continuous retract of C. $\square$

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