Induced seismicity using Dieterich’s Rate and State Theory and comparison to the Critical Pressure Theory

Friedemann Wenzel*

KIT, Geophysical Institute, Hetzstr. 16, 76187 Karlsruhe, Germany

Abstract

A theoretical frame of seismicity evolution due to stress changes in a medium is the rate- and state-dependent theory of frictional fault motion. The evolution is expressed in the form of a differential equation, which can be used for normal stress changes on faults caused by fluid pressure changes resulting from injection. If the pressures rate is high enough the seismic activity becomes proportional to the local pressure rate in accordance with the Critical Pressure Theory. Examples including constant pressure injection, shut-in and variable injection rates lead to results comparable to the Critical Pressure Theory.

Keywords: Induced Seismicity; Rate- and State-dependent Friction; Pressure Diffusion

1. Introduction

We utilize the rate- and state-dependent theory of [1], further referred to as RST for modelling fluid injection induced seismicity. For simplicity we consider the case of fluid injection at the origin in a medium that allows isotropic pressure diffusion that depends on time and distance to the injection point only. We compare the solutions obtained by RST with those provided by the Critical Pressure Theory (CPT) developed by [2]. In the RST a step-wise change in shear stress results in a step-wise change in slip-velocity on individual faults in the vicinity of the main shock. This can be derived from the rate- and state-dependent friction law for an individual fault. From later work of

* Corresponding author. Tel.: +49-721-608-44431; fax: +49-721-71173.
E-mail address: friedemann.wenzel@kit.edu
it became evident that this step in shear stress imposed on an initial small slip-rate may or may not cause instability of the fault. The latter case applies if the step is too small. If large enough, the slip-rate will increase to the point of instability, representing an earthquake. The time required for this increase depends on the initial slip-rate and the step-size. It is the time to failure. Generally it is smallest for larger stress steps.

For the study of seismicity [1] assumes, that the faults have different initial slip speeds so that the failure times are equally distributed. The complexity of the mathematics involved stems from the fact that this equal distribution of the many faults in terms of rupture proximity has to be modelled with the initial slip-rate distribution in the rate- and state-dependent friction law. It is much easier captured by the CPT where the proximity to rupture in terms of stress enters directly the equations. Nevertheless [1] derives a general equation for seismicity changes, as compared to the background seismicity if shear and/or normal stresses are changed. The RST has been used in the simulation of aftershocks and earthquake swarms [4, 5, 6, 7, 8] and recently also utilized for seismicity modelling in geothermal reservoirs [9]. In this model the change in Coulomb failure stress [10] is combined with the RST theory. However, the RST does not require a Coulomb failure mechanism, it rather replaces it conceptually. We thus develop the theory strictly within the rate- and state-dependent friction concept without taking refuge to Coulomb failure.

The CPT has emerged from the study of induced seismicity in relation to permeability for instance in [11, 12, 13, 14].

2. Rate and State Theory for fluid injection

In an area with faults under shear stress and normal stress \((\tau, \sigma_n)\) and with background seismicity per unit volume of \(v_{\text{loc}}\) and stress rate \(\dot{\tau}_{\text{loc}}\) the induced seismicity (= events per unit time) per unit volume is \(\frac{1}{\gamma(t)} \cdot \frac{v_{\text{loc}}}{\dot{\tau}_{\text{loc}}}\).

The expression \(R(t) = \frac{1}{\gamma(t)} \cdot \frac{1}{\dot{\tau}_{\text{loc}}}\) determines how much higher the induced seismicity is as compared to the natural tectonic seismicity and is called the Seismicity Enhancement Factor (SEF). The function \(\gamma(t)\) is controlled by the equation

\[
\frac{d\gamma}{dt} = \frac{1}{A \cdot \sigma_n(t)} \left( 1 - \gamma(t) \cdot \frac{d\tau}{dt} + \gamma(t) \left( \frac{\tau(t)}{\sigma_n(t)} - \alpha \right) \frac{d\sigma_n}{dt} \right) \tag{1}
\]

If stress varies in space, the SEF changes not only with time but also with location. The parameter \(A\) is dimensionless and controls the time of a tectonic system to adjust – in a steady state sense - to a new shear stress rate.

The time constant with which this state is exponentially achieved is \(A \cdot \frac{\sigma_n}{\dot{\tau}}\); [1] claims that \(A\) can range between 0.005 and 0.02 and uses a value of \(A = 0.01\). \(\alpha\) is also dimensionless and describes the dependence of the state parameter for fault evolution of normal stress. [15] determined \(\alpha\) from laboratory experiments as 0.23. Equation (1) is derived in [1] for the study of aftershocks after stress field changes resulting from a main shock. It reproduces approximately Omori’s law. However, (1) is not restricted to this case and can be used for fluid pressure injections into a seismogenic medium, thus modelling induced seismicity. When simulating an injection process in a borehole one can make the following assumptions at a given site in the medium where the pressure changes:

The shear stress in the medium does not change during the injection period because the tectonic stress rate \(\frac{d\tau}{dt} = \dot{\tau}_{\text{loc}}\) is very small compared to the pore pressure rate due to the injection of injection. Thus the variable shear
stress \( \tau(t) \) in (1) is replaced by a constant value \( \tau_{tec} \).

Only normal stress changes according to \( \sigma_n(t) = \sigma_n^0 - p(t) \).

We further assume that the pressure is small: \( p(t) << \sigma_n^0 \).

Using a Taylor expansion of terms with \( 1/\sigma_n(t) \) one can write the equation for \( \gamma(t) \) as linear ordinary differential equation

\[
\frac{d\gamma}{dt} = \frac{1}{A \cdot \sigma_n^0} \left( 1 + \frac{p(t)}{\sigma_n^0} - \gamma(t) \cdot \left( \dot{\tau}_{tec} \cdot \left( 1 + \frac{p(t)}{\sigma_n^0} \right) + g_0 \cdot \dot{p}(t) \right) \right)
\]

(2)

\[
g_0 = \frac{\tau_{tec}}{\sigma_n^0} - \alpha
\]

The limitations on the value of \( \alpha \) require \( g_0 \) being positive. Assuming usual relations of tectonic shear to normal stress of 0.4 to 0.8 it will be in the order of 0.2 to 0.6 if \( \alpha = 0.2 \). If the induced pressure rate is much higher than the tectonic stress rate and by assuming small pressures compared to crustal normal stress we can write

\[
\frac{d\gamma}{dt} = \frac{1}{A \cdot \sigma_n^0} \cdot (1 - \gamma(t) \cdot g_0 \cdot \dot{p}(t))
\]

(3)

Equation (3) shows that \( \dot{p}(t) \) must be positive in order to find solutions for \( R(t) \) that grow with time. As soon as \( \dot{p}(t) \) is negative \( R(t) \) drops off quickly. This reflects the fact that only positive pressure rates can induce seismicity. This is in agreement with observations, where the so-called back-front of seismicity (the spatial surface behind which seismicity vanishes at a given time) has been clearly seen [16]. Whereas the CPT identifies a sharp transition in space that separates the back-front defined by \( \dot{p}(t) = 0 \) the RST provides a rapid drop with a time constant.

3. Solution for the Rate and State Theory

For most of the injection history it would be sufficient to solve the first order differential equation (3). However, the complete formula (2) is needed for simulating seismicity after very long injection times. Its solution with initial condition is
\[
\frac{d\gamma(t)}{dt} + a(t) \cdot \gamma(t) = b(t)
\]

\[
\gamma(t = 0) = \gamma_0 = \frac{1}{\dot{\tau}_{tec}}
\]

\[
a(t) = \frac{1}{A \cdot \sigma_n^0} \cdot \left( \dot{\tau}_{tec} \cdot \left( 1 + \frac{p(t)}{\sigma_n^0} \right) + g_0 \cdot \dot{p}(t) \right) = a_1 \cdot \left( 1 + \frac{p(t)}{\sigma_n^0} \right) + a_2 \cdot \dot{p}(t)
\]

\[
b(t) = \frac{1}{A \cdot \sigma_n^0} \cdot \left( 1 + \frac{p(t)}{\sigma_n^0} \right) = a_1 \cdot \left( 1 + \frac{p(t)}{\sigma_n^0} \right)
\]

\[
a_1 = \frac{\dot{\tau}_{tec}}{A \cdot \sigma_n^0} \quad a_2 = \frac{g_0}{A \cdot \sigma_n^0}
\]

\[
\gamma(t) = \frac{1}{\dot{\tau}_{tec}} \cdot e^{-\int_0^t a(\tau) d\tau} + \int_0^t b(\tau) \cdot e^{-\int_0^\tau a(\vartheta) d\vartheta} d\tau
\]

4. Approximations to the Rate and State Theory

We approximate the solution in order to get insight into its general behaviour. As the pressure – at least some distance away from the injection point - becomes much smaller than the normal stress we approximate

\[
\int_\tau^t a(\vartheta) d\vartheta \approx a_1 \cdot (t - \tau) + a_2 \cdot (p(t) - p(\tau))
\]

\[
e^{-\int_0^t a(\tau) d\tau} \approx e^{-a_1 \cdot t - a_2 \cdot p(t)}
\]

\[
\int_0^t b(\tau) \cdot e^{-\int_\tau^\tau a(\vartheta) d\vartheta} d\tau \approx \frac{a_1}{\dot{\tau}_{tec}} \cdot \int_0^t e^{-a_1 \cdot (t - \tau) - a_2 \cdot (p(t) - p(\tau))} d\vartheta
\]

Obviously the integration range close to \( t \) contributes most to the integral so that we replace the full exponent by

\[
p(\vartheta) \approx p(t) - \dot{p}(t) \cdot (t - \vartheta)
\]

We consider \( \dot{p}(t) \) positive as we know already that only in this case induced seismicity evolves. Thus results in
And the SEF

\[
R(t) = \frac{N(t)}{v_{tec}} \approx \frac{1}{1 + g_0 \cdot \dot{p}(t) \cdot \dot{\tau}_{tec}} \\
\frac{1}{e^{-a_1 \cdot t - a_2 \cdot \dot{p}(t) \cdot t} + \frac{1 - e^{-a_1 \cdot t - a_2 \cdot \dot{p}(t) \cdot t}}{1 + g_0 \cdot \dot{p}(t) \cdot \dot{\tau}_{tec}}}
\]

This approximate relation between induced seismicity measured by \( \frac{N(t)}{v_{tec}} \) and the evolution of pressure at a particular site within the medium that experiences pressure changes by fluid injection is a key result of this paper. It allows the following interpretation. The denominator consists of two additive terms, both being positive. For high induced seismicity the denominator must be small, which can only be achieved if both terms are small. The first term is controlled by pressure the second by its rate. If both are zero we recover the initial condition (no induced seismicity, only tectonic background). The equivalent solution of the simplified equation (3) which provides the induced seismicity as long as injection is intense is

\[
R(t) = \frac{N(t)}{v_{tec}} \approx \frac{1}{1 + g_0 \cdot \dot{p}(t) \cdot \dot{\tau}_{tec}} \\
\frac{1}{e^{-a_2 \cdot \dot{p}(t) \cdot t} + \frac{1 - e^{-a_2 \cdot \dot{p}(t) \cdot t}}{g_0 \cdot \dot{p}(t) \cdot \dot{\tau}_{tec}}}
\]

The first term of the denominator in (6) is small compared to one if

\[
p(t) > \frac{1}{a_2} \frac{A \sigma_n^0 \tau_{tec}}{\sigma_n^0} = p_{low}
\]

We can view \( p_{low} \) as minimum pressure level below which no seismicity can be induced. Frequently assumed lower values from aftershock observations are 1 to 100 kPa and lead to A-values between \( 10^{-5} \) and \( 10^{-3} \). The parameter A control thus the lower pressure threshold below which no seismicity is induced. The second term of the denominator in (6) is only small if

\[
\dot{p}(t) \cdot t > \frac{1}{a_2} \frac{A \sigma_n^0 \tau_{tec}}{\sigma_n^0} = p_{low}
\]
Even if the pressure is high enough \( p(t) > p_{low} \) there may be no seismicity induced as the rate is not high enough. The RST causes a pressure-rate- and pressure-state-dependent seismicity. In order to better understand the both condition on seismicity generation we consider the case of constant pressure injection.

\[
p(r,t) = \frac{q_0}{4\pi Dr} \left(1 - \text{erf} \left( \frac{r}{\sqrt{4Dt}} \right) \right)
\]

with diffusion constant D. For large times steady state is reached with

\[
p(r,t \to \infty) = p_{\text{max}}(r) = \frac{q_0}{4\pi Dr}
\]

and the radius below which no increased seismicity should be observed is

\[
r_{\text{max}} = \frac{q_0}{4\pi D \cdot p_{low}}
\]

At locations \( r < r_{\text{max}} \) pressure always exceeds the required minimum after some time and seismicity will be triggered. However, because of the second condition on the pressure rate

\[
\frac{\partial p(r,t)}{\partial t} \cdot t = \frac{q_0}{\sqrt{t} \cdot (4\pi D)^{3/2}} \cdot \exp \left( -\frac{r^2}{4Dt} \right)
\]

and because of

\[
[\dot{p} \cdot t]_{t \to \infty} = 0
\]

induced seismicity will always vanish after some time as the rate becomes too small. The conditions on the pressure rate allows induced seismicity only around time \( t_f \) which scales with \( r^2 / D \). This can be viewed as the ‘arrival time’ of the pressure ‘front’. If both conditions (7a,b) apply (6) can be simplified to

\[
R(t) = \frac{N(t)}{v_{tec}} \approx g_{0} \cdot \frac{\dot{p}(t)}{\dot{t}_{tec}}
\]

5. **Comparison to the Critical Pressure Theory**

\( N(t) \) in (8) refers to the seismicity per unit volume at a given site where the pressure evolves as \( \dot{p}(t) \). For an entire injection volume one has to write
where the pressure evolution is provided by the diffusion equation with proper initial and boundary conditions and the integration is restricted to the volume $V_p$ defined such that all parts of the volume are excluded where $\dot{p}(t) \leq 0$. $v(t)$ is the rate of induced seismic events exceeding magnitude $m_1$ in the entire volume.

These expressions are very similar to the formulae developed by [2] for constant fluid mass injection rate. The RST thus predicts an evolution of seismicity that is in accordance with the well-established CPT. In the simplest case of a constant injection rate $Q_0$ in a homogeneous space with homogeneously distributed permeability and with homogeneously and isotropically distributed faults the seismicity rate for events above a lower magnitude of $m_1$ would be expressed in the approximate RST as

$$v_1 = g_0 \cdot \frac{V_{tec}}{\dot{\tau}_{tec}} \cdot \frac{Q_0}{S} \approx \frac{V_{tec}}{\dot{\tau}_{tec}} \cdot \frac{Q_0}{S}$$

with the seismicity rate for magnitude in excess of $m_1$, a tectonic shear stress rate of $\dot{\tau}_{tec}$, and the injection rate $Q_0$. We used the assumption that $g_0$ is of order 1. The equivalent expression in the CPT is

$$v_1 = \frac{n_1}{\Delta C} \cdot Q_0$$

with $n_1$ density of faults that can be triggered as earthquakes above magnitude $m_1$ and $\Delta C$ the stress range parameter. Comparison of both expressions shows that if we postulate

$$\frac{V_{tec}}{\dot{\tau}_{tec}} \approx \frac{\dot{n}_1}{\Delta C} \approx \frac{n_1}{\Delta C}$$

the expressions become equivalent. Indeed, the tectonic background seismicity rate per unit volume in the RST theory is nothing but the available faults triggered per unit time and unit volume in the CPT. Similarly the tectonic background shear stress rate can be equated with the change of stress per unit time.

6. Numerical Examples

The numerical tests all assume a point source injection with variable pressure history in a homogeneously permeable medium. The pressure evolution in the medium follows the diffusion equation (diffusion constant is 0.1 m/s²) in radial symmetry. The evolution of seismicity with time is derived using the integration according to (9). Formula (3) is tested for several injection time histories and compared to CPT solutions. For a constant pressure injection we find no difference to the CPT. For the case of constant pressure injection for 1½ hours and shut-in after 8½ hours Fig. 1 shows pressure at distances 30, 50, and 100m and the total evolution of seismicity (9) with time with the almost constant seismicity after the maximum injection pressure is reached until shut-in and an Omori-type decay [16] thereafter. The CPT solutions do not include a lower pressure threshold for seismicity generation.
Fig. 1 Left: Pressure evolution at tree distances from the injection point. Input pressure at injection point raising from zero to 10 MPa 1 ½ hours, staying constant for 7 hours and dropping rapidly to zero thereafter. Note the reduction of maximum pressure with distance and the shift of the location of the maximum. Right: Seismicity calculated with RST, equations (3) and (9) shows increase of seismicity with increasing pressure, an almost constant level of activity before shut-in at 8 ½ hours and an Omori-type of decay after this.

The slight decay of seismicity after t reached its maximum is caused by the minimum pressure below which no seismicity is triggered. Differences between RST and CPT emerge when more complex injection time histories are used. For the results shown in Fig. 2 the pressure was increased and decreased between 0 to 10 MPa in two cycles during 27 hours so that the first maximum occurred at 8 hours, the second at 21 hours. The CPT solution reproduces the periodic injection history in the seismicity evolution with the maxima at the around the same times and on equal level (right panel). The RST solution shows similar behaviour for the first cycle, but shifts the maximum of seismicity of the second cycle from 21 to 23 hours and also reduces the level of seismicity. The difference is related to the fact that CPT, in the form we used here, has infinite recharging time of triggered faults. A once triggered location will not be triggered again. RST does not make this assumption and leads to a smaller and delayed second peak.

Fig. 2. Left: RST solution for periodic fluid injection showing the first seismicity maximum at the first maximum of injection pressure, but the second maximum shifted to higher time and lowered in amplitude. Right: CPT solution for periodic fluid injection showing maxima of seismicity very close to maxima of injection pressure.
7. Summary and Conclusions

The Rate and State Theory of [1] has been utilized for simulating fluid-induced seismicity without using the concept of Coulomb failure stress. When we approximate the solutions of the differential equations we recover the Critical Pressure Theory of [2]. Numerical tests indicate that the similarity of rate and state approach and Critical Pressure Theory holds beyond the approximations, although for more complex injection histories we also find differences. They are most likely caused by the different recharging options used in Rate and State Theory as compared to Critical Pressure Theory.

Acknowledgements

I am indebted to Serge Shapiro who has provided a number of valuable comments, which significantly improved the manuscript.

References