# Some Aspects of the Theory of Norms

Chi-Kwong Li*

*Department of Mathematics*
*The College of William and Mary*
*Williamsburg, Virginia 23187*

Submitted by Leiba Rodman

ABSTRACT

Let $\mathcal{V}$ be a finite dimensional vector space. Motivated by theory or applications, one might want to consider different kinds of norms on $\mathcal{V}$. In this paper we discuss some results and problems involving different classes of norms on a vector space studied by this author in the past few years. The paper consists of five sections. Section 1 concerns the conditions on two vectors $x, y \in \mathcal{V}$ satisfying $\|x\| \le \|y\|$ for all $\|\cdot\|$ in a certain class of norms. Section 2 concerns the isometry groups of $G$-invariant norms, i.e., norms $\|\cdot\|$ that satisfy $\|g(x)\| = \|x\|$ for all $x \in \mathcal{V}$ and for all $g \in G$, where $G$ is a group of unitary (orthogonal) operators on $\mathcal{V}$. Section 3 concerns $G$-invariant norms that satisfy some special properties. Section 4 concerns the best approximation(s) $x_0 \in \mathcal{T}$ of $y$, where $y \notin \mathcal{T} \subseteq \mathcal{V}$, with respect to different kinds of norms. Additional open problems, topics, and references are mentioned in Section 5.

## 1. COMPARISON OF NORMS OF TWO VECTORS

The content of this section is based on the talk of the author presented in the Third ILAS (International Linear Algebra Society) Conference held at Pensacola, Florida in March 1993. Several personal remarks on mathematics research are also included. This informal style of presentation will be changed after this section.

*Research supported by NSF grant DMS 91 00344 and a faculty research grant of The College of William and Mary. This research was done while the author was spending his sabbatical leave at the University of Toronto in the academic year 1992–1993.




655 Avenue of the Americas, New York, NY 10010 0024-3795/94/$7.00

### 1.1. *Beginning Remarks*

It is a great honor for me to be invited to give a talk in the Third ILAS Conference. In the invitation letter, I was asked to

> attempt to maintain an exposition that will be understandable to a broad spectrum of linear algebraists, be broad based in scope, and tie in with as many other areas as is possible.

In fact, choosing a topic with the several features mentioned in the invitation letter is not just the desire of the organizer; it is also my favorite idea for doing research. Some of the reasons why I like to do linear algebra research are: most linear algebra problems are very accessible, can be approached by both elementary and advanced techniques, and have interactions with and applications to many other branches of mathematics and sciences. [For further discussion and elaboration along this line, see Brualdi (1992), Li (1994a), Li and Tsing (1992), Fuhrmann (1992) Thompson (1992), Tucker (1993).]

In the past few years, I have been working on the topics of *linear preserver problems* and *numerical ranges,* which seem to fit the request of the organizer very well. However, the first topic was the main theme of a mini-symposium in the Fourth International Matrix Theory Conference held at Auburn in 1990, and has been discussed by me in the Second ILAS Conference held at Lisbon in 1992. Moreover, a monograph on the topic has appeared as a special issue of *Linear and Multilinear Algebra* (Pierce et al., 1992). The second topic was the main theme of a mini-symposium in the Fourth SIAM Conference on Applied Linear Algebra held at Minneapolis in 1991, and a workshop held in Williamsburg in 1992. Furthermore, a special issue of *Linear and Multilinear Algebra*[1] will be devoted to that topic as well. For these reasons, I am convinced that another topic should be considered now. Here is one problem that might be appropriate for our purpose: Let $\mathcal{V}$ be a finite dimensional vector space, and let $\mathcal{S}$ be a class of norms defined on $\mathcal{V}$. Determine the conditions on a pair of vectors $x, y \in \mathcal{V}$ such that

$$\|y\| \leq \|x\| \qquad \textit{for all } \|\cdot\| \textit{in } \mathcal{S}.$$

### 1.2. *Motivation*

Let $\mathcal{V} = M_{m,n}(\mathbb{F})$, the set of all $m \times n$ matrices over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. It is convenient to assume $m \geq n$ in our discussion. If $m = n$, we use the notation $M_n(\mathbb{F})$. A norm $\|\cdot\|$ on $\mathcal{V}$ is *unitarily invariant* if $\|UAV\| = \|A\|$ for all $A \in \mathcal{V}$ and all unitary (orthogonal) matrices $U$ and $V$. This topic

[1]The numerical range and numerical radius, *Linear and Multilinear Algebra* 37(1–3) 1994, Special Editors: T. Ando and C. K. Li.

has been extensively studied in the last few decades [e.g., see Horn and Johnson (1991), Li and Tsing (1987), Marshall and Olkin (1979), Mirsky (1960), and von Neumann (1937)]. In particular, it is known that for any unitarily invariant norm $\|\cdot\|$, $\|A\|$ depends solely on the singular values $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$ of $A$. Recall that the singular values of $A \in M_{m,n}(\mathbb{F})$ are the nonnegative square roots of the eigenvalues of the matrix $A^*A$. Moreover, Fan (1951) proved the following interesting result.

THEOREM 1.1. *Let $A, B \in M_{m,n}(\mathbb{F})$. Then $\|B\| \leq \|A\|$ for all unitarily invariant norms $\|\cdot\|$ if and only if $\Sigma_{i=1}^k \sigma_i(A)$ for all $k = 1, \ldots, n$.*

It is not hard to check that for any positive integer $k \leq n$ the function on $\mathcal{V}$ defined by $\|X\|_k = \Sigma_{i=1}^k \sigma_i(X)$ is a unitarily invariant norm; it is sometimes referred to as the *Ky Fan k-norm*

The result of Ky Fan naturally suggests the following problems:

Let $\mathcal{V}$ be a finite dimensional vector space, and $\mathcal{S}$ be a class of norms defined on $\mathcal{V}$.

(I) Determine the conditions on a pair of vectors $x, y \in \mathcal{V}$ such that

$$\|y\| \leq \|x\| \qquad \textit{for all } \|\cdot\| \textit{in } \mathcal{S}. \tag{1.1}$$

(II) Find a "small" subset $\mathcal{T}$ of $\mathcal{S}$ such that (1) holds whenever

$$\|y\| \leq \|x\| \qquad \textit{for all } \|\cdot\| \textit{in } \mathcal{T}.$$

Let

$$\mathcal{B}_{\mathcal{S}}(x) = \{y \in \mathcal{V} : \|y\| \leq \|x\| \quad \textit{for all } \|\cdot\| \textit{in } \mathcal{S}\}.$$

Using this notation, problems (I) and (II) can be restated as:

(I) Determine $\mathcal{B}_{\mathcal{S}}(x)$ for a given $x \in \mathcal{V}$.
(II) Find a "small" subset $\mathcal{T}$ of $\mathcal{S}$ such that $\mathcal{B}_{\mathcal{S}}(x) = \mathcal{B}_{\mathcal{T}}(x)$ for all $x \in \mathcal{V}$.

When $\mathcal{V} = M_{m,n}(\mathbb{F})$ and $\mathcal{S}$ is the set of all unitarily invariant norms, the result of Ky Fan gives very nice answers to questions (I) and (II). Let us look at some more results of this type in the following.

Let $\mathcal{V} = \mathbb{F}^n$. A *symmetric gauge function* on $\mathcal{V}$ is a norm $\|\cdot\|$ that satisfies

$$\|(x_1, \ldots, x_n)^t\| = \|(\mu_1 x_{i_1}, \ldots, \mu_n x_{i_n})^t\|$$

for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$, and for any $\mu_i \in \mathbb{F}$ satisfying $|\mu_i| = 1$. For $p \geq 1$, the $\ell_p$ norms defined by $\|(x_1, \ldots, x_n)^t\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ are examples of symmetric gauge functions. We have the following result (see 7.4.45 of [Horn and Johnson (1991)] that gives a nice answer to our question.

THEOREM 1.2. *Let $x, y \in \mathbb{F}^n$. Then $\|y\| \leq \|x\|$ for all symmetric gauge functions $\|\cdot\|$ if and only if for all $k = 1, \ldots, n$,*

$$\max\left\{\sum_{j=1}^{k} |y_{i_j}| : 1 \leq i_1 < \cdots < i_k \leq n\right\} \leq \max\left\{\sum_{j=1}^{k} |x_{i_j}| : 1 \leq i_1 < \cdots < i_k \leq n\right\}.$$

Similar to the result of Ky Fan, one can find a finite subset $\mathcal{T}$ of $\mathcal{S}$ such that (1) holds in this case. However, such a nice situation may not occur in general, as shown in the next example.

Let $\mathcal{V} = M_n(\mathbb{C})$ or $\mathcal{H}_n$, where $\mathcal{H}_n$ is the real linear space of $n \times n$ complex hermition matrices. A norm on $\mathcal{V}$ is *unitary similarity invariant* or *weakly unitarily invariant* if $\|U^*AU\| = \|A\|$ for all $A \in \mathcal{V}$ and for all unitary $U$ [e.g., see Bhatia and Holbrook (1987), Fong and Holbrook (1983), Li and Tsing (1989a)]. Clearly, every unitarily invariant norm on $M_n(\mathbb{C})$ is unitary similarity invariant. The *numerical radius* defined by

$$\begin{aligned} r(A) &= \max\{|x^*Ax| : x \in \mathbb{C}^n, x^*x = 1\} \\ &= \max\{|\mathrm{tr}(E_{11}U^*AU)| : U \textit{unitary}\} \end{aligned}$$

is an example of unitary similarity invariant norm that is not unitarily invariant. More generally, for a given $C \in \mathcal{V}$ satisfying $C \neq \mu I$ and tr $C \neq 0$, the *C-numerical radius* defined by

$$r_C(A) = \max\{|\mathrm{tr}(CU^*AU)| : U \textit{unitary}\}$$

is a unitary similarity invariant norm that is not unitarily invariant [see Goldberg and Straus (1979), Marcus and Sandy (1982), Tam (1986), and Li and Tsing (1989a)]. In fact, the concept of $C$-numerical radius is very useful in the study of unitary similarity invariant norms [e.g., see Mathias (1989) and Li and Tsing (1989a)]. Related to problems (I) and (II), we have the following result [see Li and Tsing (1989a)].

THEOREM 1.3. *Let $\mathcal{V} = M_n(\mathbb{C})$ or $\mathcal{H}_n$, and let $\mathcal{S}$ be the set of all unitary similarity invariant norms on $\mathcal{V}$.*

(i) *For any* $A \in \mathcal{V}, \mathcal{B}_{\mathcal{S}}(A) = \text{conv}\,\{\mu U^*AU : \mu \in \mathbb{F}, |\mu| = 1, U$ *unitary*$\}$, *where* conv *denotes the "convex hull of."*
(ii) *If* $\mathcal{T}$ *is the collection of all C-numerical radii with* $C \in \mathcal{V}$ *satisfying* $C \neq \mu I$ *and tr* $C \neq 0$, *then* $\mathcal{B}_{\mathcal{T}}(A) = \mathcal{B}_{\mathcal{S}}(A)$ *for any* $A \in \mathcal{V}$.
(iii) *There is no finite set* $\mathcal{T}$ *such that* $\mathcal{B}_{\mathcal{T}}(A) = \mathcal{B}_{\mathcal{S}}(A)$ *for any* $A \in \mathcal{V}$.

### *1.3. A General Result on G-Invariant Norms*

In this section, we describe a general result that embraces all the examples mentioned in Section 1.2. The discussion is based on some results in Li and Tsing (1991b).

Let $G$ be a subgroup of the unitary (or orthogonal) group on the vector space $\mathcal{V}$. A norm $\|\cdot\|$ on $\mathcal{V}$ is *G-invariant* if $\|g(x)\| = \|x\|$ for all $x \in \mathcal{V}$ and all $g \in G$. For example, if $G$ is the group of all $n \times n$ generalized permutation matrices, i.e., those matrices of the form $DP$, where $D$ is a diagonal unitary (or orthogonal) matrix and $P$ is a permutation matrix, then $G$-invariant norms on $\mathbb{F}^n$ are just symmetric gauge functions. If $G$ is the collection of all the linear operators on $M_{m,n}(\mathbb{F})$ of the form $A \mapsto UAV$ for some fixed unitary (or orthogonal) $U$ and $V$, then $G$-invariant norms on $M_{m,n}(\mathbb{F})$ are just unitarily invariant norms. Similarly, if $G$ is the collection of all linear operators on $\mathcal{V} = M_n(\mathbb{C})$ or $\mathcal{H}_n$ of the form $A \mapsto U^*AU$ for some fixed unitary $U$, then $G$-invariant norms are just unitary similarity invariant norms.

Let $c \in \mathcal{V}$. Denote by $G(c) = \{g(c) : g \in G\}$ the orbit of $c$ under $G$. If $c \in \mathcal{V}$ is such that span $G(c) = V$, then the *G(c)-radius* defined by

$$r_{G(c)}(x) = \max\{|\langle x, u\rangle| : u \in G(c)\}$$

is a $G$-invariant norm; otherwise, it is just a (semi)norm. As shown in Li and Tsing (1991b), $G(c)$-radii can be regarded as the building blocks of $G$-invariant (seminorms, because every $G$-invariant (semi)norm $\|\cdot\|$ admits a representation of the form

$$\|x\| = \sup\{r_{G(c)}(x) : c \in \mathcal{E}\} \qquad \text{for all} \quad x \in \mathcal{V},$$

where $\mathcal{E}$ is a subset of $\mathcal{V}$ depending on the given (semi)norm $\|\cdot\|$. In fact, $\mathcal{E}$ can be chosen to be a compact set if $\|\cdot\|$ is a norm.

Related to problems (I) and (II), we have the following general result.

THEOREM 1.4. *Let* $G$ *be a subgroup of the unitary* (*or orthogonal*) *group acting on the vector space* $\mathcal{V}$. *Suppose* $\mathcal{S}$ *is the collection of all G-invariant norms.*

(i) *For any* $x \in \mathcal{V}, \mathcal{B}_{\mathcal{S}}(x) = \mathrm{conv}\{\mu g(x) : \mu \in \mathbb{F}, |\mu| = 1, g \in G\}$.
(ii) *If* $\mathcal{T}$ *is the set of all* $G(c)$*-radii in* $\mathcal{S}$ *and it is not empty, then* $\mathcal{B}_{\mathcal{S}}(x) = \mathcal{B}_{\mathcal{T}}(x)$ *for all* $x \in \mathcal{V}$.
(iii) *If* $\mathcal{V}$ *is reducible under the action of the group* $G$, *there is no finite* $\mathcal{T}$ *such that* $\mathcal{B}_{\mathcal{S}}(x) = \mathcal{B}_{\mathcal{T}}(x)$ *for all* $x \in \mathcal{V}$.

One easily checks that under the action of unitary similarity the space $\mathcal{V} = M_n(\mathbb{C})$ or $\mathcal{H}_n$ consists of two irreducible subspaces, namely, the space $\mathcal{W}$ of all matrices with zero trace and the space $\mathcal{W}^{\perp}$ of scalar matrices. That is why condition (iii) in Theorem 1.3 holds.

It is of interest to determine the condition on $G$ such that none of the $G(c)$-radii is a norm. Also, it is known [see Li and Tsing, (1991b)] that the converse of (iii) in Theorem 1.4 does not hold in general. It would be nice to know the condition on $G$ for the existence of a finite $\mathcal{T} \subseteq \mathcal{S}$ such that $\mathcal{B}_{\mathcal{S}}(x) = \mathcal{B}_{\mathcal{T}}(x)$ for all $x \in \mathcal{V}$. A related problem is: prove or disprove the claim that if there exists a finite $\mathcal{T}$ such that $\mathcal{B}_{\mathcal{S}}(x) = \mathcal{B}_{\mathcal{T}}(x)$ for all $x \in \mathcal{V}$, then there is a finite $\tilde{\mathcal{T}}$ consisting of $G(c)$-radii such that $\mathcal{B}_{\tilde{\mathcal{T}}}(x) = \mathcal{B}_{\mathcal{S}}(x)$ for all $x \in \mathcal{V}$.

### *1.4. Some New Developments*

The content of this section is based on Li Tsing, and Zhang (1994).

Suppose $G$ is the trivial group consisting of the identity transformation on $\mathcal{V}$ as the only element. Then every norm on $\mathcal{V}$ can be viewed as a $G$-invariant norm. By Theorem 1.4, we have the following result.

PROPOSITION 1.5. *Let* $\mathcal{S}$ *be the set of all norms on* $\mathcal{V}$. *Then*

$$\mathcal{B}_{\mathcal{S}}(x) = \{\mu x : \mu \in \mathbb{F}, |\mu| \le 1\}.$$

F. Z. Zhang and C. R. Johnson have a proof of the special case of Proposition 1.5 when $\mathcal{V}$ is a matrix space using the theory of linear functionals. In fact, this result holds on a Banach space of any dimension [e.g., see Lemma 5 in Kovarik (1975)].

Using elementary techniques, this author and N. K. Tsing improved Proposition 1.5 and obtained Proposition 1.6 [see Li, Tsing, and Zhang (1993)]. Recall that for $p \ge 1$ the $\ell_p$-norm on $\mathcal{V}$ with respect to the basis $\{v_1, \ldots, v_n\}$ is defined by $\|x\|_p = (\sum_{i=1}^n |\lambda_i|^p)^{1/p}$ for any $x = \sum_{i=1}^n \lambda_i v_i \in \mathcal{V}$. A norm on $\mathcal{V}$ is induced by an inner product if and only if it is an $\ell_2$-norm with respect to some basis. In particular, if we use the standard bases, then the $\ell_2$-norm is known as the *Euclidean* norm on $\mathbb{F}^n$, and the *Frobenius* norm on $M_{m,n}(\mathbb{F})$.

PROPOSITION 1.6. *Let $\mathcal{S}$ be the set of all norms on $\mathcal{V}$, and let $\mathcal{T}_p$ be the set of all $\ell_p$-norms with respect to different bases for a fixed $p \geq 1$. Then $\mathcal{B}_\mathcal{S}(x) = \mathcal{B}_{\mathcal{T}_p}(x) = \{\mu x : \mu \in \mathbb{F}, |\mu| \leq 1\}$.*

Next we turn to a problem which is not covered by the $G$-invariant norm result.

Recall that a norm $\|\cdot\|$ on $M_n(\mathbb{F})$ is called a *ring norm* (or *algebra norm*) if it satisfies $\|I\| = 1$ and $\|AB\| \leq \|A\|\|B\|$ for all $A, B \in M_n(\mathbb{F})$. For the basic properties and results on ring norms, we refer the reader to Belitskii and Lyubich (1988). A norm $\|\cdot\|$ on $M_n(\mathbb{F})$ is an *induced norm* if it is defined by $\|A\| = \sup\{|Ax| : x \in \mathbb{F}^n, |x| \leq 1\}$ for a given norm $|\cdot|$ on $\mathbb{F}^n$. It is not hard to check that every induced norm is a ring norm. We are interested in studying problems (I) and (II) when $\mathcal{V} = M_n(\mathbb{F})$, and $\mathcal{S}$ is the set of all ring norms. It turns out that one can choose $\mathcal{T}$ to be the set of all induced norms, so that $\mathcal{B}_\mathcal{S}(A) = \mathcal{B}_\mathcal{T}(A)$ for all $A \in M_n(\mathbb{F})$ except when $A \in M_2(\mathbb{R})$ has no real eigenvalues. The detailed proofs of our results are contained in Li, Tsing, and Zhang (1994). In the following, we describe the process of our discovery, which is quite amusing.

First of all, if $A$ is a scalar matrix or a nilpotent matrix, one only needs to check those norms on $M_n(\mathbb{F})$ induced by $\ell_1$-norms on $\mathbb{F}^n$ with respect to different bases to obtain the following result.

PROPOSITION 1.7. *Let $\mathcal{S}$ be the set of all ring norms, and let $\mathcal{T}$ be the set of all induced norms on $M_n(\mathbb{F})$. If $A$ is a scalar matrix or a nilpotent matrix, then*

$$\mathcal{B}_\mathcal{S}(A) = \mathcal{B}_\mathcal{T}(A) = \{\mu A : \mu \in \mathbb{F}, |\mu| \leq 1\}.$$

For a general matrix $A$, the problem is more difficult. In the complex case, F. Z. Zhang observed that:

Suppose $B \in \mathcal{B}_\mathcal{S}(A)$, and suppose $S \in M_n(\mathbb{C})$ is invertible such that $S^{-1}AS = J_1 \oplus \cdots \oplus J_k$, where $J_i$ is an upper triangular Jordan block with eigenvalue $\lambda_i(A)$ for $i = 1, \ldots, k$. Then $S^{-1}BS = \hat{J}_1 \oplus \cdots \oplus \hat{J}_k$, and for each $i = 1, \ldots, k$, the $(p, q)$ entry of $\hat{J}_i$ equals $\lambda_i(B)$ if $p = q$, equals $\mu$ (independent of $i$) with $|\mu| \leq 1$ if $q = p + 1$, and equals 0 otherwise. Furthermore, $|\lambda_i(B) - \lambda_j(B)| \leq |\lambda_i(A) - \lambda_j(A)|$ for all $i, j$.

This observation clearly suggests that there should be a nice theorem behind the picture. In my past research experience one of the following situations has often arisen when I tried to prove a new theorem.

*Situation 1.* The statement of the theorem is clear, but it is hard to find the proof. In this situation, one may need to find the right idea to bring in some advanced techniques to solve the problem [this notion is due to S. Friedland; e.g. see Thompson (1992)] or to find the clever trick or observation needed in the proof (this notion probably occurs to all mathematicians); or to find the right way to fit things together (in some casual discussions with other linear algebraists, including Bapat and Uhlig, we all agreed that an interesting aspect of linear algebra research is: a nontrivial theorem can be obtained by fitting elementary results in a nice and clever way).

*Situation 2.* One has all the knowledge of a subject needed to extend the existing results. However, it is not easy to find the right formulation to state the definitions and theorems.

*Situation 3.* One knows that there *is* a theorem. The statement of the result is not quite clear; it is not even clear what type of techniques should be used. One really has to search for the result as well as for the proof.

In the study of $\mathcal{B}_S(A)$ for a general $A \in M_n(\mathbb{F})$ we were and still are in situation 3. It would be nice if one could handle the problem by the situation 1 scenario, i.e., bring in some advanced techniques or clever ideas. Anyway, let me describe our progress on the problem in the following.

Suppose $A$ is not a scalar matrix or a nilpotent matrix. There is no harm in assuming that the spectral radius $\rho(A)$ of $A$ is 1 in our problem. Under this assumption, one easily verifies that:

(a) For any ring norm $\|\cdot\|$, one has $\|A\| \geq \rho(A) = 1 = \|I\|$. Hence $I \in \mathcal{B}_S(A)$.
(b) Suppose $\|\cdot\|$ is a ring norm such that $\|rA\| = 1$ with $r \in (0,1]$. Then $\|B\| \leq \|A\|$ for any $B \in K(r,A)$, where

$$K(r,A) = \text{conv}\{\mu_i (rA)^i / r : \mu_i \in \mathbb{F}, |\mu_i| = 1, i = 0,1,2,\ldots\}.$$

Thus

$$\bigcap_{r \in (0,1]} K(r,A) \subseteq \mathcal{B}_S(A).$$

It is also clear that $\mathcal{B}_S(A) \subseteq \mathcal{B}_T(A)$. It is somewhat unexpected that the following theorem holds.

THEOREM 1.8. *Suppose $A$ is not a scalar matrix satisfying $\rho(A) = 1$.*

*Then*

$$\bigcap_{r\in(0,1]} K(r,A) = \mathcal{B}_{\mathcal{S}}(A) = \mathcal{B}_{\mathcal{T}}(A)$$

unless $A \in M_2(\mathbb{R})$ has no real eigenvalues.

Even though we have proved $\mathcal{B}_{\mathcal{S}}(A) = \mathcal{B}_{\mathcal{T}}(A)$ and have a description of this set, it is unsatisfactory that the observation of Zhang does not follow easily. We therefore took a closer look of the set $\cap_{r\in(0,1]}K(r,A)$. Notice that if $A$ has minimal polynomial $f(t) = t^m + a_{m-1}t^{m-1} + \cdots + a_0$, then all $K(r,A) \subseteq \text{span}\{I, A, \ldots, A^{m-1}\}$, the smallest algebra generated by $I$ and $A$. It turns out that

(c) $\mathcal{B}_{\mathcal{S}}(A) \subseteq \text{span}\{I, A\}$.

This is another unexpected discovery in our study. Furthermore, one can show that if $r > 0$ is sufficiently small, then $K(r,A) \subseteq \text{conv}\{\mu_i(rA)^i/r : \mu_i \in \mathbb{F}, |\mu_i| = 1, i = 0, 1, \ldots, m-1\}$. It follows that for any $\widehat{r} < r$, $K(\widehat{r}, A) \cap \text{span}\{I, A\} \subseteq K(r,A) \cap \text{span}\{I, A\}$. Consequently, we have the following result, from which the observation of Zhang follows.

THEOREM 1.9. *Suppose $A$ satisfies the hypothesis of Theorem* 1.8. *Then*

$$\bigcap_{r\in(0,1]} K(r,A) = \bigcap_{r\in[r_0,1]} K(r,A) \bigcap \text{span } \{I, A\},$$

*where $r_0$ is the smallest real number $r \in (0,1]$ such that $K(r,A) \subseteq$* conv $\{\mu_i\ (rA)^i/r : \mu_i \in \mathbb{F},\ |\mu_i| = 1, i = 0, 1, \ldots, m-1\}$.

We remark that $r_0$ can be determined by the minimal polynomial of $A$. Moreover, we suspect that the following is true:

Suppose $A$ satisfies the hypothesis of Theorem 1.8. Then

$$\bigcap_{r\in(0,1]} K(r,A) = K(r_0, A) \cap K(1, A) \cap \text{span}\{I, A\},$$

where $r_0$ satisfies the condition mentioned in Theorem 1.9.

It would be nice to prove or disprove this statement. At this point, we are still working on this problem. This conjecture has been disproved by Li, Tsing and Zhang (1994).

## 2. ISOMETRY GROUPS OF $G$-INVARIANT NORMS

Given a norm $\|\cdot\|$ on $\mathcal{V}$, the collection of linear operators $L$ satisfying $\|L(x)\| = \|x\|$ for all $x$ forms a group, which is known as the isometry group for $\|\cdot\|$. One may see Horn and Johnson (1985) for some basic properties of isometries and isometry groups. In this section, our focus is on the isometry groups of $G$-invariant norms. We shall always assume that $G$ is a closed subgroup of the unitary (orthogonal) group of $\mathcal{V}$ such that $\mu G = G$ for all $\mu \in \mathbb{F}$ with $|\mu| = 1$, unless specified otherwise.

Let us first describe some general results concerning the isometries and isometry groups of norms.

THEOREM 2.1. *A linear operator $L$ is an isometry for a norm $\|\cdot\|$ on $V$ if and only if one of the following conditions holds:*

(a) *$L(\mathcal{B}) = \mathcal{B}$, where $\mathcal{B}$ is the unit norm ball with respect to $\|\cdot\|$ in $\mathcal{V}$.*
(b) *$L(\mathcal{E}) = \mathcal{E}$, where $\mathcal{E}$ is the set of extreme points of the unit norm ball with respect to $\|\cdot\|$ in $\mathcal{V}$.*
(c) *The dual transformation $L^*$ of $L$ is an isometry for the dual norm of $\|\cdot\|$.*

By this theorem, one can use the geometry of $\mathcal{B}$ and $\mathcal{E}$ to help study $L$. Also, if $\|\cdot\|, \mathcal{B}$, and $\mathcal{E}$ are complicated, one can consider the dual norm, the dual norm ball and its extreme points, etc., to get information about $L^*$. For example, if a $G(c)$-radius is a norm, then the set of extreme points of the dual norm ball is just $G(c)$ [e.g., see the proof of the special case in Li and Mehta (1994, Proposition 4.3)]. Thus, studying the isometries for the $G(c)$-radius is the same as studying those linear operators mapping $G(c)$ onto itself [see Li and Tsing (1988c, 1988d, 1991a)]. In fact, this is true even if $r_{G(c)}$ is not a norm as shown in the following result [see Li and Tsing (1991b)].

THEOREM 2.2. *Let $c \in V$. A linear operator $L$ on $\mathcal{V}$ is an isometry for the $G(c)$-radius if and only if its dual transformation $L^*$ satisfies $L^*(G(c)) = G(c)$.*

Sometimes it is easier to study the whole isometry group instead of an individual isometry for a given norm. The following result [e.g., see Deutsch and Schneider (1974) and Li and Tsing (1990a)] is very useful if one uses this approach.

THEOREM 2.3. *Suppose $\mathcal{K}$ is a bounded group of linear operators acting on $\mathcal{V}$. There exists an invertible linear operator $S$ such that $S^{-1}\mathcal{K}S$ is a subgroup of the group of unitary operators acting on $\mathcal{V}$. Moreover, if $\mathcal{K}$ contains an irreducible subgroup of unitary (orthogonal) operators, then $S$ can be chosen to be the identity operator, i.e., $\mathcal{K}$ is a subgroup of the group* *of unitary operators.*

We shall make further comments on how the above results are used in our recent work in the next few sections.

It is interesting to note that the problem of determining which subgroup, of the orthogonal (unitary) group on $\mathcal{V}$ can be the isometry group of a norm is still open [see Gordon and Loewy (1979, Theorem 3.1) for a partial answer].

### *2.1. Unitarily Invariant Norms*

In this section, we concentrate on the isometry group of a unitarily invariant norm on $M_{m,n}(\mathbb{F})$. We always assume that $G$ is the collection of all linear operators on $M_{m,n}(\mathbb{F})$ of the form $A \mapsto UAV$ for some fixed unitary (orthogonal) $U$ and $V$. Denote by $\tau$ the transposition operator on $M_n(\mathbb{F})$, i.e., $\tau(A) = A^t$. On $M_4(\mathbb{R})$ denote by $\phi$ the special linear operator defined by $A \mapsto (B_1AC_1 + B_2AC_2 + B_3AC_3 + A)/2$, where

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We have the following result.

THEOREM 2.4. *The isometry group $\mathcal{K}$ of a unitarily invariant norm $\|\cdot\|$ on $\mathcal{V} = M_{m,n}(\mathbb{F})$ must be of one of the following forms:*

(a) *$\mathcal{K}$ is the unitary group on $\mathcal{V}$.*
(b) *$\mathcal{V} = M_4(\mathbb{R})$ and $\mathcal{K} = \langle G, \tau, \phi \rangle$.*
(c) $\mathcal{K} = \begin{cases} G & \text{if } m \neq n, \\ \langle G, \tau \rangle & \text{if } m = n. \end{cases}$

*Furthermore, condition* (a) *holds if and only if $\|\cdot\|$ is a multiple of the Frobenius norm; on $\mathcal{V} = M_4(\mathbb{R})$ condition* (b) *holds if and only if* (a) *does not hold and $\phi$ is an isometry of $\|\cdot\|$; and condition* (c) *holds if and only if* (a), (b) *do not hold.*

The study of isometries of unitarily invariant norms has an extended history. Schur (1925) showed that Theorem 2.4 holds for the spectral norm

$\|\cdot\|$ on $M_n(\mathbb{C})$. Russo (1969) obtained the result for the trace norm, i.e., the Ky Fan $n$-norm, on $M_n(\mathbb{C})$. Arazy (1975) studied the isometries of Schatten $p$-norms on symmetrically normed ideals. In the finite dimensional case, his result reduces to Theorem 2.4 with $\mathcal{V} = M_n(\mathbb{C})$ and $\|A\| = \{\sum_{i=1}^n \sigma_i(A)^p\}^{1/p}$, where $p \geq 1$. Grone and Marcus (1977) [see also Grone (1979), (1980)] proposed the study of the $(p,k)$-norm defined by $\|A\| = \{\sum_{i=1}^k \sigma_i(A)^p\}^{1/p}$ with $1 \leq k \leq n$ and $p \geq 1$ on $M_{m,n}(\mathbb{C})$, and obtained some partial results using geometrical techniques (cf. Theorem 2.1). A complete answer of the problem raised by Grone and Marcus was given in Li and Tsing (1988b). In his thesis Grone (1976) made the ambitious conjecture that $\mathcal{K}$ satisfies condition (c) unless $\|\cdot\|$ is a multiple of the Frobenius norm. For $\mathcal{V} = M_n(\mathbb{C})$ the conjecture was confirmed by Sourour (1981), whose result actually holds for the infinite dimensional case. In Johnson, Laffey, and Li (1988), it was shown that the linear operator $\phi$ defined in Theorem 2.4 is an isometry of the Ky Fan 2-norm on $M_4(\mathbb{R})$, using some theory of real quaternions [see also Đoković (1990) and Chang and Li (1991)]. Thus the conjecture of Grone is not valid on $M_n(\mathbb{R})$ in general. In Li and Tsing (1988d), the authors obtained the result for $G(c)$-radius on $M_{m,n}(\mathbb{F})$ and showed that $\phi$ is the only exceptional case. The main tools in the proofs is Theorem 2.2 and some differential geometry techniques. This approach was further refined by the same authors to obtain Theorem 2.4 in Li and Tsing (1990a). Later Đoković and Li (1993) used a group theoretic approach [see Li (1993a)] to obtain a general result on linear operators leaving invariant functions on singular values of matrices that covers Theorem 2.4.

### *2.2. Symmetric Gauge Functions*

In this subsection, we consider the isometries for symmetric gauge functions on $\mathbb{F}^n$. We use $G$ to denote the group of generalized permutation matrices. Since $G$ is irreducible, it follows from Theorem 2.3 that the isometry group of a given symmetric gauge function is always a subgroup of the unitary (orthogonal) group. In Chang and Li (1992) the authors characterized the isometries for $G(c)$-radii and some other special symmetric gauge functions on $\mathbb{R}^n$, using geometric techniques (cf. Theorem 2.1). It was conjectured that except when $n = 2, 4$, the isometry group is either the whole unitary group or the group $G$. This was confirmed in Đoković, Li, and Rodman (1991). The main idea of the proof is to show that except for $n = 2, 4$, there is no closed subgroup lying strictly between $G$ and the orthogonal group. Since the isometry group of a symmetric gauge function must contain $G$ and must be a subgroup of the orthogonal group, the desired conclusion follows. In the paper, we actually characterized all the possible isometry groups of symmetric gauge functions on $\mathbb{F}^n$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or the skew field of real quaternions.

THEOREM 2.5. *Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or the skew field of real quaternions. The isometry group $\mathcal{K}$ of a symmetric gauge function $\|\cdot\|$ on $\mathcal{V} = \mathbb{F}^n$ must be of one of the following forms*:

(a) *$\mathcal{K}$ is the orthogonal (unitary or sympletic) group.*
(b) *$\mathcal{V} = \mathbb{R}^4$ and $\mathcal{K} = \langle G, A\rangle$, where*

$$A = \frac{1}{\sqrt{2}}\left\{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right\}.$$

(c) *$\mathcal{V} = \mathbb{R}^4$ and $\mathcal{K} = \langle G, B\rangle$, where*

$$B = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

(d) *$\mathcal{V} = \mathbb{R}^2$, and $\mathcal{K}$ is the dihedral group with $8k$ elements.*
(e) *$\mathcal{K} = G$.*

*Furthermore, condition* (a) *holds if and only if $\|\cdot\|$ is a multiple of the Frobenius norm; on $\mathcal{V} = \mathbb{R}^4$ condition* (b) *holds if and only if* (a) *does not hold and $A$ is an isometry of $\|\cdot\|$, and condition* (c) *holds if and only if* (a) *and* (b) *do not hold and $B$ is an isometry of $\|\cdot\|$; on $\mathcal{V} = \mathbb{R}^2$ condition* (d) *holds if and only if the unit ball of $\|\cdot\|$ is a regular $4k$-sided polygon; and condition* (e) *holds if and only if* (a)–(d) *do not hold.*

Notice that in the above theorem, one can replace $\|\cdot\|$ by any function $f$ on $\mathbb{F}^n$ such that $f(Px) = f(x)$ for any $P \in G$. If the group $\mathcal{K}$ of invertible matrices $A \in M_n(\mathbb{F})$ satisfying $f(Ax) = f(x)$ for all $x$ is compact, then $\mathcal{K}$ must be one of the groups described in Theorem 2.5.

An infinite dimensional version of the Theorem 2.5 can be found in Rolewicz (1985, Theorem 9.8.3). As pointed out in Đoković, Li, and Rodman (1991), Theorem 2.5 can be deduced from the theory of reflection groups. In fact, Gordon and Lewis (1977) have used this approach to solve certain isometry problems (cf. Section 2.5) that cover Theorem 2.5 for $n \geq 13$.

### *2.3. Unitary Similarity Invariant Norms*

In this subsection, we focus on unitary similarity invariant norms on $\mathcal{V} = M_n(\mathbb{C})$ or $\mathcal{H}_n$. We let $G$ be the collection of all linear operators on $\mathcal{V}$

of the form $A \mapsto U^*AU$ for some fixed unitary $U$. As mentioned before, in this case the $G(c)$-radius reduces to the $C$-numerical radius. Concerning the isometric for $C$-numerical radius, we have the following result [see Li (1987a, b), Li and Tsing (1988a, c), Man (1991), Li, Mehta, and Rodman (1994)].

THEOREM 2.6. *Suppose $C \in M_n(\mathbb{C})$ is a normal matrix or satisfies* rank $C = 1$. *If $r_C$ is a norm on $M_n(\mathbb{C})$ and $L$ is an isometry for $r_C$, then there exist $\mu \in \mathbb{C}$ with $|\mu| = 1$ and a unitary $U$ such that $L$ is of the form $A \mapsto \mu[U^*AU + (\bar{\eta} - 1)(\text{tr } A)/I]$ or $A \mapsto \mu[U^*A^tU + (\bar{\eta} - 1)(\text{tr } A)/I]$, where $\eta$ is a $k$th root of unity such that $\eta\widehat{C}$ is unitarily similar to $\widehat{C} := C - (\text{tr } C)I/n$.*

Whether the same conclusion in Theorem 2.6 holds for general $C$ is not known. A general result concerning the isometry group of a unitary similarity invariant norm on $M_n(\mathbb{C})$ is not available. If $\mathcal{V} = \mathcal{H}_n$, we have the complete information as shown in Theorem 2.7 [see Li and Tsing(1990b)]. Note that if a unitary similarity invariant norm is induced by an inner product, then there exists a positive operator $P$ defined by $P(A) = \mu(\text{tr } A)I + \nu[nA - (\text{tr } A)I]$, where $\mu, \nu > 0$ such that $\|A\| = \langle P(A), P(A)\rangle^{1/2}$.

THEOREM 2.7. *The isometry group $\mathcal{K}$ of a unitary similarity invariant norm $\|\cdot\|$ on $\mathcal{H}_n$ must be of one of the following forms:*

(a) *There exists a positive operator $P$ on $\mathcal{H}_n$ defined by $P(A) = \mu(\text{tr } A)I + \nu[nA - (tr\ A)I]$, where $\mu, \nu > 0$ such that $P\mathcal{K}P^{-1}$ is the orthogonal group on $\mathcal{H}_n$.*
(b) *$\mathcal{K}$ is the collection of all orthogonal operators on $\mathcal{H}_n$ mapping $I$ to $\pm I$.*
(c) *$\mathcal{K} = \langle G, \tau, L\rangle$, where $L(A) = A - 2(\text{tr } A)I/n$ for all $A \in \mathcal{H}_n$.*
(d) *$\mathcal{K} = \langle G, \tau\rangle$.*

*Furthermore, condition* (a) *holds if and only if $\|\cdot\|$ is induced by an inner product on $\mathcal{H}_n$ with $\|A\|^2 = \langle P(A), P(A)\rangle$; condition* (b) *holds if and only if* (a) *does not hold and $\|A\|$ depends only on* $|\text{tr } A|$ *and* tr $(A^*A)$; *condition* (c) *holds if and only if* (a), (b) *do not hold and $L$ is an isometry of $\|\cdot\|$; condition* (d) *holds if and only if conditions* (a)–(c) *do not hold.*

We remark that Theorem 2.7 also holds if we replace $\mathcal{H}_n$ by the linear space of all $n \times n$ real symmetric matrices (see that next section).

All the results mentioned in this section were obtained by computation and geometrical techniques. It was proposed in Li (1994a) that the problem can be studied by a group theoretic approach, namely, enumerating all the subgroups between $G$ and the unitary group on $M_n(\mathbb{C})$.

*2.4. Unitary Congruence Invariant Norms*

In this subsection, we consider those norms $\|\cdot\|$ on $\mathcal{V} = M_n(\mathbb{F}), S_n(\mathbb{F})$, or $K_n(\mathbb{F})$ that satisfy $\|U^tAU\| = \|A\|$ for all $A \in \mathcal{V}$ and all unitary (orthogonal) $U$, where $S_n(\mathbb{F})$ is the linear space of all symmetric matrices and $K_n(\mathbb{F})$ is the linear space of all skew-symmetric matrices. These norms are referred to as *unitary congruence invariant norms or unitary consimilarity invariant norms.* We let $G$ be the group of linear operators on $\mathcal{V}$ of the form $A \mapsto U^tAU$ for some fixed unitary (orthogonal) $U$. The behavior of unitary congruence invariant norms on $S_n(\mathbb{R})$ is the same as that of unitary similarity invariant norms on $\mathcal{H}_n$. In particular, one can easily modify Theorem 2.7 to get a corresponding result concerning the isometry groups of unitary congruence invariant norms on $S_n(\mathbb{R})$. For the isometry group of unitary congruence invariant norms on $S_n(\mathbb{C})$ and $K_n(\mathbb{F})$, we have the following results [see Li and Tsing (1991a) and also Morita (1941, 1944)].

THEOREM 2.8. *Let $\mathcal{K}$ be the isometry group of a unitary congruence invariant norm on $S_n(\mathbb{C})$ which is not a multiple of the Frobenius norm. Then $\mathcal{K} = G$.*

THEOREM 2.9. *Let $\mathcal{K}$ be the isometry group of a unitary congruence invariant norm on $K_n(\mathbb{F})$ which is not a multiple of the Frobenius norm. Then one of the following holds:*

(a) $n = 4$ *and*

$$\mathcal{K} = \begin{cases} \langle G, \psi \rangle & \text{if} \quad \mathbb{F} = \mathbb{C}, \\ \langle G, \tau, \psi \rangle & \text{if} \quad \mathbb{F} = \mathbb{R}, \end{cases}$$

*where $\psi(A)$ is obtained from $A$ by interchanging its* (1, 4) *and* (2, 3) *entries, and interchanging its* (4,1 ) *and* (2, 3) *entries accordingly.*

(b) $\mathcal{K} = \begin{cases} \langle G \rangle & \text{if} \quad \mathbb{F} = \mathbb{C}, \\ \langle G, \tau \rangle & \text{if} \quad \mathbb{F} = \mathbb{R}. \end{cases}$

General results concerning the isometry groups of unitary congruence invariant norms on $M_n(\mathbb{F})$ are not available. The result is not known even for a $G(c)$-radius.

Notice that if $\mathbb{F} = \mathbb{C}$, the $G(c)$-radius reduces to the $c$-congruence numerical radius. It is known [see Cheng, (1990) and Li and Tsing (1991b)] that in this case a $G(c)$-radius is a norm if and only if $c \in M_n(\mathbb{C})$ is neither symmetric nor skew-symmetric. If $\mathbb{F} = \mathbb{R}$, the $G(c)$-radius reduces to the

$c$-numerical radius, and it is a norm if and only if tr $c \neq 0$, and $c - (\text{tr } c)I/n$ is neither symmetric nor skew-symmetric. While the isometry group of a $G(c)$-radius for general $c \in M_n(\mathbb{F})$ is not known, we have the characterization of those linear operators $L$ leaving the $G(C)$-radius invariant when $c \in S_n(\mathbb{F}) \cup K_n(\mathbb{F})$. Notice that in this case, the $G(c)$ radius is a seminorm.

Suppose $c \in S_n(\mathbb{C})$ is nonzero, and suppose $L$ is a linear operator on $M_n(\mathbb{C})$ leaving the $G(c)$-radius invariant. Then $r_{G(c)}(A) = 0$ if and only if $A \in K_n(\mathbb{C})$. Therefore, if $L$ is decomposed into $L_1 + L_2$, where $L_1(A) = L(A + A^t)/2$ and $L_2 = L(A - A^t)/2$, then $L_2(K_n(\mathbb{C})) \subseteq K_n(\mathbb{C})$. Furthermore, if $L_0(A) = [L_1(A) + L_1(A)^t]/2$ for all $A \in S_n(\mathbb{C})$, then $L_0$ is a linear operator on $S_n(\mathbb{C})$ leaving the $G(c)$-radius invariant, and hence Theorem 2.8 applies. Consequently, one has

THEOREM 2.10. *Suppose $c \in S_n(\mathbb{C})$, and suppose $L$ is a linear operator on $M_n(\mathbb{C})$ leaving the $G(c)$-radius invariant. Then there exist a unitary $U$ and a linear operator $\widehat{L}$ on $M_n(\mathbb{C})$ with its range lying in $K_n(\mathbb{C})$ such that $L(A) = U^t(A + A^t)U/2 + \widehat{L}(A)$ for all $A$.*

Similarly, if $c \in S_n(\mathbb{R})$, we have the following result.

THEOREM 2.11. *Suppose $c \in S_n(\mathbb{R})$, and suppose $L$ is a linear operator on $M_n(\mathbb{R})$ leaving the $G(c)$-radius invariant. Let*

$$\mathcal{W} = \begin{cases} \mathbb{R} \cdot I & \text{if } c \text{ is a scalar matrix,} \\ \{X \in S_n(\mathbb{R}) : \text{tr } X = 0\} & \text{if tr } c = 0, \\ S_n(\mathbb{R}) & \text{otherwise,} \end{cases}$$

*and let $\mathcal{W}^{\perp}$ be the orthogonal complement of $\mathcal{W}$ in $M_n(\mathbb{R})$. Then there exist an orthogonal $U$ and a linear operator $\widehat{L}$ on $M_n(\mathbb{R})$ with its range lying in $\mathcal{W}^{\perp}$ such that $L$ is of the form $A \mapsto \mu[U^t(A - A^t)U/2 + (\bar{\eta} - 1)(\text{tr } A)I/n] + \widehat{L}(A)$, where $\mu = \pm 1$ and $\eta = \pm 1$ such that $\eta\widehat{A}$ has the same spectrum as $\widehat{A} := (A - A^t)/2 - (\text{tr } A)I/n$.*

By the same argument, if $c \in K_n(\mathbb{F})$, we have the following result.

THEOREM 2.12. *Suppose $c \in K_n(\mathbb{F})$, and suppose $L$ is a linear operator on $M_n(\mathbb{F})$ leaving the $G(c)$-radius invariant. Then there exist a unitary $U$ and a linear operator $\widehat{L}$ on $M_n(\mathbb{F})$ with its range lying in $S_n(\mathbb{F})$ such that $L$ is of the form*

$$A \mapsto \mu U^t(A - A^t)U/2 + \widehat{L}(A)$$

*or*

$$A \mapsto \mu U^t \psi(A - A^t)U/2 + \widehat{L}(A),$$

*where* $\mu = \begin{cases} 1 & \text{if } \mathbb{F} = \mathbb{C}, \\ \pm 1 & \text{if } \mathbb{F} = \mathbb{R}. \end{cases}$

It would be nice to solve the general problem on $M_n(\mathbb{F})$.

*2.5. Permutation Invariant Norms*

In this subsection, we consider those norms $\|\cdot\|$ on $\mathcal{V} = \mathbb{F}^n$ that satisfy $\|Qx\| = \|x\|$ for all $x \in \mathcal{V}$ and all permutation matrices $Q$. We shall let $G$ be the group generated by all permutation matrices and the scalar matrices $\mu I$ with $|\mu| = 1$. Suppose $c \in \mathbb{F}^n$. The $G(c)$-radius is defined by

$$r_{G(c)}(x) = \max\{|c^*Px| : P \text{ a permutation matrix}\}.$$

It is a norm on $\mathbb{F}^n$ if and only if $c \neq \mu e$ and $c^*e \neq 0$, where $e = (1, \ldots, 1)^t \in \mathbb{F}^n$. Note much is known about the isometry group of this type of $G$-invariant norm. Let $e_i$ denote the $i$th column of $I_n$ for $i = 1, \ldots, n$. Denote by $J_n$ the $n \times n$ matrix with all entries equal to one. We have the following result [see Li and Mehta (1994)].

THEOREM 2.13. *Let* $c(c_1, \ldots, c_n)^t \in \mathbb{R}^n$ *with entries arranged in descending order. Suppose the* $G(c)$*-radius is a norm on* $\mathbb{R}^n$, *and* $\mathcal{K}$ *is the corresponding isometry group. Then one of the following holds*:

(a) *There exists* $S = \alpha I + \beta J$ *such that* $Sc = e_1$ *and* $S^{-1}\mathcal{K}S$ *is the group of* $n \times n$ *real generalized permutation matrices.*
(b) $n$ *is odd, and there exists* $S$ *of the form* $\alpha I + \beta J$ *such that* $Sc = \sum_{i \le n/2} e_i$ *and* $S^{-1}\mathcal{K}S = \langle G, B\rangle$, *where*

$$B = \begin{pmatrix} & & 0 \\ I_{n-2} - \frac{2}{n-1} J_{n-1} & & \vdots \\ & & 0 \\ \frac{n-3}{n-1} \cdots \frac{n-3}{n-1} & & 1 \end{pmatrix}.$$

(c) $c_i + c_{n-i+1}$ *are all equal for* $i = 1, \ldots, n$, *and* $\mathcal{K} = \langle G, I, -(2/n)J\rangle$.
(d) $c$ *does not satisfy the conditions described in* (a)–(c), *and* $\mathcal{K} = G$.

The proof of this result in Li and Mehta (1994) is purely computational. R. Loewy has informed this author that Gordon and Lewis (1977) have used the theory of reflection groups [see Benson and Grove (1985) for basic definitions] to characterize isometry groups of general permutation invariant norms on $\mathbb{R}^n$ for $n \geq 13$. Currently, this author and W. Whitney are extending the techniques in Gordon and Lewis (1977) to give a complete solution of the problem [see Li and Whitney (1994)].

## 3. SPECIAL TYPES OF $G$-INVARIANT NORMS

### *3.1. Norms Induces by Inner Products*

In this subsection, we consider those $G$-invariant norms that are induced by inner products. We have the following general result (Li and Tsing, 1989c) (cf. Theorem 3.1).

THEOREM 3.1. *Let $\mathcal{V} = \mathcal{W}_1 \oplus + \cdots + \oplus \mathcal{W}_k$ be mutually orthogonal $G$-irreducible subspaces. Then $\|\cdot\|$ is a $G$-invariant norm on $\mathcal{V}$ induced by an inner product if and only if there is positive definite $H = (h_{ij}) \in S_k(\mathbb{R})$, where $h_{ij} = 0$ whenever $\mathcal{W}_i$ and $\mathcal{W}_j$ are not isomorphic, such that for any $v = w_1 + \cdots + w_k$ with $w_i \in \mathcal{W}_i$ we have $\|v\|^2 = \sum_{i,j} h_{ij} \langle x_i, x_j \rangle$.*

Notice that by Theorem 3.1, if $\mathcal{V}$ is irreducible under the action of $G$, then a $G$-invariant norm on $\mathcal{V}$ is induced by an inner product if and only if it is a multiple of the Frobenius norm.

Using Theorem 3.1, Li and Tsing (1989c) re-proved several known results in Li and Tsing (1987) and Bhatia and Holbrook (1987), and obtained some new ones. In the following we mention some of the results related to the $G$-invariant norms mentioned in the previous sections.

THEOREM 3.2.

(a) *Suppose $\|\cdot\|$ is a symmetric gauge function on $\mathbb{F}^n$, or a unitarily invariant (unitary congruence) norm on $M_{m,n}(\mathbb{F})$, ($S_n(\mathbb{C})$ or $K_n(\mathbb{F})$). Then $\|\cdot\|$ is induced by an inner product if and only if it is a positive multiple of the Frobenius norm.*
(b) *A permutation invariant norm $\|\cdot\|$ on $\mathbb{F}^n$ is induced by an inner product if and only if there exist $\mu, \eta \in \mathbb{R}$ with $\eta > 0$ and $n\mu + \eta > 0$ such that*

$$\|x\|^2 = \mu |x^t e|^2 + \eta (x^* x) \qquad \text{for all} \quad x \in \mathbb{F}^n.$$

(c) *A unitary similarity invariant norm $\|\cdot\|$ on $\mathcal{V} = M_n(\mathbb{C}), \mathcal{H}_n$, or $S_n(\mathbb{R})$ is induced by an inner product if and only if there exist $\mu, \eta \in \mathbb{R}$ with*

*$\eta > 0$ and $n\mu + \eta > 0$ such that*

$$\|A\|^2 = \mu|\mathrm{tr}\, A|^2 + \eta\, \mathrm{tr}\, (AA^*) \quad \textit{for all} \quad A \in \mathcal{V}.$$

(d) *An orthogonal similarity invariant norm $\|\cdot\|$ on $M_n(\mathbb{R})$ is induced by an inner product if and only if there exists $\mu, \eta, \nu \in \mathbb{R}$ with $\eta > |\nu|$ and $n\mu + \eta + \nu > 0$ such that*

$$\|A\|^2 = \mu(tr\ A)^2 + \eta\, tr\ (AA^t) + \nu\, tr\ (A^2) \qquad \textit{for all} \quad A \in M_n(\mathbb{R}).$$

*3.2. Norms Induces by Other G-Invariant Norms*

Suppose $\|\cdot\|$ is a norm on $\mathcal{V}$. For any invertible linear operator $S$ on $\mathcal{V}$, one can consider the norm $\|x\|_S$ defined by $\|\cdot\|_S = \|S(x)\|$. One would like to know when $\|\cdot\|_S$ is a $G$-invariant norm. We have the following result (Hemasinha, Weaver, and Li, 1992, Theorem 2.1).

THEOREM 3.3. *Let $\|\cdot\|$ be a norm on $\mathcal{V}$ with $\mathcal{K}$ as the isometry group. Suppose $S$ is an invertible linear operator on $\mathcal{V}$. Then $\|\cdot\|$ is a $G$-invariant norm if and only if $SGS^{-1} \subseteq \mathcal{K}$.*

In Hemasinha, Weaver, and Li (1992), the authors studied the cases when $\|\cdot\|$ is a symmetric gauge function on $\mathbb{F}^n$ and $G$ is the group of diagonal unitary (orthogonal) matrices, the group of permutation matrices, or the group of generalized permutation matrices. In other words, for a given symmetric gauge function $\|\cdot\|$, the authors determined the condition on $S \in M_n(\mathbb{F})$ such that the induced norm $\|\cdot\|_S$ is an absolute norm, a permutation invariant norm, or a symmetric gauge function. R. Hemasinha has been studying the problem for a full rank $S \in M_{m,n}(\mathbb{F})$, i.e., $\|\cdot\|$ is a norm on $\mathbb{F}^m$ [see Hemasinha (1993)].

Note that a norm on $\mathbb{F}^n$ is a symmetric gauge function if and only if it is absolute and is permutation invariant. A few years ago, Cheng raised the question whether a norm on $M_n(\mathbb{C})$ is unitarily invariant if and only if it is unitary similarity invariant and unitary congruence invariant. The author of this paper gave an affirmative answer to this question, and communicated the result to Cheng (see [Cheng (1991b); cf. Theorem 4.4.1]).

THEOREM 3.4. *A norm on $M_n(\mathbb{C})$ is unitarily invariant if and only if it is both unitary similarity invariant and unitary congruence invariant.*

*Proof.* The necessity part is clear. To prove the converse, suppose $\|\cdot\|$ on $M_n(\mathbb{C})$ is both unitary similarity invariant and unitary congruence invariant. Suppose $A \in M_n(\mathbb{C})$ and $U, V$ are unitary. We show that there exist unitary $X, Y, D$ such that $D$ is diagonal, $U = X^*DY^*$, and $V = YDX$. It will then follow that $\|UAV\| = \|X^*D^tY^*AYDX\| = \|A\|$.

Now, since $UV$ is unitary, there exist a unitary matrix $W$ and a diagonal unitary matrix $D$ such that $UV = WD^2W^*$. One easily check that $X = (DW^*)$ and $Y = (U^*W)$ satisfy the required conditions. ■

By Theorem 2.5, we know that all the possible isometry groups of unitarily invariant norms. With Theorem 3.3, one can deduce the following result.

THEOREM 3.5. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{V} = M_{m,n}(\mathbb{F})$ with isometry group $\mathcal{K}$, and let $S$ be an invertible linear operator on $M_{m,n}(\mathbb{F})$. Then $\|\cdot\|$ is a unitarily invariant norm if and only if $S$ belongs to the normalizer of $\mathcal{K}$ in the group of invertible linear operators on $\mathcal{V}$.*

One may further the study of this problem for other types of $G$-invariant norms.

## 4. APPROXIMATION PROBLEMS

Let $\mathcal{T}$ be a subset of $\mathcal{V}$. Suppose $y \in \mathcal{V}\backslash\mathcal{T}$. It is of interest to determine $x_0 \in \mathcal{T}$ such that

$$\|y - x_0\| \leq \|y - x\| \qquad \text{for all} \quad x \in \mathcal{T}$$

for a given norm or a given class of norms. The element $x_0$ is known as the *best approximation* of $y$ from the set $\mathcal{T}$. There is a lot of research on this topic; e.g., see Marshall and Olkin (1979), Singer (1970), and their references.

To give a concrete example of this type of results, we mention the following result, which is a slight extension of Theorem 4.1 in Li and Tsing (1987).

THEOREM 4.1. *Let $\|\cdot\|$ be a norm on $\mathcal{V}$. Suppose $f : \mathcal{V} \to \mathcal{V}$ such that $f(x+y) = f(x) + f(y), f \circ f(x) = x$, and $\|f(x)\| = \|x\|$, for all $x$ and $y$ in $\mathcal{V}$. If $\mathcal{T} = \{x \in \mathcal{V} : f(x) = x\}$, then for any $y \in \mathcal{V}\backslash\mathcal{T}$*

$$\|y - x_0\| \leq \|y - x\| \qquad \textit{for all} \quad x \in \mathcal{T},$$

*where*

$$x_0 = [f(x) + x]/2 \in \mathcal{T}.$$

One may take $\|\cdot\|$ to be any unitarily invariant norm on $M_n(\mathbb{C})$ or any $C$-numerical radius on $M_n(\mathbb{C})$ with $C = C^*$; the conclusion of Theorem 4.1 is valid if $f(A) = \pm A^*$, $\pm\overline{A}$, or $\pm A^t$. In fact, for a given $G$-invariant norm $\|\cdot\|$ and a given transformation $f$ on $\mathcal{V}$, one has $\|f(x)\| = \|x\|$ for all $x \in \mathcal{V}$ if and only if there is a compact set $\mathcal{E} \subseteq \mathcal{V}$ such that $f(\mathcal{E}) = \mathcal{E}$ and

$$\|x\| = \max\left\{r_{G(c)}(x) : x \in \mathcal{E}\right\} \qquad \text{for all} \quad x \in \mathcal{V}.$$

Another situation of interest is when $\|\cdot\|$ is a $G$-invariant norm and $\mathcal{T} = G(x)$ or conv $G(x)$. We list some results and open problems in the following.

THEOREM 4.2. *Let $G$ be the group of generalized permutation matrices over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Suppose $x, y \in \mathbb{F}^n$ are such that $y \notin G(x)$. Let $P, Q \in G$ be such that $Py$ and $Qx$ have nonnegative entries arranged in descending order. Then for any symmetric function $\|\cdot\|$ on $\mathbb{F}^n$,*

$$\|y - P^{-1}Qx\| \le \|y - z\| \le \|y - (-P^{-1}Q)x\| \qquad \textit{for all} \quad z \in G(x)$$

*and*

$$\|y - x_0\| \le \|y - z\| \qquad \textit{for all} \quad z \in \text{conv } G(x),$$

*where $x_0 \in$ conv $G(x)$ is constructed by the following algorithm:*

Step 1. *Set $\Delta = Py - Qx$.*
Step 2. *If $\Delta_1 \ge \cdots \ge \Delta_n$, then go to step 4. Otherwise go to step 3.*
Step 3. *Let $1 < k \le \ell \le n$ be such that*

$$\Delta_1 \ge \cdots \ge \Delta_{k-1} \le \Delta_k = \cdots = \Delta_\ell \neq \Delta_{l+1},$$

*and let $j$ be the smallest integer such that $\Delta_j = \Delta_{k-1}$. Then for $i = j, \ldots, \ell$, replace $\Delta_i$ by $(\sum_{p=\ell}^{l} \Delta_p)/(\ell - j + 1)$. Go to step 2.*
Step 4. *For $i = 1, \ldots, n$, replace $\Delta_i$ by 0 if $\Delta_i < 0$. Then set $x_0 = y - P^{-1}\Delta$.*

Theorem 4.2 was proved in Cheng (1991a). By the fact that there is a one-one correspondence between unitarily invariant norms on $M_{m,n}(\mathbb{F})(m \ge n)$ and symmetric gauge functions on $\mathbb{R}^n$ [e.g., see Mirsky (1960)], one can deduce the following result from Theorem 4.2.

THEOREM 4.3. *Let $G$ be the group of linear operators on $M_{m,n}(\mathbb{F})(m \ge n)$ of the form $A \mapsto UAV$ for some unitary $U$ and $V$. Suppose $X, Y \in$*

*$M_{m,n}(\mathbb{F})$, and suppose $P, Q, R, S$ are unitary matrices such that $PXQ = \sum_{i=1}^n \sigma_i(X)E_{ii}$ and $RYS = \sum_{i=1}^n \sigma_i(Y)E_{ii}$. Then for any unitarily invariant norm $\|\cdot\|$ on $M_{m,n}(\mathbb{F})$,*

$$\|Y - R^*PXQS^*\| \le \|Y - Z\| \le \|Y - (-R^*PXQS^*)\| \qquad \text{for all } Z \in G(X),$$

*and*

$$\|Y - R^* \left(\sum_{i=i}^n \mu_i E_{ii}\right) S^*\| \le \|Y - Z\| \qquad \text{for all } Z \in \operatorname{conv} G(X),$$

*where $x_0 = (\mu_1, \ldots, \mu_n)^t$ is obtained by the algorithm of Theorem 4.2 on setting $x = \sigma(X)$ and $y = \sigma(Y)$ in step 1.*

By the same idea, one can apply Theorem 4.2 to obtain results for unitary congruence invariant norms on $S_n(\mathbb{C})$ and $K_n(\mathbb{F})$ [see Cheng (1991a)]. In fact, the study of Cheng (1991a) was motivated by an earlier paper by Li and Tsing (1989b). In that paper, the authors considered unitary similarity invariant norms on $\mathcal{V} = \mathcal{H}_n$ or $S_n(\mathbb{R})$. The proofs actually cover the problem of permutation invariant norms on $\mathbb{R}^n$. We have the following results.

THEOREM 4.4. *Let $G$ be the group of permutation matrices. Suppose $x, y \in \mathbb{R}^n$ are such that $y \notin G(x)$. Let $P, Q, \widehat{Q} \in G$ be such that $Py$ and $Qx$ have entries arranged in descending order, and $\widehat{Q}x$ has entries arranged in ascending order. Then for any permutation invariant norm $\|\cdot\|$ on $\mathbb{F}^n$,*

$$\|y - P^{-1}Qx\| \le \|y - z\| \le \|y - (P^{-1}\widehat{Q})x\| \qquad \text{for all } z \in G(x),$$

*and*

$$\|y - x_0\| \le \|y - z\| \qquad \text{for all } z \in \operatorname{conv} G(x),$$

*where $x_0 \in \operatorname{conv} G(x)$ is constructed by the following algorithm:*

Step 1. *Set $\Delta = Py - Qx$.*

Step 2. *If $\Delta_1 \ge \cdots \ge \Delta_n$, then go to step 4. Otherwise go to step 3.*

Step 3. *Let $1 < k \le \ell \le n$ be such that*

$$\Delta_1 \ge \cdots \ge \Delta_{k-1} < \Delta_k = \cdots = \Delta_\ell \ne \Delta_{\ell+1},$$

*and let $j$ be the smallest integer such that $\Delta_j = \Delta_{k-1}$. Then for $i = j, \ldots, l$, replace $\Delta_i$ by $(\sum_{p=j}^{\ell} \Delta_p)/(\ell - j + 1)$. Go to step 2.*

Step 4. *Set* $x_0 = y - P^{-1}\Delta$ .

Since there is a one-one correspondence between permutation invariant norms and unitary similarity invariant norms on $\mathcal{V} = \mathcal{H}_n$ or $S_n(\mathbb{R})$, one can apply Theorem 4.4 to obtain the following result. We shall use $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))^t$ to denote the vector of eigenvalues of $A \in \mathcal{V}$ with entries arranged in descending order.

THEOREM 4.5. *Let* $G$ *be the group for linear operators on* $\mathcal{V} = \mathcal{H}_n$ *or* $S_n(\mathbb{R})$ *of the form* $A \mapsto U^*AU$ *for some unitary* $U$. *Suppose* $X, Y \in \mathcal{V}$, *and suppose* $V, W, R$ *are unitary matrices such that* $V^*XV = \sum_{i=1}^n \lambda_i(X) E_{ii}$, $W^*XW = \sum_{i=1}^n \lambda_{n-i+1}(X)E_{ii}$, *and* $R^*YR = \sum_{i=1}^n \lambda_i(Y)E_{ii}$. *Then for any unitary similarity invariant norm* $\|\cdot\|$ *on* $\mathcal{V}$,

$$\|Y - RV^*XVR^*\| \leq \|Y - Z\| \leq \|Y - RW^*XWR^*\| \qquad \text{for all} \quad Z \in G(X)$$

*and*

$$\left\|Y - R\left(\sum_{i=1}^n \mu_i E_{ii}\right) R^*\right\| \leq \|Y - Z\| \qquad \text{for all} \quad Z \in \operatorname{conv} G(X),$$

*where* $x_0 = (\mu_1, \ldots, \mu_n)^t$ *is obtained by the algorithm of Theorem* 4.4 *on setting* $x = \lambda(X)$ *and* $y = \lambda(Y)$ *in step* 1.

The problems are open if we consider unitary congruence invariant norms on $M_n(\mathbb{C})$ or unitary similarity invariant norms on $M_n(\mathbb{F})$. The problems are much more difficult if we consider $\mathcal{T}$ to be a union of orbits, or the convex hull of a union of orbits. For example, if $\mathcal{V} = M_n(\mathbb{C}), G$ is the group in Theorem 4.5, and $\mathcal{T}$ is the union of $G(D)$ for all diagonal $D$, then finding $D_0$ in $\mathcal{T}$ that best approximates a given $A \in \mathcal{V}$ is the same as finding the best normal approximation of $A$. This is known to be a difficult problem even using the Frobenius norm [see Ruhe (1987)].

There are many other interesting approximation problems on matrix spaces. For example, if

$$\mathcal{T} = \left\{ \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \in M_{m,n}(\mathbb{F}) : X \in M_{p,q}(\mathbb{F}) \right\},$$

then finding a best approximation in $\mathcal{T}$ for a given

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in M_{m,n}(\mathbb{F}) \qquad \text{with} \quad A_2 \in M_{p,q}(\mathbb{F})$$

is equivalent to finding a minimum norm completion of the partial matrix

$$\begin{pmatrix} A_1 & A_2 \\ X & A_4 \end{pmatrix},$$

where $X \in M_{p,q}(\mathbb{F})$ is unspecified. It is known [e.g., see Davis, Kahan, and Weinberger (1982) and also Paulsen, Power, and Smith (1989)] that if $\|\cdot\|$ is the spectral norm, i.e., the Ky Fan 1-norm, then the minimum value of

$$\left\| \begin{pmatrix} A_1 & A_2 \\ X & A_4 \end{pmatrix} \right\| \qquad \text{with} \quad X \in M_{p,q}(\mathbb{F})$$

coincides with the obvious lower bound

$$\max \left\{ \|(A_1 \; A_2)\|, \left\| \begin{pmatrix} A_2 \\ A_4 \end{pmatrix} \right\| \right\}.$$

One may consider the minimum norm completion problem for other norms. For example, finding a minimum numerical radius completion for a given partial matrix in $M_n(\mathbb{C})$ is highly nontrivial even for the $2 \times 2$ case [see Choi and Li (1993)]. In fact, the problem is still open for $n \geq 3$.

Also, characterizing the best approximation elements is of interest [see Zietak (1993) and its references].

## 5. OTHER PROBLEMS AND RELATED TOPICS

In this section, we list a few more general problems on norms that the author has been involved in the last few years. We only give a brief description, either because better references on the problems are available elsewhere, or because not many results are available yet.

### *5.1. Inequalities*

Given two norms $|\cdot|$ and $\|\cdot\|$ on $\mathcal{V}$, it is of interest to find the largest $\alpha$ and the smallest $\beta$ such that

$$\alpha|x| \leq \|x\| \leq \beta|x| \qquad \text{for all} \quad x \in \mathcal{V}.$$

Suppose $\circ$ is a product on $\mathcal{V}$, i.e., a binary operation on $\mathcal{V}$ that is continuous with respect to the given norm $\|\cdot\|$. One may be interested in knowing the smallest $\nu$ such that

$$\nu\|x \circ y\| \leq \nu\|x\|\nu\|y\| \qquad \text{for all} \quad x, y \in \mathcal{V}.$$

A related problem is to determine the smallest $\nu$ such that

$$\nu\|x^k\| \leq (\nu\|x\|)^k \qquad \text{for all} \quad x \in \mathcal{V}, \quad k \geq 2,$$

where $x^k = \underbrace{x \circ \cdots \circ x}_{k}$.

There are many results and open problems on this topic. The work of this author is mainly on unitary similarity invariant norms and has been summarized in Li (1994b). For related topics, one may see Stone (1962), Marcus and Sandy (1985), Goldberg and Straus (1982, 1983), Gottlieb, Johnson, and Spitkovsky (1994), Johnson and Li (1988), Li (1986, 1991), R. C. Li (1993) and Okubo (1993).

### *5.2. Geometrical Structure of the Unit Ball*

To study problems related to a norm, it is very useful to know the geometrical structure of the unit norm balls. For examples, in the study of isometry problems or inequalities involving norms, knowing the characterization of the extreme points of the unit balls is very useful [see Chang and Li (1992), Li and Mehta (1994a, b, & c), and Li and Tsing (1988a, d)]. Recently, some authors, including So (1990), Zietak (1988), and de Sa (1994a, b, c) have studied the facial structure of the norm balls. While the problems on unitarily invariant norms are quite well studied, not much is known for other types of $G$-invariant norms.

### *5.3. Other Sources of Problems and References*

There are many other results and problems involving different types of norms. See, for example, the excellent monographs by Belitskii and Lyubich (1988), Bhatia (1987), Schatten (1950, 1961), and Stewart and Sun (1992), as well as the lecture note series by Lindenstrauss and Milman (1985–90).

*Note added in proof*: Dr. Beata Randrianantoanina has brought our attention to the excellent survey on isometry problems by Fleming and Jamison (1994). These are several references, namely Braverman and Semenov (1974) and Schneider and Turner (1973), that are closely related to our discussion in Section 2.

## REFERENCES

1 Arazy, J. 1975. The isometry of $C_p$, *Israel J. Math.* 22:247–256.
2 Belitskii, G. R. and Lyubich, Yu. I. 1988. *Matrix Norms and Their Applications*, Oper. Theory Adv. Appl. 36, Birkhäuser.
3 Benson, C. T. and Grove, L. C. 1985. *Finite Reflection Groups*, 2nd ed. Springer-Verlag, New York.

4 Bhatia, R. 1987. *Perturbation Bounds for Matrix Eigenvalues*, Pitman Res. Notes Math. Ser., Longman.
5 Bhatia, R. and Holbrook, J. A. R. 1987. Unitary invariance and spectral variation, *Linear Algebra Appl.* 95:43–68.
6 Brualdi, R. A. 1992. The symbiotic relations of combinatorics and matrix theory, *Linear Algebra Appl.* 162–164:65–105.
7 Chang, S. S. and Li, C. K. 1991. A special linear operator on $M_4(\mathbb{R})$, *Linear and Multilinear Algebra* 30:65–75.
8 Chang, S. S. and Li, C. K. 1992. Certain isometries on $\mathbb{R}^n$, *Linear Algebra Appl.* 165:251–261.
9 Cheng, C. M. 1990. Two results on $C$-congruence numerical radii, *Tamkang J. Math.* 21:59–64.
10 Cheng, C. M. 1991a. An approximation problem with respect to symmetric gauge functions with application to matrix spaces, *Linear and Multilinear Algebra* 29:169–180.
11 Cheng, C. M. 1991b. *Some results on Eigenvalues, Singular Values and Orthostochastic Matrices*, Ph.D. Thesis, Univ. of Hong Kong.
12 Choi, M. D. and Li, C. K. Some basic properties of the numerical radius, in preparation.
13 Davis, C., Kahan, W. and Weinberger, H. 1982. Norm-preserving dilations and their applications to optimal error bounds, *SIAM J. Numer. Anal.* 19:455–469.
14 de Sa, E. M. 1994a., Faces and Traces of the unit ball of a symmetric gauge function, *Linear Algebra Appl.*, 197–198:349–396.
15 de Sa, E. M. 1994b., Exposed faces and duality for synnetric and unitarily invariant norms, LAA 197–198:429–450.
16 de Sa, E. M. 1994c., Faces of the unit ball of a unitarily Invariant norm, LAA 197–198:451–494.
17 Deutsch, E. and Schneider, H. 1974. Bounded groups and norm-hermitian matrices, *Linear Algebra Appl.* 9:9–27.
18 Đoković, D. Ž. 1990. On isometries of matrix spaces, *Linear and Multilinear Algebra* 27:73–78.
19 Đoković, D. Ž. and Li, C. K. 1993. Overgroups of some classical linear groups with applications to linear preserver problems, *Linear Algebra Appl.*, to appear.
20 Đoković, D. Ž., Li, C. K. and Rodman, L. 1991. Isometries of symmetric gauge functions, *Linear and Multilinear Algebra* 30:81–92.
21 Fan, K. 1951. Maximum properties and inequalities for the eigenvalues of completely continuous operators, *Proc. Nat. Acad. Sci. U.S.A.* 37:760–766.
22 Fleming, R. J. and Jamison, J. E., 1994. Isometries on Bauach spaces—a Survey, preprint.
23 Fong, C. K. and Holbrook, J. A. R. 1983. Unitarily invariant operator

norms, *Canad. J. Math.*, 35(2):274–299.

24 Fuhrmann, P. A. 1992. Functional models in linear algebra, *Linear Algebra Appl.* 162–164:107–151.

25 Goldberg, M. and Straus, E. 1979. Norm properties of $C$-numerical radii, *Linear Algebra Appl.* 24:113–131.

26 Goldberg, M. and Straus, E. 1982. Operator norms, multiplicativity factors and $C$-numerical radii, *Linear Algebra Appl.* 43:137–159.

27 Goldberg, M. and Straus, E. 1983. Multiplicativity factors and $C$-numerical radii, *Linear Algebra Appl.* 54:1-16.

28 Gordon, Y. and Lewis, D. R. 1977. Isometries of diagonally symmetric Banach spaces, *Israel J. Math.* 28:45–67.

29 Gordon, Y. and Loewy, R. 1979. Uniqueness of $(\Delta)$ bases and isometries of Banach spaces, *Math Ann.* 241:159–180.

30 Grone, R. 1976. *Isometries of Matrix Algebras*, Ph.D. Thesis, Univ. of California, Santa Barbara.

31 Grone, R. 1979. The invariance and partial isomatries, *Indiana Math. J.* 28:445–449.

32 Grone R., 1980. Certain isometries of rectangular complex matrices, *Linear Algebra Appl.* 29:161–171.

33 Grone, R. and Marcus, M. 1977. Isometries of matrix algebras, *J. Algebra* 47:180–189.

34 Hemasinha, R. 1993. The sign invariance of certain norms on $\mathbb{R}^n$, *Linear and Multilinear Algebra* 35:135–151.

35 Hemasinha, R., Weaver, J., and Li, C. K. 1992. Norms induced by symmetric gauge functions, *Linear and Multilinear Algebra* 31:217–224.

36 Horn, R. A. and Johnson, C. R. 1985. *Matrix Analysis*, Cambridge U.P., New York.

37 Horn, R. A. and Johnson, C. R. 1991. *Topics in Matrix Analysis*, Cambridge U.P., New York.

38 Johnson, C. R., Laffey, T. J., and Li, C. K. 1988. Linear transformations on $M_n(\mathbf{R})$ that preserve the Ky Fan $k$-norm and a remarkable special case when $(n,k) = (4,2)$, *Linear and Multilinear Algebra* 23:285–298.

39 Johnson, C. R. and Li, C. K. 1988. Inequalities relating unitarily invariant norms and the numerical radius, *Linear and Multilinear Algebra* 23:183–191.

40 Johnson, C. R., Spitkovsky, and Gottlieb, S., I. M. 1994. Some inequalities involving the numerical radius, *Linear and Multilinear Algebra*, 37, 13–24.

41 Kovarik, Z. V. 1975. Spectrum localization in Banach spaces *II, Linear Algebra Appl.* 12:223–229.

42 Li, C. K. 1986. On the higher numerical radius and spectral norm, *Linear Algebra Appl.* 80:55–70.

43 Li, C. K. 1987a. A generalization of spectral radius, numerical radius, and

spectral norm, *Linear Algebra Appl.* 90:105–118.

44 Li, C. K. 1987b. Linear operators preserving the numerical radius of matrices, *Proc. Amer. Math. Soc.* 99(4):601–608.

45 Li, C. K. 1987c. Linear operators preserving the higher numerical radius of matrices, *Linear and Multilinear Algebra* 21:63–73.

46 Li, C. K. 1991. Inequalities relating norms invariant under unitary similarities, *Linear and Multilinear Algebra* 29:155–167.

47 Li, C. K. 1994a. A class of linear preserver problems and some useful approaches, *Linear Algebra Appl.*, 197–198:817–824.

48 Li, C. K. 1994b. $C$-numerical ranges and $C$-numerical radii, *Linear and Multilinear Algebra*, 37:051–82.

49 Li, C. K. and Mehta, P. P. 1994. Permutation invariant norms, *Linear Algebra Appl.*, to appear.

50 Li, C. K., Mehta, P., and Rodman, L. 1994. A generalized numerical range: The range of a constrained sesquilinear form, *Linear and Multilinear Algebra*, 37:25–50.

51 Li, C. K. and Tsing, N. K. 1987. On unitarily invariant norms and related results, *Linear and Multilinear Algebra* 20:107–119.

52 Li, C. K. and Tsing, N. K. 1988a. Linear operators that preserve the $C$-numerical range or radius of matrices, *Linear and Multilinear Algebra* 23:27–46.

53 Li, C. K. and Tsing, N. K. 1988b. Some isometries of rectangular complex matrices, *Linear and Multilinear Algebra* 23:47–53.

54 Li, C. K. and Tsing, N. K. 1988c. Duality between some linear preservers problems:The invariance of the $C$-numerical range, the $C$-numerical radius and certain matrix sets, *Linear and Multilinear Algebra* 23:353–362.

55 Li, C. K. and Tsing, N. K. 1988d. Duality between some linear preserver problems II: Isometries with respect to $c$-spectral norms and matrices with fixed singular values, *Linear Algebra Appl.* 110:181–212.

56 Li, C. K. and Tsing, N. K. 1989a. Norms that are invariant under unitary similarities and the $C$-numerical radii, *Linear and Multilinear Algebra* 24:209–222.

57 Li, C. K. and Tsing, N. K. 1989b. Distance to the convex hull of unitary orbit with respect to unitary similarity invariant norms, *Linear and Multilinear Algebra* 25:93–103.

58 Li, C. K. and Tsing, N. K. 1989c. $G$-invariant hermitian forms and $G$-invariant elliptical norms, *SIAM J. Matrix Anal. Appl.* 10:435–445.

59 Li, C. K. and Tsing, N. K., 1990a. Linear operators preserving unitarily invariant norms of matrices, *Linear and Multilinear Algebra* 26:119–132.

60 Li, C. K. and Tsing, N. K. 1990b. Linear operators preserving unitary similarity invariants norms on matrices, *Linear and Multilinear Algebra* 27:213–224.

61 Li, C. K. and Tsing, N. K. 1991a. Duality between some linear preserver problems III: $c$-spectral norms on (skew)-symmetric matrices and matrices with fixed singular values, *Linear Algebra Appl.* 143:67–97.

62 Li, C. K. and Tsing, N. K. 1991b. $G$-invariant norms and $G(c)$-radii, *Linear Algebra Appl.* 150:179–194.

63 Li, C. K. and Tsing, N. K. 1992. Linear preserver problems: A brief introduction and some special techniques, *Linear Algebra Appl.* 162–164:217–236.

64 Li, C. K. and Tsing, N. K., and Zhang F. Z. 1994. Comparison of norms of two vectors, preprint.

65 Li, C. K. and Whitney, W. 1994. Symmetric closed overgroups of $S_n$ in $O_n$, preprint.

66 Li, R. C. 1993. Norms of certain matrices with applications to variations of the spectra of matrices and matrix pencils, *Linear Algebra Appl.* 182:199–234.

67 Lindenstrauss, J. and Milman, V. 1985–90. *Geometric Aspects of Functional Analysis*, Springer-Verlag Notes 1267, 1317, 1376, 1469.

68 Man, W. Y. 1991. The invariance of $C$-numerical range, $C$-numerical radius and their dual problems, *Linear and Multilinear Algebra* 30:117–128.

69 Marcus, M. and Sandy, M. 1982. Three elementary proofs of the Goldberg-Straus theorm on numerical radii, *Linear and Multilinear Algebra* 11:243–252.

70 Marcus, M. and Sandy, M. 1985. Singular values and numerical radii, *Linear and Multilinear Algebra* 18:337–353.

71 Marshall, A. W. and Olkin, I. 1979. *Inequalities: Theory of Majorization and its Applications*, Academic, New York.

72 Mathias, R. 1989. A Representation of Unitary Similarity Invariant Norms, Technical Report 515, Dept. of Mathematical Sciences, Johns Hopkins Univ., Blatimore.

73 Mirsky, L. 1960. Symmetric gauge functions and unitarily invariant norms, *Quart. J. Math.* 11:50–59.

74 Morita, K. 1941. Analytical characterization of displacements in general Poubcaré space, *Proc. Imperial Acad.* 17:489–494.

75 Morita, K. 1944. Schwarz's lemma in a homogeneous space of higher dimension, *Japan J. Math.* 19:45–56.

76 Okubo, K. 1993. Some equality conditions with respect to the dual norm of the numerical radius, *Linear and Multilinear Algebra* 34:365–376.

77 Paulsen, V., Power, S., and Smith, R. 1989. Schur product and matrix completions, *J. Funct. Anal.* 85:151–178.

78 Pierce, S. et al., 1992. *A survey of linear preserver problems, Linear and Multilinear Algebra* 33, Nos. 1–2.

79 Rolewicz, S. 1985. *Metric Linear Spaces*, 2nd ed., PWN—Polish Scientific

Publications.
80 Ruhe, A. 1987. Closest normal matrix finally found, BIT 27:585–598
81 Russo, B. 1969. Trace preserving mappings of matrix algebra, *Duke Math. J.* 36:297–300.
82 Schatten, R. 1950. *A Theory of Cross-Spaces*, Princeton U.P., Princeton, N.J.
83 Schatten, R. 1961. *Norm Ideals of Completely Continuous Operators*, Ergeb. Math. N.F. 27, Springer-Verlag, Berlin.
84 Schur, I. 1925. Einige Bemerkungen zur Determinanten Theorie, *Sitzungsber. Preuss. Akad. Wiss. Berlin* 25:454–463.
85 Singer, I. 1970. *Best Approximation in Normal Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag, Berlin.
86 So, W. 1990. Facial structures of Schatten $p$-norms, *Linear and Multilinear Algebra* 27:207–212.
87 Sourour, A. R. 1981. Isometries of norm ideals of compact operators. *J. Funct. Anal.* 43:69–77.
88 Stewart, G. W. and Sun, J. G. 1992. *Matrix Perturbation Theory*, Academic, New York.
89 Stone, B. J. 1962. Best possible ratios of certain matrix norms, *Numer. Math.* 4:114–116.
90 Tam, B. S. 1986. A simple proof of Goldberg-Straus theorem on numerical radii, *Glasgow Math. J.* 28:139–141.
91 Thompson, R. C. 1980. The congruence numerical range, *Linear and Multilinear Algebra* 18:197–206.
92 Thompson, R. C. 1992. High, low, and quantitative roads in linear algebra, *Linear Algebra Appl.* 162–164:23–64.
93 Tucker, A. 1993. The growing importance of linear algebra in undergraduate mathematics, *College Math. J.* 24:3–9.
94 von Neumann, J. 1937. Some matrix-inequalities and metrization of matrix-space, *Tomsk Univ. Rev.* 1:286–300.
95 Zietak, K. 1988. On the characterization of the extremal points of the unit sphere of matrices, *Linear Algebra Appl.* 106:57–75.
96 Zietak, K. 1993. Properties of linear approximations of matrices in the spectral norm, *Linear Algebra Appl.* 183:41–60.

*Received 9 June 1993; final manuscript accepted 30 October 1993*