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Edge Reductions in Cyclically k-Connected Cubic Graphs

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This paper examines edge reductions in cyclically k-connected cubic graphs, focusing on when they preserve the cyclic k-connectedness. For a cyclically k-connected cubic graph G, we denote by $N_k(G)$ the set of edges whose reduction gives a cubic graph which is not cyclically k-connected. With the exception of three graphs, $N_k(G)$ consists of the edges in independent k-edge cuts. For this reason we examine the properties and interactions between independent k-edge cuts in cyclically k-connected cubic graphs. These results lead to an understanding of the structure of $G[N_k]$. For every k, we prove that $G[N_k]$ is a forest with at least k trees if G is a cyclically k-connected cubic graph with girth at least $k+1$ and $N_k \neq \emptyset$. Let $f_k(v)$ be the smallest integer such that $|N_k(G)| \leqslant f_k(v)$ for all cyclically k-connected cubic graphs G on v vertices. For all cyclically 3-connected cubic graphs G such that $6 \leqslant v(G)$ and $N_3 \neq \emptyset$, we prove that $G[N_3]$ is a forest with at least three trees. We determine f_3 and state a characterization of the extremal graphs. We define a very restricted subset N_4^b of N_4 and prove that if $N_4^g = N_4 - N_4^b \neq \emptyset$, then $G[N_4^g]$ is a forest with at least four trees. We determine f_4 and state a characterization of the extremal graphs. There exist cyclically 5-connected cubic graphs such that $E(G) = N_5(G)$, for every v such that $10 \leqslant v$ and $16 \neq v$. We characterize these graphs. Let $g_k(v)$ be the smallest integer such that $|N_k(G)| \leqslant g_k(v)$ for all cyclically k-connected cubic graphs G with v vertices and girth at least $k+1$. For $k \in \{3, 4, 5\}$, we determine g_k and state a characterization of the extremal graphs.

1. INTRODUCTION

Edge reductions in cubic graphs were first used in a significant way by Steinitz and Rademacher [25] to prove that a graph is planar, cubic, and 3-connected if and only if it is the graph of a simple 3-polytope. In fact, they chose edge reductions which preserved 3-connectedness.

The definition of cyclic connectivity first appears in Tutte [30]. The concept of cyclic k-connectivity in cubic graphs has appeared in the theory

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developed to solve the Four Colour Conjecture. In 1852, Guthrie [5] conjectured that every planar bridgeless graph was 4-face colourable. Tait [27] in 1880 showed that this conjecture was equivalent to the statement that every planar bridgeless cubic graph is 3-edge coulourable. As well, Tait [26] showed that the Four Colour Conjecture would follow if every 3-connected cubic graph was hamiltonian. But in 1946 Tutte [29] constructed a nonhamiltonian planar 3-connected cubic graph. Later Tutte [30] and Walther [31] constructed nonhamiltonian planar cubic graphs that were cyclically 4-connected and cyclically 5-connected, respectively. The work of Isaacs [16] and Goldberg [14] has shown that 4-edge chromatic bridgeless cubic graphs which are not cyclically 5-connected can be constructed using smaller 4-edge chromatic bridgeless cubic graphs. Thus, the nontrivial 4-edge chromatic bridgeless cubic graphs are cyclically 5-connected.

Edge reductions of cubic graphs which preserve the cyclic k-connectedness are useful as a proof technique. For example, such edge reductions can be used to prove that any S of three independent edges in a 3-connected cubic graph is contained in the edge set of some cyclc if S is not an edge cut.

In the rest of the introduction we give some background and prove some results needed in later sections. In Section 2 we examine the structure of $G[N_k]$, where G is a cyclically k-connected cubic graph and N_k is the set of edges which are in independent k-edge cuts. In section k we examine edge reductions in cyclically k-connected cubic graphs, $k = 3, 4, 5$. Edge reductions in cyclically k-connected cubic graphs with girth at least $k + 1$, where $3 \leqslant k$, are examined in Section 6.

We use the notation and terminology of Bondy and Murty [7] in this paper. In particular, for a graph G, $\nu(G)$ is the number of vertices of G, $\varepsilon(G)$ is the number of edges of G, $\omega(G)$ is the number of components of G, $d_G(x)$ is the degree of x in G, and δ_G is the minimum degree of G. In addition, define $\nu_i(G)$ to be the number of vertices of degree i in graph G, $0 \leqslant i$, define p_j to be the path with j vertices, $1 \leqslant j$, and define $d_G(e, f)$ to be the shortest distance in graph G between an end of edge e and an end of edge f.

The graph in Fig. $x.y.z$ will be referred to as $G_{(x.y.z)}$. In the figures we will use large circles and ovals to represent subgraphs and we will refer to them as *clouds*. For example, Fig. 5.2 has five clouds. Frequently, we will use notation in a proof which is defined by an accompanying figure.

Let A_i be a graph such that $\nu_2 = k$ and $\nu_3 = \nu - k$, and let $x_{i_1}, ... x_{i_k}$ be its vertices of degree two, $i = 1, 2$. We define $H^k(A_1, A_2)$ to be $A_1 + A_2 + \{x_{1_j} x_{2_j} \mid j = 1, ..., k\}$. We note that A_1 and A_2 do not always uniquely determine $H^k(A_1, A_2)$. Depending on the context, we will use $H^k(A_1, A_2)$ to refer to an arbitrary such graph or to all such graphs.

Let G be a graph and let $X \subseteq V(G)$ and $S \subseteq E(G)$. If $|X| = k$ and $2 \leqslant \omega(G - X)$, then X is called a *k-vertex cut.* If $|S| = k$ and $2 \leqslant \omega(G - S)$, then S is called a *k-edge cut.* If $|X| = k$ and G has subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$, $V(G_1 \cap G_2) = X$, and both G_1 and G_2 contain cycles, then X is called a *cycle-separating k-vertex cut.* If $|S| = k$ and $G - S$ has at least two components which contain cycles, then S is called a *cycle-separating k-edge cut.*

If G has a pair of nonadjacent vertices and every vertex cut of G contains at least k vertices, then G is said to be *k-vertex connected.* If G has a spanning complete subgraph and $k - 1 \leqslant v(G)$, then G is also said to be *k-vertex connected.* If every edge cut of G contains at least k edges, then G is said to be *k-edge connected.* If every cycle-separating vertex (edge) cut of G has size at least k, then G is said to be *cyclically k-vertex (edge) connected.*

If G is a cubic graph, then G has a cycle C. Since $G = C \cup G$, $V(C \cap G) = V(C)$, and both C and G have cycles, $V(C)$ is a cycle-separating vertex cut. Hence there is a largest k such that G is cyclically k-vertex connected, and k is at most the girth of G.

Let e be in $E(G)$. If H is a subgraph of graph G and e has one end in $V(H)$ and one end in $V(G) - V(H)$, then we say that *e is incident with H.* If e is incident with disjoint subgraphs H_1 and H_2 of G, then we say that *e joins* H_1 and H_2.

Let θ be the loopless graph on two vertices with exactly three edges.

The following two theorems demonstrate that various forms of connectivity are equivalent for all but a finite number of cubic graphs. Both have routine proofs. The first statement of Theorem 1.2 is a corollary of a result of Lovász [18; 19, Exercise 10.4].

THEOREM 1.1. *If G is a connected cubic graph such that $G \neq \theta$ and $1 \leqslant k \leqslant 3$, then the following statements are equivalent:*

(i) *G is k-edge connected.*

(ii) *G is k-vertex connected.*

(iii) *G is cyclically k-vertex connected.*

THEOREM 1.2. *Let G be a connected cubic graph. G has two disjoint cycles if and only if $G \notin \{\theta, K_4, K_{3,3}\}$. If $G \notin \{\theta, K_4, K_{3,3}\}$, then the following three conditions are equivalent:*

(i) *G is cyclically k-vertex connected.*

(ii) *Every independent edge cut of G has size at least k.*

(iii) *G is cyclically k-edge connected.*

Since θ, K_4, and $K_{3,3}$ do not have two disjoint cycles, they have no cycle-

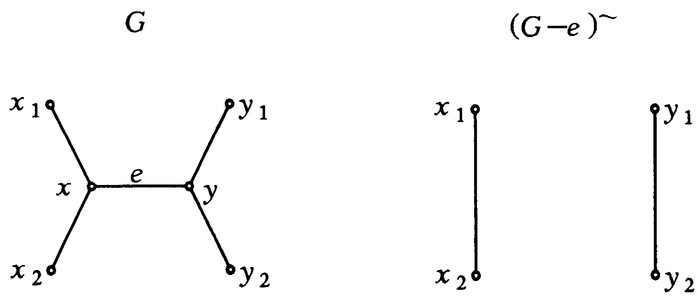


FIGURE 1.1

separating or independent edge cut. Hence, for these three graphs, (ii) and (iii) are vacuously satisfied for all values of k. We will say that a graph G is *cyclically k-connected* if G is cyclically k-vertex connected.

Let e be an edge joining distinct vertices x and y in a cubic graph G. Let $N(x) = \{y, x_1, x_2\}$ and $N(y) = \{x, y_1, y_2\}$. Suppose that e is the only edge joining x and y and that x and y are not incident with a loop, that is, suppose $\{x, y\} \cap \{x_1, x_2, y_1, y_2\} = \emptyset$. Define $(G-e)^{\sim}$ to be $(G - \{x, y\}) + \{x_1x_2, y_1y_2\}$ (see Fig. 1.1). By our assumptions $(G-e)^{\sim}$ is a well defined cubic graph with $v(G)-2$ vertices. We say that $(G-e)^{\sim}$ is obtained from G by an edge reduction at e, and that G is obtained from $(G-e)^{\sim}$ by adding an edge e across x_1x_2 and y_1y_2.

Since a cyclically 3-connected cubic graph has no loops or multiple edges, there is an edge reduction at every edge.

If G is a cyclically k-connected cubic graph and $G \notin \{\theta, K_4, K_{3,3}\}$, then we define $N_k(G)$ to be the set of edges which are in some independent k-edge cut. For G in $\{\theta, K_4, K_{3,3}\}$, we define $N_k(G)$ to be $E(G)$. We define $R_k(G)$ to be $E(G) - N_k(G)$. The next theorem, phrased differently, is proven in a paper by Wormald [32]. It demonstrates the significance of preceeding definitions.

THEOREM 1.3. *If G is a cyclically k-connected cubic graph, where $3 \leqslant k$, then $N_k(G)$ is the set of edges e such that $(G-e)^{\sim}$ is not cyclically k-connected.*

The next theorem is stated without proof in a paper by Wormald [32].

THEOREM 1.4. *Let G be obtained by adding an edge e across two edges in a cyclically k-connected cubic graph G'. Then G is cyclically k-connected if and only if G has girth at least k.*

Proof. If G has girth less than k, then G has a cycle C with less than k vertices. Then $v(C)$ is a cycle-separating vertex cut with less than k vertices, and so G is not cyclically k-connected.

Suppose G is not cyclically k-connected. Since $4 \leqslant v(G')+2=v(G)$, $G \neq \theta$. If G is K_4 or $K_{3,3}$, then G' is θ or K_4, respectively, and the result follows. Suppose $G \notin \{\theta, K_4, K_{3,3}\}$. Then Theorem 1.2 implies that G has an independent edge cut S such that $|S|<k$ and S is cycle-separating. Let A_i be a component of $G-S$ having a cycle C_i, $i=1, 2$.

Suppose $e \in S$. Since $e \notin E(A_i)$ there is a cycle C_i' in G' corresponding to C_i, $i=1, 2$. Now $S-\{e\}$ is a cycle-separating edge cut in G' of size less than k, a contradiction.

Thus, we may assume that $e \in A_1$. If A_1-e has a cycle, then G' again has two cycles separated by an edge cut of size less than k. Hence A_1-e is acyclic. Since $\delta_{A_1}=2$, A_1-e has at most two leaves. Hence, A_1-e is a path, and so A_1 is a cycle. But the number of vertices of degree two in A_1 is $|S|$, and so A_1 is a cycle of length less than k. Thus, the girth of G is less than k. ∎

We end the section with two lemmas on the structure of the components of a graph $G-S$, where S is an independent k-edge cut in a cyclically k-connected cubic graph G.

LEMMA 1.5. *Let S be an independent k-edge cut in a cyclically k-connected cubic graph G. Then $G-S$ has exactly two components. If $3 \leqslant k$ and A is a component of $G-S$, then A is 2-vertex connected.*

Proof. By definition $G-S$ has at least two components. If $G-S$ has three components, then $S-\{e\}$ is a cycle-separating $(k-1)$-edge cut for any e in S, a contradiction.

Suppose A is not 2-vertex connected. Then $V(A)$ contains a vertex x such that $A-x$ is not connected. Since $d_A(x) \leqslant 3$, x is incident with a cut edge e of A. Let A_1 and A_2 be the components of $A-e$. Let S_i be the set of edges in S incident with A_i, $i=1, 2$.

A vertex in A_i has degree one in A_i if and only if it is incident with e and an edge in S_i, $i=1, 2$. Hence, A_1 and A_2 each have at most one vertex of degree one, and so A_1 and A_2 both have cycles. Therefore, $S_i \cup \{e\}$ is cycle-separating, and so $k \leqslant |S_i \cup \{e\}|$, $i=1, 2$. Thus, $2(k-1) \leqslant |S_1|+|S_2|=|S|=k$, and so $k \leqslant 2$. ∎

LEMMA 1.6. *Let A be a subgraph of a cubic graph G, let S be the set of edges incident with A, and let $B=G-V(A)$. Suppose $|S|=k$, where $1 \leqslant k$, and the edges in S have distinct ends in $V(A)$. If G is cyclically $(k+1)$-connected, then B is acyclic. If $1 \leqslant k \leqslant 2$, then B has a cycle. If $3 \leqslant k \leqslant 5$ and B is acyclic, then $B=p_{k-2}$.*

Proof. Suppose G is cyclically $(k+1)$-connected. Then $|S|<k+1$ implies that A or B is acyclic. Since the edges of S have distinct ends in $V(A)$, $\delta_A=2$. Then A has a cycle, and so B is acyclic.

If $1 \leqslant k \leqslant 2$, then at most one vertex of B has degree at most one, and so B has a cycle.

Suppose B is acyclic and $3 \leqslant k \leqslant 5$. If B is not connected, then some component B' of B is incident with at most 2 edges in S. But then B' would have a cycle. Hence, B is connected. If $B = p_1$, then all edges in S are incident with the vertex of B, and so $k = 3$. Suppose $B \neq p_1$. Then $2 \leqslant v_1(B)$ and $v_3(B) = v_1(B) - 2$ because B is a tree. Also, since G is cubic, $2v_1 + v_2 = |S| = k$. Hence $4 \leqslant k$. If $k = 4$, then $v_1 = 2$, $v_2 = 0$, and $v_3 = 0$, and so $B = p_2$. If $k = 5$, then $v_1 = 2$, $v_2 = 1$, and $v_3 = 0$; so $B = p_3$. ▮

2. Crossing Edge Cuts and the Subgraph Induced by the Edges in Independent k-Edge Cuts

In this section we define crossing edge cuts and prove a lemma which gives the structure of a cyclically k-connected cubic graph having two independent k-edge cuts which cross. This main lemma provides insights on the structure of $G[N_k]$, where G is a cyclically k-connected cubic graph. In particular, we show that the edges in N_k which are not on k-cycles induce a forest. If in addition, G has girth $k+1$ and $N_k \neq \varnothing$, then this forest has at least k trees. This last result is reminiscent of a theorem of Mader [6, p. 24; 20]: the vertices of degree at least $k+1$ in a minimally k-connected graph induce a forest. Mader's proof uses the notion of crossing cuts. We also define a method of constructing new cubic graphs from given cubic graphs and prove several results needed in later sections.

Let S and S' be independent k-edge cuts in a graph G. We say that S and S' *cross* if there are components A and B of $G - S$ and components A' and B' of $G - S'$ such that $A \cap A'$, $A \cap B'$, $B \cap B'$, and $B \cap A'$ are nonempty.

Define θ_8 to be the graph obtained by replacing every edge of θ by a path of length three.

Lemma 2.1. *Let S and S' be crossing independent k-edge cuts in a cyclically k-connected cubic graph G. Let A and B (respectively, A' and B') be the components of $G - S$ (respectively, $G - S'$), and let $G_1 = A \cap A'$, $G_2 = A \cap B'$, $G_3 = B \cap B'$, and $G_4 = B \cap A'$. Let S_i be the set of edges in $S \cup S'$ incident with G_i, $1 \leqslant i \leqslant 4$, and let E_{ij} be the edges in $S \cup S'$ joining G_i and G_j, $1 \leqslant i < j \leqslant 4$ (see Fig. 2.1). Then $|S_1| + |S_3| = 2k - 2\,|E_{24}|$ and $|S_2| + |S_4| = 2k - 2\,|E_{13}|$. Also, if $|S_i| \leqslant k$ and G_i is acyclic, $i = 1, 2$, then A is a k-cycle, or $k = 6$ and $A = \theta_8$.*

Proof. First, $|S_1| + |S_3| = |S_1 \cup S_3| + |S_1 \cap S_3| = |(S \cup S') - E_{24}| + |E_{13}| = |S \cup S'| - |E_{24}| + |E_{13}| = |S| + |S'| - |S \cap S'| - |E_{24}| +$

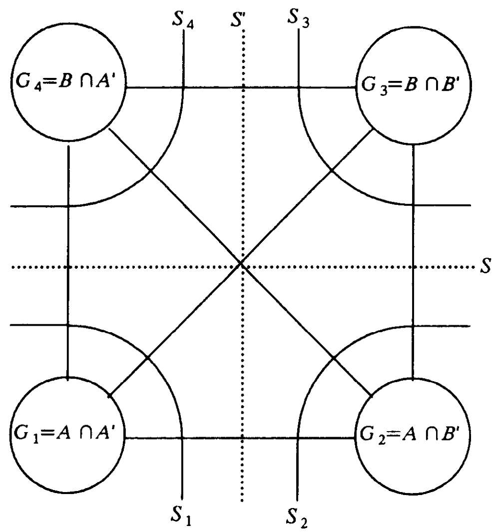


FIGURE 2.1

$|E_{13}| = 2k - |E_{24} \cup E_{13}| - |E_{24}| + |E_{13}| = 2k - 2\,|E_{24}|$. Similarly, $|S_2| + |S_4| = 2k - 2\,|E_{13}|$.

Suppose $|S_i| \leqslant k$ and G_i is acyclic, $i = 1, 2$.

Let $x \in V(G_i)$, where $1 \leqslant i \leqslant 4$. Then x is incident with at most one edge in each of S and S'. Also, $d_{G_i}(x) = 3 - j$ if and only if x is incident with j edges in $S \cup S'$. Thus, for G_i, $1 \leqslant \delta$ and $|S_i| = 2v_1 + v_2$. Since G_i is a forest, $v_1 - 2\omega = v_3$.

Thus, for G_i we have

$$v = v_1 + v_2 + [v_1 - 2\omega] = [2v_1 + v_2] - 2\omega = |S_i| - 2\omega. \tag{2.1}$$

Since $S_1 \cap S_2 = E_{12}$ and $S_1 \cup S_2 = S \cup E_{12}$,

$$|S_1| + |S_2| = |S| + 2\,|E_{12}| = k + 2\,|E_{12}|. \tag{2.2}$$

Since $|S_i| \leqslant k$, $i = 1, 2$, (2.2) implies

$$2\,|E_{12}| \leqslant k. \tag{2.3}$$

From (2.1), (2.2), and (2.3) we obtain

$$\begin{aligned} v(A) &= v(G_1) + v(G_2) = |S_1| - 2\omega(G_1) + |S_2| - 2\omega(G_2) \\ &= k + 2\,|E_{12}| - 2[\omega(G_1) + \omega(G_2)] \leqslant 2k - 4. \end{aligned} \tag{2.4}$$

Since S is independent and G is cubic, A is a k-cycle or A is a subdivision of a cubic graph H. Suppose we have the second case.

Since G is cyclically k-connected, A has girth at least k. If H has two disjoint cycles, then so does A. But then $2k \leqslant v(A)$ and we contradict (2.4); so H does not have two disjoint cycles. Hence, H has no cycle-separating edge cut, and so Theorem 1.2 implies $H \in \{\theta, K_4, K_{3,3}\}$.

Suppose $H \in \{K_4, K_{3,3}\}$. Then A has a subgraph H' which is a subdivision of K_4. Let $C_{H'}$ be the set of cycles of H' using exactly 3 vertices of degree 3 in H'. Let a be the number of ordered pairs (x, C) where $C \in C_{H'}$ and $x \in V(C)$. Since H' has girth at least k and $|C_{H'}| = 4$, $4k \leqslant a$. For $l = 2, 3$, $d_{H'}(x) = l$ if and only if x is on l cycles in $C_{H'}$; so $a = 2v_2(H') + 3v_3(H') = 2v(H') + 4$. Finally, using (2.4) we obtain $4k \leqslant a = 2v(H') + 4 \leqslant 2v(A) + 4 \leqslant 2[2k-4] + 4 = 4k - 4 < 4k$, a contradiction.

Suppose $H = \theta$. Let C_A be the set of cycles of A. Let b be the number of ordered pairs (x, C), where $C \in C_A$ and $x \in V(C)$. Since A has girth at least k and $|C_A| = 3$, $3k \leqslant b$. If $d_A(x) = l$, then x is on l cycles in C_A, $l = 2, 3$, and so $b = 2v_2(A) + 3v_3(A) = 2k + 6$. Therefore, $3k \leqslant b = 2k + 6$, so $k \leqslant 6$. By (2.4), $k + 2 = v(A) \leqslant 2k - 4$, so $6 \leqslant k$. Thus, $k = 6$. For $k = 6$, $v_2(A) = 6$ and A has girth at least 6; so $A = \theta_8$. ∎

Let $N'_k(H)$ be the set of edges in $N_k(H)$ which are not on a k-cycle.

THEOREM 2.2. *Let G be a cyclically k-connected cubic graph. Then $G[N'_k]$ is acyclic.*

Proof. Suppose C is a cycle of $G[N'_k]$. Let $e_1 \in E(C)$ and let S be an independent k-edge cut such that $e_1 \in S$. Then S contains another edge $e_2 \in E(C)$. Suppose e_1, e_2, and S are chosen so that $d_C(e_1, e_2)$ is minimal. Let p be an (e_1, e_2)-path in C with length $d_C(e_1, e_2)$.

If there exists $e \in E(p) \cap S$, then we have contradicted the choice of e_1, e_2, and S because $d_C(e_1, e) < d_C(e_1, e_2)$. Therefore, p is a subgraph of a component A of $G - S$. By Lemma 1.5, $G - S$ has only one other component; call it B.

Choose $f_1 \in E(p)$. Let S' be an independent k-edge cut containing f_1. If $E(p) \cup \{e_1, e_2\}$ contains an edge f in $S' - \{f_1\}$, then f, f_1, and S' contradict the choice of e_1, e_2, and S.

Let q be an (e_1, e_2)-path in B. Let C' be the cycle $G[E(p) \cup E(q) \cup \{e_1, e_2\}]$. Since $f_1 \in E(C')$, there exists another edge f_2 in $S' \cap E(C')$. By the previous paragraph, f_2 is in $E(q)$. Hence, S and S' cross.

We now adopt the notation of Lemma 2.1 (see Fig. 2.1). Since $\{f_1\} = (E(p) \cup \{e_1, e_2\}) \cap S'$, we may assume that $e_1 \in A'$ and $e_2 \in B'$. Thus, not all the edges in A' (respectively, B') are on k-cycles. The edges in $E(C)$ which are incident with e_1 are in $E(A)$ and $E(B)$. Thus, not all the edges in A (respectively, B) are on k-cycles.

Suppose $k < |S_1|$. Then by Lemma 2.1, $|S_3| = 2k - 2\,|E_{24}| - |S_1| < k$. Since G is cyclically k-connected, G_3 is acyclic. If $k < |S_2|$, then $|S_4| = 2k - 2\,|E_{13}| - |S_2| < k$; so G_4 is acyclic. Then Lemma 2.1 implies that every edge of B is on a k-cycle, a contradiction. Hence, $|S_2| \leqslant k$. If G_2 is acyclic then Lemma 2.1 implies that every edge of B' is on a k-cycle, a contradiction. Therefore, G_2 has a cycle. Let V_a be the set of ends in G_2 of edges in S_2. Since V_a is a cycle-separating vertex cut and G is cyclically k-connected, $k \leqslant |V_a| \leqslant |S_2|$. Thus, $k = |S_2|$ and the edges in S_2 have distinct ends in G_2. Hence, S_2 is an independent k-edge cut. But now e_2, f_1, and S_2 contradict the choice of e_1, e_2, and S. Thus, $|S_1| \leqslant k$. Similarly, $|S_2| \leqslant k$.

If G_i contains a cycle, then we can again show that S_i is an independent k-edge cut, $i = 1, 2$. This will again contradict the choice of e_1, e_2, and S. Therefore, G_1 and G_2 are acyclic. Applying Lemma 2.1 again we obtain our final contradiction: every edge of A is on a k-cycle. ∎

For a cyclically k-connected cubic graph G such that $N'_k \neq \emptyset$, the forest $G[N'_k]$ does not necessarily have more than one component. Fouquet and Thuillier [12], have constructed a cyclically 5-connected cubic graph G such that $G[N'_5]$ has only one component.

THEOREM 2.3. *Let G be a cyclically k-connected cubic graph such that $N_k = N'_k$. Then we cannot have two edges in N_k in the same independent k-edge cut and in the same component of $G[N_k]$. If $N_k \neq \emptyset$, then $G[N_k]$ is a forest with at least k trees.*

Proof. Suppose e_1 and e_2 are in an independent k-edge cut S and there exists an (e_1, e_2)-path p in $G[N_k]$. Assume e_1 and e_2 are chosen so that the length of p is minimal. Let $f_1 \in E(p)$ and let S' be an independent k-edge cut containing f_1. As in the proof of Theorem 2.2, we can show that S and S' cross. Let A, B, A', B' be as in Lemma 2.1 (see Fig. 2.1). Since $E(A) \cap N_k \neq \emptyset$, we cannot have all the edges of A on k-cycles. Similarly, this is the case for A', B', and B. We now proceed exactly as in the proof of Theorem 2.2 to derive a contradiction.

Now we show that if $N_k \neq \emptyset$, then $G[N_k]$ is a forest with at least k trees. Since $N_k = N'_k$, Theorem 2.2 implies that $G[N_k]$ is acyclic. Since $N_k \neq \emptyset$, there exists an independent k-edge cut T. Then all the edges of T are in different components of $G[N_k]$; so we have at least k trees. ∎

COROLLARY 2.4. *Let G be a cyclically k-connected cubic graph with girth at least $k+1$. If $N_k \neq \emptyset$, $G[N_k]$ is a forest with at least k trees.*

Proof. The girth of G is at least $k+1$, so $N_k = N'_k$. The result now follows from Theorem 2.3. ∎

THEOREM 2.5. *Let G_0 be a cyclically k-connected cubic graph, where $5 \leqslant k$. Let G_{i+1} be obtained from G_i by edge addition, $i = 0, 1$. If G_2 has girth at least k, then G_2 is cyclically k-connected.*

Proof. If G_1 has girth less than $k-1$, then G_2 can have girth at most $k-1$, a contradiction. Hence, G_1 has girth at least $k-1$. Using Theorem 1.4 twice we derive that G_2 is cyclically $(k-1)$-connected.

Suppose G_2 has an independent $(k-1)$-edge cut S_2. Let A_2 and B_2 be the components of $G_2 - S_2$. If $v(A_2) = k-1$, then A_2 is a $(k-1)$-cycle, a contradiction. If $v(A_2) = k+1$, then A_2 has two vertices of degree 3 and $k-1$ vertices of degree 2. Using the methods in the proof of Lemma 2.1, we have $3k \leqslant 2v_2(A_2) + 3v_3(A_2) = 2k-4$. Hence, $k \leqslant 4$, a contradiction. Thus $k+3 \leqslant v(A_2)$. Similarly, $k+3 \leqslant v(B_2)$.

Let $A_0 = G_0[V(A_2) \cap V(G_0)]$ and $B_0 = G_0[V(B_2) \cap V(G_0)]$. Let S_0 be the set of edges joining A_0 and B_0. Since G_0 is obtained from G_2 by two edge reductions and since $|S_2| = k-1$, we have $k-1 \leqslant v(A_0)$, $k-1 \leqslant v(B_0)$, and $|S_0| \leqslant k-1$. Counting the incidences in A_0 in two ways we obtain $2v(A_0) \leqslant 3v(A_0) - (k-1) \leqslant 3v(A_0) - |S_0| = 2\varepsilon(A_0)$. Hence, $v(A_0) \leqslant \varepsilon(A_0)$, and so A_0 has a cycle. Similarly, B_0 has a cycle. But now S_0 is a cycle-separating edge cut with at most $k-1$ edges, and the cyclic k-connectedness of G_0 is contradicted. Therefore, G_2 is cyclically k-connected. ▮

LEMMA 2.6. *Let $3 \leqslant k \leqslant 5$. Let S and S′ be crossing independent k-edge cuts in a cyclically k-connected cubic graph G. Then $k = 3$ is not possible. If $k = 4$, then G has the form $G_{(2.2.a)}$. If $k = 5$, then G has the form $G_{(2.2.b)}$ or $G_{(2.2.c)}$. (see Fig. 2.2.)*

Proof. We use the notation of Lemma 2.1 (see Fig. 2.1). By Lemma 1.5, E_{12}, E_{23}, E_{34}, E_{14} all have at least two edges. Hence $4 \leqslant k$. If $k = 5$ and $|S_i| = 4$, then $G_i = K_2$ by Lemma 1.6, $i = 1, 2$. The result now follows. ▮

THEOREM 2.7. *Let S be an independent k-edge cut in a cyclically k-connected cubic graph G, let A be a component of $G - S$, and let e be in $E(A) \cap N_k(G)$. Then e is in an independent k-edge cut contained in $E(A) \cup S$ if any of the following conditions hold*:

(i) $k = 3$;

(ii) $k = 4$ *and* $6 \leqslant v(A)$;

(iii) *G has girth at least* $k+1$.

Proof. Suppose e is in the independent k-edge cut S'. If $S' \subseteq E(A) \cup S$

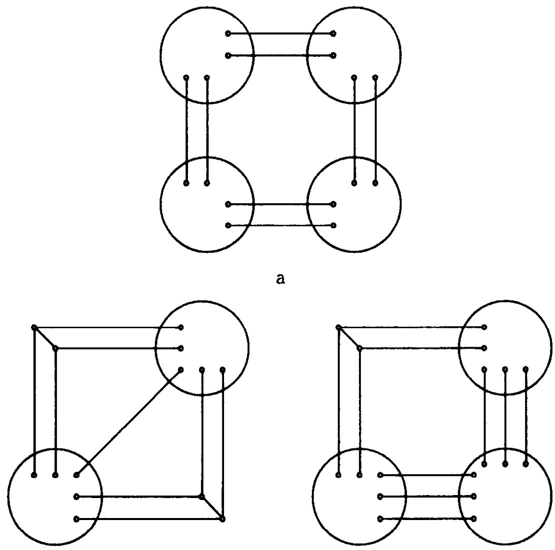


FIGURE 2.2

we are done. By Lemma 2.6, this is the case if $k=3$. So we may assume $4 \leqslant k$ and S and S' cross. We use the notation of Lemma 2.1 (see Fig. 2.1).

If $k=4$, then G has the form $G_{(2.2.a)}$ by Lemma 2.6. If S_1 or S_2 is an independent 4-edge cut, then we are done. If not, then $G_i = K_2$ by Lemma 1.6, $i=1, 2$. But then $\nu(A)=4$ and we have a contradiction.

We now prove the theorem when G has girth at least $k+1$. Suppose $|S_1| < k$. Then G_1 is acyclic. If $|S_2| < k$, then G_2 is also acyclic; but now we have a contradiction because A has a k-cycle by Lemma 2.1. Hence, $k \leqslant |S_2|$. Similarly, $k \leqslant |S_4|$. By Lemma 2.1, $|S_2| + |S_4| = 2k - 2\,|E_{13}|$; so $|S_2| = |S_4| = k$.

If G_2 is acyclic, then A again has a k-cycle; so G_2 has a cycle. Let V_a be the set of ends in G_2 of edges in S_2. Since V_a is a cycle-separating vertex cut and G is cyclically k-connected, $k \leqslant |V_a| \leqslant |S_2|$. Thus, $k = |S_2|$ and the edges in S_2 have distinct ends in G_2. Hence, S_2 is an independent k-edge cut. Since $e \in S_2$ and $S_2 \subseteq E(A) \cup S$, we are done. Similarly, if $|S_i| < k$, for some $i \in \{2, 3, 4\}$, we are done.

Suppose $k \leqslant |S_i|$, $i=1, 2, 3, 4$. By Lemma 2.1, $|S_1| + |S_3| \leqslant 2k$ and $|S_2| + |S_4| \leqslant 2k$, so $|S_i| = k$, $i=1, 2, 3, 4$. If G_2 contains a cycle, then as before we can show that S_2 is the required independent k-edge cut. Similarly, if G_1 contains a cycle we are done. If G_1 and G_2 are acyclic, then we have a contradiction because A has a k-cycle by Lemma 2.1. ∎

3. Edge Reductions in Cyclically 3-Connected Cubic Graphs

Throughout this section, G will denote a cyclically 3-connected cubic graph such that $G \neq K_4$.

In this section we will examine edge reductions in cyclically 3-connected cubic graphs. The first theorem proves that $G[N_3]$ is a forest with at least three trees and that $|N_3| \leq v-3$. We then state a characterization of those graphs G with $|N_3| = v-3$.

It is a classical theorem that every 3-connected graph except K_4 can be reduced to a smaller 3-connected graph by means of a more general form of edge reduction. This was proven for planar 3-connected graphs by Steinitz and Rademacher [25] and later for all 3-connected graphs by Barnette and Grünbaum [3], and independently, Titov [28]. This result implies that every cyclically 3-connected cubic graph except K_4 has an edge reduction which gives a smaller cyclically 3-connected cubic graph.

THEOREM 3.1. *If $N_3(G) \neq \emptyset$, then $N_3 = N_3'$, $G[N_3]$ is a forest with at least three trees, and $|N_3| \leq v-3$. If $|N_3| = v-3$, then $G[N_3]$ is a spanning forest with exactly three trees.*

Proof. A 3-edge cut of a cyclically 3-connected cubic graph G which contains an edge on a 3-cycle C must necessarily contain two adjacent edges on C. Therefore, no independent 3-edge cut can contain an edge on a 3-cycle. Hence, $N_3 = N_3'$. Then Theorem 2.3 implies that $G[N_3]$ is a forest with at least three trees if $N_3 \neq \emptyset$. Hence, $|N_3| = \varepsilon(G[N_3]) = v(G[N_3]) - \omega(G[N_3]) \leq v-3$. Hence, if $|N_3| = v-3$, then $G[N_3]$ is a spanning forest with exactly three trees. ▮

Suppose H_i is a cubic graph equal to $H^3(A_i, K_3)$, $i = 1, 2$. We define H_1 ③ H_2 to be $H^3(A_1, A_2)$. We note that H_1 and H_2 do not uniquely determine H_1 ③ H_2. We will use H_1 ③ H_2 to refer to all such graphs.

We now recursively define a set $\mathscr{T}$. Let $H_{(3.1.a)}$, $H_{(3.1.b)}$, and $H_{(3.1.c)}$ be $\mathscr{T}$. If H_1 and H_2 are in $\mathscr{T} - \{H_{(3.1.a)}\}$ then H_1 ③ H_2 is in $\mathscr{T}$. See Fig. 3.1.

The following theorem has a routine proof by induction on v. It is similar to the methods used in [21]. The proof is given in [22].

THEOREM 3.2. *$G \in \mathscr{T}$ if and only if $|N_3(G)| = v(G) - 3$.*

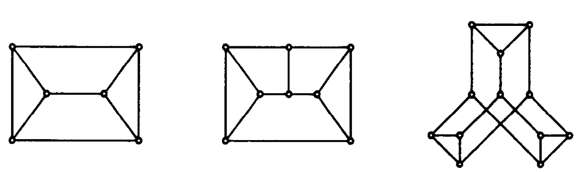


FIGURE 3.1

4. Edge Reductions in Cyclically 4-Connected Cubic Graphs

Throughout this section, G will denote a cyclically 4-connected cubic graph.

Every cyclically 4-connected cubic graph, except $K_{3,3}$ and the 3-cube, has an edge reduction which gives a smaller cyclically 4-connected cubic graph. This was proven for planar graphs by Kotzig [17] and, independently, by Faulkner and Younger [9]; later it was proven in general by Wormald [32] and, independently, by Fontet [10].

In this section we define a very restricted subset N_4^b of N_4. Unlike $H[N_3]$ for a 3-connected cubic graph H, $G[N_4]$ is not necessarily a forest. But if we consider $N_4^g = N_4 - N_4^b$, then we are able to show that $G[N_4^g]$ is a forest with at least four trees when $N_4^g \neq \emptyset$. We next define a function f_4 and show that it is an upper bound for $|N_4(G)|$.

In [21], f_4 is shown to be a sharp upper bound and the extremal graphs are characterized. When G is restricted to being planar, the upper bound for $|N_4(G)|$ can be improved, as is shown in [15]. In [21], the upper bound of [15] is shown to be sharp and the extremal graphs are characterized. If G is a plane graph, then its dual graph G^* is a 4-connected plane triangulation. An edge reduction of G results in a smaller cyclically 4-connected graph if and only if the contraction of the dual edge in G^* results in a smaller 4-connected graph. Hence, the results of [21] give a sharp lower bound on the number of edge contractions of a 4-connected plane triangulation which give a smaller 4-connected triangulation. The extremal graphs are the duals of the extremal graphs for edge reductions.

Let $p_n = x_1 x_2 \cdots x_n$ and $p'_n = y_1 y_2 \cdots y_n$ be disjoint paths. The *2n-ladder*, L_{2n}, is defined to be $p_n + p'_n + \{x_i y_i \mid i = 1, ..., n\}$. Define p_n and p'_n to be the *sides* S_1 and S_2 of L_{2n}, define the edges in $E(S_1) \cup E(S_2)$ to be the *side edges* of L_{2n}, define the edges in $\{x_i y_i \mid i = 1, ..., n\}$ to be the *rungs* of L_{2n}, define $x_2, ..., x_{n-1}, y_2, ..., y_{n-1}$ to be the *intermediate vertices* of L_{2n}, and define x_1, y_1, x_n, y_n to be the *end vertices* of L_{2n}. Two side edges on the same 4-cycle are said to be *corresponding side edges.* Two end vertices incident with the same rung are said to be *corresponding end vertices.* Let $E_s(L)$, $E_r(L)$, and $I(L)$, be the sets of side edges, rungs, and intermediate vertices, respectively, of ladder L. Note that for L_4 we may choose which two independent edges in $E(L_4)$ are the side edges. The other two independent edges will then be the rungs.

Let $N_4^b(G)$ be the set of edges e which satisfy each of the following conditions:

(i) $e \in N_4$.

(ii) e is on a 4-cycle.

(iii) For every independent 4-edge cut S containing e, one of the components of $G-S$ is a ladder.

Let $N_4^g(G)=N_4(G)-N_4^b(G)$.

Let $C_{2n}=v_1v_2\cdots v_{2n}v_1$, and define M_{2n} to be $C_{2n}+\{v_iv_{n+i} \mid i=1,...,n\}$, for every $n\geqslant 3$. Let $E_s(M_{2n})=E(C_{2n})$. Let $C_n=x_1x_2\cdots x_nx_1$ and $C'_n=y_1y_2\cdots y_ny_1$ be disjoint cycles and define Q_{2n} to be $C_n+C'_n+\{x_iy_i \mid i=1,...,n\}$, for every $n\geqslant 4$. Let $E_s(Q_{2n})=E(C_n)\cup E(C'_n)$. Let $\mathcal{M}=\{M_{2n} \mid 3\leqslant n\}$ and $\mathcal{Q}=\{Q_{2n} \mid 4\leqslant n\}$.

The following lemma has a routine proof using induction and Theorem 1.4.

LEMMA 4.1. *The graphs in $\mathcal{M}\cup\mathcal{Q}$ are cyclically 4-connected. $N_4^b(M_6)=E(M_6)$ and $N_4^b(Q_8)=E(Q_8)$. For $n\geqslant 4$, $N_4(M_{2n})=N_4^b(M_{2n})=E_s(M_{2n})$. For $n\geqslant 5$, $N_4(Q_{2n})=N_4^b(Q_{2n})=E_s(Q_{2n})$.*

LEMMA 4.2. *Let $C=v_1v_2v_3v_4v_1$ be a 4-cycle of G. If v_1v_2, $v_2v_3\in N_4$, then $G\in\{M_6, Q_8\}$.*

Proof. Suppose $G\neq M_6$. There exist independent 4-edge cuts S and T such that $v_1v_2\in S$ and $v_2v_3\in T$. Then $v_3v_4\in S$ and $v_4v_1\in T$, and so S and T cross. Thus, G has the form $G_{(2.2.a)}$ by Lemma 2.6. Lemma 1.6 implies that each of the components of $G-(S\cup T)$ is a K_2. Thus, $G=Q_8$. ▮

The following lemma has a routine proof.

LEMMA 4.3. *Let S be an independent 4-edge cut of G. If both components of $G-S$ are ladders, then $N_g^4\subseteq S$.*

LEMMA 4.4. *Let e_1 and e_2 be in $N_4^g(G)$. If e_1 and e_2 are both in some independent 4-edge cut S and there is an (e_1, e_2)-path p in $G[N_4^g]$, then one of the components of $G-S$ is a ladder having p as a side.*

Proof. Suppose the result is false. Choose e_1, e_2, S, and p so that the length of p is minimum.

Suppose some edge e in $S-\{e_1, e_2\}$ is in $E(p)$. Let p_i be the subpath of p joining e and e_i, $i=1, 2$. Then p_i is shorter than p and $e\in N_4^g$. Since e, e_i, S and p_i cannot contradict the choice of e_1, e_2, S, and p, one of the components of $G-S$ is a ladder having p_i as a side, $i=1, 2$. Since $V(p_1)$ and $V(p_2)$ contain different ends of e, the component of $G-S$ containing p_1 is distinct from the one containing p_2. Hence, both components of $G-S$ are ladders. Thus, $N_4^g\subseteq S$ by Lemma 4.3. But then $E(p)\nsubseteq N_4^g$, and so we have a contradiction. Hence, $(S-\{e_1, e_2\}) \cap E(p)=\emptyset$ and p is a subgraph of one of the components, say G_1, of $G-S$.

Let f_1 be in $E(p)$ and let f_1 be in an independent 4-edge cut T. Suppose $[T-\{f_1\}] \cap [E(p) \cup \{e_1, e_2\}]=\emptyset$. Then by Lemma 2.6, G has the form $G_{(2.2.a)}$. Let p_i' be the subpath of p joining f_1 and e_i, $i=1, 2$. Let G_{1i} be the component of $G-(S \cup T)$ incident with e_i and f_1, $i=1, 2$. Let S_i be the edges incident with G_{1i}, $i=1, 2$. By Lemma 1.6, either $G_{1i}=K_2$ or S_i is independent. If S_i is independent, then, by the choice of e_1, e_2, S, and p, one of the components of $G-S_i$ is a ladder having side p_i'. Since $V(p_i') \cap V(G_{1i}) \neq \emptyset$, this component is G_{1i}. In all cases G_1 is a ladder L. Since $G \neq Q_8$, Lemma 4.2 implies that N_4 cannot contain two adjacent edges of a 4-cycle of L. Hence, $E(p) \subseteq N_4 \cap E_s(L)$; so p is a side of L, a contradiction.

Suppose $f_2 \in [T-\{f_1\}] \cap [E(p) \cup \{e_1, e_2\}]$. Let p' be the subpath of p joining f_1 and f_2. By the choice of e_1, e_2, S, and p, one of the components of $G-T$ is a ladder with side p'.

Let f_1' be in $E(p')$. Then f_1' is on a 4-cycle and $f_1' \in N_4$. But $f_1' \notin N_4^b$, and so f_1' does not satisfy condition (iii) in the definition of N_4^b. Therefore, there exists an independent 4-edge cut T' containing f_1' such that neither component of $G-T'$ is a ladder.

As before with f_1 and T, if $[T'-\{f_1'\}] \cap [E(p) \cup \{e_1, e_2\}]=\emptyset$, then G_1 is a ladder having p as a side, a contradiction. Also, as before with f_1 and T, if $[T'-\{f_1'\}] \cap [E(p) \cup \{e_1, e_2\}] \neq \emptyset$, then one of the components of $G-T'$ is a ladder, a contradiction. ∎

THEOREM 4.5. *If $N_4^g(G) \neq \emptyset$, then $G[N_4^g]$ is a forest with at least four trees.*

Proof. Suppose there exists a cycle C in $G[N_4^g]$. Let S be a cut containing an edge of C. Since C is a cycle, $E(C)$ contains two or four edges in $E(S)$. Therefore, each component of $G-S$ contains a path in $G[N_4^g]$ connecting two edges in $N_4^g \cap S$. By Lemma 4.4, each component is a ladder. Then $N_4^g \subseteq S$ by Lemma 4.3, a contradiction. Thus, $G[N_4^g]$ is a forest.

We now prove that $G[N_4^g]$ has at least four components. Suppose G has no 4-cycles. There is an independent 4-edge cut T because $N_4^g \neq \emptyset$. Since $N_4^b \neq \emptyset$, $T \subseteq N_4^g$. Since G has no ladders, no two edges in T are in the same component of $G[N_4^g]$ by Lemma 4.4.

Suppose G has a 4-cycle. Choose a maximal ladder L. Let S' be the set of edges incident with L. If L is a spanning subgraph of G, then $|S'|=2$ and $G \in \mathscr{M} \cup \mathscr{Q}$. But $N_4^g=\emptyset$ for graphs in $\mathscr{M} \cap \mathscr{Q}$ by Lemma 4.1, a contradiction. If the edges of S' are not independent, then $G-V(L)=K_2$ by

Lemma 1.6. But then corresponding end vertices of L are adjacent to distinct adjacent vertices of $G - V(L)$, and we contradict the maximality of L. Hence, S' is an independent 4-edge cut. Since L is maximal, none of the edges in S' is on a 4-cycle, and so $S' \subseteq N_4^g$.

If the edges in S' are in different components of $G[N_4^g]$ we are done; so suppose not. By Lemma 4.4, one of the components of $G - S'$ is a ladder with a side having its edges in N_4^g. Therefore, some edge e in N_4^g is on a 4-cycle. Since $e \in N_4^g$ and e is on a 4-cycle, e is in a 4-edge cut T' such that neither component of $G - T'$ is a ladder. Then $T' \subseteq N_4^g$ and, by Lemma 4.4, the edges of T' are in different components of $G[N_4^g]$. ∎

If $G \notin \mathscr{M} \cup \mathscr{Q}$, we define $\mathscr{L}(G)$ to be the set of all maximal ladders L of G such that $E_s(L) \subseteq N_4^b$. Let $E(\mathscr{L}) = \bigcup_{L \in \mathscr{L}} E(L)$, $E_s(\mathscr{L}) = \bigcup_{L \in \mathscr{L}} E_s(L)$, $E_r(\mathscr{L}) = \bigcup_{L \in \mathscr{L}} E_r(L)$, and $I(\mathscr{L}) = \bigcup_{L \in \mathscr{L}} I(L)$.

LEMMA 4.6. *Suppose $G \notin \mathscr{M} \cup \mathscr{Q}$. The maximal ladders of G are disjoint. If $L \in \mathscr{L}$, then $6 \leqslant v(L)$. $E(\mathscr{L}) \cap N_4 = E_s(\mathscr{L}) = N_4^b$.*

Proof. Let L_a be a maximal ladder of G, and let S' be the set of edges incident with L_a. As in the proof of Theorem 4.5, we can show that S' is an independent 4-edge cut. Suppose $V(L_a) \cap V(L_b) \neq \varnothing$ for some maximal ladder L_b such that $L_a \neq L_b$. Then some edge in S' is on a 4-cycle C of L_b. Then L_a is a proper subgraph of the ladder $L_a \cup C$, a contradiction.

Suppose $L \in \mathscr{L}$ and $e \in E_s(L)$. Then $e \in N_4^b$. Hence, e is on a 4-cycle and incident with a 4-cycle by conditions (ii) and (iii), respectively, of the definition of N_4^b. Therefore, $6 \leqslant v(L)$.

By definition, $E_s(\mathscr{L}) \subseteq E(\mathscr{L}) \cap N_4$. Since $G \notin \{M_6, Q_8\}$, Lemma 4.2 implies that $E_r(\mathscr{L}) \cap N_4 = \varnothing$. Hence, $E_s(\mathscr{L}) = E(\mathscr{L}) \cap N_4$.

If $6 \leqslant v(L)$ for a ladder L, then every edge in $E_s(L)$ is incident with a 4-cycle of L. Hence, $E_s(L) \subseteq N_4$. Hence, in order to prove that $N_4^b \subseteq E_s(\mathscr{L})$, it suffices to show that either $E_s(L) \subseteq N_4^g$ or $E_s(L) \subseteq N_4^b$, for every maximal ladder L such that $6 \leqslant v(L)$.

Suppose $6 \leqslant v(L)$ and $e_1 \in E_s(L) \cap N_4^g$. Since e_1 is on a 4-cycle and $e_1 \notin N_4^b$, e_1 is in some independent 4-edge cut T such that neither component of $G - T$ is a ladder. The corresponding edge e_2 of e_1 is necessarily in T. Let $T = \{e_1, e_2, e_3, e_4\}$.

Let f_1 and f_2 be corresponding side edges of L. Since neither component of $G - T$ is a ladder, e_3 and e_4 are not incident with any vertices in $V(L)$. Hence, $T' = \{f_1, f_2, e_3, e_4\}$ is an independent 4-edge cut. If some component of $G - T'$ is a ladder, then it is a subgraph of L. But then e_3 and e_4 are incident with vertices in $V(L)$, a contradiction. Therefore, f_1, $f_2 \in N_4^g$. Hence, $E_s(L) \subseteq N_4^g$.

Thus, $E_s(L) \subseteq N_4^b$ or $E_s(L) \subseteq N_4^g$. Thus, $E_s(\mathscr{L}) = N_4^b$. ∎

Let

$$f_4(v)=\begin{cases} 9, & \text{if } v=6\\ 12, & \text{if } v=8\\ 10, & \text{if } v=10\\ \dfrac{6v-15}{5}, & \text{if } v\equiv 0 \pmod{10} \text{ and } v>10\\ \dfrac{6v-12}{5}, & \text{if } v\equiv 2 \pmod{10} \text{ and } v>10\\ \dfrac{6v-14}{5}, & \text{if } v\equiv 4 \pmod{10} \text{ and } v>10\\ \dfrac{6v-16}{5}, & \text{if } v\equiv 6 \pmod{10} \text{ and } v>10\\ \dfrac{6v-18}{5}, & \text{if } v\equiv 8 \pmod{10} \text{ and } v>10. \end{cases}$$

THEOREM 4.7. *Suppose $G\notin \mathcal{M}\cup\mathcal{Q}$. Then $|N_4(G)|\leqslant f_4(v(G))$. If $v\leqslant 10$, then $|N_4|<f_4(v)$.*

Proof. Suppose $N_4^b=\varnothing$. If $N_4^g=\varnothing$ we are done. If $N_4^g\neq\varnothing$, then $G[N_4^g]$ is a forest with at least four trees by Theorem 4.5, and so $|N_4|\leqslant v-4<f_4(v)$.

Suppose $N_4^b\neq\varnothing$. Let $|\mathcal{L}|=l$ and $|I(\mathcal{L})|=i$. By Lemma 4.6, $E_s(\mathcal{L})=N_4^b$, and so $|N_4^b|=2l+i$. If $L\in\mathcal{L}$, then the set S of edges incident with L is an independent 4-edge cut. Since L is maximal, no edge in S is on a 4-cycle. Therefore, $S\subseteq N_4^g$ and $N_4^g\neq\varnothing$. Lemma 4.6 implies $N_4^g\cap E(\mathcal{L})=\varnothing$, and so $G[N_4^g]$ is a subgraph of $G-I(\mathcal{L})$. Let $a_v=v(G-I(\mathcal{L}))-v(G[N_4^g])$. By Theorem 4.5, $G[N_4^g]$ is a forest with $4+a_t$ trees, for some $a_t\geqslant 0$. Hence, $|N_4^g|=v(G[N_4^g])-(4+a_t)=v(G-I(\mathcal{L}))-a_v-4-a_t=v-i-a_v-4-a_t$. Thus,

$$|N_4|=v+2l-4-a_v-a_t. \tag{4.1}$$

Since $3\leqslant|E_r(L)|$, for every L in $\mathcal{L}$, $E_r(\mathcal{L})=3l+a_r$, for some $a_r\geqslant 0$. Let $a_e=\varepsilon(G)-|E_r(\mathcal{L})\cup N_4|$. Then

$$|N_4|=\varepsilon(G)-|E_r(L)|-a_e=\frac{3v}{2}-3l-a_r-a_e. \tag{4.2}$$

Thus, for $v \geqslant 12$,

$$|N_4| \leqslant \min\left\{v+2l-4, \frac{3v}{2}-3l\right\} \leqslant \max_l\left(\min\left\{v+2l-4, \frac{3v}{2}-3l\right\}\right)=f_4(v). \tag{4.3}$$

For $v \leqslant 10$, the previous equation gives $|N_4| < f_4(v)$. ▮

5. Edge Reductions in Cyclically 5-Connected Cubic Graphs

Throughout this section, G will denote a cyclically 5-connected cubic graph.

In this section we examine edge reductions in cyclically 5-connected cubic graphs. We found that f_3 and f_4 were linear in v. In contrast there exists a graph G on v vertices such that $N_5(G)=E(G)$, for all $v \geqslant 10$ such that $v \neq 16$. In this section we characterize such graphs G. As with N_4 we will define a restricted subset N_5^b of N_5. The characterization is then found mainly by examining the structure of $G[N_5^b]$.

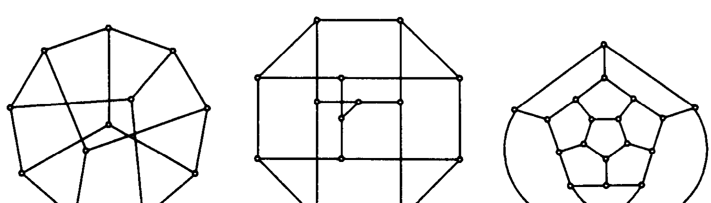
Define S_{12}, S_{14}, and S_{18} to be $G_{(5.1.a)}$, $G_{(5.1.b)}$, and $G_{(5.1.c)}$, respectively. See Fig. 5.1. Let $\mathscr{S}=\{S_{12}, S_{14}, S_{18}\}$.

For every odd integer $n \geqslant 5$, we define P_{2n} as follows. Let $C_u=u_1u_2\cdots u_n$ and $C_v=v_1v_3v_5\cdots v_{n-2}v_nv_2v_4v_6\cdots v_{n-3}v_{n-1}$ be disjoint cycles. Let $P_{2n}=C_u+C_v+\{u_iv_i \mid i=1, ..., n\}$. Let $\mathscr{P}$ to be the set of all such graphs. The Petersen graph is P_{10}.

For every $n \geqslant 5$, we define D_{4n} as follows. Let $C_x=x_1x_2\cdots x_n$, $C_y=y_1y_2\cdots y_n$, and $C_z=z_1z_2\cdots z_{2n}$ be disjoint cycles. Let $D_{4n}=C_x+C_y+C_z+(\{x_iz_{2i-1} \mid i=1, ..., n\} \cup \{y_iz_{2i} \mid i=1, ..., n\})$. Let $\mathscr{D}$ be the set of all such graphs. The dodecahedron graph is D_{20}.

For every cyclically 5-connected cubic graph G, define $N_5^b(G)$ to be the set of edges in N_5 which are in the edge set of a 5-cycle.

Let C be a 5-cycle of D_{20} and let $D=D_{20}-V(C)$. Let $\mathscr{A}$ be the set of



a b c

Figure 5.1

cyclically 5-connected cubic graphs such that $N_5(G)=E(G)$ and every component of $G[N_5^b]$ is isomorphic to D.

Let $\mathscr{H}=\mathscr{S}\cup\mathscr{P}\cup\mathscr{D}\cup\mathscr{A}$.

LEMMA 5.1. *The graphs in $\mathscr{H}$ are cyclically 5-connected.*

Proof. It is easy to show P_{10}, S_{12}, S_{14}, S_{18}, and D_{20} are cyclically 5-connected. The graphs in $\mathscr{A}$ are cyclically 5-connected by definition.

We prove by induction that the graphs in $\mathscr{P}$ and $\mathscr{D}$ are cyclically 5-connected. Suppose $n>5$. Then $P_{2(n-2)}=((P_{2n}-u_nv_n)^{\sim}-u_{n-1}v_{n-1})^{\sim}$ and $D_{4(n-1)}=((D_{4n}-x_nz_{2n-1})^{\sim}-y_nz_{2n})^{\sim}$. Thus, P_{2n} and D_{4n} are obtained from $P_{2(n-2)}$ and $D_{4(n-1)}$, respectively, by two edge additions. Since $P_{(2n-2)}$ and $D_{4(n-1)}$ are cyclically 5-connected, Theorem 2.5 implies that P_{2n} and D_{2n} are cyclically 5-connected. ∎

For any graph G in $\mathscr{S}\cup\mathscr{P}\cup\mathscr{D}$, every edge is incident with a 5-cycle; so $N_5(G)=E(G)$. For any graph G in $\mathscr{A}$, $N_5(G)=E(G)$ by definition. Thus, $N_5(G)=E(G)$ for all graphs G in $\mathscr{H}$.

LEMMA 5.2. *If G has the form $G_{(5.2.a)}$ and $e\in N_5(G)$, then e is incident with a 5-cycle. (See Fig. 5.2.)*

Proof. Let e be the independent 5-edge cut T. By Lemma 1.5, G_i is 2-connected, $i=1,2$. Furthermore, the vertices x_i and y_i of G_i are in different components of $G-T$, $i=1,2$. Hence $2\leqslant|E(G_i)\cap T|$, $i=1,2$. Thus, if we consider the disjoint, crossing, independent 5-edge cuts T and $\{f_1,f_2,f_3,f_4,f_5\}$, Lemma 2.6 implies G has the form $G_{(5.2.b)}$. (We note that the role of x and y in Figure 5.2.b. can be assumed without loss of generality.)

Let K be the component of $G-T$ such that $x\in V(K)$. By Lemma 1.6, $B-x$ is isomorphic to K_2. Therefore, $v(K)=5$. K is also simple and 2-regular, and so K is a 5-cycle. ∎

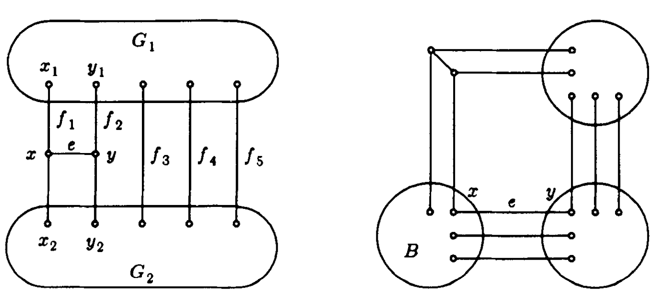


FIGURE 5.2

LEMMA 5.3. *Suppose $E(G)=N_5(G)$. If $e \in E(G)$ and e is on a 5-cycle C, then e is incident with a 5-cycle.*

Proof. Since $E(G)=N_5(G)$, e is in an independent 5-edge cut S. S must necessarily include another edge e' in $E(C)$. Hence, G has the form $G_{(5.3.a)}$.

If y_1, y_2, y_3, y_4, and y_5 are not distinct, then $G_2=p_3$ by Lemma 1.6. Then $K=G[V(G_2)\cup\{z_1,z_2\}]$ has 5 vertices. K is also simple and 2-regular; so K is a 5-cycle and we are done.

Suppose y_1, y_2, y_3, y_4, and y_5 are distinct. Then G has the form $G_{(5.2.a)}$. Therefore, Lemma 5.2 implies that z_1z_2 is incident with a 5-cycle. Thus, G has the form $G_{(5.3.b)}$. (We note that the role of e and e' could be interchanged in Fig. 5.3.b. We will show that e and e' are both incident with some 5-cycle.)

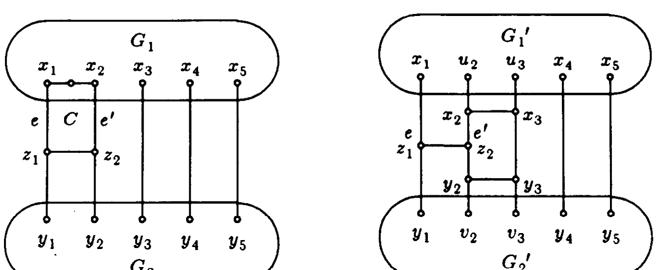
If y_1, v_2, v_3, y_4, and y_5 are not distinct, then $G_2'=p_3$ by Lemma 1.6. Then G_2 is a 5-cycle. Hence, y_1 and y_2 are both adjacent to some vertex w. Then e and e' are incident with the 5-cycle $z_1z_2y_2wy_1$. Similarly, if x_1, u_2, u_3, x_4, and x_5 are not distinct then e and e' are incident with a 5-cycle.

Suppose $\{x_1,u_2,u_3,x_4,x_5\}$ and $\{y_1,v_2,v_3,y_4,y_5\}$ are both sets of distinct vertices. Lemma 5.2 then implies that y_2y_3 is incident with a 5-cycle. Since x_1, u_2, u_3, x_4, and x_5 are distinct, the only possibility for this 5-cycle is $z_1z_2y_2v_2y_1$ and we are done. ∎

LEMMA 5.4. *Suppose $E(G)=N_5(G)$. If G has two 5-cycles whose intersection is p_3, then $G\in\{P_{10},S_{12},S_{14}\}$.*

Proof. If G has two 5-cycles whose intersection is p_3, then G has a $G_{(5.4.a)}$ subgraph.

Figure 5.4 shows a proof of the theorem. Let r be in $\{a,c,d,e\}$. By Lemma 5.3, the dashed edge of $G_{(5.4.r)}$ is incident with a 5-cycle. Then G must have as a subgraph one of the graphs to which there is an arrow from $G_{(5.4.r)}$. We assume that from $G_{(5.4.c)}$ (respectively, $G_{(5.4.d)}$) we do not obtain



a b

FIGURE 5.3

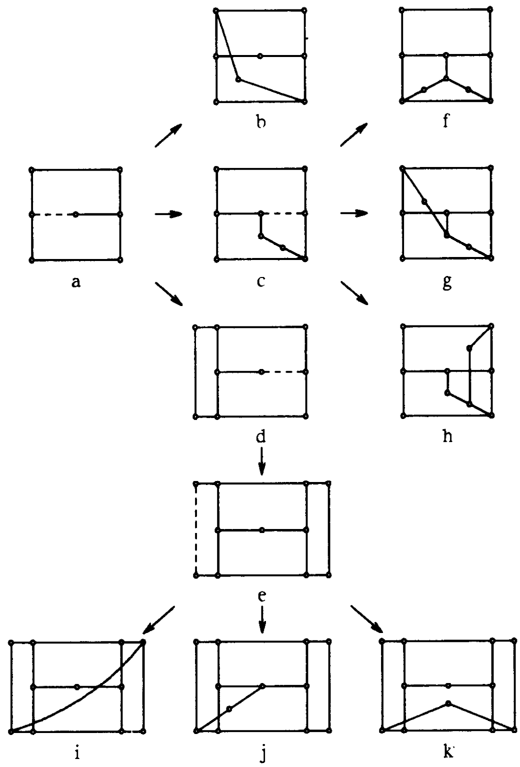


FIGURE 5.4

a subgraph of G with a $G_{(5.4.b)}$ subgraph (respectively, $G_{(5.4.b)}$ or $G_{(5.4.c)}$ subgraph).

Lemma 1.6 and the fact that G has girth 5 imply that $G = P_{10}$ if G has a $G_{(5.4.b)}$ or $G_{(5.4.f)}$ subgraph. Similarly, $G = S_{12}$ if G has a $G_{(5.4.s)}$ subgraph, where $s \in \{g, h, i, j\}$, and $G = S_{14}$ if G has a $G_{(5.4.k)}$ subgraph. ▌

THEOREM 5.5. *If $E(G) = N_5(G)$, then $G \in \mathscr{H}$.*

Proof. If the intersection of two 5-cycles is p_3, then $G \in \{P_{10}, S_{12}, S_{14}\}$ by Lemma 5.4. Therefore, we may assume that the intersection of any two 5-cycles is empty or p_2.

Let K be a component of $G[N_5^b]$. Figure 5.5 gives a proof that K has a $G_{(5.5.e)}$ subgraph. Let r be in $\{a, b, c, d\}$. By Lemma 5.3, the dashed edge of $G_{(5.5.r)}$ is incident with a 5-cycle C. The possibilities for C are limited by the fact that G has girth five and by the assumption that the intersection of any two 5-cycles is empty or p_2. As well as insuring that the intersection of C and any other 5-cycle is not p_3, we must insure that this is also true for the

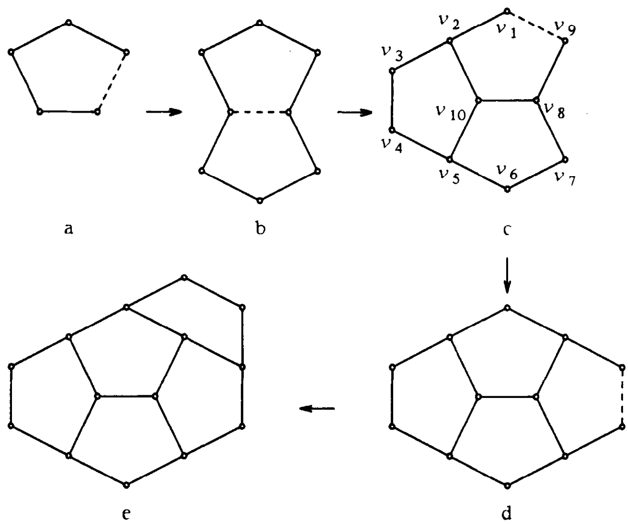


FIGURE 5.5

5-cycles using exactly one edge in $E(C)-E(G_{(5.5.r)})$. For $G_{(5.5.c)}$, this is the case: if $C=v_1v_2v_3v_7v_6$, then $(v_3v_4v_5v_6v_7)\cap(v_2v_3v_4v_5v_{10})=p_3$. In all possibilities one of the arrows from $G_{(5.5.r)}$ points to a graph which is also a subgraph of K.

For every even integer $m\geqslant 14$, we define B_m as follows. We use the notation used in the definitions of the graphs in $\mathscr{P}\cup\mathscr{D}$. If $m=2n$ and n is odd, then $B_m=P_{2n}-\{u_1u_2, v_1v_3, v_2v_n\}$. If $m=4n$, then $B_m=D_{4n}-\{x_1x_2, z_1z_2, y_1y_n\}$. Since $G_{(5.5.e)}=B_{14}$, we may choose a B_m subgraph of K such that $14\leqslant m$ and m is maximal.

Suppose $m\equiv 0 \pmod 4$ and consider $B_m=G_{(5.6.a)}$. See Fig. 5.6. By Lemma 5.3, x_1z_1 is incident with a 5-cycle C. By assumption, the intersection of C and any other 5-cycle is not p_3. Then K has a $G_{(5.6.b)}$, $G_{(5.6.c)}$, $G_{(5.6.d)}$, or $G_{(5.6.e)}$ subgraph. If K has a $G_{(5.6.b)}$ subgraph, then we have contradicted the maximality of m since $G_{(5.6.b)}=B_{m+2}$. If K has a $G_{(5.6.c)}$ subgraph, then $G=G_{(5.6.c)}+x_1x_2$. But then y_1z_2 is not incident with a 5-cycle, contradicting Lemma 5.3. If K has a $G_{(5.6.d)}$ subgraph, then $20\leqslant m$ and $G=G_{(5.6.d)}+x_1x_2=D_m$. If K has a $G_{(5.6.e)}$ subgraph then consider y_1z_2. Since y_1z_2 and x_1z_1 are isomorphic edges of B_m, when we consider the 5-cycle incident with y_1z_2, we can show that either $G=D_m$ or $y_1y_n\in E(G)$. If $y_1y_n\in E(G)$, then $G=G_{(5.6.e)}+\{y_1y_2, z_1z_2\}=D_{20}$.

Suppose $m\equiv 2 \pmod 4$ and consider $B_m=G_{(5.7.a)}$. See Fig. 5.7. By Lemma 5.3, u_1v_1 is incident with a 5-cycle C. By assumption, the intersection of C and any other 5-cycle is not p_3. Then K has a $G_{(5.7.b)}$, $G_{(5.7.c)}$, $G_{(5.7.d)}$, $G_{(5.7.e)}$, or $G_{(5.7.f)}$ subgraph. If K has a $G_{(5.7.b)}$ subgraph, then we

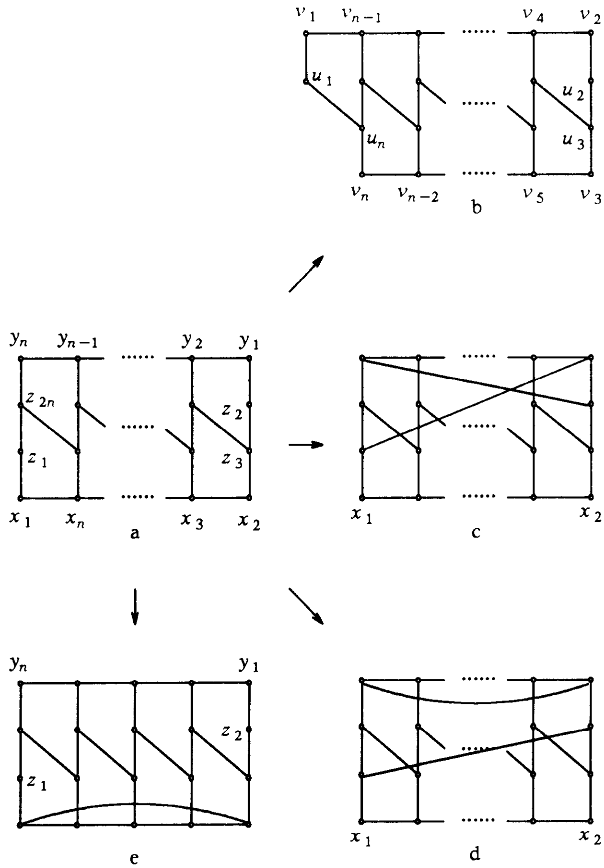


FIGURE 5.6

have contradicted the maximality of m since $G_{(5.7.b)} = B_{m+2}$. If K has a $G_{(5.7.c)}$ subgraph, then $G = G_{(5.7.c)} + v_1v_3$. By Lemma 5.3, u_2v_2 is incident with a 5-cycle. This is only possible if $m = 14$. Then $G = S_{14}$. If K has a $G_{(5.7.d)}$ subgraph, then $G = G_{(5.7.d)} + v_1v_3 = P_m$.

Suppose K has the subgraph $D = G_{(5.7.e)}$. If none of the edges incident with D is in the edge set of a 5-cycle, then $K = D$. If not, then K has a $G_{(5.8.a)}$ subgraph. See Fig. 5.8. By Lemma 5.3, e is incident with a 5-cycle; so K has a $G_{(5.8.b)}$ subgraph. Then Lemma 1.6 and the girth of G imply that $G \in \{S_{18}, D_{20}, G_{(5.8.c)}\}$. If $G = G_{(5.8.c)}$, then f is not incident with a 5-cycle and we have contradicted Lemma 5.3. Finally, if K has a $G_{(5.7.f)}$ subgraph, then we are done since $G_{(5.7.f)} = G_{(5.8.b)}$.

Thus, we have shown that either $G \in \mathscr{S} \cup \mathscr{P} \cup \mathscr{D}$ or every component of $G[N_5^b]$ is D. In the latter case, $G \in \mathscr{A}$. ▮

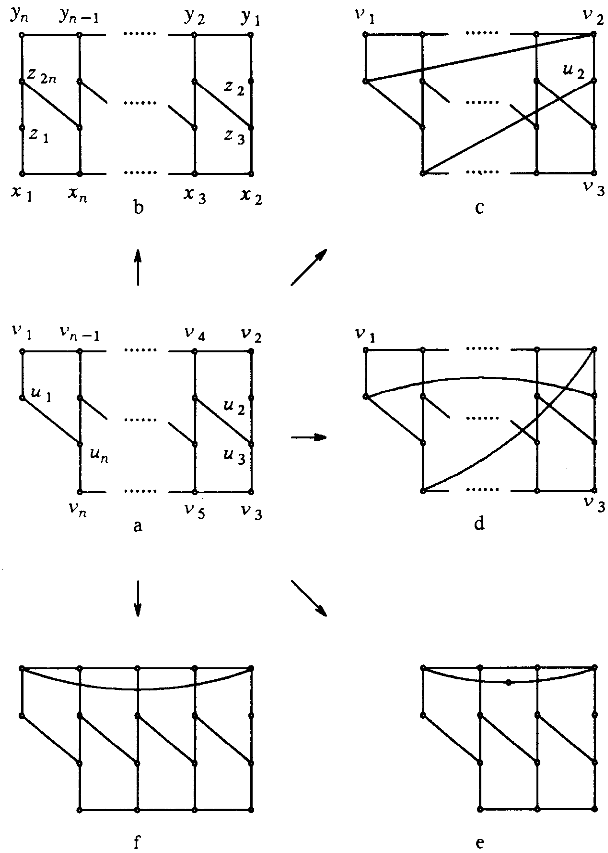


FIGURE 5.7

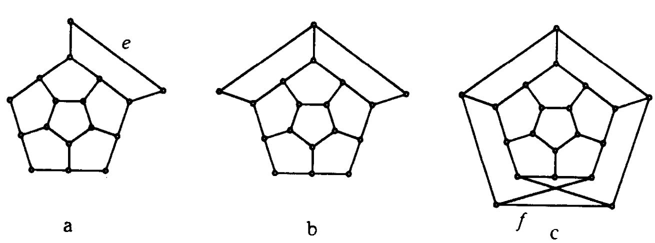


FIGURE 5.8

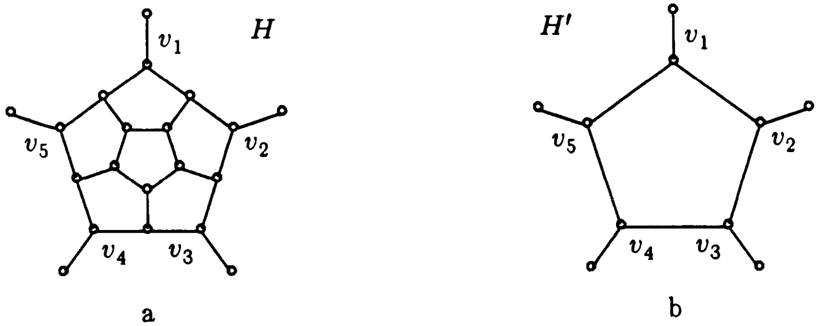


FIGURE 5.9

We end this section by stating a theorem on the structure of graphs in $\mathscr{A}$. A proof can be found in [22].

Let H be a cubic graph equal to $H^5(A, D)$. We say that $H' = H^5(A, C_5)$ is the *D-reduction of H at D* (see Fig. 5.9). Define N'_5 to be the set of edges in N_5 which are not on a 5-cycle.

THEOREM 5.6. *Let G be in $\mathscr{A}$. G has at least two D subgraphs. Any D-reduction of G is cyclically* 5*-connected. If $e \in N'_5$, then there exists an independent* 5*-edge cut T contained in N'_5 such that $e \in T$. $G[N'_5]$ is a forest with at least five trees.*

6. EDGE REDUCTIONS IN CYCLICALLY k-CONNECTED CUBIC GRAPHS WITHOUT k-CYCLES

Throughout this section G will represent a cyclically k-connected cubic graph with girth at least $k+1$, where $3 \leqslant k$.

In Section 2 we proved that for every k, if $N_k(G) \neq \varnothing$, then $G[N_k]$ is a forest with at least k trees. In this section we give more results about the structure of G and derive a theorem of Nedela and Škoviera. For $k \in \{3, 4, 5\}$, we give a sharp upper bound $g_k(v)$ on $|N_k(G)|$ for all G with v vertices and a complete characterization of the extremal graphs obtaining this bound.

A subgraph A of a cyclically k-connected graph H is a *k-end* if $E(A) \cap N_k(H) = \varnothing$ and A is a component of $H - S$, for some independent k-edge cut S.

THEOREM 6.1. *If A and B are k-ends of G, then either $A \cap B = \varnothing$ or $A = B$.*

Proof. Suppose $A \cap B \neq \emptyset$. Let S be the independent k-edge cut incident with A. Since A and B are connected and $A \cap B \neq \emptyset$, $A \cup B$ is connected. Since $A \cup B$ is connected and $E(A \cup B) \cap N_k(G) = \emptyset$, $A \cup B$ is a subgraph of some component of $G - N_k$. Therefore, $A \cup B$ is a subgraph of some component of $G - S$, and so $A \cup B = A$. Similarly, $A \cup B = B$. Thus, $A = B$. ▌

THEOREM 6.2. *If A is a component of $G - S$, for some independent k-edge cut S, then there is a k-end which is a subgraph of A. If $N_k(G) \neq \emptyset$, then G has at least two k-ends.*

Proof. Let S_1 be an independent k-edge cut such that $G - S_1$ has a component A_1 which is a subgraph of A. Suppose S_1 is chosen so that A_1 is minimal. Let B_1 be the other component of $G - S_1$.

Suppose $e \in E(A_1) \cap N_k(G)$. By Theorem 2.7, $E(A_1) \cup S_1$ contains an independent k-edge cut S_2 which includes e. Since $S_2 \cap E(B_1) = \emptyset$, B_1 is a subgraph of a component B_2 of $G - S_2$. Since $S_1 \neq S_2$, B_1 is a proper subgraph of B_2. Therefore, the other component A_2 of $G - S_2$ is a proper subgraph of A_1. But now we have contradicted the minimality of A_1. Therefore, $E(A_1) \cap N_k(G) = \emptyset$ and A_1 is a k-end.

Suppose $N_k(G) \neq \emptyset$. Then G has an independent k-edge cut S. Both components of $G - S$ contain a k-end. ▌

We now derive a result due to Škoviera and Nedela [23, 24].

THEOREM 6.3. *If H is a vertex-transitive cyclically k-connected cubic graph such that $N_k(H) \neq \emptyset$, then H has girth k.*

Proof. Suppose $N_k(G) \neq \emptyset$. Then G has a k-end A by Lemma 6.2. Since G has girth at least $k+1$, A has more than k vertices. Thus, the definition of k-end implies that A has a vertex x which is not incident with any edge in $N_k(G)$. Since $N_k(G) \neq \emptyset$, G also has a vertex y incident with an edge in $N_k(G)$. Since no automorphism can map x to y, G is not vertex-transitive. ▌

Let $A_3 = K_{3,2}$, $A_4 = G_{(6.1.b)}$, and $A_5 = G_{(6.1.e)}$. See Fig. 6.1. Let H_i be a cubic graph equal to $H^k(A_k, B_i)$, $i = 1, 2$. Define $H_1 \circledk H_2$ to be $H^k(B_1, B_2)$, $k = 3, 4, 5$. We note that H_1 and H_2 do not uniquely determine $H_1 \circledk H_2$. We will use $H_1 \circledk H_2$ to denote all such graphs.

We now recursively define three sets of graphs, $\mathcal{H}_3$, $\mathcal{H}_4$, and $\mathcal{H}_5$. Let $\mathcal{A}_3 = \{A_3, G_{(6.1.a)}\}$, $\mathcal{A}_4 = \{A_4, G_{(6.1.c)}, G_{(6.1.d)}\}$, and $\mathcal{A}_5 = \{A_5, G_{(6.1.f)}, G_{(6.1.g)}\}$. Let $\mathcal{H}_k$ contain all graphs $H^k(B_1, B_2)$ of girth $k+1$, where B_1 and B_2 are

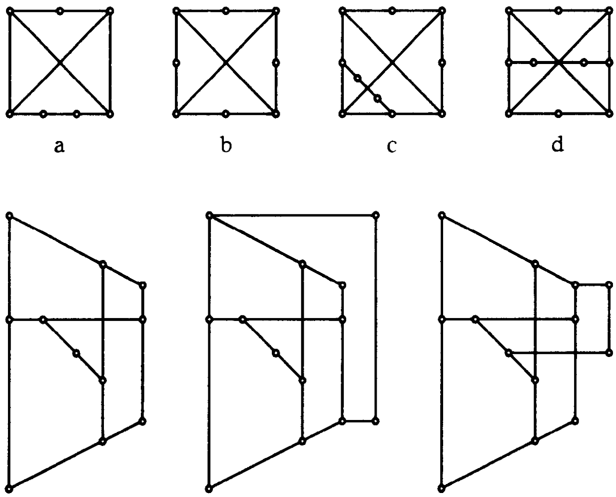


FIGURE 6.1

in $\mathscr{A}_k$, $k=3, 4, 5$. For $k=3, 4, 5$, if $H_i \in \mathscr{H}_k$ and $6k-6 \leqslant v(H_i)$, $i=1, 2$, and $H_1 \circledast H_2$ has girth at least $k+1$, then $H_1 \circledast H_2 \in \mathscr{H}_k$.

Define g_3, g_4, and g_5 as follows.

$$g_3(v)=\begin{cases} 0, & \text{if } 6 \leqslant v \leqslant 8 \\ v-7, & \text{if } 10 \leqslant v \end{cases}$$

$$g_4(v)=\begin{cases} 0, & \text{if } 10 \leqslant v \leqslant 14 \\ v-12, & \text{if } 16 \leqslant v \end{cases}$$

$$g_5(v)=\begin{cases} 0, & \text{if } 14 \leqslant v \leqslant 20 \\ v-17, & \text{if } 22 \leqslant v. \end{cases}$$

We now give a sharp upper bound for $|N_k(G)|$ and characterize the extremal graphs, for $k \in \{3, 4, 5\}$. We will only prove the upper bound for $|N_5(G)|$. The rest of the proof is similar to the methods used in [21] and Section 3. A complete proof can be found in [22].

THEOREM 6.4. *Let k be in $\{3, 4, 5\}$. Then $|N_k(G)| \leqslant g_k(v(G))$, and $|N_k(G)| = g_k(v(G))$ if and only if $G \in \mathscr{H}_k$. $\mathscr{H}_k$ contains a graph on v vertices, for every possible v.*

Proof. If G has an independent 5-edge cut S, then it is a routine exercise to show that both components of $G-S$ have at least 11 vertices. Hence, $N_5 = \varnothing$ if $v \leqslant 20$.

Suppose $N_5 \neq \varnothing$. By Theorem 2.3, $G[N_5]$ is a forest with at least five trees. Let r be the number of 5-ends of G. Let V_e be the set of vertices which are not incident with an edge in $N_k(G)$. Each 5-end has at least 11 vertices and the 5-ends are disjoint by Theorem 6.1, and so $6r \leqslant |V_e|$. From Theorem 6.2 we know that $2 \leqslant r$. Thus, $|N_5| = v(G[N_5]) - \omega(G[N_5]) \leqslant |V - V_e| - 5 = v - |V_e| - 5 \leqslant v - 6r - 5 \leqslant v - 17$. ∎

Acknowledgments

The results in this paper are from the author's doctoral thesis [22]. The author thanks the referee for helpful comments.

The results of Section 3 were obtained independently by Fouquet and Thuillier [11]. Andersen, Fleischner, and Jackson [2] have independently proven that $|N_4(G)| \leqslant (6v - 12)/5$, for all cyclically 4-connected graphs G. Consider $G_{(5.5.d)}$ to be a subgraph of a cyclically 5-edge connected cubic graph G and let $e = v_2v_{10}$ and $f = v_7v_8$, where we have used the notation of Fig. 5.5.c. We refer to an edge reduction at e followed by an edge reduction at f as a special double edge reduction of G. Barnette [4] and Butler [8] have independently proven that every planar cyclically 5-connected cubic graph except D_{20} has an edge reduction, a D-reduction, or a special double edge reduction to a smaller planar cyclically 5-connected cubic graph. The set of graphs G in $\mathscr{A}$ such that every component of $G[N_5']$ is isomorphic to K_2 was first discovered by Wormald [32]. The characterization of cyclically 5-connected cubic graphs with no edge reduction to a smaller cyclically 5-connected cubic graph was independently discovered by Aldred and Holton [1]. Fouquet and Thuillier [13] have independently proven Theorem 6.4 for $k = 3, 4$. The proof of Theorem 6.3 given by Nedela and Škoviera [24] roughly follows the version given in this paper, that is, they prove versions of the relevant parts of Lemma 2.1 and Theorems 2.7 and 6.2.

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