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Riccati inequality and other results for discrete symplectic systems

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Abstract

In this paper we establish several new results regarding the positivity and nonnegativity of discrete quadratic functionals $\mathcal{F}$ associated with discrete symplectic systems. In particular, we derive (i) the Riccati inequality for the positivity of $\mathcal{F}$ with separated endpoints, (ii) a characterization of the nonnegativity of $\mathcal{F}$ for the case of general (jointly varying) endpoints, and (iii) several perturbation-type inequalities regarding the nonnegativity of $\mathcal{F}$ with zero endpoints. Some of these results are new even for the special case of discrete Hamiltonian systems.

Keywords: Discrete symplectic system; Quadratic functional; Nonnegativity; Positivity; Riccati inequality; Riccati equation; Conjoined basis; Sturmian theorem

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1. Introduction and motivation

Recently, several papers appeared regarding the definiteness of the discrete quadratic functional

$$\mathcal{F}_0(x,u) := \sum_{k=0}^{N} \{x_k^T \mathcal{C}_k^T \mathcal{A}_k x_k + 2 x_k^T \mathcal{C}_k^T \mathcal{B}_k u_k + u_k^T \mathcal{D}_k^T \mathcal{B}_k u_k\},$$

see, for example, [6–8,10,12,15,16], where $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$ are $n \times n$ matrices and $x = \{x_k\}_{k=0}^{N+1}$, $u = \{u_k\}_{k=0}^{N}$ are sequences of n-vectors. The standing hypothesis about the coefficients is that the $2n \times 2n$ matrix $\mathcal{S}_k := \left(\begin{smallmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{smallmatrix}\right)$ is symplectic, that is, $\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}$ where $\mathcal{J} := \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ is the $2n \times 2n$ skew-symmetric matrix.

With the functional $\mathcal{F}_0$ we associate a linear system, called the *discrete symplectic system*,

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \qquad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k, \tag{S}$$

whose name is derived from the fact that its transition matrix is the symplectic matrix $\mathcal{S}_k$.

Discrete symplectic systems were introduced in [1] and they cover a large variety of linear difference equations, in particular discrete *Hamiltonian systems*. The latter are of the form

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \qquad \Delta u_k = C_k x_{k+1} - A_k^T u_k \tag{H}$$

with $A_k := I - \mathcal{A}_k^{-1}$ and symmetric $B_k := \mathcal{A}_k^{-1}\mathcal{B}_k$ and $C_k := \mathcal{C}_k \mathcal{A}_k^{-1}$, see, e.g., [1, Section 3.4]. That is, system (S) reduces to a Hamiltonian system (H) if (and only if) the matrix $\mathcal{A}_k$ is invertible.

The functional above arises as second variation in the discrete calculus of variations and control problems, so it is important to understand conditions characterizing its nonnegativity and positivity.

In this paper, we establish new results regarding the positivity and nonnegativity of certain discrete quadratic functionals $\mathcal{F}$ associated with $\mathcal{F}_0$ with *variable endpoints*. In particular, we solve an open problem pertaining the characterization of the *positivity* of $\mathcal{F}$ in terms of a discrete *Riccati inequality* (Section 3)—a result which was known only for the special case of discrete Hamiltonian systems, see [14, Section 4]. Furthermore, we derive a characterization of the *nonnegativity* of $\mathcal{F}$ with *jointly varying* endpoints (Section 4), thus extending the fixed and separable endpoints results in [7,8]. Finally, we establish new perturbation-type inequalities related to the nonnegativity of $\mathcal{F}_0$ when the initially zero endpoint x_0 becomes restricted to a subspace (Section 5). These inequalities are of the same fashion as inequalities known for the positivity of $\mathcal{F}_0$, where x_0 can be taken free. The results of the last two sections are new even for the special case of discrete Hamiltonian systems.

2. Prerequisities

2.1. Symplectic systems

The property that $\mathcal{S}_k$ (and hence $\mathcal{S}_k^T$, $\mathcal{S}_k^{-1}$, $\mathcal{S}_k^{T-1}$) is a symplectic matrix means that the coefficients satisfy

$$\begin{aligned} &\mathcal{A}_k^T \mathcal{D}_k - \mathcal{C}_k^T \mathcal{B}_k = \mathcal{A}_k \mathcal{D}_k^T - \mathcal{B}_k \mathcal{C}_k^T = I, \\ &\mathcal{A}_k \mathcal{B}_k^T,\ \mathcal{C}_k \mathcal{D}_k^T,\ \mathcal{C}_k^T \mathcal{A}_k,\ \mathcal{D}_k^T \mathcal{B}_k \ \text{ symmetric.} \end{aligned}$$

Solutions of (S) are uniquely determined by their values at one index k because any symplectic matrix is invertible.

A *conjoined basis* of (S) is a matrix solution (X, U) such that $X_k^T U_k$ is symmetric and $\operatorname{rank}\binom{X_k}{U_k} = n$ at some (and hence at any) index $k \in [0, N+1]$. The *principal solution* is the conjoined basis $(\hat{X}, \hat{U})$ of (S) starting with the initial values $\hat{X}_0 = 0$ and $\hat{U}_0 = I$. According to [6], a conjoined basis (X, U) of (S) has *no focal points in* $(m, m+1]$ if

$$\operatorname{Ker} X_{m+1} \subseteq \operatorname{Ker} X_m, \qquad P_m := X_m X_{m+1}^{\dagger} \mathcal{B}_m \geqslant 0, \tag{1}$$

where † stands for the Moore–Penrose generalized inverse of the given matrix. Two conjoined bases (X, U), $(\tilde{X}, \tilde{U})$ of (S) are *normalized* if their (constant) Wronskian matrix is the identity matrix, that is, $X_k^T \tilde{U}_k - U_k^T \tilde{X}_k = I$ for some (and hence for all) $k \in [0, N+1]$.

A pair (x, u) is *admissible* (for a quadratic functional) if $x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k$ for all $k \in [0, N]$. We will study the definiteness of quadratic functionals over such admissible pairs satisfying in addition certain *boundary conditions*. Namely, we will consider *separated* boundary conditions $\mathcal{M}_0 x_0 = 0$, $\mathcal{M}_1 x_{N+1} = 0$ with $n \times n$ projections $\mathcal{M}_0$, $\mathcal{M}_1$ and the associated (symmetric) $n \times n$ endpoints cost matrices Γ_0, Γ_1 satisfying $\Gamma_i = (I - \mathcal{M}_i)\Gamma_i(I - \mathcal{M}_i)$, $i = 0, 1$. In this context, the principal solution of (S) is replaced by the *natural conjoined basis* (X, U) of (S) which is given by the initial conditions $X_0 = I - \mathcal{M}_0$, $U_0 = \Gamma_0 + \mathcal{M}_0$. Note that $(X, U) = (\hat{X}, \hat{U})$ when the left endpoint is fixed, i.e., when $\mathcal{M}_0 = I$. Finally, we will deal with general *joint* boundary conditions $\mathcal{M}\binom{x_0}{x_{N+1}} = 0$, with $2n \times 2n$ projection $\mathcal{M}$, and the associated (symmetric) $2n \times 2n$ cost matrix Γ satisfying $\Gamma = (I - \mathcal{M})\Gamma(I - \mathcal{M})$.

We say that the functional $\mathcal{F}_0$ or $\mathcal{F}$ is *nonnegative* if it takes nonnegative values on all admissible pairs (x, u) satisfying the given boundary conditions, while $\mathcal{F}_0$ or $\mathcal{F}$ is *positive* (or *positive definite*) if it takes positive values on all such admissible pairs (x, u) with $x \not\equiv 0$. Considering the nonnegativity and positivity of $\mathcal{F}_0$ or $\mathcal{F}$, we will always assume that the corresponding pairs (x, u) are admissible without specifying this any further.

System (S) is called $(\mathcal{M}_0 : I)$*-normal on* $[0, N+1]$ if the only solution of the system $u_{k+1} = \mathcal{D}_k u_k$, $\mathcal{B}_k u_k = 0$, for $k \in [0, N]$, with $u_0 = \mathcal{M}_0 \gamma_0$ for some $\gamma_0 \in \mathbb{R}^n$, is the zero solution $u_k \equiv 0$ on $[0, N+1]$.

The *Riccati operator* and the explicit discrete *Riccati equation* associated with the system (S) is

$$R[Q]_k := Q_{k+1}(\mathcal{A}_k + \mathcal{B}_k Q_k) - (\mathcal{C}_k + \mathcal{D}_k Q_k) = 0. \tag{RE}$$

For any pair of normalized conjoined bases (X, U), $(\tilde{X}, \tilde{U})$ of (S) we define on $[0, N+1]$ the symmetric $n \times n$ matrix

$$Q_k := U_k X_k^{\dagger} + \big(U_k X_k^{\dagger} \tilde{X}_k - \tilde{U}_k\big)\big(I - X_k^{\dagger} X_k\big) U_k^T. \tag{2}$$

This matrix then satisfies the identity $Q_k X_k = U_k X_k^{\dagger} X_k$. Furthermore, for any symmetric matrix Q_k we set

$$\mathcal{P}_k := \mathcal{B}_k^T \mathcal{D}_k - \mathcal{B}_k^T Q_{k+1} \mathcal{B}_k.$$

For any conjoined basis (X, U) of (S) we define the $n \times n$ matrices

$$M_k := \big(I - X_{k+1} X_{k+1}^{\dagger}\big)\mathcal{B}_k, \qquad T_k := I - M_k^{\dagger} M_k. \tag{3}$$

It is known [16] that $M_k = 0$ if and only if the kernel condition $\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k$ holds, and that this kernel condition is not necessary for the nonnegativity of $\mathcal{F}$, see [8]. On the other hand, it is proven in the same paper and in [7] that the image condition $x_k \in \operatorname{Im} X_k$ should be used in the characterization of the nonnegativity of $\mathcal{F}$. The above matrices satisfy the identities

$$\begin{aligned} &X_{k+1}^T M_k = 0, \qquad M_k T_k = 0, \qquad M_k^\dagger X_{k+1} = 0, \\ &\mathcal{B}_k T_k = X_{k+1} X_{k+1}^\dagger \mathcal{B}_k T_k, \qquad T_k X_k = T_k X_k X_{k+1}^\dagger X_{k+1}. \end{aligned} \tag{4}$$

Moreover, $T_k P_k T_k = T_k \mathcal{P}_k T_k$ is always symmetric, see, e.g., [10, Section 2.5]. Observe that the last identity in (4) is new and, in view of the equivalence $\operatorname{Ker} V \subseteq \operatorname{Ker} W \Leftrightarrow W = W V^\dagger V$, see, e.g., [3, Lemma A.5], it is equivalent to $\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} T_k X_k$.

2.2. Roundabout and comparison theorems

Next we present main tools which are needed in order to prove the results of this paper. Consider the quadratic functional with *separable endpoints*

$$\mathcal{F}(x,u) := x_0^T \Gamma_0 x_0 + x_{N+1}^T \Gamma_1 x_{N+1} + \mathcal{F}_0(x,u).$$

The following result characterizes the positivity of $\mathcal{F}$ and can be found in [15, Theorems 6, 7].

Proposition 1 *(Roundabout theorem). The following statements are equivalent.*

(i) $\mathcal{F}(x,u) > 0$ *over* $\mathcal{M}_0 x_0 = 0$, $\mathcal{M}_1 x_{N+1} = 0$, *and* $x \not\equiv 0$.

(ii) *There exists a conjoined basis* (X,U) *of* (S) *with no focal points in* $(0, N+1]$ *such that* X_k *is invertible for all* $k \in [0, N+1]$ *and satisfying the final endpoint inequality*

$$X_{N+1}^T (\Gamma_1 X_{N+1} + U_{N+1}) > 0 \quad \text{on } \operatorname{Ker} \mathcal{M}_1 X_{N+1}, \tag{5}$$

and one of the initial endpoint constraints

$$(I - \mathcal{M}_0)(\Gamma_0 X_0 - U_0) = 0 \quad \text{if (S) is } (\mathcal{M}_0 : I)\text{-normal on } [0, N+1], \tag{6}$$

$$X_0^T (\Gamma_0 X_0 - U_0) > 0 \quad \text{on } \operatorname{Ker} \mathcal{M}_0 X_0. \tag{7}$$

(iii) *There exists a symmetric solution* Q_k *on* $[0, N+1]$ *of the Riccati equation* (RE) *with*

$$\mathcal{A}_k + \mathcal{B}_k Q_k \text{ invertible and } \mathcal{P}_k := (\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1} \mathcal{B}_k \geqslant 0 \quad \text{for all } k \in [0, N], \tag{8}$$

and satisfying the final endpoint inequality

$$\Gamma_1 + Q_{N+1} > 0 \quad \text{on } \operatorname{Ker} \mathcal{M}_1, \tag{9}$$

and one of the initial endpoint constraints

$$(I - \mathcal{M}_0) Q_0 - \Gamma_0 = 0 \quad \text{if (S) is } (\mathcal{M}_0 : I)\text{-normal on } [0, N+1], \tag{10}$$

$$\Gamma_0 - Q_0 > 0 \quad \text{on } \operatorname{Ker} \mathcal{M}_0. \tag{11}$$

Positivity of $\mathcal{F}$ above is characterized in [15, Theorem 5] also in terms of the natural conjoined basis of (S), conjugate intervals, or an implicit Riccati equation. However, these results are not needed in the present paper.

In the following comparison theorem we use the approach from [12] and [13, Section 3.2]. Let $\mathcal{E}_k$ be any symmetric $n \times n$ matrix satisfying $\mathcal{D}_k^T \mathcal{B}_k = \mathcal{B}_k^T \mathcal{E}_k \mathcal{B}_k$, for example, $\mathcal{E}_k = \mathcal{B}_k \mathcal{B}_k^\dagger \mathcal{D}_k \mathcal{B}_k^\dagger$.

Using the admissibility equation $\mathcal{B}_k u_k = x_{k+1} - \mathcal{A}_k x_k$, the quadratic functional $\mathcal{F}$ can be written in the form

$$\mathcal{F}(x,u) = \sum_{k=0}^{N} \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}^T \mathcal{G}_k \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix},$$

where $\mathcal{G}_k$ is the symmetric $2n \times 2n$ matrix

$$\mathcal{G}_k := \begin{pmatrix} \mathcal{A}_k^T \mathcal{E}_k \mathcal{A}_k - \mathcal{A}_k^T \mathcal{C}_k + \delta_{k,0} \Gamma_0 & \mathcal{C}_k^T - \mathcal{A}_k^T \mathcal{E}_k \\ \mathcal{C}_k - \mathcal{E}_k \mathcal{A}_k & \mathcal{E}_k + \delta_{k,N} \Gamma_1 \end{pmatrix}$$

with $\delta_{k,j}$ being the Kronecker delta function, that is, $\delta_{k,k} = 1$ and $\delta_{k,j} = 0$ for $k \neq j$.

Consider another symplectic system ($\underline{\mathrm{S}}$) with coefficients $\underline{\mathcal{A}}_k, \underline{\mathcal{B}}_k, \underline{\mathcal{C}}_k, \underline{\mathcal{D}}_k, \underline{\mathcal{S}}_k$ and another discrete quadratic functional $\underline{\mathcal{F}}$ with data $\underline{\Gamma}_0, \underline{\Gamma}_1, \underline{\mathcal{M}}_0, \underline{\mathcal{M}}_1, \underline{\mathcal{E}}_k, \underline{\mathcal{G}}_k$ satisfying the same assumptions as the coefficients and data of the system (S) and functional $\mathcal{F}$, respectively.

Proposition 2 *(Comparison theorem). Assume that* $\operatorname{Ker} \mathcal{M}_0 \subseteq \operatorname{Ker} \underline{\mathcal{M}}_0$, $\operatorname{Ker} \mathcal{M}_1 \subseteq \operatorname{Ker} \underline{\mathcal{M}}_1$, *and*

$$\mathcal{G}_k \geqslant \underline{\mathcal{G}}_k, \quad \operatorname{Im}(\mathcal{A}_k - \underline{\mathcal{A}}_k \quad \mathcal{B}_k) \subseteq \operatorname{Im} \underline{\mathcal{B}}_k \quad \textit{for all } k \in [0, N]. \tag{12}$$

Then the following implications hold.

(i) *If* $\underline{\mathcal{F}}(\underline{x}, \underline{u}) \geqslant 0$ *over* $\underline{\mathcal{M}}_0 \underline{x}_0 = 0$, $\underline{\mathcal{M}}_1 \underline{x}_{N+1} = 0$, *then also* $\mathcal{F}(x,u) \geqslant 0$ *over* $\mathcal{M}_0 x_0 = 0$, $\mathcal{M}_1 x_{N+1} = 0$.
(ii) *If* $\underline{\mathcal{F}}(\underline{x}, \underline{u}) > 0$ *over* $\underline{\mathcal{M}}_0 \underline{x}_0 = 0$, $\underline{\mathcal{M}}_1 \underline{x}_{N+1} = 0$, *and* $\underline{x} \not\equiv 0$, *then also* $\mathcal{F}(x,u) > 0$ *over* $\mathcal{M}_0 x_0 = 0$, $\mathcal{M}_1 x_{N+1} = 0$, *and* $x \not\equiv 0$.

The next result characterizes the nonnegativity of $\mathcal{F}$ with separable endpoints and can be found in [7, Theorem 2].

Proposition 3 *(Roundabout theorem). The functional* $\mathcal{F}(x,u) \geqslant 0$ *over* $\mathcal{M}_0 x_0 = 0$ *and* $\mathcal{M}_1 x_{N+1} = 0$ *if and only if the natural conjoined basis* (X, U) *of* (S) *satisfies the* P*-condition*

$$T_k P_k T_k \geqslant 0 \quad \textit{for all } k \in [0, N], \tag{13}$$

the image condition

$$x_k \in \operatorname{Im} X_k \quad \textit{for all } k \in [0, N+1],$$
$$\textit{for all admissible } (x,u) \textit{ with } \mathcal{M}_0 x_0 = 0,\ \mathcal{M}_1 x_{N+1} = 0,$$

and the final endpoint inequality

$$X_{N+1}^T (\Gamma_1 X_{N+1} + U_{N+1}) \geqslant 0 \quad \textit{on } \operatorname{Ker} \mathcal{M}_1 X_{N+1}. \tag{14}$$

Remark 1. Condition (14) is equivalent to the inequality

$$Q_{N+1} + \Gamma_1 \geqslant 0 \quad \text{on } \operatorname{Ker} \mathcal{M}_1 \cap \operatorname{Im} X_{N+1},$$

where the matrix Q_k is defined by (2) via the natural conjoined basis (X, U).

3. Riccati inequality and positivity

In this section we establish one of the main results of this paper—a characterization of the positivity of $\mathcal{F}$ with separable endpoints in terms of the *Riccati inequality*

$$R[Q]_k(\mathcal{A}_k+\mathcal{B}_kQ_k)^{-1}\leqslant 0,\quad k\in[0,N]. \tag{RI}$$

This result generalizes the Hamiltonian Riccati inequality in [14, Section 4] to discrete symplectic systems. A specific difficulty in this extension lies in finding the system ($\underline{\mathrm{S}}$) to which the comparison theorem (Proposition 2) can be applied. Once we derive the correct form of the coefficients $\underline{\mathcal{A}}$, $\underline{\mathcal{B}}$, $\underline{\mathcal{C}}$, $\underline{\mathcal{D}}$, $\underline{\mathcal{E}}$ (see the proof of Theorem 1 below), the application of the comparison theorem for symplectic systems yields the result in a similar way as in the Hamiltonian systems case.

Theorem 1 *(Riccati inequality). The functional $\mathcal{F}$ is positive definite, that is, conditions* (i)–(iii) *in Proposition* 1 *hold, if and only if either of the following equivalent conditions is satisfied.*

(iv) *The system*

$$X_{k+1}=\mathcal{A}_kX_k+\mathcal{B}_kU_k,$$
$$N_k:=X_{k+1}^T(U_{k+1}-\mathcal{C}_kX_k-\mathcal{D}_kU_k)\leqslant 0, \tag{15}$$

$k\in[0,N]$, has a solution (X,U) such that $X_k^TU_k$ is symmetric and X_k is invertible for all $k\in[0,N+1]$, $P_k=X_kX_{k+1}^{-1}\mathcal{B}_k\geqslant 0$ on $[0,N]$*, condition* (5) *holds, and one of the conditions* (6) *or* (7) *is satisfied.*

(v) *The discrete Riccati inequality* (RI)*, $k\in[0,N]$, has a symmetric solution Q_k on* $[0,N+1]$ *such that conditions* (8) *and* (9) *hold, and one of the conditions* (10) *or* (11) *is satisfied.*

Riccati inequality is often used in nonoscillation criteria for differential and difference equations, since it is easier to find a solution of the *inequality* (which corresponds to a solution of *some* majorant equation) than a solution to the *equality*. In the following examples we show a situation when a symmetric Q_k solves the Riccati inequality (RI) and satisfies condition (v) in Theorem 1, but it does not solve the Riccati equation (R) so that condition (iii) in Proposition 1 is not satisfied with this Q_k.

Example 1.

(a) Let $\mathcal{A}_k\equiv 0$, $\mathcal{B}_k\equiv -\mathcal{C}^{T-1}$, $\mathcal{C}_k\equiv\mathcal{C}$, $\mathcal{D}_k\equiv-\mathcal{C}^{T-1}-\mathcal{C}-K$, where $\mathcal{C}$ is a constant nonsingular matrix, $K\neq 0$, and $\mathcal{C}K^T=K\mathcal{C}^T\geqslant 0$, and consider the zero endpoints, i.e. $\mathcal{M}_i=I$ and $\Gamma_i=0$ $(i=0,1)$. Then $Q_k\equiv I$ satisfies condition (v) in Theorem 1, since $\mathcal{A}_k+\mathcal{B}_kQ_k=-\mathcal{C}^{T-1}$ is invertible, $\mathcal{P}_k\equiv I>0$, and $R[Q]_k(\mathcal{A}_k+\mathcal{B}_kQ_k)^{-1}=-K\mathcal{C}^T\leqslant 0$, while the Riccati equation is $R[Q]_k=K\neq 0$. Another (more specific) example can be obtained when we take, e.g., $\mathcal{C}=K=I$. Note also that since $\mathcal{A}_k$ is not invertible, the Hamiltonian Riccati inequality in [14, Corollary 4.1] cannot be applied to this setting.

(b) Let $\mathcal{A}_k$ and $\mathcal{C}_k$ be invertible, $\mathcal{B}_k\equiv 0$, and $\mathcal{D}_k=\mathcal{A}_k^{T-1}$, with $\mathcal{C}_k^T\mathcal{A}_k>0$, and with free endpoints, i.e. $\mathcal{M}_i=0$ and $\Gamma_i>0$ $(i=0,1)$. Then $Q_k\equiv 0$ satisfies condition (v) in Theorem 1, since $\mathcal{A}_k+\mathcal{B}_kQ_k=\mathcal{A}_k$ is invertible, $\mathcal{P}_k\equiv 0$, and $R[Q]_k(\mathcal{A}_k+\mathcal{B}_kQ_k)^{-1}=-\mathcal{C}_k\mathcal{A}_k^{-1}<0$, while the Riccati equation is $R[Q]_k=-\mathcal{C}_k\neq 0$. However, in this simple example we can directly verify that $\mathcal{F}>0$ over free endpoints.

The proof of Theorem 1 is shown below after some auxiliary identities. In these identities, we do not work with solutions of (S), but instead with arbitrary matrices X_k, U_k satisfying the given properties. However, it is still assumed that the coefficients $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{D}_k$ fulfill the requirement that the matrix $\mathcal{S}_k$ is symplectic.

Lemma 1. *Let $k \in [0, N]$ be fixed and assume that, for $j \in [k, k+1]$, X_j and U_j are $n \times n$ matrices such that $X_{k+1} = \mathcal{A}_k X_k + \mathcal{B}_k U_k$. Then the following conditions hold.*

(i) *If $X_j^T U_j$ is symmetric for $j \in [k, k+1]$, then the matrix*

$$X_{k+1}^T (U_{k+1} - \mathcal{C}_k X_k - \mathcal{D}_k U_k) \\ = \Delta\big(X_k^T U_k\big) - \big(X_k^T \mathcal{C}_k^T \mathcal{A}_k X_k + 2 X_k^T \mathcal{C}_k^T \mathcal{B}_k U_k + U_k^T \mathcal{D}_k^T \mathcal{B}_k U_k\big)$$

is symmetric as well.

(ii) *If $X_{k+1}^T U_{k+1}$ is symmetric and if Q_j is symmetric with $Q_j X_j = U_j X_j^\dagger X_j$ for $j \in [k, k+1]$, then*

$$X_{k+1}^T R[Q]_k X_k = X_{k+1}^T (U_{k+1} - \mathcal{C}_k X_k - \mathcal{D}_k U_k) X_k^\dagger X_k.$$

(iii) *If $X_j^T U_j$ and Q_j are symmetric with $Q_j X_j = U_j X_j^\dagger X_j$ for $j \in [k, k+1]$ and if X_k is invertible, then the matrix*

$$X_{k+1}^T R[Q]_k X_k = X_{k+1}^T (U_{k+1} - \mathcal{C}_k X_k - \mathcal{D}_k U_k)$$

is symmetric.

Proof. Part (i) is a simple calculation. For part (ii), we first derive

$$R[Q]_k X_k = \big[Q_{k+1} X_{k+1} - (\mathcal{C}_k X_k + \mathcal{D}_k U_k)\big] X_k^\dagger X_k \\ = \big[U_{k+1} - \mathcal{C}_k X_k - \mathcal{D}_k U_k - U_{k+1}\big(I - X_{k+1}^\dagger X_{k+1}\big)\big] X_k^\dagger X_k.$$

Then, after multiplying by X_{k+1}^T from the left and by using the symmetry of $X_{k+1}^T U_{k+1}$ we obtain the required identity. Part (iii) follows directly from (i) and (ii). □

Proof of Theorem 1. In this proof, conditions (i)–(iii) refer to Proposition 1 unless otherwise specified. Condition (ii) implies (iv) trivially, since (X, U) satisfying part (ii) is a solution of (S). Condition (iv) implies (v) by the Riccati substitution $Q_k := U_k X_k^{-1}$ on $[0, N+1]$. Next, we show that condition (v) implies (iv). Let $F_k := R[Q]_k (\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1} \leqslant 0$ be the matrix defining the inequality (RI), where Q_k satisfies condition (v). Let X be the solution of the equation $X_{k+1} = (\mathcal{A}_k + \mathcal{B}_k Q_k) X_k$, $k \in [0, N]$, given by the initial condition $X_0 = I$. Then X_k is invertible on $[0, N+1]$. If we set $U_k := Q_k X_k$ on $[0, N+1]$, then (X, U) satisfies $X_{k+1} = \mathcal{A}_k X_k + \mathcal{B}_k U_k$ and

$$N_k = X_{k+1}^T \big[Q_{k+1} - (\mathcal{C}_k + \mathcal{D}_k Q_k) X_k X_{k+1}^{-1}\big] X_{k+1} = X_{k+1}^T F_k X_{k+1} \leqslant 0$$

for all $k \in [0, N]$, that is, (X, U) solves system (15). Note that the matrices N_k and $F_k = X_{k+1}^{T-1} N_k X_{k+1}^{-1}$ are symmetric, by Lemma 1(iii).

The rest of the proof is about showing that condition (iv) implies (i). With N_k as in (15) we put $F_k := X_{k+1}^{T-1} N_k X_{k+1}^{-1} \leqslant 0$. Define $\underline{\mathcal{A}}_k := \mathcal{A}_k$, $\underline{\mathcal{B}}_k := \mathcal{B}_k$, $\underline{\mathcal{C}}_k := \mathcal{C}_k + F_k \mathcal{A}_k$, $\underline{\mathcal{D}}_k := \mathcal{D}_k + F_k \mathcal{B}_k$, and

$$\underline{\mathcal{S}}_k := \begin{pmatrix} \underline{\mathcal{A}}_k & \underline{\mathcal{B}}_k \\ \underline{\mathcal{C}}_k & \underline{\mathcal{D}}_k \end{pmatrix} = \mathcal{S}_k + \mathcal{R}_k \quad \text{with } \mathcal{R}_k := \begin{pmatrix} 0 & 0 \\ F_k \mathcal{A}_k & F_k \mathcal{B}_k \end{pmatrix}. \tag{16}$$

The proof will be finished by showing the following claims.

Claim 1. The matrix $\underline{\mathcal{S}}_k$ is symplectic.

This follows from the observation that $\mathcal{S}_k^T \mathcal{J} \mathcal{R}_k = (\mathcal{A}_k \quad \mathcal{B}_k)^T F_k (\mathcal{A}_k \quad \mathcal{B}_k)$ is symmetric, $\mathcal{R}_k^T \mathcal{J} \mathcal{R}_k = 0$, and from the calculation

$$\begin{aligned}\underline{\mathcal{S}}_k^T \mathcal{J} \underline{\mathcal{S}}_k &= (\mathcal{S}_k + \mathcal{R}_k)^T \mathcal{J} (\mathcal{S}_k + \mathcal{R}_k) = \mathcal{S}_k^T \mathcal{J} \mathcal{S}_k + \mathcal{S}_k^T \mathcal{J} \mathcal{R}_k + \mathcal{R}_k^T \mathcal{J} \mathcal{S}_k + \mathcal{R}_k^T \mathcal{J} \mathcal{R}_k \\ &= \mathcal{J} + \mathcal{R}_k^T \mathcal{J}^T \mathcal{S}_k + \mathcal{R}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}.\end{aligned}$$

Claim 2. The pair (X, U) solves the system ($\underline{\mathrm{S}}$), hence it is a conjoined basis of ($\underline{\mathrm{S}}$) with no focal points in $(0, N+1]$.

This follows from the invertibility of X, the calculations

$$\begin{aligned}\underline{\mathcal{A}}_k X_k + \underline{\mathcal{B}}_k U_k &= \mathcal{A}_k X_k + \mathcal{B}_k U_k = X_{k+1}, \\ \underline{\mathcal{C}}_k X_k + \underline{\mathcal{D}}_k U_k &= \mathcal{C}_k X_k + \mathcal{D}_k U_k + F_k (\mathcal{A}_k X_k + \mathcal{B}_k U_k) \\ &= \mathcal{C}_k X_k + \mathcal{D}_k U_k + X_{k+1}^{T-1} N_k = U_{k+1},\end{aligned}$$

and from $\underline{P}_k := X_k X_{k+1}^{-1} \underline{\mathcal{B}}_k = X_k X_{k+1}^{-1} \mathcal{B}_k \geqslant 0$.

Claim 3. $\mathcal{F}(x, u) > 0$ over $\mathcal{M}_0 x_0 = 0$, $\mathcal{M}_1 x_{N+1} = 0$, and $x \not\equiv 0$, that is, condition (i) holds.

We have by Proposition 1 applied to ($\underline{\mathrm{S}}$) that the functional $\underline{\mathcal{F}}(x, u) > 0$ over $\underline{\mathcal{M}}_0 x_0 = 0$, $\underline{\mathcal{M}}_1 x_{N+1} = 0$, and $x \not\equiv 0$, where $\underline{\mathcal{M}}_0 := \mathcal{M}_0$, $\underline{\mathcal{M}}_1 := \mathcal{M}_1$, $\underline{\Gamma}_0 := \Gamma_0$, and $\underline{\Gamma}_1 := \Gamma_1$. Next, the definition of $\underline{\mathcal{A}}_k$ and $\underline{\mathcal{B}}_k$ implies that $\operatorname{Im}(\mathcal{A}_k - \underline{\mathcal{A}}_k \quad \mathcal{B}_k) = \operatorname{Im} \mathcal{B}_k = \operatorname{Im} \underline{\mathcal{B}}_k$. Furthermore, the symmetric matrix $\underline{\mathcal{E}}_k := \mathcal{E}_k + F_k$ satisfies $\underline{\mathcal{D}}_k^T \underline{\mathcal{B}}_k = \mathcal{D}_k^T \mathcal{B}_k + \mathcal{B}_k^T F_k \mathcal{B}_k = \mathcal{B}_k^T (\mathcal{E}_k + F_k) \mathcal{B}_k = \underline{\mathcal{B}}_k^T \underline{\mathcal{E}}_k \underline{\mathcal{B}}_k$, and

$$\begin{aligned}&\mathcal{G}_k - \underline{\mathcal{G}}_k \\ &\quad = \begin{pmatrix} \mathcal{A}_k^T \mathcal{E}_k \mathcal{A}_k - \mathcal{A}_k^T \mathcal{C}_k + \delta_{k,0} \Gamma_0 & \mathcal{C}_k^T - \mathcal{A}_k^T \mathcal{E}_k \\ \mathcal{C}_k - \mathcal{E}_k \mathcal{A}_k & \mathcal{E}_k + \delta_{k,N} \Gamma_1 \end{pmatrix} \\ &\qquad - \begin{pmatrix} \mathcal{A}_k^T (\mathcal{E}_k + F_k) \mathcal{A}_k - \mathcal{A}_k^T (\mathcal{C}_k + F_k \mathcal{A}_k) + \delta_{k,0} \Gamma_0 & \mathcal{C}_k^T + \mathcal{A}_k^T F_k - \mathcal{A}_k^T (\mathcal{E}_k + F_k) \\ \mathcal{C}_k + F_k \mathcal{A}_k - (\mathcal{E}_k + F_k) \mathcal{A}_k & \mathcal{E}_k + F_k + \delta_{k,N} \Gamma_1 \end{pmatrix} \\ &\quad = \begin{pmatrix} 0 & 0 \\ 0 & -F_k \end{pmatrix} \geqslant 0.\end{aligned}$$

Consequently, assumption (12) is satisfied and Proposition 2(ii) yields the statement of Claim 3. Hence, the proof of this theorem is now complete. □

Remark 2. In the proof above we used a matrix $\underline{\mathcal{S}}$ of the form $\underline{\mathcal{S}} = \mathcal{S} + \mathcal{R}$ with $\mathcal{R} = \left(\begin{smallmatrix} 0 & 0 \\ G & H \end{smallmatrix}\right)$. This matrix $\underline{\mathcal{S}}$ is symplectic if and only if $G^T \mathcal{A}$ and $H^T \mathcal{B}$ are symmetric, and the identity $H^T \mathcal{A} = \mathcal{B}^T G$ holds. The choice $G := F \mathcal{A}$ and $H := F \mathcal{B}$ with symmetric F is then natural, which was first observed in [9] in connection with an eigenvalue problem associated with system (S).

4. Nonnegativity for joint endpoints

In this section we present a characterization of the *nonnegativity* of the quadratic functional

$$\mathcal{F}(x,u) := \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix}^T \Gamma \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix} + \mathcal{F}_0(x,u)$$

with general *jointly varying* endpoints, namely the endpoints constraint $\mathcal{M}\binom{x_0}{x_{N+1}} = 0$. The properties of the $2n \times 2n$ matrices Γ, $\mathcal{M}$ are given in Section 2. The following result is a generalization of [8, Theorem 1.1], where the functional $\mathcal{F}$ has fixed endpoints, i.e. it has the form of $\mathcal{F}_0$ over $x_0 = 0 = x_{N+1}$. Its proof is displayed after some auxiliary results.

Let $(\hat{X}, \hat{U})$ be the principal solution of (S), i.e. $\hat{X}_0 = 0$ and $\hat{U}_0 = I$. Let $(\tilde{X}, \tilde{U})$ be the conjoined basis of (S) given by the initial conditions $(\tilde{X}_0, \tilde{U}_0) = (I, 0)$, so that $(\tilde{X}, \tilde{U})$ and $(\hat{X}, \hat{U})$ are normalized, and define the $2n \times 2n$ matrices

$$\hat{X}_k^* := \begin{pmatrix} 0 & I \\ \hat{X}_k & \tilde{X}_k \end{pmatrix}, \qquad \hat{U}_k^* := \begin{pmatrix} -I & 0 \\ \hat{U}_k & \tilde{U}_k \end{pmatrix}. \tag{17}$$

In the theorem below, the matrices P_k, M_k, and T_k are defined via the principal solution $(\hat{X}, \hat{U})$ of (S), i.e. $P_k := \hat{X}_k \hat{X}_{k+1}^\dagger \mathcal{B}_k$, $M_k := (I - \hat{X}_{k+1} \hat{X}_{k+1}^\dagger)\mathcal{B}_k$, and $T_k := I - M_k^\dagger M_k$.

Theorem 2. *The functional $\mathcal{F}(x,u) \geqslant 0$ over $\mathcal{M}\binom{x_0}{x_{N+1}} = 0$ if and only if the principal solution $(\hat{X}, \hat{U})$ of* (S) *satisfies the P-condition* (13)*, the image condition*

$$\begin{aligned} &x_k - \tilde{X}_k x_0 \in \operatorname{Im} \hat{X}_k \quad \textit{for all } k \in [0, N+1], \\ &\quad \textit{for all admissible } (x,u) \textit{ with } \mathcal{M}\tbinom{x_0}{x_{N+1}} = 0, \end{aligned} \tag{18}$$

and the final endpoint inequality

$$\big(\hat{X}_{N+1}^*\big)^T \big(\Gamma \hat{X}_{N+1}^* + \hat{U}_{N+1}^*\big) \geqslant 0 \quad \textit{on } \operatorname{Ker} \mathcal{M} \hat{X}_{N+1}^*. \tag{19}$$

Remark 3. Condition (19) is equivalent to the inequality

$$\hat{Q}_{N+1}^* + \Gamma \geqslant 0 \quad \text{on } \operatorname{Ker} \mathcal{M} \cap \operatorname{Im} \hat{X}_{N+1}^*,$$

where the symmetric matrix $\hat{Q}_k^*$ is defined by (skipping the index k)

$$\hat{Q}^* := \begin{pmatrix} \hat{X}^\dagger \tilde{X} \hat{X}^\dagger \hat{X} & -\hat{X}^\dagger + \hat{X}^\dagger \tilde{X}(I - \hat{X}^\dagger \hat{X})\hat{U}^T \\ \star & \hat{U}\hat{X}^\dagger - (\hat{U}\hat{X}^\dagger \tilde{X} - \tilde{U})(I - \hat{X}^\dagger \hat{X})\hat{U}^T \end{pmatrix}. \tag{20}$$

The proof of Theorem 2 is based on transforming the quadratic functional $\mathcal{F}$ and system (S) into a problem in dimension $2n$, see, e.g., [15] modified to the setting of this paper or [5]. Hence, introduce the $2n \times 2n$ matrices $\mathcal{A}_k^* := \begin{pmatrix} I & 0 \\ 0 & \mathcal{A}_k \end{pmatrix}$, $\mathcal{B}_k^* := \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{B}_k \end{pmatrix}$, $\mathcal{C}_k^* := \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{C}_k \end{pmatrix}$, $\mathcal{D}_k^* := \begin{pmatrix} I & 0 \\ 0 & \mathcal{D}_k \end{pmatrix}$, and the $4n \times 4n$ symplectic matrix $\mathcal{S}_k^* := \begin{pmatrix} \mathcal{A}_k^* & \mathcal{B}_k^* \\ \mathcal{C}_k^* & \mathcal{D}_k^* \end{pmatrix}$, which defines a new symplectic system denoted here by (S*). Now with the $2n \times 2n$ projections $\mathcal{M}_0^* := \frac{1}{2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$ and $\mathcal{M}_1^* := \mathcal{M}$, and with the symmetric $2n \times 2n$ matrices $\Gamma_0^* := 0$ and $\Gamma_1^* := \Gamma$ we consider the quadratic functional

$$\mathcal{F}^*(x^*, u^*) := \big(x_0^*\big)^T \Gamma_0^* x_0^* + \big(x_{N+1}^*\big)^T \Gamma_1^* x_{N+1}^* + \mathcal{F}_0^*(x^*, u^*)$$

over the separated endpoints $\mathcal{M}_0^* x_0^* = 0$ and $\mathcal{M}_1^* x_{N+1}^* = 0$.

Now as in (1) and (3) in Section 2, for any matrix solution (X^*, U^*) of (S*) we can define the $2n \times 2n$ matrices $P_k^* := X_k^* (X_{k+1}^*)^\dagger \mathcal{B}_k^*$, $M_k^* := [I - X_{k+1}^* (X_{k+1}^*)^\dagger] \mathcal{B}_k^*$, and $T_k^* := I - (M_k^*)^\dagger M_k^*$. In particular, for

$$X_k^* := \begin{pmatrix} 0 & I \\ X_k & \tilde{X}_k \end{pmatrix}, \qquad U_k^* := \begin{pmatrix} -I & 0 \\ U_k & \tilde{U}_k \end{pmatrix}, \tag{21}$$

where (X, U) and $(\tilde{X}, \tilde{U})$ are arbitrary solutions of (S), we have

$$M_k^* = \begin{pmatrix} 0 & -(I + \tilde{X}_{k+1}^T \tilde{X}_{k+1})^{-1} \tilde{X}_{k+1}^T M_k \\ 0 & [I - \tilde{X}_{k+1} (I + \tilde{X}_{k+1}^T \tilde{X}_{k+1})^{-1} \tilde{X}_{k+1}^T] M_k \end{pmatrix}. \tag{22}$$

The next lemma displays a formula for the Moore–Penrose generalized inverse of this M_k^*. It is given via its full-rank factorization, see, e.g., [2, pp. 26, 48].

Lemma 2. *Let (X, U) and $(\tilde{X}, \tilde{U})$ be any solutions of* (S) *and let the matrices P_k, M_k, and T_k be defined by* (1), (3). *Let $M_k = F_k R_k$ be a full rank factorization of M_k, i.e., $F_k \in \mathbb{R}^{n \times r_k}$ and $R_k \in \mathbb{R}^{r_k \times n}$ with $r_k := \operatorname{rank} M_k = \operatorname{rank} F_k = \operatorname{rank} R_k$. Let X_k^*, U_k^*, and M_k^* be as in* (21), (22). *Then the Moore–Penrose inverse of M_k^* is given by*

$$(M_k^*)^\dagger = \begin{pmatrix} 0 & 0 \\ -R_k^T (R_k R_k^T)^{-1} H_k \tilde{X}_{k+1} (I + \tilde{X}_{k+1}^T \tilde{X}_{k+1})^{-1} & R_k^T (R_k R_k^T)^{-1} H_k (I + \tilde{X}_{k+1} \tilde{X}_{k+1}^T)^{-1} \end{pmatrix},$$

where $H_k := [F_k^T (I + \tilde{X}_{k+1} \tilde{X}_{k+1}^T)^{-1} F_k]^{-1} F_k^T \in \mathbb{R}^{r_k \times n}$. Consequently, we have

$$(M_k^*)^\dagger M_k^* = \begin{pmatrix} 0 & 0 \\ 0 & M_k^\dagger M_k \end{pmatrix}, \qquad T_k^* = \begin{pmatrix} I & 0 \\ 0 & T_k \end{pmatrix}, \qquad T_k^* P_k^* T_k^* = \begin{pmatrix} 0 & 0 \\ 0 & T_k P_k T_k \end{pmatrix}. \tag{23}$$

Proof. The proof of the formula for $(M_k^*)^\dagger$ consists from a number of calculations, which are summarized in the following claims.

Claim 1. For any matrix A, the following identity holds (we shall use this identity with $A := \tilde{X}_{k+1}$)

$$(I + AA^T)^{-1} = I - A(I + A^T A)^{-1} A^T. \tag{24}$$

This follows from [11] or by a direct calculation.

Claim 2. $M_k^* = F_k^* R_k^*$ is a full rank factorization of M_k^*, where the matrices $F_k^* \in \mathbb{R}^{2n \times r_k}$ and $R_k^* \in \mathbb{R}^{r_k \times 2n}$ are defined by

$$F_k^* := \begin{pmatrix} -(I + \tilde{X}_{k+1}^T \tilde{X}_{k+1})^{-1} \tilde{X}_{k+1}^T F_k \\ [I - \tilde{X}_{k+1} (I + \tilde{X}_{k+1}^T \tilde{X}_{k+1})^{-1} \tilde{X}_{k+1}^T] F_k \end{pmatrix}, \qquad R_k^* := \begin{pmatrix} 0 & R_k \end{pmatrix}.$$

Clearly, $F_k^* R_k^* = M_k^*$, $r_k = \operatorname{rank} R_k^* = \operatorname{rank} M_k^*$, while $\operatorname{rank} F_k^* = r_k$ follows from the invertibility of the matrix on the right-hand side of (24).

Claim 3. $(M_k^*)^\dagger$ has the form as in the lemma above and identities (23) hold.

Since for any matrix M with a full rank factorization $M = FR$ we have $M^\dagger = R^T(RR^T)^{-1} \times (F^TF)^{-1}F^T$ and $M^\dagger M = R^T(RR^T)^{-1}R$, see [2, p. 48], the results follow by applying these identities to the matrices M_k^*, F_k^*, and R_k^*. □

The next result shows that the multiplication of a solution (X, U) of (S) by a constant nonsingular matrix does not change the associated matrices $T_k P_k T_k$, M_k, and T_k.

Lemma 3. *Let (X, U) be any solution of* (S) *and let P_k, M_k, T_k be defined by* (1), (3). *Then for any constant nonsingular $n \times n$ matrix E and the solution $(\tilde{X}, \tilde{U}) := (XE, UE)$ of* (S) *we have*

(i) $\tilde{X}_k \tilde{X}_k^\dagger = X_k X_k^\dagger$,
(ii) $\tilde{M}_k = M_k$ *and* $\tilde{T}_k = T_k$, *where* $\tilde{M}_k := (I - \tilde{X}_{k+1}\tilde{X}_{k+1}^\dagger)\mathcal{B}_k$ *and* $\tilde{T}_k := I - \tilde{M}_k^\dagger \tilde{M}_k$,
(iii) $\tilde{T}_k \tilde{P}_k = T_k P_k$, *where* $\tilde{P}_k := \tilde{X}_k \tilde{X}_{k+1}^\dagger \mathcal{B}_k$.

Proof. Part (i) follows by a direct calculation or from [3, p. 93]. Part (ii) is a consequence of (i) at the index $k+1$. To show part (iii), we use the last identity in (4), $\tilde{T}_k = T_k$, and part (i) at the index $k+1$ to get

$$\begin{aligned}\tilde{T}_k \tilde{P}_k &= T_k X_k E \tilde{X}_{k+1}^\dagger \mathcal{B}_k = T_k X_k X_{k+1}^\dagger X_{k+1} E \tilde{X}_{k+1}^\dagger \mathcal{B}_k = T_k X_k X_{k+1}^\dagger \tilde{X}_{k+1} \tilde{X}_{k+1}^\dagger \mathcal{B}_k \\ &= T_k X_k X_{k+1}^\dagger X_{k+1} X_{k+1}^\dagger \mathcal{B}_k = T_k X_k X_{k+1}^\dagger \mathcal{B}_k = T_k P_k.\end{aligned}$$

Thus, this lemma is proven. □

Proof of Theorem 2. Since the nonnegativity of the functional $\mathcal{F}$ over $\mathcal{M}\binom{x_0}{x_{N+1}} = 0$ is equivalent to the nonnegativity of $\mathcal{F}^*$ over $\mathcal{M}_0^* x_0^* = 0$ and $\mathcal{M}_1^* x_{N+1}^* = 0$, see, e.g., [15, Lemma 4], we can apply Proposition 3 to this transformed augmented functional. Thus, we get that the augmented natural conjoined basis (X^*, U^*) of (S^*) given by the initial conditions

$$X_0^* = I - \mathcal{M}_0^* = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix} \quad \text{and} \quad U_0^* = \Gamma_0^* + \mathcal{M}_0^* = \frac{1}{2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

satisfies the P^*-condition

$$T_k^* P_k^* T_k^* \geqslant 0 \quad \text{for all } k \in [0, N], \tag{25}$$

the image* condition

$$\begin{aligned}&x_k^* \in \operatorname{Im} X_k^* \quad \text{for all } k \in [0, N+1], \\ &\quad \text{for all admissible (for } \mathcal{F}^*\text{) pairs } (x^*, u^*) \text{ with } \mathcal{M}_0^* x_0^* = 0 \text{ and } \mathcal{M}_1^* x_{N+1}^* = 0,\end{aligned} \tag{26}$$

and the corresponding augmented final endpoint inequality.

Now take the nonsingular $2n \times 2n$ matrix $E := \begin{pmatrix} -I & I \\ I & I \end{pmatrix}$ and consider the solution (X^*E, U^*E) of (S^*) that has $\left(\begin{pmatrix} 0 & I \\ 0 & I \end{pmatrix}, \begin{pmatrix} -I & 0 \\ I & 0 \end{pmatrix}\right)$ as initial conditions. Then, by (17), $(X^*E, U^*E) = (\hat{X}^*, \hat{U}^*)$ and, by Lemma 3 applied to this setting, the P^*-condition (25) for (X^*, U^*) is equivalent to the $\hat{P}^*$-condition for $(\hat{X}^*, \hat{U}^*)$. In turn, Lemma 2 applied to the conjoined basis $(\hat{X}^*, \hat{U}^*)$ yields that $0 \leqslant \hat{T}_k^* \hat{P}_k^* \hat{T}_k^* = \operatorname{diag}\{0, T_k P_k T_k\}$, where the matrices T_k and P_k are defined via the principal solution $(\hat{X}, \hat{U})$ of (S).

Next, since $\operatorname{Im} X_k^* = \operatorname{Im} \hat{X}_k^*$, the image* condition (26) for X_k^* is equivalent to the image* condition for $\hat{X}_k^*$, while the latter one is equivalent to (18).

Finally, the final endpoint inequality

$$\left(X_{N+1}^*\right)^T\left(\Gamma_1^* X_{N+1}^* + U_{N+1}^*\right) \geqslant 0 \quad \text{on } \operatorname{Ker}\mathcal{M}_1^* X_{N+1}^*$$

for (X^*, U^*) is equivalent to (19) in terms of $(\hat{X}^*, \hat{U}^*)$. The proof is now complete. □

At the end of this section we show that under certain conditions the Riccati quotient Q_k in (2) is invariant under the multiplication by a constant nonsingular matrix. Although this result is not directly needed in this paper, it fits well into the global theory of discrete symplectic systems.

Lemma 4. *Let (X, U), $(\tilde{X}, \tilde{U})$ be any normalized conjoined bases of* (S) *and let Q_k be defined by* (2). *Let E be any nonsingular $n \times n$ matrix with $E^{-1} = cE^T$ for some $c \in \mathbb{R}$ and set $(\underline{X}, \underline{U}) := (XE, UE)$ and $(\underline{\tilde{X}}, \underline{\tilde{U}}) := (c\tilde{X}E, c\tilde{U}E)$. Then $(\underline{X}, \underline{U})$ and $(\underline{\tilde{X}}, \underline{\tilde{U}})$ are also normalized conjoined bases of* (S) *and the matrix*

$$\underline{Q}_k := \underline{U}_k \underline{X}_k^{\dagger} + \left(\underline{U}_k \underline{X}_k^{\dagger} \underline{\tilde{X}}_k - \underline{\tilde{U}}_k\right)\left(I - \underline{X}_k^{\dagger} \underline{X}_k\right)\underline{U}_k^T$$

satisfies $\underline{Q}_k = Q_k$ on $[0, N+1]$.

Proof. The conjoined bases $(\underline{X}, \underline{U})$ and $(\underline{\tilde{X}}, \underline{\tilde{U}})$ satisfy

$$\underline{X}_k^T \underline{\tilde{U}}_k - \underline{U}_k^T \underline{\tilde{X}}_k = cE^T\left(X_k^T \tilde{U}_k - U_k^T \tilde{X}_k\right)E = cE^T E = I,$$

so that they are normalized. Next, the properties of the Moore–Penrose inverses imply that $\underline{X}_k^{\dagger} = cE^T X_k^{\dagger}$. Using $cEE^T = I$, a simple calculation then yields $\underline{Q}_k = Q_k$. □

5. Inequalities and nonnegativity

In this section we derive some inequalities related to the nonnegativity of discrete quadratic functionals. First result says that the nonnegativity of $\mathcal{F}_0$ (where $x_0 = 0$) is equivalent to the nonnegativity of a certain perturbed functional where x_0 is a restricted to a subspace. Its proof is shown below after some comments and auxiliary lemmas.

In this section (as in Theorem 2), we shall denote by $(\tilde{X}, \tilde{U})$ the conjoined basis of (S) given by the initial conditions $(\tilde{X}_0, \tilde{U}_0) = (I, 0)$. Also, recall that $(\hat{X}, \hat{U})$ denotes the principal solution of (S), i.e. $(\hat{X}_0, \hat{U}_0) = (0, I)$, and that $(\tilde{X}, \tilde{U})$ and $(\hat{X}, \hat{U})$ are normalized.

Theorem 3. *The functional $\mathcal{F}_0(x, u) \geqslant 0$ over $x_0 = 0 = x_{N+1}$ if and only if there exists $\alpha > 0$ such that the functional $\mathcal{F}(x, u) := \alpha\|x_0\|^2 + \mathcal{F}_0(x, u) \geqslant 0$ over $\tilde{X}_{N+1}x_0 = x_{N+1}$.*

This result has a simple but important consequence.

Corollary 1. *The following statements are equivalent.*

(i) $\mathcal{F}_0(x, u) \geqslant 0$ *over* $x_0 = 0 = x_{N+1}$.
(ii) *There exists $\alpha > 0$ such that the functional $\mathcal{F}(x, u) := \alpha\|x_0\|^2 + \mathcal{F}_0(x, u) \geqslant 0$ over $\tilde{X}_{N+1}x_0 = 0 = x_{N+1}$.*
(iii) *There exists $\alpha > 0$ such that the conjoined basis (X, U) of* (S) *given by the initial conditions*

$$X_0 = I - \tilde{X}_{N+1}^{\dagger}\tilde{X}_{N+1}, \qquad U_0 = \alpha I + \tilde{X}_{N+1}^{\dagger}\tilde{X}_{N+1}$$

satisfies P-condition (13) *and the image condition*

$$x_k \in \operatorname{Im} X_k \quad \text{for all } k \in [0, N+1],$$
$$\text{for all admissible } (x,u) \text{ with } \tilde{X}_{N+1}x_0 = 0 = x_{N+1}.$$

Proof. The equivalence of (i) and (ii) follows directly from Theorem 3, while the equivalence of (ii) and (iii) follows from Proposition 3 with $\mathcal{M}_0 := \tilde{X}_{N+1}^{\dagger}\tilde{X}_{N+1}$, $\mathcal{M}_1 := I$, $\Gamma_0 := \alpha I$, and $\Gamma_1 := 0$. Note that in this case $\operatorname{Ker}\mathcal{M}_0 = \operatorname{Ker}\tilde{X}_{N+1}$. □

The equivalence of (i) and (ii) in Corollary 1 is an analog of the corresponding result for the *positivity* of $\mathcal{F}_0$, namely

> $\mathcal{F}_0(x,u) > 0$ *over* $x_0 = 0 = x_{N+1}$ *and* $x \not\equiv 0$ *if and only if there exists* $\alpha > 0$ *such that* $\mathcal{F}(x,u) := \alpha\|x_0\|^2 + \mathcal{F}_0(x,u) > 0$ *over* x_0 *free*, $x_{N+1} = 0$, *and* $x \not\equiv 0$,

which is a part of the proof of [13, Theorem 7] or [4, Theorem 4]. We see that in case of the *nonnegativity* of $\mathcal{F}_0$, the endpoint x_0 cannot free, but must be restricted to a subspace ($= \operatorname{Ker}\tilde{X}_{N+1}$). This is also shown in the following example where $\mathcal{F}_0 \geqslant 0$ over $x_0 = 0 = x_{N+1}$, but there is no $\alpha > 0$ such that $\mathcal{F}(x,u) = \alpha\|x_0\|^2 + \mathcal{F}_0(x,u) \geqslant 0$ over $x_{N+1} = 0$.

Example 2. Consider the coefficients $\mathcal{S}_k \equiv \mathcal{J}$, that is, $\mathcal{A}_k = \mathcal{D}_k \equiv 0$ and $\mathcal{B}_k = -\mathcal{C}_k \equiv I$ for all $k \in [0, N]$. Then the solution $\tilde{X}_k$ is

$$\{\tilde{X}_k\}_{k=0}^{N+1} = \{I, 0, -I, 0, I, 0, -I, 0, \ldots\},$$

and the functional $\mathcal{F}_0$ takes the form

$$\mathcal{F}_0(x,u) = -2\left\{x_0^T u_0 + \sum_{k=1}^{N-1} u_{k-1}^T u_k\right\}$$

for admissible (x,u), i.e. $x_{k+1} = u_k$ on $[0, N]$, with $x_{N+1} = 0$.

If we take $N = 1$, then $\mathcal{F}_0(x,u) = -2x_0^T u_0$ for admissible (x,u) with $x_2 = 0$ and, in particular, $\mathcal{F}_0(x,u) = 0$ ($\geqslant 0$) when also $x_0 = 0$. Note that in this case $\mathcal{F}_0$ is not positive definite. On the other hand, $\mathcal{F}(x,u) = \alpha\|x_0\|^2 - 2x_0^T u_0 \ngeqslant 0$ over $x_2 = 0$ and x_0 free, which follows for example by choosing $u_0 := \alpha x_0 \neq 0$, so that $\mathcal{F}(x,u) = -\alpha\|x_0\|^2 < 0$. Note that, by Corollary 1, the initial endpoint x_0 cannot be in the kernel of $\tilde{X}_2$, which is verified by observing that $\tilde{X}_2 = -I$.

Finally, observe that when $N \geqslant 2$, then $\mathcal{F}_0(x,u) \ngeqslant 0$ over $x_0 = 0 = x_{N+1}$, which can be shown, e.g., by choosing $u_1 := u_0 \neq 0$ and $u_2 = \cdots = u_N := 0$, so that $\mathcal{F}_0(x,u) = -2\|u_0\|^2 < 0$.

For the proof of Theorem 3 we will need the following results.

Lemma 5. *The image condition* $\bar{x}_k \in \operatorname{Im}\hat{X}_k$ *on* $[0, N+1]$ *holds for all admissible* $(\bar{x}, \bar{u})$ *with* $\bar{x}_0 = 0 = \bar{x}_{N+1}$ *if and only if the image condition* $x_k - \tilde{X}_k x_0 \in \operatorname{Im}\hat{X}_k$ *on* $[0, N+1]$ *holds for all admissible* (x,u) *with* $x_{N+1} = \tilde{X}_{N+1}x_0$.

Proof. "⇒" Let (x,u) be admissible with $x_{N+1} = \tilde{X}_{N+1}x_0$ and set $(\bar{x}_k, \bar{u}_k) := (x_k, u_k) - (\tilde{X}_k, \tilde{U}_k)x_0$ on $[0, N+1]$. Then, because $\tilde{X}_0 = I$, we have $\bar{x}_0 = 0 = \bar{x}_{N+1}$ and $(\bar{x}_k, \bar{u}_k)$ is admissible, so that the assumption implies $x_k - \tilde{X}_k x_0 = \bar{x}_k \in \operatorname{Im}\hat{X}_k$ on $[0, N+1]$.

"⇐" Let $(\bar{x}, \bar{u})$ be admissible with $\bar{x}_0 = 0 = \bar{x}_{N+1}$. Then $\bar{x}_{N+1} = 0 = \tilde{X}_{N+1}\bar{x}_0$, so that $\bar{x}_k = \bar{x}_k - \tilde{X}_{N+1}\bar{x}_0 \in \operatorname{Im}\hat{X}_k$ on $[0, N+1]$. □

Lemma 6. *The image condition $\bar{x}_k \in \operatorname{Im}\hat{X}_k$ on $[0, N+1]$ holds for all admissible $(\bar{x}, \bar{u})$ with $\bar{x}_0 = 0 = \bar{x}_{N+1}$ if and only if the image condition $x_k - \tilde{X}_k x_0 \in \operatorname{Im}\hat{X}_k$ on $[0, N+1]$ holds for all admissible (x, u) with $\tilde{\mathcal{M}}\binom{x_0}{x_{N+1}} = 0$, where $\tilde{\mathcal{M}}$ is the $2n \times 2n$ projection*

$$\tilde{\mathcal{M}} := \begin{pmatrix} \tilde{X}_{N+1}^T(I + \tilde{X}_{N+1}\tilde{X}_{N+1}^T)^{-1}\tilde{X}_{N+1} & -\tilde{X}_{N+1}^T(I + \tilde{X}_{N+1}\tilde{X}_{N+1}^T)^{-1} \\ -(I + \tilde{X}_{N+1}\tilde{X}_{N+1}^T)^{-1}\tilde{X}_{N+1} & (I + \tilde{X}_{N+1}\tilde{X}_{N+1}^T)^{-1} \end{pmatrix}. \tag{27}$$

Proof. By Lemma 5, it suffices to show that $\tilde{\mathcal{M}}\binom{x_0}{x_{N+1}} = 0$ if and only if $x_{N+1} = \tilde{X}_{N+1}x_0$. However, this follows trivially since $\operatorname{Ker}\tilde{\mathcal{M}} = \operatorname{Im}\binom{I}{\tilde{X}_{N+1}}$. □

Proof of Theorem 3. Part "⇐" is trivial. Let us prove "⇒." Since $\mathcal{F}_0 \geqslant 0$, we have from Proposition 3 that the image condition $x_k \in \operatorname{Im}\hat{X}_k$ on $[0, N+1]$ holds for all admissible (x, u) with $x_0 = 0 = x_{N+1}$, and the P-condition (13) holds for the principal solution $(\hat{X}, \hat{U})$. From Lemma 6, we get that the image condition $x_k - \tilde{X}_k x_0 \in \operatorname{Im}\hat{X}_k$ on $[0, N+1]$ holds for all admissible (x, u) with $\tilde{\mathcal{M}}\binom{x_0}{x_{N+1}} = 0$, where $\tilde{\mathcal{M}}$ is given by (27). Put $\Gamma := \left(\begin{smallmatrix}\alpha I & 0\\ 0 & 0\end{smallmatrix}\right)$, where $\alpha > 0$ will be specified later. Then the functional $\mathcal{F}(x, u)$ can be regarded as being with jointly varying endpoints, that is, of the form in Theorem 2. Hence, by the same theorem, $\mathcal{F}(x, u)$ will be nonnegative over $\tilde{\mathcal{M}}\binom{x_0}{x_{N+1}} = 0$, hence over $x_{N+1} = \tilde{X}_{N+1}x_0$, once we show that the final endpoint inequality

$$\Gamma + \hat{Q}^*_{N+1} \geqslant 0 \quad \text{on } \operatorname{Ker}\tilde{\mathcal{M}} \cap \operatorname{Im}\hat{X}^*_{N+1} \tag{28}$$

is satisfied, where the (symmetric) matrix $\hat{Q}^*_{N+1}$ is defined by (20) and $\hat{X}^*_{N+1}$ is given by (17). To prove (28), note that $\operatorname{Ker}\tilde{\mathcal{M}} \subseteq \operatorname{Im}\hat{X}^*_{N+1}$, because any $\binom{c}{d} \in \operatorname{Ker}\tilde{\mathcal{M}}$ has $d = \tilde{X}_{N+1}c$, so that $\binom{c}{d} = \hat{X}^*_{N+1}\binom{0}{c}$. Denote by λ_0 the smallest eigenvalue of the symmetric matrix $\binom{I}{\tilde{X}_{N+1}}^T \hat{Q}^*_{N+1}\binom{I}{\tilde{X}_{N+1}}$. Then for any $\binom{c}{d} \in \operatorname{Ker}\tilde{\mathcal{M}}$ and for $\alpha := 1/\varepsilon$ we have

$$\begin{aligned}\begin{pmatrix} c \\ d \end{pmatrix}^T (\Gamma + \hat{Q}^*_{N+1}) \begin{pmatrix} c \\ d \end{pmatrix} &= (1/\varepsilon)\|c\|^2 + c^T \begin{pmatrix} I \\ \tilde{X}_{N+1} \end{pmatrix}^T \hat{Q}^*_{N+1} \begin{pmatrix} I \\ \tilde{X}_{N+1} \end{pmatrix} c \\ &\geqslant (1/\varepsilon)(1 + \varepsilon\lambda_0)\|c\|^2 \geqslant (1/\varepsilon)\big(1 - \varepsilon|\lambda_0|\big)\|c\|^2 \geqslant 0,\end{aligned}$$

where $\varepsilon > 0$ is small enough, e.g., $\varepsilon = 1/(2|\lambda_0|)$. Thus, the proof is complete. □

Next we wish to present a result regarding the *nonnegativity* of $\mathcal{F}_0$ in a parallel way to Theorem 1. In order to establish this statement, we shall use the associated symplectic system ($\underline{\mathrm{S}}$), defined in the proof of Theorem 1, and the results of Corollary 1(iii) and Proposition 2(i).

Theorem 4. *The functional $\mathcal{F}_0(x, u) \geqslant 0$ over $x_0 = 0 = x_{N+1}$ if and only if there exist $\alpha > 0$, symmetric matrices $F_k \leqslant 0$, $k \in [0, N]$, and a solution (X, U) of the system*

$$\begin{aligned} X_{k+1} &= \mathcal{A}_k X_k + \mathcal{B}_k U_k, \\ F_k X_{k+1} &= U_{k+1} - \mathcal{C}_k X_k - \mathcal{D}_k U_k, \end{aligned}$$

$k \in [0, N]$, *satisfying the initial conditions*

$$X_0 = I - \underline{\tilde{X}}^\dagger_{N+1}\underline{\tilde{X}}_{N+1}, \qquad U_0 = \alpha I + \underline{\tilde{X}}^\dagger_{N+1}\underline{\tilde{X}}_{N+1},$$

such that $X_k^T U_k$ *is symmetric for all* $k \in [0, N+1]$, *the* P*-condition* (13) *holds, and the image condition*

$$x_k \in \operatorname{Im} X_k \quad \textit{for all } k \in [0, N+1]$$
$$\textit{for all admissible } (x,u) \textit{ with } \tilde{\underline{X}}_{N+1} x_0 = 0 = x_{N+1}$$

holds, where $(\tilde{\underline{X}}, \tilde{\underline{U}})$ *is the conjoined basis of the system* ($\underline{\mathrm{S}}$), *which has the coefficient matrix* $\underline{\mathcal{S}}_k$ *defined in* (16), *with the initial conditions* $(\tilde{\underline{X}}_0, \tilde{\underline{U}}_0) = (I, 0)$.

Proof. Assume $\mathcal{F}_0(x,u) \geqslant 0$ over $x_0 = 0 = x_{N+1}$. Then the conclusion follows from Corollary 1(iii) upon taking $F_k \equiv 0$. Note that in this case $(\tilde{\underline{X}}, \tilde{\underline{U}}) \equiv (\tilde{X}, \tilde{U})$, because system ($\underline{\mathrm{S}}$) is identical with (S).

Conversely, assume that (X,U) satisfies the conditions of this theorem. Then

$$\underline{\mathcal{C}}_k X_k + \underline{\mathcal{D}}_k U_k = \mathcal{C}_k X_k + \mathcal{D}_k U_k + F_k(\mathcal{A}_k X_k + \mathcal{B}_k U_k) = U_{k+1},$$

that is, (X,U) is a conjoined basis of the system ($\underline{\mathrm{S}}$), where the coefficient matrix $\underline{\mathcal{S}}_k$ is defined in (16). Note that $\operatorname{rank}\binom{X_0}{U_0} = n$ because $X_0^T X_0 + U_0^T U_0 = (\alpha^2+1)I + 2\alpha \tilde{\underline{X}}_{N+1}^{\dagger} \tilde{\underline{X}}_{N+1} > 0$. Thus, by Corollary 1 applied to the system ($\underline{\mathrm{S}}$), we get that the functional $\underline{\mathcal{F}}_0(x,u) \geqslant 0$ over $x_0 = 0 = x_{N+1}$. Now with the same data $\mathcal{E}_k$, $\underline{\mathcal{E}}_k$, $\mathcal{G}_k$, $\underline{\mathcal{G}}_k$ as in Claim 3 in the proof of Theorem 1 and with $\mathcal{M}_i = \underline{\mathcal{M}}_i = I$, $\Gamma_i = \underline{\Gamma}_i = 0$ $(i = 0,1)$, we obtain from Proposition 2(i) that $\mathcal{F}_0(x,u) \geqslant 0$ over $x_0 = 0 = x_{N+1}$. The proof is now complete. □

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