Edge Reductions in Cyclically $k$-Connected Cubic Graphs

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This paper examines edge reductions in cyclically $k$-connected cubic graphs, focusing on when they preserve the cyclic $k$-connectedness. For a cyclically $k$-connected cubic graph $G$, we denote by $N_k(G)$ the set of edges whose reduction gives a cubic graph which is not cyclically $k$-connected. With the exception of three graphs, $N_k(G)$ consists of the edges in independent $k$-edge cuts. For this reason we examine the properties and interactions between independent $k$-edge cuts in cyclically $k$-connected cubic graphs. These results lead to an understanding of the structure of $G[N_k]$. For every $k$, we prove that $G[N_k]$ is a forest with at least $k$ trees if $G$ is a cyclically $k$-connected cubic graph with girth at least $k+1$ and $N_k \neq \emptyset$. Let $f_k(v)$ be the smallest integer such that $|N_k(G)| \leq f_k(v)$ for all cyclically $k$-connected cubic graphs $G$ on $v$ vertices. For all cyclically 3-connected cubic graphs $G$ such that $6 \leq v(G)$ and $N_3 \neq \emptyset$, we prove that $G[N_3]$ is a forest with at least three trees. We determine $f_3$ and state a characterization of the extremal graphs. We define a very restricted subset $N_4^k$ of $N_4$ and prove that if $N_4^k = N_4 - N_4^k \neq \emptyset$, then $G[N_4^k]$ is a forest with at least four trees. We determine $f_4$ and state a characterization of the extremal graphs. There exist cyclically 5-connected cubic graphs such that $E(G) = N_5(G)$, for every $v$ such that $10 \leq v$ and $16 \neq v$. We characterize these graphs. Let $g_k(v)$ be the smallest integer such that $|N_k(G)| < g_k(v)$ for all cyclically $k$-connected cubic graphs $G$ with $v$ vertices and girth at least $k + 1$. For $k \in \{3, 4, 5\}$, we determine $g_k$ and state a characterization of the extremal graphs.

1. INTRODUCTION

Edge reductions in cubic graphs were first used in a significant way by Steinitz and Rademacher [25] to prove that a graph is planar, cubic, and 3-connected if and only if it is the graph of a simple 3-polytope. In fact, they chose edge reductions which preserved 3-connectedness.

The definition of cyclic connectivity first appears in Tutte [30]. The concept of cyclic $k$-connectivity in cubic graphs has appeared in the theory

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developed to solve the Four Colour Conjecture. In 1852, Guthrie [5] conjectured that every planar bridgeless graph was 4-face colourable. Tait [27] in 1880 showed that this conjecture was equivalent to the statement that every planar bridgeless cubic graph is 3-edge colourable. As well, Tait [26] showed that the Four Colour Conjecture would follow if every 3-connected cubic graph was hamiltonian. But in 1946 Tutte [29] constructed a nonhamiltonian planar 3-connected cubic graph. Later Tutte [30] and Walther [31] constructed nonhamiltonian planar cubic graphs that were cyclically 4-connected and cyclically 5-connected, respectively. The work of Isaacs [16] and Goldberg [14] has shown that 4-edge chromatic bridgeless cubic graphs which are not cyclically 5-connected can be constructed using smaller 4-edge chromatic bridgeless cubic graphs. Thus, the nontrivial 4-edge chromatic bridgeless cubic graphs are cyclically 5-connected.

Edge reductions of cubic graphs which preserve the cyclic $k$-connectedness are useful as a proof technique. For example, such edge reductions can be used to prove that any $S$ of three independent edges in a 3-connected cubic graph is contained in the edge set of some cycle if $S$ is not an edge cut.

In the rest of the introduction we give some background and prove some results needed in later sections. In Section 2 we examine the structure of $G[N_k]$, where $G$ is a cyclically $k$-connected cubic graph and $N_k$ is the set of edges which are in independent $k$-edge cuts. In section $k$ we examine edge reductions in cyclically $k$-connected cubic graphs, $k = 3, 4, 5$. Edge reductions in cyclically $k$-connected cubic graphs with girth at least $k + 1$, where $3 \leq k$, are examined in Section 6.

We use the notation and terminology of Bondy and Murty [7] in this paper. In particular, for a graph $G$, $v(G)$ is the number of vertices of $G$, $e(G)$ is the number of edges of $G$, $\omega(G)$ is the number of components of $G$, $d_G(x)$ is the degree of $x$ in $G$, and $\delta_G$ is the minimum degree of $G$. In addition, define $v_i(G)$ to be the number of vertices of degree $i$ in graph $G$, $0 \leq i$, define $p_j$ to be the path with $j$ vertices, $1 \leq j$, and define $d_G(e, f)$ to be the shortest distance in graph $G$ between an end of edge $e$ and an end of edge $f$.

The graph in Fig. x.y.z will be referred to as $G_{(x,y,z)}$. In the figures we will use large circles and ovals to represent subgraphs and we will refer to them as clouds. For example, Fig. 5.2 has five clouds. Frequently, we will use notation in a proof which is defined by an accompanying figure.

Let $A_i$ be a graph such that $v_2 = k$ and $v_3 = v - k$, and let $x_{i_1}, \ldots x_{i_k}$ be its vertices of degree two, $i = 1, 2$. We define $H^k(A_1, A_2)$ to be $A_1 + A_2 + \{x_{j_1}x_{j_2} | j = 1, \ldots, k\}$. We note that $A_1$ and $A_2$ do not always uniquely determine $H^k(A_1, A_2)$. Depending on the context, we will use $H^k(A_1, A_2)$ to refer to an arbitrary such graph or to all such graphs.
Let $G$ be a graph and let $X \subseteq V(G)$ and $S \subseteq E(G)$. If $|X| = k$ and $2 \leq \omega(G - X)$, then $X$ is called a $k$-vertex cut. If $|S| = k$ and $2 \leq \omega(G - S)$, then $S$ is called a $k$-edge cut. If $|X| = k$ and $G$ has subgraphs $G_1$ and $G_2$ such that $G = G_1 \cup G_2$, $V(G_1 \cap G_2) = X$, and both $G_1$ and $G_2$ contain cycles, then $X$ is called a cycle-separating $k$-vertex cut. If $|S| = k$ and $G - S$ has at least two components which contain cycles, then $S$ is called a cycle-separating $k$-edge cut.

If $G$ has a pair of nonadjacent vertices and every vertex cut of $G$ contains at least $k$ vertices, then $G$ is said to be $k$-vertex connected. If $G$ has a spanning complete subgraph and $k - 1 \leq v(G)$, then $G$ is also said to be $k$-vertex connected. If every edge cut of $G$ contains at least $k$ edges, then $G$ is said to be $k$-edge connected. If every cycle-separating vertex (edge) cut of $G$ has size at least $k$, then $G$ is said to be cyclically $k$-vertex (edge) connected.

If $G$ is a cubic graph, then $G$ has a cycle $C$. Since $G = C \cup G$, $V(C \cap G) = V(C)$, and both $C$ and $G$ have cycles, $V(C)$ is a cycle-separating vertex cut. Hence there is a largest $k$ such that $G$ is cyclically $k$-vertex connected, and $k$ is at most the girth of $G$.

Let $e$ be in $E(G)$. If $H$ is a subgraph of graph $G$ and $e$ has one end in $V(H)$ and one end in $V(G) - V(H)$, then we say that $e$ is incident with $H$. If $e$ is incident with disjoint subgraphs $H_1$ and $H_2$ of $G$, then we say that $e$ joins $H_1$ and $H_2$.

Let $\theta$ be the loopless graph on two vertices with exactly three edges.

The following two theorems demonstrate that various forms of connectivity are equivalent for all but a finite number of cubic graphs. Both have routine proofs. The first statement of Theorem 1.2 is a corollary of a result of Lovász [18; 19, Exercise 10.4].

**Theorem 1.1.** If $G$ is a connected cubic graph such that $G \neq \emptyset$ and $1 \leq k \leq 3$, then the following statements are equivalent:

(i) $G$ is $k$-edge connected.
(ii) $G$ is $k$-vertex connected.
(iii) $G$ is cyclically $k$-vertex connected.

**Theorem 1.2.** Let $G$ be a connected cubic graph. $G$ has two disjoint cycles if and only if $G \notin \{\emptyset, K_4, K_{3,3}\}$. If $G \notin \{\emptyset, K_4, K_{3,3}\}$, then the following three conditions are equivalent:

(i) $G$ is cyclically $k$-vertex connected.
(ii) Every independent edge cut of $G$ has size at least $k$.
(iii) $G$ is cyclically $k$-edge connected.

Since $\emptyset$, $K_4$, and $K_{3,3}$ do not have two disjoint cycles, they have no cycle-
separating or independent edge cut. Hence, for these three graphs, (ii) and (iii) are vacuously satisfied for all values of \( k \). We will say that a graph \( G \) is cyclically \( k \)-connected if \( G \) is cyclically \( k \)-vertex connected.

Let \( e \) be an edge joining distinct vertices \( x \) and \( y \) in a cubic graph \( G \). Let \( N(x) = \{ y, x_1, x_2 \} \) and \( N(y) = \{ x, y_1, y_2 \} \). Suppose that \( e \) is the only edge joining \( x \) and \( y \) and that \( x \) and \( y \) are not incident with a loop, that is, suppose \( \{ x, y \} \cap \{ x_1, x_2, y_1, y_2 \} = \emptyset \). Define \((G - e)^{\sim}\) to be \((G - \{ x, y \}) + \{ x_1, x_2, y_1, y_2 \} \) (see Fig. 1.1). By our assumptions \((G - e)^{\sim}\) is a well defined cubic graph with \( v(G) - 2 \) vertices. We say that \((G - e)^{\sim}\) is obtained from \( G \) by an edge reduction at \( e \), and that \( G \) is obtained from \((G - e)^{\sim}\) by adding an edge \( e \) across \( x_1x_2 \) and \( y_1y_2 \).

Since a cyclically 3-connected cubic graph has no loops or multiple edges, there is an edge reduction at every edge.

If \( G \) is a cyclically \( k \)-connected cubic graph and \( G \not\in \{ \emptyset, K_4, K_3, 3 \} \), then we define \( N_k(G) \) to be the set of edges which are in some independent \( k \)-edge cut. For \( G \) in \( \{ \emptyset, K_4, K_3, 3 \} \), we define \( N_k(G) \) to be \( E(G) \). We define \( R_k(G) \) to be \( E(G) - N_k(G) \). The next theorem, phrased differently, is proven in a paper by Wormald [32]. It demonstrates the significance of preceding definitions.

**Theorem 1.3.** If \( G \) is a cyclically \( k \)-connected cubic graph, where \( 3 \leq k \), then \( N_k(G) \) is the set of edges \( e \) such that \((G - e)^{\sim}\) is not cyclically \( k \)-connected.

The next theorem is stated without proof in a paper by Wormald [32].

**Theorem 1.4.** Let \( G \) be obtained by adding an edge \( e \) across two edges in a cyclically \( k \)-connected cubic graph \( G' \). Then \( G \) is cyclically \( k \)-connected if and only if \( G \) has girth at least \( k \).

**Proof.** If \( G \) has girth less than \( k \), then \( G \) has a cycle \( C \) with less than \( k \) vertices. Then \( v(C) \) is a cycle-separating vertex cut with less than \( k \) vertices, and so \( G \) is not cyclically \( k \)-connected.
Suppose \( G \) is not cyclically \( k \)-connected. Since \( 4 \leq \nu(G') + 2 = \nu(G) \), \( G \neq \emptyset \). If \( G \) is \( K_4 \) or \( K_{3,3} \), then \( G' \) is \( \emptyset \) or \( K_4 \), respectively, and the result follows. Suppose \( G \neq \{ \emptyset, K_4, K_{3,3} \} \). Then Theorem 1.2 implies that \( G \) has an independent edge cut \( S \) such that \( |S| < k \) and \( S \) is cycle-separating. Let \( A_i \) be a component of \( G - S \) having a cycle \( C_i, i = 1, 2 \).

Suppose \( e \in S \). Since \( e \notin E(A_i) \) there is a cycle \( C_i \) in \( G' \) corresponding to \( C_i, i = 1, 2 \). Now \( S - \{ e \} \) is a cycle-separating edge cut in \( G' \) of size less than \( k \), a contradiction.

Thus, we may assume that \( e \in A_1 \). If \( A_1 - e \) has a cycle, then \( G' \) again has two cycles separated by an edge cut of size less than \( k \). Hence \( A_1 - e \) is acyclic. Since \( \delta_{A_1} = 2 \), \( A_1 - e \) has at most two leaves. Hence, \( A_1 - e \) is a path, and so \( A_1 \) is a cycle. But the number of vertices of degree two in \( A_1 \) is \( |S| \), and so \( A_1 \) is a cycle of length less than \( k \). Thus, the girth of \( G \) is less than \( k \). 1

We end the section with two lemmas on the structure of the components of a graph \( G - S \), where \( S \) is an independent \( k \)-edge cut in a cyclically \( k \)-connected cubic graph \( G \).

**Lemma 1.5.** Let \( S \) be an independent \( k \)-edge cut in a cyclically \( k \)-connected cubic graph \( G \). Then \( G - S \) has exactly two components. If \( 3 < k \) and \( A \) is a component of \( G - S \), then \( A \) is 2-vertex connected.

Proof. By definition \( G - S \) has at least two components. If \( G - S \) has three components, then \( S - \{ e \} \) is a cycle-separating \((k - 1)\)-edge cut for any \( e \) in \( S \), a contradiction.

Suppose \( A \) is not 2-vertex connected. Then \( V(A) \) contains a vertex \( x \) such that \( A - x \) is not connected. Since \( d_A(x) \leq 3 \), \( x \) is incident with a cut edge \( e \) of \( A \). Let \( A_1 \) and \( A_2 \) be the components of \( A - e \). Let \( S \), be the set of edges in \( S \) incident with \( A_i, i = 1, 2 \).

A vertex in \( A_i \) has degree one in \( A_i \) if and only if it is incident with \( e \) and an edge in \( S_i, i = 1, 2 \). Hence, \( A_1 \) and \( A_2 \) each have at most one vertex of degree one, and so \( A_1 \) and \( A_2 \) both have cycles. Therefore, \( S_i \cup \{ e \} \) is cycle-separating, and so \( k \leq |S_i \cup \{ e \}|, i = 1, 2 \). Thus, \( 2(k - 1) \leq |S_1| + |S_2| = |S| = k \), and so \( k \leq 2 \).

**Lemma 1.6.** Let \( A \) be a subgraph of a cubic graph \( G \), let \( S \) be the set of edges incident with \( A \), and let \( B = G - V(A) \). Suppose \( |S| = k \), where \( 1 \leq k \), and the edges in \( S \) have distinct ends in \( V(A) \). If \( G \) is cyclically \((k + 1)\)-connected, then \( B \) is acyclic. If \( 1 \leq k \leq 2 \), then \( B \) has a cycle. If \( 3 \leq k \leq 5 \) and \( B \) is acyclic, then \( B = p_{k-2} \).

Proof. Suppose \( G \) is cyclically \((k + 1)\)-connected. Then \( |S| < k + 1 \) implies that \( A \) or \( B \) is acyclic. Since the edges of \( S \) have distinct ends in \( V(A) \), \( \delta_A = 2 \). Then \( A \) has a cycle, and so \( B \) is acyclic.
If $1 \leq k \leq 2$, then at most one vertex of $B$ has degree at most one, and so $B$ has a cycle.

Suppose $B$ is acyclic and $3 \leq k \leq 5$. If $B$ is not connected, then some component $B'$ of $B$ is incident with at most 2 edges in $S$. But then $B'$ would have a cycle. Hence, $B$ is connected. If $B = p_1$, then all edges in $S$ are incident with the vertex of $B$, and so $k = 3$. Suppose $B \neq p_1$. Then $2 \leq v_1(B)$ and $v_3(B) = v_1(B) - 2$ because $B$ is a tree. Also, since $G$ is cubic, $2v_1 + v_2 = |S| = k$. Hence $4 \leq k$. If $k = 4$, then $v_1 = 2$, $v_2 = 0$, and $v_3 = 0$, and so $B = p_2$. If $k = 5$, then $v_1 = 2$, $v_2 = 1$, and $v_3 = 0$; so $B = p_3$. 

\[ \]

2. Crossing Edge Cuts and the Subgraph Induced by the Edges in Independent $k$-Edge Cuts

In this section we define crossing edge cuts and prove a lemma which gives the structure of a cyclically $k$-connected cubic graph having two independent $k$-edge cuts which cross. This main lemma provides insights on the structure of $G[N_k]$, where $G$ is a cyclically $k$-connected cubic graph. In particular, we show that the edges in $N_k$ which are not on $k$-cycles induce a forest. If in addition, $G$ has girth $k + 1$ and $N_k \neq \emptyset$, then this forest has at least $k$ trees. This last result is reminiscent of a theorem of Mader [6, p. 24; 20]: the vertices of degree at least $k + 1$ in a minimally $k$-connected graph induce a forest. Mader's proof uses the notion of crossing cuts. We also define a method of constructing new cubic graphs from given cubic graphs and prove several results needed in later sections.

Let $S$ and $S'$ be independent $k$-edge cuts in a graph $G$. We say that $S$ and $S'$ cross if there are components $A$ and $B$ of $G - S$ and components $A'$ and $B'$ of $G - S'$ such that $A \cap A'$, $A \cap B'$, $B \cap B'$, and $B \cap A'$ are nonempty.

Define $\theta_8$ to be the graph obtained by replacing every edge of $\theta$ by a path of length three.

**Lemma 2.1.** Let $S$ and $S'$ be crossing independent $k$-edge cuts in a cyclically $k$-connected cubic graph $G$. Let $A$ and $B$ (respectively, $A'$ and $B'$) be the components of $G - S$ (respectively, $G - S'$), and let $G_1 = A \cap A'$, $G_2 = A \cap B'$, $G_3 = B \cap B'$, and $G_4 = B \cap A'$. Let $S_i$ be the set of edges in $S \cup S'$ incident with $G_i$, $1 \leq i \leq 4$, and let $E_{ij}$ be the edges in $S \cup S'$ joining $G_i$ and $G_j$, $1 \leq i < j \leq 4$ (see Fig. 2.1). Then $|S_1| + |S_3| = 2k - 2 |E_{24}|$ and $|S_2| + |S_4| = 2k - 2 |E_{13}|$. Also, if $|S| \leq k$ and $G_i$ is acyclic, $i = 1, 2$, then $A$ is a $k$-cycle, or $k = 6$ and $A = \theta_8$.

**Proof.** First, $|S_1| + |S_3| = |S_1 \cup S_3| + |S_1 \cap S_3| = (|S \cup S'| - E_{24}| + |E_{13}| = |S \cup S'| - |E_{24}| + |E_{13}| = |S| + |S'| - |S \cap S'| - |E_{24}| +$
\[ |E_{13}| = 2k - |E_{24} \cup E_{13}| - |E_{24}| + |E_{13}| = 2k - 2|E_{24}|. \] Similarly,
\[ |S_2| + |S_4| = 2k - 2|E_{13}|. \]

Suppose \( |S_i| \leq k \) and \( G_i \) is acyclic, \( i = 1, 2 \).

Let \( x \in V(G_i) \), where \( 1 \leq i \leq 4 \). Then \( x \) is incident with at most one edge in each of \( S \) and \( S' \). Also, \( d_{G_i}(x) = 3 - j \) if and only if \( x \) is incident with \( j \) edges in \( S \cup S' \). Thus, for \( G_i \), \( 1 \leq \delta \) and \( |S_i| = 2v_1 + v_2 \). Since \( G_i \) is a forest, \( v_1 - 2\omega = v_3 \).

Thus, for \( G_i \) we have
\[
v = v_1 + v_2 + [v_1 - 2\omega] = [2v_1 + v_2] - 2\omega = |S_i| - 2\omega. \tag{2.1}
\]

Since \( S_1 \cap S_2 = E_{12} \) and \( S_1 \cup S_2 = S \cup E_{12} \),
\[
|S_1| + |S_2| = |S| + 2|E_{12}| = k + 2|E_{12}|. \tag{2.2}
\]

Since \( |S_i| \leq k \), \( i = 1, 2 \), (2.2) implies
\[
2|E_{12}| \leq k. \tag{2.3}
\]

From (2.1), (2.2), and (2.3) we obtain
\[
v(A) = v(G_1) + v(G_2) = |S_1| - 2\omega(G_1) + |S_2| - 2\omega(G_2)
= k + 2|E_{12}| - 2[\omega(G_1) + \omega(G_2)] \leq 2k - 4. \tag{2.4}
\]
Since $S$ is independent and $G$ is cubic, $A$ is a $k$-cycle or $A$ is a subdivision of a cubic graph $H$. Suppose we have the second case.

Since $G$ is cyclically $k$-connected, $A$ has girth at least $k$. If $H$ has two disjoint cycles, then so does $A$. But then $2k \leq v(A)$ and we contradict (2.4); so $H$ does not have two disjoint cycles. Hence, $H$ has no cycle-separating edge cut, and so Theorem 1.2 implies $H \in \{ \emptyset, K_4, K_{3,3}, K_4 \}$.

Suppose $H \in \{ K_4, K_{3,3} \}$. Then $A$ has a subgraph $H'$ which is a subdivision of $K_4$. Let $C_H'$ be the set of cycles of $H'$ using exactly 3 vertices of degree 3 in $H$. Let $a$ be the number of ordered pairs $(x, C)$ where $C \in C_H'$ and $x \in V(C)$. Since $H'$ has girth at least $k$ and $|C_H'| = 4$, $4k \leq a$. For $l = 2, 3$, $d_{C_H'}(x) = l$ if and only if $x$ is on $l$ cycles in $C_H'$; so $a = 2v_2(H') + 3v_3(H') = 2v(H') + 4$. Finally, using (2.4) we obtain $4k < a = 2v(H') + 4 \leq 2v(A) + 4 \leq 2[2k - 4] + 4 = 4k - 4 < 4k$, a contradiction.

Suppose $H = \emptyset$. Let $C_A$ be the set of cycles of $A$. Let $b$ be the number of ordered pairs $(x, C)$ where $C \in C_A$ and $x \in V(C)$. Since $A$ has girth at least $k$ and $|C_A| = 3$, $3k \leq b$. If $d_A(x) = l$, then $x$ is on $l$ cycles in $C_A$, $l = 2, 3$, and so $b - 2v_2(A) + 3v_3(A) = 2k + 6$. Therefore, $3k \leq b = 2k + 6$, so $k \leq 6$. By (2.4), $k + 2 = v(A) \leq 2k - 4$, so $6 \leq k$. Thus, $k = 6$. For $k = 6$, $v_2(A) = 6$ and $A$ has girth at least 6; so $A = \emptyset$.

Let $N_k^c(H)$ be the set of edges in $N_k^c(H)$ which are not on a $k$-cycle.

**Theorem 2.2.** Let $G$ be a cyclically $k$-connected cubic graph. Then $G[N_k^c]$ is acyclic.

**Proof.** Suppose $C$ is a cycle of $G[N_k^c]$. Let $e_1 \in E(C)$ and let $S$ be an independent $k$-edge cut such that $e_1 \in S$. Then $S$ contains another edge $e_2 \in E(C)$. Suppose $e_1, e_2$, and $S$ are chosen so that $d_C(e_1, e_2)$ is minimal. Let $p$ be an $(e_1, e_2)$-path in $C$ with length $d_C(e_1, e_2)$.

If there exists $e \in E(p) \cap S$, then we have contradicted the choice of $e_1, e_2$, and $S$ because $d_C(e_1, e) < d_C(e_1, e_2)$. Therefore, $p$ is a subgraph of a component $A$ of $G - S$. By Lemma 1.5, $G - S$ has only one other component; call it $B$.

Choose $f_1 \in E(p)$. Let $S'$ be an independent $k$-edge cut containing $f_1$. If $E(p) \cup \{ e_1, e_2 \}$ contains an edge $f$ in $S' - \{ f_1 \}$, then $f, f_1$, and $S'$ contradict the choice of $e_1, e_2$, and $S$.

Let $q$ be an $(e_1, e_2)$-path in $B$. Let $C'$ be the cycle $G[E(p) \cup E(q) \cup \{ e_1, e_2 \}]$. Since $f_1 \in E(C')$, there exists another edge $f_2$ in $S' \cap E(C')$. By the previous paragraph, $f_2$ is in $E(q)$. Hence, $S$ and $S'$ cross.

We now adopt the notation of Lemma 2.1 (see Fig. 2.1). Since $\{ f_1 \} = (E(p) \cup \{ e_1, e_2 \}) \cap S'$, we may assume that $e_1 \in A'$ and $e_2 \in B'$. Thus, not all the edges in $A'$ (respectively, $B'$) are on $k$-cycles. The edges in $E(C)$ which are incident with $e_1$ are in $E(A)$ and $E(B)$. Thus, not all the edges in $A$ (respectively, $B$) are on $k$-cycles.
Suppose $k < |S_1|$. Then by Lemma 2.1, $|S_3| = 2k - 2|E_3| = |S_1| < k$. Since $G$ is cyclically $k$-connected, $G_3$ is acyclic. If $k < |S_2|$, then $|S_4| = 2k - 2|E_4| = |S_2| < k$, so $G_4$ is acyclic. Then Lemma 2.1 implies that every edge of $B$ is on a $k$-cycle, a contradiction. Hence, $|S_2| \leq k$. If $G_2$ is acyclic then Lemma 2.1 implies that every edge of $B'$ is on a $k$-cycle, a contradiction. Therefore, $G_2$ has a cycle. Let $V_a$ be the set of ends in $G_2$ of edges in $S_2$. Since $V_a$ is a cycle-separating vertex cut and $G$ is cyclically $k$-connected, $k \leq |V_a| \leq |S_2|$. Thus, $k = |S_2|$ and the edges in $S_2$ have distinct ends in $G_2$. Hence, $S_2$ is an independent $k$-edge cut. But now $e_2$, $f_1$, and $S_2$ contradict the choice of $e_1$, $e_2$, and $S$. Thus, $|S_1| \leq k$. Similarly, $|S_2| \leq k$.

If $G_i$ contains a cycle, then we can again show that $S_i$ is an independent $k$-edge cut, $i = 1, 2$. This will again contradict the choice of $e_1$, $e_2$, and $S$. Therefore, $G_1$ and $G_2$ are acyclic. Applying Lemma 2.1 again we obtain our final contradiction: every edge of $A$ is on a $k$-cycle.

For a cyclically $k$-connected cubic graph $G$ such that $N'_k \neq \emptyset$, the forest $G[N'_k]$ does not necessarily have more than one component. Fouquet and Thuillier [12], have constructed a cyclically $5$-connected cubic graph $G$ such that $G[N'_3]$ has only one component.

**THEOREM 2.3.** Let $G$ be a cyclically $k$-connected cubic graph such that $N_k = N'_k$. Then we cannot have two edges in $N_k$ in the same independent $k$-edge cut and in the same component of $G[N_k]$. If $N_k \neq \emptyset$, then $G[N_k]$ is a forest with at least $k$ trees.

**Proof.** Suppose $e_1$ and $e_2$ are in an independent $k$-edge cut $S$ and there exists an $(e_1, e_2)$-path $p$ in $G[N_k]$. Assume $e_1$ and $e_2$ are chosen so that the length of $p$ is minimal. Let $f_1 \in E(p)$ and let $S'$ be an independent $k$-edge cut containing $f_1$. As in the proof of Theorem 2.2, we can show that $S$ and $S'$ cross. Let $A$, $B$, $A'$, $B'$ be as in Lemma 2.1 (see Fig. 2.1). Since $E(A) \cap N_k \neq \emptyset$, we cannot have all the edges of $A$ on $k$-cycles. Similarly, this is the case for $A'$, $B'$, and $B$. We now proceed exactly as in the proof of Theorem 2.2 to derive a contradiction.

Now we show that if $N_k \neq \emptyset$, then $G[N_k]$ is a forest with at least $k$ trees. Since $N_k = N'_k$, Theorem 2.2 implies that $G[N_k]$ is acyclic. Since $N_k \neq \emptyset$, there exists an independent $k$-edge cut $T$. Then all the edges of $T$ are in different components of $G[N_k]$; so we have at least $k$ trees.

**COROLLARY 2.4.** Let $G$ be a cyclically $k$-connected cubic graph with girth at least $k + 1$. If $N_k \neq \emptyset$, $G[N_k]$ is a forest with at least $k$ trees.

**Proof.** The girth of $G$ is at least $k + 1$, so $N_k = N'_k$. The result now follows from Theorem 2.3.
Theorem 2.5. Let $G_0$ be a cyclically $k$-connected cubic graph, where $5 \leq k$. Let $G_{i+1}$ be obtained from $G_i$ by edge addition, $i = 0, 1$. If $G_2$ has girth at least $k$, then $G_2$ is cyclically $k$-connected.

Proof. If $G_1$ has girth less than $k - 1$, then $G_2$ can have girth at most $k - 1$, a contradiction. Hence, $G_1$ has girth at least $k - 1$. Using Theorem 1.4 twice we derive that $G_2$ is cyclically $(k - 1)$-connected.

Suppose $G_2$ has an independent $(k-1)$-edge cut $S_2$. Let $A_2$ and $B_2$ be the components of $G_2 - S_2$. If $v(A_2) = k - 1$, then $A_2$ is a $(k-1)$-cycle, a contradiction. If $v(A_2) = k + 1$, then $A_2$ has two vertices of degree 3 and $k - 1$ vertices of degree 2. Using the methods in the proof of Lemma 2.1, we have $3k \leq 2v_2(A_2) + 3v_3(A_2) = 2k - 4$. Hence, $k \leq 4$, a contradiction. Thus $k + 3 < v(A_2)$. Similarly, $k + 3 < v(B_2)$.

Let $A_0 = G_0[V(A_2) \cap V(G_0)]$ and $B_0 = G_0[V(B_2) \cap V(G_0)]$. Let $S_0$ be the set of edges joining $A_0$ and $B_0$. Since $G_0$ is obtained from $G_2$ by two edge reductions and since $|S_2| = k - 1$, we have $k - 1 \leq v(A_0)$, $k - 1 \leq v(B_0)$, and $|S_0| \leq k - 1$. Counting the incidences in $A_0$ in two ways we obtain $2v(A_0) \leq 3v_2(A_0) + (k - 1)v_3(A_0) - |S_0| - 2v(A_0)$. Hence, $v(A_0) \leq v(A_0)$, and so $A_0$ has a cycle. Similarly, $B_0$ has a cycle. But now $S_0$ is a cycle-separating edge cut with at most $k - 1$ edges, and the cyclic $k$-connectedness of $G_0$ is contradicted. Therefore, $G_2$ is cyclically $k$-connected.

Lemma 2.6. Let $3 \leq k \leq 5$. Let $S$ and $S'$ be crossing independent $k$-edge cuts in a cyclically $k$-connected cubic graph $G$. Then $k = 3$ is not possible. If $k = 4$, then $G$ has the form $G_{(2,2,a)}$. If $k = 5$, then $G$ has the form $G_{(2,2,b)}$ or $G_{(2,2,c)}$. (see Fig. 2.2.)

Proof. We use the notation of Lemma 2.1 (see Fig. 2.1). By Lemma 1.5, $E_1, E_2, E_3, E_4, E_4$ all have at least two edges. Hence $4 \leq k$. If $k = 5$ and $|S_2| = 4$, then $G_i = K_2$ by Lemma 1.6, $i = 1, 2$. The result now follows.

Theorem 2.7. Let $S$ be an independent $k$-edge cut in a cyclically $k$-connected cubic graph $G$, let $A$ be a component of $G - S$, and let $e$ be in $E(A) \cap N_k(G)$. Then $e$ is in an independent $k$-edge cut contained in $E(A) \cup S$ if any of the following conditions hold:

(i) $k = 3$;

(ii) $k = 4$ and $6 \leq v(A)$;

(iii) $G$ has girth at least $k + 1$.

Proof. Suppose $e$ is in the independent $k$-edge cut $S'$. If $S' \subseteq E(A) \cup S$
we are done. By Lemma 2.6, this is the case if \( k = 3 \). So we may assume \( 4 \leq k \) and \( S \) and \( S' \) cross. We use the notation of Lemma 2.1 (see Fig. 2.1).

If \( k = 4 \), then \( G \) has the form \( G_{1,2,2} \) by Lemma 2.6. If \( S_1 \) or \( S_2 \) is an independent 4-edge cut, then we are done. If not, then \( G_i = K_2 \) by Lemma 1.6, \( i = 1, 2 \). But then \( v(A) = 4 \) and we have a contradiction.

We now prove the theorem when \( G \) has girth at least \( k + 1 \). Suppose \( |S_1| < k \). Then \( G_1 \) is acyclic. If \( |S_2| < k \), then \( G_2 \) is also acyclic; but now we have a contradiction because \( A \) has a \( k \)-cycle by Lemma 2.1. Hence, \( k < |S_2| \). Similarly, \( k < |S_4| \). By Lemma 2.1, \( |S_2| + |S_4| = 2k - 2 |E_1| \); so \( |S_2| = |S_4| = k \).

If \( G_2 \) is acyclic, then \( A \) again has a \( k \)-cycle; so \( G_2 \) has a cycle. Let \( V_\alpha \) be the set of ends in \( G_2 \) of edges in \( S_2 \). Since \( V_\alpha \) is a cycle-separating vertex cut and \( G \) is cyclically \( k \)-connected, \( k \leq |V_\alpha| \leq |S_2| \). Thus, \( k = |S_2| \) and the edges in \( S_2 \) have distinct ends in \( G_2 \). Hence, \( S_2 \) is an independent \( k \)-edge cut. Since \( e \in S_2 \) and \( S_2 \subseteq E(A) \cup S \), we are done. Similarly, if \( |S_i| < k \), for some \( i \in \{2, 3, 4\} \), we are done.

Suppose \( k \leq |S_i|, i = 1, 2, 3, 4 \). By Lemma 2.1, \( |S_1| + |S_3| \leq 2k \) and \( |S_2| + |S_4| \leq 2k \), so \( |S_i| = k, i = 1, 2, 3, 4 \). If \( G_2 \) contains a cycle, then as before we can show that \( S_2 \) is the required independent \( k \)-edge cut. Similarly, if \( G_1 \) contains a cycle we are done. If \( G_1 \) and \( G_2 \) are acyclic, then we have a contradiction because \( A \) has a \( k \)-cycle by Lemma 2.1.
3. EDGE REDUCTIONS IN CYCLICALLY 3-CONNECTED CUBIC GRAPHS

Throughout this section, $G$ will denote a cyclically 3-connected cubic graph such that $G \neq K_4$.

In this section we will examine edge reductions in cyclically 3-connected cubic graphs. The first theorem proves that $G[N_3]$ is a forest with at least three trees and that $|N_3| \leq v - 3$. We then state a characterization of those graphs $G$ with $|N_3| = v - 3$.

It is a classical theorem that every 3-connected graph except $K_4$ can be reduced to a smaller 3-connected graph by means of a more general form of edge reduction. This was proven for planar 3-connected graphs by Steinitz and Rademacher [25] and later for all 3-connected graphs by Barnette and Grünbaum [3], and independently, Titov [28]. This result implies that every cyclically 3-connected cubic graph except $K_4$ has an edge reduction which gives a smaller cyclically 3-connected cubic graph.

**Theorem 3.1.** If $N_3(G) \neq \emptyset$, then $N_3 = N'_3$, $G[N_3]$ is a forest with at least three trees, and $|N_3| \leq v - 3$. If $|N_3| = v - 3$, then $G[N_3]$ is a spanning forest with exactly three trees.

**Proof.** A 3-edge cut of a cyclically 3-connected cubic graph $G$ which contains an edge on a 3-cycle $C$ must necessarily contain two adjacent edges on $C$. Therefore, no independent 3-edge cut can contain an edge on a 3-cycle. Hence, $N_3 = N'_3$. Then Theorem 2.3 implies that $G[N_3]$ is a forest with at least three trees if $N_3 \neq \emptyset$. Hence, $|N_3| = e(G[N_3]) = v(G[N_3]) - \omega(G[N_3]) \leq v - 3$. Hence, if $|N_3| = v - 3$, then $G[N_3]$ is a spanning forest with exactly three trees.

Suppose $H_i$ is a cubic graph equal to $H^3(A_i, K_3)$, $i = 1, 2$. We define $H_1 \circ H_2$ to be $H^3(A_1, A_2)$. We note that $H_1$ and $H_2$ do not uniquely determine $H_1 \circ H_2$. We will use $H_1 \circ H_2$ to refer to all such graphs.

We now recursively define a set $\mathcal{T}$. Let $H_{(3.1.a)}$, $H_{(3.1.b)}$, and $H_{(3.1.c)}$ be $\mathcal{T}$. If $H_1$ and $H_2$ are in $\mathcal{T} - \{H_{(3.1.a)}\}$ then $H_1 \circ H_2$ is in $\mathcal{T}$. See Fig. 3.1.

The following theorem has a routine proof by induction on $v$. It is similar to the methods used in [21]. The proof is given in [22].

**Theorem 3.2.** $G \in \mathcal{T}$ if and only if $|N_3(G)| = v(G) - 3$.

![Figure 3.1](image-url)
4. EDGE REDUCTIONS IN CYCLICALLY 4-CONNECTED CUBIC GRAPHS

Throughout this section, \( G \) will denote a cyclically 4-connected cubic graph.

Every cyclically 4-connected cubic graph, except \( K_3,3 \) and the 3-cube, has an edge reduction which gives a smaller cyclically 4-connected cubic graph. This was proven for planar graphs by Kotzig [17] and, independently, by Faulkner and Younger [9]; later it was proven in general by Wormald [32] and, independently, by Fontet [10].

In this section we define a very restricted subset \( N_4^b \) of \( N_4 \). Unlike \( H[N_3] \) for a 3-connected cubic graph \( H \), \( G[N_4] \) is not necessarily a forest. But if we consider \( N_4^b = N_4 - N_4^b \), then we are able to show that \( G[N_4^b] \) is a forest with at least four trees when \( N_4^b \neq \emptyset \). We next define a function \( f_4 \) and show that it is an upper bound for \( |N_4(G)| \).

In [21], \( f_4 \) is shown to be a sharp upper bound and the extremal graphs are characterized. When \( G \) is restricted to being planar, the upper bound for \( |N_4(G)| \) can be improved, as is shown in [15]. In [21], the upper bound of [15] is shown to be sharp and the extremal graphs are characterized. If \( G \) is a plane graph, then its dual graph \( G^* \) is a 4-connected plane triangulation. An edge reduction of \( G \) results in a smaller cyclically 4-connected graph if and only if the contraction of the dual edge in \( G^* \) results in a smaller 4-connected graph. Hence, the results of [21] give a sharp lower bound on the number of edge contractions of a 4-connected plane triangulation which give a smaller 4-connected triangulation. The extremal graphs are the duals of the extremal graphs for edge reductions.

Let \( p_n = x_1x_2 \cdots x_n \) and \( p'_n = y_1y_2 \cdots y_n \) be disjoint paths. The \( 2n \)-ladder, \( L_{2n} \), is defined to be \( p_n + p'_n + \{x_i y_i \mid i = 1, \ldots, n\} \). Define \( p_n \) and \( p'_n \) to be the sides \( S_1 \) and \( S_2 \) of \( L_{2n} \), define the edges in \( E(S_1) \cup E(S_2) \) to be the side edges of \( L_{2n} \), define the edges in \( \{x_i y_i \mid i = 1, \ldots, n\} \) to be the rungs of \( L_{2n} \), define \( x_2, \ldots, x_{n-1}, y_2, \ldots, y_{n-1} \) to be the intermediate vertices of \( L_{2n} \), and define \( x_1, y_1, x_n, y_n \) to be the end vertices of \( L_{2n} \). Two side edges on the same 4-cycle are said to be corresponding side edges. Two end vertices incident with the same rung are said to be corresponding end vertices. Let \( E_1(L), E_2(L), \) and \( I(L) \), be the sets of side edges, rungs, and intermediate vertices, respectively, of ladder \( L \). Note that for \( L_4 \) we may choose which two independent edges in \( E(L_4) \) are the side edges. The other two independent edges will then be the rungs.

Let \( N_4^b(G) \) be the set of edges \( e \) which satisfy each of the following conditions:

(i) \( e \in N_4 \).

(ii) \( e \) is on a 4-cycle.
(iii) For every independent 4-edge cut $S$ containing $e$, one of the components of $G - S$ is a ladder.

Let $N_4^c(G) = N_4(G) - N_4^b(G)$.

Let $C_{2n} = v_1 v_2 \cdots v_{2n} v_1$, and define $M_{2n}$ to be $C_{2n} + \{v_i v_{i+1} \mid i = 1, \ldots, n\}$, for every $n \geq 3$. Let $E_s(M_{2n}) = E(C_{2n})$. Let $C_n = x_1 x_2 \cdots x_n x_1$ and $C_n' = y_1 y_2 \cdots y_n y_1$ be disjoint cycles and define $Q_{2n}$ to be $C_n + C_n' + \{x_i y_i \mid i = 1, \ldots, n\}$, for every $n \geq 4$. Let $E_s(Q_{2n}) = E(C_n) \cup E(C_n')$. Let $\mathcal{M} = \{M_{2n} \mid 3 \leq n\}$ and $\mathcal{Q} = \{Q_{2n} \mid 4 \leq n\}$.

The following lemma has a routine proof using induction and Theorem 1.4.

**Lemma 4.1.** The graphs in $\mathcal{M} \cup \mathcal{Q}$ are cyclically 4-connected. $N_4^b(M_6) = E(M_6)$ and $N_4^b(Q_8) = E(Q_8)$. For $n \geq 4$, $N_4(M_{2n}) = N_4^b(M_{2n}) = E_s(M_{2n})$. For $n \geq 5$, $N_4(Q_{2n}) = N_4^b(Q_{2n}) = E_s(Q_{2n})$.

**Lemma 4.2.** Let $C = v_1 v_2 v_3 v_4 v_1$ be a 4-cycle of $G$. If $v_1 v_2, v_2 v_3 \in N_4$, then $G \in \{M_6, Q_8\}$.

**Proof.** Suppose $G \neq M_6$. There exist independent 4-edge cuts $S$ and $T$ such that $v_1 v_2 \in S$ and $v_2 v_3 \in T$. Then $v_3 v_4 \in S$ and $v_4 v_1 \in T$, and so $S$ and $T$ cross. Thus, $G$ has the form $G_{(3,2,n)}$ by Lemma 2.6. Lemma 1.6 implies that each of the components of $G - (S \cup T)$ is a $K_2$. Thus, $G = Q_8$.

The following lemma has a routine proof.

**Lemma 4.3.** Let $S$ be an independent 4-edge cut of $G$. If both components of $G - S$ are ladders, then $N_8^4 \subseteq S$.

**Lemma 4.4.** Let $e_1$ and $e_2$ be in $N_4^c(G)$. If $e_1$ and $e_2$ are both in some independent 4-edge cut $S$ and there is an $(e_1, e_2)$-path $p$ in $G[N_4^c]$, then one of the components of $G - S$ is a ladder having $p$ as a side.

**Proof.** Suppose the result is false. Choose $e_1, e_2, S$, and $p$ so that the length of $p$ is minimum.

Suppose some edge $e$ in $S - \{e_1, e_2\}$ is in $E(p)$. Let $p_i$ be the subpath of $p$ joining $e$ and $e_i$, $i = 1, 2$. Then $p_i$ is shorter than $p$ and $e \in N_4^c$. Since $e$, $e_1$, $S$ and $p_i$ cannot contradict the choice of $e_1$, $e_2$, $S$, and $p$, one of the components of $G - S$ is a ladder having $p_i$ as a side, $i = 1, 2$. Since $V(p_1)$ and $V(p_2)$ contain different ends of $e$, the component of $G - S$ containing $p_1$ is distinct from the one containing $p_2$. Hence, both components of $G - S$ are ladders. Thus, $N_8^4 \subseteq S$ by Lemma 4.3. But then $E(p) \not\subseteq N_4^c$, and so we
have a contradiction. Hence, \((S - \{e_1, e_2\}) \cap E(p) = \emptyset\) and \(p\) is a subgraph of one of the components, say \(G_1\), of \(G - S\).

Let \(f_1\) be in \(E(p)\) and let \(f_1'\) be in an independent 4-edge cut \(T\). Suppose \([T - \{f_1\}] \cap \left[ E(p) \cup \{e_1, e_2\} \right] = \emptyset\). Then by Lemma 2.6, \(G\) has the form \(G_{2,2,2,2}\). Let \(p'_1\) be the subpath of \(p\) joining \(f_1\) and \(e_i\), \(i = 1, 2\). Let \(G_{1i}\) be the component of \(G - (S \cup T)\) incident with \(e_i\) and \(f_1\), \(i = 1, 2\). Let \(S_i\) be the edges incident with \(G_{1i}\), \(i = 1, 2\). By Lemma 1.6, either \(G_{1i} = K_2\) or \(S_i\) is independent. If \(S_i\) is independent, then, by the choice of \(e_1, e_2, S_i\), and \(p\), one of the components of \(G - S_i\) is a ladder having side \(p'_1\). Since \(V(p'_1) \cap V(G_{1i}) \neq \emptyset\), this component is \(G_{1i}\). In all cases \(G_1\) is a ladder \(L\). Since \(G \neq Q_8\), Lemma 4.2 implies that \(N_4\) cannot contain two adjacent edges of a 4-cycle of \(L\). Hence, \(E(p) \subseteq N_4 \cap E_4(L)\); so \(p\) is a side of \(L\), a contradiction.

Suppose \(f_2 \in [T - \{f_1\}] \cap \left[ E(p) \cup \{e_1, e_2\} \right]\). Let \(p'\) be the subpath of \(p\) joining \(f_1\) and \(f_2\). By the choice of \(e_1, e_2, S, p\), one of the components of \(G - T\) is a ladder with side \(p'\).

Let \(f'_1\) be in \(E(p')\). Then \(f'_1\) is on a 4-cycle and \(f'_1 \in N_4\). But \(f'_1 \notin N_4\), and so \(f'_1\) does not satisfy condition (iii) in the definition of \(N_4\). Therefore, there exists an independent 4-edge cut \(T'\) containing \(f'_1\) such that neither component of \(G - T'\) is a ladder.

As before with \(f_1\) and \(T\), if \([T' - \{f'_1\}] \cap \left[ E(p) \cup \{e_1, e_2\} \right] = \emptyset\), then \(G_1\) is a ladder having \(p\) as a side, a contradiction. Also, as before with \(f_1\) and \(T\), if \([T' - \{f'_1\}] \cap \left[ E(p) \cup \{e_1, e_2\} \right] \neq \emptyset\), then one of the components of \(G - T'\) is a ladder, a contradiction.

**Theorem 4.5.** If \(N_4^x(G) \neq \emptyset\), then \(G[N_4^x]\) is a forest with at least four trees.

**Proof:** Suppose there exists a cycle \(C\) in \(G[N_4^x]\). Let \(S\) be a cut containing an edge of \(C\). Since \(C\) is a cycle, \(E(C)\) contains two or four edges in \(E(S)\). Therefore, each component of \(G - S\) contains a path in \(G[N_4^x]\) connecting two edges in \(N_4^x \cap S\). By Lemma 4.4, each component is a ladder. Then \(N_4^x \subseteq S\) by Lemma 4.3, a contradiction. Thus, \(G[N_4^x]\) is a forest.

We now prove that \(G[N_4^x]\) has at least four components. Suppose \(G\) has no 4-cycles. There is an independent 4-edge cut \(T\) because \(N_4^x \neq \emptyset\). Since \(N_4^x \neq \emptyset\), \(T \subseteq N_4^x\). Since \(G\) has no ladders, no two edges in \(T\) are in the same component of \(G[N_4^x]\) by Lemma 4.4.

Suppose \(G\) has a 4-cycle. Choose a maximal ladder \(L\). Let \(S'\) be the set of edges incident with \(L\). If \(L\) is a spanning subgraph of \(G\), then \(|S'| = 2\) and \(G \in \mathcal{M} \cup \mathcal{B}\). But \(N_4 = \emptyset\) for graphs in \(\mathcal{M} \cap \mathcal{B}\) by Lemma 4.1, a contradiction. If the edges of \(S'\) are not independent, then \(G - V(L) = K_2\) by
Lemma 1.6. But then corresponding end vertices of \( L \) are adjacent to distinct adjacent vertices of \( G - V(L) \), and we contradict the maximality of \( L \). Hence, \( S' \) is an independent 4-edge cut. Since \( L \) is maximal, none of the edges in \( S' \) is on a 4-cycle, and so \( S' \subseteq N^b_4 \).

If the edges in \( S' \) are in different components of \( G[N^b_4] \) we are done; so suppose not. By Lemma 4.4, one of the components of \( G - S' \) is a ladder with a side having its edges in \( N^b_4 \). Therefore, some edge \( e \in N^b_4 \) is on a 4-cycle. Since \( e \in N^b_4 \) and \( e \) is on a 4-cycle, \( e \) is in a 4-edge cut \( T' \) such that neither component of \( G - T' \) is a ladder. Then \( T' \subseteq N^b_4 \) and, by Lemma 4.4, the edges of \( T' \) are in different components of \( G[N^b_4] \).

If \( G \notin \mathcal{M} \cup \mathcal{O} \), we define \( \mathcal{L}(G) \) to be the set of all maximal ladders \( L \) of \( G \) such that \( E_s(L) \subseteq N^a_4 \). Let \( E(\mathcal{L}) = \bigcup_{L \in \mathcal{L}} E(L) \), \( E_s(\mathcal{L}) = \bigcup_{L \in \mathcal{L}} E_s(L) \), \( E_r(\mathcal{L}) = \bigcup_{L \in \mathcal{L}} E_r(L) \), and \( I(\mathcal{L}) = \bigcup_{L \in \mathcal{L}} I(L) \).

**Lemma 4.6.** Suppose \( G \notin \mathcal{M} \cup \mathcal{O} \). The maximal ladders of \( G \) are disjoint. If \( L \in \mathcal{L} \), then \( 6 \leq v(L) \). \( E(\mathcal{L}) \cap N_4 = E_s(\mathcal{L}) = N^b_4 \).

**Proof.** Let \( L_a \) be a maximal ladder of \( G \), and let \( S' \) be the set of edges incident with \( L_a \). As in the proof of Theorem 4.5, we can show that \( S' \) is an independent 4-edge cut. Suppose \( V(L_a) \cap V(L_b) \neq \emptyset \) for some maximal ladder \( L_b \) such that \( L_a \neq L_b \). Then some edge in \( S' \) is on a 4-cycle \( C \) of \( L_b \). Then \( L_a \) is a proper subgraph of the ladder \( L_a \cup C \), a contradiction.

Suppose \( L \in \mathcal{L} \) and \( e \in E_s(L) \). Then \( e \in N^b_4 \). Hence, \( e \) is on a 4-cycle and incident with a 4-cycle by conditions (ii) and (iii), respectively, of the definition of \( N^b_4 \). Therefore, \( 6 \leq v(L) \).

By definition, \( E_s(\mathcal{L}) \subseteq E(\mathcal{L}) \cap N_4 \). Since \( G \notin \{ M_6, Q_8 \} \), Lemma 4.2 implies that \( E_s(\mathcal{L}) \cap N_4 = \emptyset \). Hence, \( E_s(\mathcal{L}) = E(\mathcal{L}) \cap N_4 \).

If \( 6 \leq v(L) \) for a ladder \( L \), then every edge in \( E_s(L) \) is incident with a 4-cycle of \( L \). Hence, \( E_s(L) \subseteq N_4 \). Hence, in order to prove that \( N^b_4 \subseteq E_s(\mathcal{L}) \), it suffices to show that either \( E_s(L) \subseteq N^b_4 \) or \( E_s(L) \subseteq N^b_4 \), for every maximal ladder \( L \) such that \( 6 \leq v(L) \).

Suppose \( 6 \leq v(L) \) and \( e_1 \in E_s(L) \cap N^b_4 \). Since \( e_1 \) is on a 4-cycle and \( e_1 \notin N^b_4 \), \( e_1 \) is in some independent 4-edge cut \( T \) such that neither component of \( G - T \) is a ladder. The corresponding edge \( e_2 \) of \( e_1 \) is necessarily in \( T \). Let \( T = \{ e_1, e_2, e_3, e_4 \} \).

Let \( f_1 \) and \( f_2 \) be corresponding side edges of \( L \). Since neither component of \( G - T \) is a ladder, \( e_3 \) and \( e_4 \) are not incident with any vertices in \( V(L) \). Hence, \( T' = \{ f_1, f_2, e_3, e_4 \} \) is an independent 4-edge cut. If some component of \( G - T' \) is a ladder, then it is a subgraph of \( L \). But then \( e_3 \) and \( e_4 \) are incident with vertices in \( V(L) \), a contradiction. Therefore, \( f_1, f_2 \in N^b_4 \). Hence, \( E_s(L) \subseteq N^b_4 \).

Thus, \( E_s(L) \subseteq N^b_4 \) or \( E_s(L) \subseteq N^b_4 \). Thus, \( E_s(\mathcal{L}) = N^b_4 \).
Let

\[
f_4(v) = \begin{cases} 
9, & \text{if } v = 6 \\
12, & \text{if } v = 8 \\
10, & \text{if } v = 10 \\
\frac{6v - 15}{5}, & \text{if } v \equiv 0 \pmod{10} \text{ and } v > 10 \\
\frac{6v - 12}{5}, & \text{if } v \equiv 2 \pmod{10} \text{ and } v > 10 \\
\frac{6v - 14}{5}, & \text{if } v \equiv 4 \pmod{10} \text{ and } v > 10 \\
\frac{6v - 16}{5}, & \text{if } v \equiv 6 \pmod{10} \text{ and } v > 10 \\
\frac{6v - 18}{5}, & \text{if } v \equiv 8 \pmod{10} \text{ and } v > 10 .
\end{cases}
\]

**Theorem 4.7.** Suppose \( G \notin \mathcal{M} \cup \mathcal{O} \). Then \( |N_4(G)| \leq f_4(v(G)) \). If \( v \leq 10 \), then \( |N_4| < f_4(v) \).

**Proof.** Suppose \( N_4^b = \emptyset \). If \( N_4^g = \emptyset \) we are done. If \( N_4^g \neq \emptyset \), then \( G[N_4^g] \) is a forest with at least four trees by Theorem 4.5, and so \( |N_4^b| \leq v - 4 < f_4(v) \).

Suppose \( N_4^b \neq \emptyset \). Let \( |\mathcal{L}| = l \) and \( |I(\mathcal{L})| = i \). By Lemma 4.6, \( E_4(\mathcal{L}) = N_4^b \), and so \( |N_4^b| = 2l + i \). If \( L \in \mathcal{L} \), then the set \( S \) of edges incident with \( L \) is an independent 4-edge cut. Since \( L \) is maximal, no edge in \( S \) is on a 4-cycle. Therefore, \( S \subseteq N_4^g \) and \( N_4^g \neq \emptyset \). Lemma 4.6 implies \( N_4^g \cap E(\mathcal{L}) = \emptyset \), and so \( G[N_4^g] \) is a subgraph of \( G - I(\mathcal{L}) \). Let \( a_v = v(G - I(\mathcal{L})) - v(G[N_4^g]) \). By Theorem 4.5, \( G[N_4^g] \) is a forest with \( 4 + a_t \) trees, for some \( a_t \geq 0 \). Hence, \( |N_4^g| = v(G[N_4^g]) - (4 + a_t) = v(G - I(\mathcal{L})) - a_v - 4 - a_t - v - i - a_v - 4 - a_t \). Thus,

\[
|N_4| = v + 2l - 4 - a_v - a_t . \tag{4.1}
\]

Since \( 3 \leq |E_r(L)| \), for every \( L \) in \( \mathcal{L} \), \( E_r(\mathcal{L}) = 3l + a_r \), for some \( a_r \geq 0 \). Let \( a_v = \epsilon(G) - |E_r(\mathcal{L}) \cup N_4| \). Then

\[
|N_4| = \epsilon(G) - |E_r(\mathcal{L})| - a_v = \frac{3v}{2} - 3l - a_v - a_c . \tag{4.2}
\]
Thus, for $v \geq 12$,
\[
|N_4| \leq \min \left\{ v + 2l - 4, \frac{3v}{2} - 3l \right\} = \max \left( \min \left\{ v + 2l - 4, \frac{3v}{2} - 3l \right\} \right) = f_4(v).
\]
(4.3)

For $v \leq 10$, the previous equation gives $|N_4| < f_4(v)$. 

5. Edge Reductions in Cyclically 5-Connected Cubic Graphs

Throughout this section, $G$ will denote a cyclically 5-connected cubic graph.

In this section we examine edge reductions in cyclically 5-connected cubic graphs. We found that $f_3$ and $f_4$ were linear in $v$. In contrast there exists a graph $G$ on $v$ vertices such that $N_5(G) = E(G)$, for all $v \geq 10$ such that $v \neq 16$. In this section we characterize such graphs $G$. As with $N_4$ we will define a restricted subset $N^b_5$ of $N_5$. The characterization is then found mainly by examining the structure of $G[N^b_5]$.

Define $S_{12}$, $S_{14}$, and $S_{18}$ to be $G_{(5.1.a)}$, $G_{(5.1.b)}$, and $G_{(5.1.c)}$, respectively. See Fig. 5.1. Let $\mathcal{P} = \{S_{12}, S_{14}, S_{18}\}$.

For every odd integer $n \geq 5$, we define $P_{2n}$ as follows. Let $C_u = u_1 u_2 \ldots u_n$ and $C_v = v_1 v_3 v_5 \ldots v_{n-2} v_{n-4} v_{n-6} \ldots v_3 v_2 v_4 v_6 \ldots v_n - 1$ be disjoint cycles. Let $P_{2n} = C_u + C_v + \{u_i v_i | i = 1, \ldots, n\}$. Let $\mathcal{D}$ be the set of all such graphs. The Petersen graph is $P_{10}$.

For every $n \geq 5$, we define $D_{4n}$ as follows. Let $C_x = x_1 x_2 \ldots x_n$, $C_y = y_1 y_2 \ldots y_n$, and $C_z = z_1 z_2 \ldots z_{2n}$ be disjoint cycles. Let $D_{4n} = C_x + C_y + C_z + \{x_i z_{2i-1} | i = 1, \ldots, n\} \cup \{y_i z_{2i} | i = 1, \ldots, n\}$. Let $\mathcal{D}$ be the set of all such graphs. The dodecahedron graph is $D_{20}$.

For every cyclically 5-connected cubic graph $G$, define $N_5^b(G)$ to be the set of edges in $N_5$ which are in the edge set of a 5-cycle.

Let $C$ be a 5-cycle of $D_{20}$ and let $D = D_{20} - V(C)$. Let $\mathcal{A}$ be the set of
cyclically 5-connected cubic graphs such that \( N_5(G) = E(G) \) and every component of \( G[N_5^0] \) is isomorphic to \( D \).

Let \( \mathcal{H} = \mathcal{S} \cup \mathcal{P} \cup \mathcal{D} \cup \mathcal{A} \).

**Lemma 5.1.** The graphs in \( \mathcal{H} \) are cyclically 5-connected.

**Proof.** It is easy to show \( P_{10}, S_{12}, S_{14}, S_{18}, \) and \( D_{20} \) are cyclically 5-connected. The graphs in \( \mathcal{A} \) are cyclically 5-connected by definition.

We prove by induction that the graphs in \( \mathcal{P} \) and \( \mathcal{D} \) are cyclically 5-connected. Suppose \( n > 5 \). Then \( P_{2(n-2)} = ((P_{2n} - u_n v_n) \sim - u_{n-1} v_{n-1}) \sim \) and \( D_{4(n-1)} = ((D_{4n} - x_n z_{2n-1}) \sim - y_n z_{2n}) \sim \). Thus, \( P_{2n} \) and \( D_{4n} \) are obtained from \( P_{2(n-2)} \) and \( D_{4(n-1)} \), respectively, by two edge additions. Since \( P_{2(n-2)} \) and \( D_{4(n-1)} \) are cyclically 5-connected, Theorem 2.5 implies that \( P_{2n} \) and \( D_{4n} \) are cyclically 5-connected. \( \square \)

For any graph \( G \) in \( \mathcal{S} \cup \mathcal{P} \cup \mathcal{D} \), every edge is incident with a 5-cycle; so \( N_5(G) = E(G) \). For any graph \( G \) in \( \mathcal{A} \), \( N_5(G) = E(G) \) by definition. Thus, \( N_5(G) = E(G) \) for all graphs \( G \) in \( \mathcal{H} \).

**Lemma 5.2.** If \( G \) has the form \( G_{(5,2.a)} \) and \( e \in N_5(G) \), then \( e \) is incident with a 5-cycle. (See Fig. 5.2.)

**Proof.** Let \( e \) be the independent 5-edge cut \( T \). By Lemma 1.5, \( G_i \) is 2-connected, \( i = 1, 2 \). Furthermore, the vertices \( x_i \) and \( y_i \) of \( G_i \) are in different components of \( G - T \), \( i = 1, 2 \). Hence \( 2 \leq |E(G_i) \cap T|, i = 1, 2 \). Thus, if we consider the disjoint, crossing, independent 5-edge cuts \( T \) and \( \{f_1, f_2, f_3, f_4, f_5\} \), Lemma 2.6 implies \( G \) has the form \( G_{(5,2.b)} \). (We note that the role of \( x \) and \( y \) in Figure 5.2.b. can be assumed without loss of generality.)

Let \( K \) be the component of \( G - T \) such that \( x \in V(K) \). By Lemma 1.6, \( B - x \) is isomorphic to \( K_2 \). Therefore, \( v(K) = 5 \). \( K \) is also simple and 2-regular, and so \( K \) is a 5-cycle. \( \square \)
Lemma 5.3. Suppose $E(G) = N_5(G)$. If $e \in E(G)$ and $e$ is on a 5-cycle $C$, then $e$ is incident with a 5-cycle.

Proof. Since $E(G) = N_5(G)$, $e$ is in an independent 5-edge cut $S$. $S$ must necessarily include another edge $e'$ in $E(C)$. Hence, $G$ has the form $G_{(5.3,a)}$.

If $y_1, y_2, y_3, y_4$, and $y_5$ are not distinct, then $G_2 = p_3$ by Lemma 1.6. Then $K = G[V(G_2) \cup \{z_1, z_2\}]$ has 5 vertices. $K$ is also simple and 2-regular; so $K$ is a 5-cycle and we are done.

Suppose $y_1, y_2, y_3, y_4$, and $y_5$ are distinct. Then $G$ has the form $G_{(5.2,a)}$. Therefore, Lemma 5.2 implies that $z_1z_2$ is incident with a 5-cycle. Thus, $G$ has the form $G_{(5.3,b)}$. (We note that the role of $e$ and $e'$ could be interchanged in Fig. 5.3.b. We will show that $e$ and $e'$ are both incident with some 5-cycle.)

If $y_1, y_2, y_3, y_4$, and $y_5$ are not distinct, then $G_2' = p_3$ by Lemma 1.6. Then $G_2$ is a 5-cycle. Hence, $y_1$ and $y_2$ are both adjacent to some vertex $w$. Then $e$ and $e'$ are incident with the 5-cycle $z_1z_2 y_2 w y_1$. Similarly, if $x_1, u_2, u_3, x_4$, and $x_5$ are not distinct then $e$ and $e'$ are incident with a 5-cycle.

Suppose $\{x_1, u_2, u_3, x_4, x_5\}$ and $\{y_1, v_2, v_3, y_4, y_5\}$ are both sets of distinct vertices. Lemma 5.2 then implies that $y_2 y_3$ is incident with a 5-cycle. Since $x_1, u_2, u_3, x_4$, and $x_5$ are distinct, the only possibility for this 5-cycle is $z_1 z_2 y_2 v_2 y_1$ and we are done.

Lemma 5.4. Suppose $E(G) = N_5(G)$. If $G$ has two 5-cycles whose intersection is $p_3$, then $G \in \{P_{10}, S_{12}, S_{14}\}$.

Proof. If $G$ has two 5-cycles whose intersection is $p_3$, then $G$ has a $G_{(5.4,a)}$ subgraph.

Figure 5.4 shows a proof of the theorem. Let $r$ be in $\{a, c, d, e\}$. By Lemma 5.3, the dashed edge of $G_{(5.4,r)}$ is incident with a 5-cycle. Then $G$ must have as a subgraph one of the graphs to which there is an arrow from $G_{(5.4,r)}$. We assume that from $G_{(5.4,c)}$ (respectively, $G_{(5.4,d)}$) we do not obtain
a subgraph of $G$ with a $G_{(5,4,b)}$ subgraph (respectively, $G_{(5,4,b)}$ or $G_{(5,4,c)}$ subgraph).

Lemma 1.6 and the fact that $G$ has girth 5 imply that $G = P_{10}$ if $G$ has a $G_{(5,4,b)}$ or $G_{(5,4,f)}$ subgraph. Similarly, $G = S_{12}$ if $G$ has a $G_{(5,4,s)}$ subgraph, where $s \in \{g, h, i, j\}$, and $G = S_{14}$ if $G$ has a $G_{(5,4,k)}$ subgraph. 1

**Theorem 5.5.** If $E(G) = N_d(G)$, then $G \in \mathcal{H}$.

*Proof.* If the intersection of two 5-cycles is $p_3$, then $G \in \{P_{10}, S_{12}, S_{14}\}$ by Lemma 5.4. Therefore, we may assume that the intersection of any two 5-cycles is empty or $p_2$.

Let $K$ be a component of $G[N^5_d]$. Figure 5.5 gives a proof that $K$ has a $G_{(5,5,e)}$ subgraph. Let $r$ be in $\{a, b, c, d\}$. By Lemma 5.3, the dashed edge of $G_{(5,5,r)}$ is incident with a 5-cycle $C$. The possibilities for $C$ are limited by the fact that $G$ has girth five and by the assumption that the intersection of any two 5-cycles is empty or $p_2$. As well as insuring that the intersection of $C$ and any other 5-cycle is not $p_3$, we must insure that this is also true for the
5-cycles using exactly one edge in $E(\overline{C}) - E(G_{(5.5.r)})$. For $G_{(5.5.c)}$, this is the case: if $C = v_1v_2v_3v_4v_5$, then $(v_3v_4v_5v_6v_7) \cap (v_2v_3v_4v_5v_6) = p_3$. In all possibilities one of the arrows from $G_{(5.5.r)}$ points to a graph which is also a subgraph of $K$.

For every even integer $m \geq 14$, we define $B_m$ as follows. We use the notation used in the definitions of the graphs in $\mathcal{P} \cup \mathcal{B}$. If $m = 2n$ and $n$ is odd, then $B_m = P_{2n} - \{u_1u_2, v_1v_2, v_2v_3\}$. If $m = 4n$, then $B_m = D_{4n} - \{x_1x_2, z_1z_2, y_1y_2\}$. Since $G_{(5.5.a)} = B_{14}$, we may choose a $B_m$ subgraph of $K$ such that $14 \leq m$ and $m$ is maximal.

Suppose $m \equiv 0 \pmod{4}$ and consider $B_m = G_{(5.6.a)}$. See Fig. 5.6. By Lemma 5.3, $x_1z_1$ is incident with a 5-cycle $C$. By assumption, the intersection of $C$ and any other 5-cycle is not $p_3$. Then $K$ has a $G_{(5.6.b)}$, $G_{(5.6.c)}$, $G_{(5.6.d)}$, or $G_{(5.6.e)}$ subgraph. If $K$ has a $G_{(5.6.b)}$ subgraph, then we have contradicted the maximality of $m$ since $G_{(5.6.b)} = B_{m+2}$. If $K$ has a $G_{(5.6.c)}$ subgraph, then $G = G_{(5.6.c)} + x_1x_2$. But then $y_1z_2$ is not incident with a 5-cycle, contradicting Lemma 5.3. If $K$ has a $G_{(5.6.d)}$ subgraph, then $20 \leq m$ and $G = G_{(5.6.d)} + x_1x_2 = D_m$. If $K$ has a $G_{(5.6.e)}$ subgraph then consider $y_1z_2$. Since $y_1z_2$ and $x_1z_1$ are isomorphic edges of $B_m$, when we consider the 5-cycle incident with $y_1z_2$, we can show that either $G = D_m$ or $y_1y_n \in E(G)$.

Suppose $m \equiv 2 \pmod{4}$ and consider $B_m = G_{(5.7.a)}$. See Fig. 5.7. By Lemma 5.3, $u_1v_1$ is incident with a 5-cycle $C$. By assumption, the intersection of $C$ and any other 5-cycle is not $p_3$. Then $K$ has a $G_{(5.7.b)}$, $G_{(5.7.c)}$, $G_{(5.7.d)}$, $G_{(5.7.e)}$, or $G_{(5.7.f)}$ subgraph. If $K$ has a $G_{(5.7.b)}$ subgraph, then we
have contradicted the maximality of $m$ since $G_{(5,7,b)} = B_{m+2}$. If $K$ has a $G_{(5,7,c)}$ subgraph, then $G = G_{(5,7,c)} + v_1v_3$. By Lemma 5.3, $u_2v_2$ is incident with a 5-cycle. This is only possible if $m = 14$. Then $G = S_{14}$. If $K$ has a $G_{(5,7,d)}$ subgraph, then $G = G_{(5,7,d)} + v_1v_3 = P_m$.

Suppose $K$ has the subgraph $D = G_{(5,7,e)}$. If none of the edges incident with $D$ is in the edge set of a cycle, then $K = D$. If not, then $K$ has a $G_{(5,8,4)}$ subgraph. See Fig. 5.8. By Lemma 5.3, $e$ is incident with a 5-cycle; so $K$ has a $G_{(5,8,b)}$ subgraph. Then Lemma 1.6 and the girth of $G$ imply that $G \in \{S_{18}, D_{20}, G_{(5,8,c)}\}$. If $G = G_{(5,8,c)}$, then $f$ is not incident with a 5-cycle and we have contradicted Lemma 5.3. Finally, if $K$ has a $G_{(5,7,f)}$ subgraph, then we are done since $G_{(5,7,f)} = G_{(5,8,b)}$.

Thus, we have shown that either $G \in \mathcal{P} \cup \mathcal{P} \cup \mathcal{D}$ or every component of $G[N_S]$ is $D$. In the latter case, $G \in \mathcal{A}$.
EDGE REDUCTIONS OF CUBIC GRAPHS

Figure 5.7

Figure 5.8
We end this section by stating a theorem on the structure of graphs in $\mathcal{F}$. A proof can be found in [22].

Let $H$ be a cubic graph equal to $H^5(A, D)$. We say that $H' = H^5(A, C_5)$ is the $D$-reduction of $H$ at $D$ (see Fig. 5.9). Define $N'_5$ to be the set of edges in $N_5$ which are not on a 5-cycle.

**Theorem 5.6.** Let $G$ be in $\mathcal{F}$. $G$ has at least two $D$ subgraphs. Any $D$-reduction of $G$ is cyclically 5-connected. If $e \in N'_5$, then there exists an independent 5-edge cut $T$ contained in $N'_5$ such that $e \in T$. $G[N'_5]$ is a forest with at least five trees.

6. **Edge Reductions in Cyclically $k$-Connected Cubic Graphs without $k$-Cycles**

Throughout this section $G$ will represent a cyclically $k$-connected cubic graph with girth at least $k + 1$, where $3 \leq k$.

In Section 2 we proved that for every $k$, if $N_k(G) \neq \emptyset$, then $G[N_k]$ is a forest with at least $k$ trees. In this section we give more results about the structure of $G$ and derive a theorem of Nedela and Škoviera. For $k \in \{3, 4, 5\}$, we give a sharp upper bound $g_k(v)$ on $|N_k(G)|$ for all $G$ with $v$ vertices and a complete characterization of the extremal graphs obtaining this bound.

A subgraph $A$ of a cyclically $k$-connected graph $H$ is a $k$-end if $E(A) \cap N_k(H) = \emptyset$ and $A$ is a component of $H - S$, for some independent $k$-edge cut $S$.

**Theorem 6.1.** If $A$ and $B$ are $k$-ends of $G$, then either $A \cap B = \emptyset$ or $A = B$. 
Proof. Suppose \( A \cap B \neq \emptyset \). Let \( S \) be the independent \( k \)-edge cut incident with \( A \). Since \( A \) and \( B \) are connected and \( A \cap B \neq \emptyset \), \( A \cup B \) is connected. Since \( A \cup B \) is connected and \( E(A \cup B) \cap N_k(G) = \emptyset \), \( A \cup B \) is a subgraph of some component of \( G - N_k \). Therefore, \( A \cup B \) is a subgraph of some component of \( G - S \), and so \( A \cup B = A \). Similarly, \( A \cup B = B \). Thus, \( A = B \).

**Theorem 6.2.** If \( A \) is a component of \( G - S \), for some independent \( k \)-edge cut \( S \), then there is a \( k \)-end which is a subgraph of \( A \). If \( N_k(G) \neq \emptyset \), then \( G \) has at least two \( k \)-ends.

Proof. Let \( S_1 \) be an independent \( k \)-edge cut such that \( G - S_1 \) has a component \( A_1 \) which is a subgraph of \( A \). Suppose \( S_1 \) is chosen so that \( A_1 \) is minimal. Let \( B_1 \) be the other component of \( G - S_1 \).

Suppose \( e \in E(A_1) \cap N_k(G) \). By Theorem 2.7, \( E(A_1) \cup S_1 \) contains an independent \( k \)-edge cut \( S_2 \) which includes \( e \). Since \( S_2 \cap E(B_1) = \emptyset \), \( B_1 \) is a subgraph of a component \( B_2 \) of \( G - S_2 \). Since \( S_1 \neq S_2 \), \( B_1 \) is a proper subgraph of \( B_2 \). Therefore, the other component \( A_2 \) of \( G - S_2 \) is a proper subgraph of \( A_1 \). But now we have contradicted the minimality of \( A_1 \). Therefore, \( E(A_1) \cap N_k(G) = \emptyset \) and \( A_1 \) is a \( k \)-end.

Suppose \( N_k(G) \neq \emptyset \). Then \( G \) has an independent \( k \)-edge cut \( S \). Both components of \( G - S \) contain a \( k \)-end.

We now derive a result due to Škoviera and Nedela [23, 24].

**Theorem 6.3.** If \( H \) is a vertex-transitive cyclically \( k \)-connected cubic graph such that \( N_k(H) \neq \emptyset \), then \( H \) has girth \( k \).

Proof. Suppose \( N_k(G) \neq \emptyset \). Then \( G \) has a \( k \)-end \( A \) by Lemma 6.2. Since \( G \) has girth at least \( k + 1 \), \( A \) has more than \( k \) vertices. Thus, the definition of \( k \)-end implies that \( A \) has a vertex \( x \) which is not incident with any edge in \( N_k(G) \). Since \( N_k(G) \neq \emptyset \), \( G \) also has a vertex \( y \) incident with an edge in \( N_k(G) \). Since no automorphism can map \( x \) to \( y \), \( G \) is not vertex-transitive.

Let \( A_3 = K_{3,2} \), \( A_4 = G_{(6,1b)} \), and \( A_5 = G_{(6,1c)} \). See Fig. 6.1. Let \( H_i \) be a cubic graph equal to \( H^k(A_k, B_i) \), \( i = 1, 2 \). Define \( H_1 \otimes H_2 \) to be \( H^k(B_1, B_2) \), \( k = 3, 4, 5 \). We note that \( H_1 \) and \( H_2 \) do not uniquely determine \( H_1 \otimes H_2 \). We will use \( H_1 \otimes H_2 \) to denote all such graphs.

We now recursively define three sets of graphs, \( \mathcal{H}_3 \), \( \mathcal{H}_4 \), and \( \mathcal{H}_5 \). Let \( \mathcal{A}_3 = \{ A_3, G_{(6.1a)} \} \), \( \mathcal{A}_4 = \{ A_4, G_{(6.1c)} \} \), \( G_{(6.1a)} \), and \( \mathcal{A}_5 = \{ A_5, G_{(6.1b)}, G_{(6.1g)} \} \). Let \( \mathcal{H}_k \) contain all graphs \( H^k(B_1, B_2) \) of girth \( k + 1 \), where \( B_1 \) and \( B_2 \) are
in $\mathcal{K}, k = 3, 4, 5$. For $k = 3, 4, 5$, if $H_i \in \mathcal{K}_k$ and $6k - 6 \leqslant v(H_i), i = 1, 2$, and $H_1 \bigcirc H_2$ has girth at least $k + 1$, then $H_1 \bigcirc H_2 \in \mathcal{K}_k$.

Define $g_3, g_4, \text{and } g_5$ as follows.

\[
g_3(v) = \begin{cases} 0, & \text{if } 6 \leqslant v \leqslant 8 \\ v - 7, & \text{if } 10 \leqslant v \end{cases}
\]

\[
g_4(v) = \begin{cases} 0, & \text{if } 10 \leqslant v \leqslant 14 \\ v - 12, & \text{if } 16 \leqslant v \end{cases}
\]

\[
g_5(v) = \begin{cases} 0, & \text{if } 14 \leqslant v \leqslant 20 \\ v - 17, & \text{if } 22 \leqslant v \end{cases}
\]

We now give a sharp upper bound for $|N_k(G)|$ and characterize the extremal graphs, for $k \in \{3, 4, 5\}$. We will only prove the upper bound for $|N_5(G)|$. The rest of the proof is similar to the methods used in [21] and Section 3. A complete proof can be found in [22].

**Theorem 6.4.** Let $k$ be in $\{3, 4, 5\}$. Then $|N_k(G)| \leqslant g_k(v(G))$, and $|N_k(G)| = g_k(v(G))$ if and only if $G \in \mathcal{K}_k$. $\mathcal{K}_k$ contains a graph on $v$ vertices, for every possible $v$.

**Proof.** If $G$ has an independent 5-edge cut $S$, then it is a routine exercise to show that both components of $G - S$ have at least 11 vertices. Hence, $N_5 = \emptyset$ if $v \leqslant 20$. 
Suppose \( N_5 \neq \emptyset \). By Theorem 2.3, \( G[N_5] \) is a forest with at least five trees. Let \( r \) be the number of 5-ends of \( G \). Let \( V_e \) be the set of vertices which are not incident with an edge in \( N_k(G) \). Each 5-end has at least 11 vertices and the 5-ends are disjoint by Theorem 6.1, and so \( 6r \leq |V_e| \). From Theorem 6.2 we know that \( 2 \leq r \). Thus, \( |N_5| = v(G[N_5]) - \omega(G[N_5]) \leq |V - V_e| - 5 = v - |V_e| - 5 \leq v - 6r - 5 \leq v - 17. \]

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The results of Section 3 were obtained independently by Fouquet and Thuillier [11]. Andersen, Fleischner, and Jackson [2] have independently proven that \( |N_6(G)| \leq (6v - 12)/5 \), for all cyclically 4-connected graphs \( G \). Consider \( G^{(5,4)} \) to be a subgraph of a cyclically 5-edge connected cubic graph \( G \) and let \( e = v_1v_2 \) and \( f = v_3v_4 \), where we have used the notation of Fig. 5.5.c. We refer to an edge reduction at \( e \) followed by an edge reduction at \( f \) as a special double edge reduction of \( G \). Barnette [4] and Butler [8] have independently proven that every planar cyclically 5-connected cubic graph except \( D_20 \) has an edge reduction, a \( D \)-reduction, or a special double edge reduction to a smaller planar cyclically 5-connected cubic graph. The set of graphs \( G \) in \( \mathcal{A} \) such that every component of \( G[N_5] \) is isomorphic to \( K_5 \) was first discovered by Wormald [32]. The characterization of cyclically 5-connected cubic graphs with no edge reduction to a smaller cyclically 5-connected cubic graph was independently discovered by Aldred and Holton [1]. Fouquet and Thuillier [11] have independently proven Theorem 6.4 for \( k = 3, 4 \). The proof of Theorem 6.3 given by Nedela and Škoviera [24] roughly follows the version given in this paper, that is, they prove versions of the relevant parts of Lemma 2.1 and Theorems 2.7 and 6.2.

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