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Existence Theorems for Some Quadratic Integral Equations

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Using the theory of measures of noncompactness, we prove a few existence theorems for some quadratic integral equations. The class of quadratic integral equations considered below contains as a special case numerous integral equations encountered in the theories of radiative transfer and neutron transport, and in the kinetic theory of gases. In particular, the well-known Chandrasekhar integral equation also belongs to this class. © 1998 Academic Press

1. INTRODUCTION

In the theory of radiative transfer, one can encounter the following quadratic integral equation:

$$x(t) = 1 + tx(t) \int_0^1 \frac{\varphi(s)}{t+s} x(s)\, ds, \tag{1}$$

where φ is a given continuous function defined on the interval [0, 1] and x is an unknown function. This equation also plays an important role in the

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theory of neutron transport and in the kinetic theory of gases (cf. [5, 7, 8, 11] and references therein).

Equations of type (1) and some of their generalizations were considered in several papers [2, 3, 6, 8, 9, 12, 13]. In those papers the authors proved that (1) (or more general equations) is solvable in some classes of Banach spaces. Some other properties, such as uniqueness, location of solutions, and convergence of successive approximations, were also studied in the mentioned papers.

Recently I. Argyros [3] investigated a class of quadratic equations of type (1) with a nonlinear perturbation. Using the theory of measures of noncompactness, he proved that those equations have solutions in some Banach function algebras.

Unfortunately, the investigations of the paper [3] contain several misprints and mistakes and are unclear in a few places. Moreover, some of the assumptions made in [3] seem to be a bit artificial.

The aim of this paper is to improve, clarify, and generalize some results obtained in the mentioned paper [3].

We will also apply the theory of measures of noncompactness, but we restrict ourselves to the space of continuous functions on an interval. This is caused by the fact that this space seems to be sufficiently general for the study of the equations in question. Nevertheless, we review the possibility of studying quadratic integral equations with a perturbation in some Banach function algebras.

The results obtained below generalize several obtained in the papers quoted above.

2. MEASURES OF NONCOMPACTNESS AND FIXED-POINT THEOREMS

Let $(E, \|\cdot\|)$ be a given Banach space and let $\mathfrak{M}_E$ denote the family of all nonempty and bounded subsets of E. By the symbol $B(x, r)$ we will denote the closed ball centered at x and with radius r.

The function χ defined on the family $\mathfrak{M}_E$ by

$$\chi(X) = \inf\{\varepsilon > 0\colon X \text{ admits a finite } \varepsilon\text{-net in } E\}$$

is called *the Hausdorff measure of noncompactness*. In the literature there exist several other definitions of the notion of a measure of noncompactness (see, e.g., [1, 4], for example). Nevertheless, the Hausdorff measure of noncompactness seems to be the most important and convenient in applications. This is caused by the fact that in some Banach spaces it is possible to express this measure with the help of handy formulas.

For example, let $C = C[0, 1]$ be the Banach space consisting of all real continuous functions defined on the interval [0, 1] and endowed with the maximum norm.

Then it may be shown [4] that for any $X \in \mathfrak{M}_C$ the following formula holds:

$$\chi(X) = \tfrac{1}{2}\omega_0(X),$$

where $\omega_0(X) = \lim_{\varepsilon \to 0}\{\sup\{\omega(x, \varepsilon): x \in X\}\}$ and $\omega(x, \varepsilon)$ denotes the modulus of continuity of the function x, i.e.,

$$\omega(x, \varepsilon) = \sup\{|x(t) - x(s)|: t, s \in [a, b], |t - s| \le \varepsilon\}.$$

For the properties of measures of noncompactness, we refer to the monographs [1, 4].

Now we recall the fixed-point theorem of Darbo [10], which enables us to prove the solvability of several operator-functional equations considered in nonlinear functional analysis.

To quote this theorem, we need the following definition from [4].

DEFINITION. Let M be a nonempty subset of a Banach space E, and let $T: M \to E$ be a continuous operator that transforms bounded sets onto bounded ones. We will say that T satisfies the Darbo condition (with a constant $k \ge 0$) if for any bounded subset X of M, we have

$$\chi(T(X)) \le k\chi(X).$$

In the case $k < 1$, the operator T is said to be a *contraction* (with respect to χ).

THEOREM 1 [10]. *Let Q be a nonempty bounded closed convex subset of E and let $T: Q \to Q$ be a contraction with respect to χ. Then T has at least one fixed point in the set Q.*

Now we are going to prove a theorem that allows us to indicate a large class of operators satisfying the Darbo condition in the case of Banach algebras. So, let us assume that E is a Banach algebra with the norm $\|\cdot\|$ and the zero element θ.

THEOREM 2. *Let M be a nonempty bounded subset of a Banach algebra E. Assume that $T: M \to E$ satisfies the Darbo condition with a constant k and $P: M \to E$ is completely continuous operator (i.e., it is continuous and the set $P(M)$ is relatively compact). Then the operator $S = PT$ satisfies the Darbo condition with the constant kb, where $b = \sup[\|P(x)\|: x \in M] < \infty$.*

Proof. Fix an arbitrary nonempty subset X of M. Let $\chi(X) = r$. Choose arbitrarily $\varepsilon > 0$. Then there exists a finite collection of points

$x_1, x_2, \ldots, x_n \in E$ such that

$$X \subset \bigcup_{i=1}^{n} B(x_i, r+\varepsilon).$$

Furthermore, let us denote $Y = P(X) \cdot T(X)$. Since $P(X)$ is compact, there exist points $z_1, z_2, \ldots, z_m \in E$ such that

$$P(X) \subset \bigcup_{i=1}^{m} B(z_i, \varepsilon).$$

In view of the assumption $\chi(T(X)) \leq k\chi(X) = kr$, we can find a finite set $\{u_1, u_2, \ldots, u_k\} \subset E$ such that

$$T(X) \subset \bigcup_{j=1}^{k} B\big(u_j, kr+\varepsilon\big).$$

Then we have

$$\begin{aligned} Y = P(X)T(X) &\subset \left(\bigcup_{i=1}^{m} B(z_i, \varepsilon)\right)\left(\bigcup_{j=1}^{k} B\big(u_j, kr+\varepsilon\big)\right) \\ &\subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{k} B(z_i, \varepsilon) B\big(u_j, kr+\varepsilon\big). \end{aligned}$$

Now, for fixed $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, k\}$ let us consider the set

$$A_{ij} = B(z_i, \varepsilon) \cdot B\big(u_j, kr+\varepsilon\big).$$

Obviously, $z_i u_j \in A_{ij}$.

Choose arbitrarily $v \in A_{ij} \cap Y$. Then $v = xy$, where $x \in B(z_i, \varepsilon) \cap P(X)$ and $y \in B(u_j, kr+\varepsilon) \cap T(X)$. Hence we get

$$\begin{aligned} \|v - z_i u_j\| &= \|xy - z_i u_j\| \leq \|xy - xu_j\| + \|xu_j - z_i u_j\| \\ &\leq \|x\| \cdot \|y - u_j\| + \|u_j\| \cdot \|x - z_i\| \leq \|x\|\big(kr+\varepsilon\big) + \|u_j\| \cdot \varepsilon. \end{aligned} \tag{2}$$

On the other hand, by virtue of the assumption, we have

$$\|x\| \leq b. \tag{3}$$

Moreover, without loss of generality, we can assume that $\{u_j\colon\ j = 1,2,\dots,k\} \subset B(T(X), kr+\varepsilon)$, where the symbol $B(\mathscr{Z}, t)$ stands for the "ball" centered at the set $\mathscr{Z}$ and with radius t. Hence we infer that the constant L, defined as

$$L = \sup\{\|w\|\colon\ w \in B(T(X), kr+\varepsilon)\},$$

is finite.

Now, taking into account the above assertion and (2) and (3), we infer that

$$\|xy - z_i u_j\| \le b(kr+\varepsilon) + \varepsilon L.$$

Thus, keeping in mind the arbitrariness of ε, we deduce that

$$\begin{aligned}\chi(Y) = \chi(P(X)T(X)) &\le \chi\left(\bigcup_{i=1}^{m}\bigcup_{j=1}^{k} A_{ij} \cap Y\right)\\ &= \max\{\chi(A_{ij}\cap Y)\colon\ i = 1,2,\dots,m;\ j = 1,2,\dots,k\}\\ &\le bkr = bk\chi(X),\end{aligned}$$

which completes the proof.

Observe that the above theorem generalizes Theorem 1 from [3]. Moreover, based on this theorem, we conclude the following fixed-point theorem.

THEOREM 3. *Let Q be a nonempty bounded closed convex subset of a Banach algebra E and $S = PT\colon\ Q \to Q$, where $P\colon\ Q \to E$ is a completely continuous operator and $T\colon\ Q \to E$ satisfies the Darbo condition with a constant k. If $bk < 1$ (where $b = \sup\{\|P(x)\|\colon\ x \in Q\}$), then S has a fixed point in the set Q.*

3. EXISTENCE THEOREMS

In this section we shall deal with the quadratic integral equation of the form

$$x(t) = 1 + (Tx)(t)\int_0^1 k(t,s)\varphi(s)x(s)\,ds, \tag{4}$$

where $t \in I = [0,1]$ and T is an operator to be described below.

Notice that Eq. (1) is a particular case of Eq. (4) (cf. Example 1).

In what follows we shall discuss Eq. (4), assuming that the following hypotheses are satisfied:

(i) $k: I \times I \setminus \{(0,0)\} \to \mathbb{R}$ is continuous, and for each $t \in I$ there exists the integral

$$\int_0^1 |k(t,s)|\, ds.$$

(ii) $\varphi \in C = C(I)$.

(iii) The operator $T: C \to C$ is continuous and satisfies the Darbo condition with a constant a.

(iv) There exists a bounded function $w: I \to \mathbb{R}_+$ ($\mathbb{R}_+ = [0,\infty)$) with the property $w(0) = \lim_{t \to 0^+} w(t) = 0$ and such that

$$\int_0^1 |k(t_2,s) - k(t_1,s)|\, ds \le w(|t_2 - t_1|)$$

for all $t_1, t_2 \in I$.

Remark. Observe that from assumptions (i) and (iv) it follows that

$$\int_0^1 |k(t,s)|\, ds \le \int_0^1 |k(t,s) - k(0,s)|\, ds + \int_0^1 |k(0,s)|\, ds$$
$$\le w(t) + \int_0^1 |k(0,s)|\, ds.$$

This implies that $q < \infty$, where

$$q = \sup\left\{ \int_0^1 |k(t,s)|\, ds \colon t \in I \right\}.$$

To ensure the existence of a ball being invariant under the operator determined by the right-hand side of Eq. (4), we further assume the following condition:

(v) There exists a constant b such that

$$\|T(x)\| \le b\|x\|$$

for each $x \in C$.

Finally, let us denote by Q the quantity

$$Q = \sup\left\{ \int_0^1 |k(t,s)||\varphi(s)|\, ds \colon t \in I \right\}.$$

Since $Q \le q\|\varphi\|$, this quantity is finite.

Then we can formulate our main existence result.

THEOREM 4. *Let the assumptions* (*i*)–(*v*) *be satisfied and* $bQ < 1/4$. *Moreover, assume that either*

1° $a < 2b$

or

2° $a \geq 2b$ *and* $a - b \geq a^2 Q$.

Then Eq. (4) *is solvable in the space* $C = C(I)$.

Proof. For an arbitrary $x \in C$ let us denote by Ax the function defined by the right-hand side of Eq. (4), i.e.,

$$(Ax)(t) = 1 + (Tx)(t)\int_0^1 k(t,s)\varphi(s)x(s)\,ds, \qquad t \in I.$$

In view of assumptions (ii)–(iv), we infer that $Ax \in C$. Moreover, we have

$$\begin{aligned}|(Ax)(t)| &\leq 1 + |(Tx)(t)|\int_0^1 |k(t,s)\varphi(s)|\,|x(s)|\,ds \\ &\leq 1 + bQ\|x\|^2.\end{aligned}$$

Since $bQ < 1/4$, it is easily seen that A transforms the ball $B(\theta, r)$ into itself for $r_0 \leq r \leq r_1$, where

$$r_0 = \left[1 - (1 - 4bQ)^{1/2}\right]/2bQ, \qquad r_1 = \left[1 + (1 - 4bQ)^{1/2}\right]/2bQ.$$

For further reasons, we assume that $r = r_0$.

Next, let us take a nonempty subset X of the ball $B(\theta, r)$. Fix $x \in X$. Then, for arbitrarily chosen $t_1, t_2 \in I$, we have

$$\begin{aligned}&\left|(Ax)(t_2) - (Ax)(t_1)\right| \\ &\leq \left|(Tx)(t_2)\int_0^1 k(t_2,s)\varphi(s)x(s)\,ds - (Tx)(t_1)\int_0^1 k(t_1,s)\varphi(s)x(s)\,ds\right| \\ &\leq \left|(Tx)(t_2)\int_0^1 k(t_2,s)\varphi(s)x(s)\,ds - (Tx)(t_2)\int_0^1 k(t_1,s)\varphi(s)x(s)\,ds\right| \\ &\quad + \left|(Tx)(t_2)\int_0^1 k(t_1,s)\varphi(s)x(s)\,ds - (Tx)(t_1)\int_0^1 k(t_1,s)\varphi(s)x(s)\,ds\right| \\ &\leq |(Tx)(t_2)|\int_0^1 |k(t_2,s) - k(t_1,s)|\,|\varphi(s)|\,|x(s)|\,ds \\ &\quad + |(Tx)(t_2) - (Tx)(t_1)|\int_0^1 |k(t_1,s)\varphi(s)|\,|x(s)|\,ds \\ &\leq b\|x\|^2\|\varphi\|w(|t_2 - t_1|) + \omega(Tx, |t_2 - t_1|)Q\|x\| \\ &\leq b\|\varphi\|r^2 w(|t_2 - t_1|) + rQ\omega(Tx, |t_2 - t_1|).\end{aligned}$$

Hence, in view of assumption (iv), we get

$$\omega_0(AX) \le rQ\omega_0(TX).$$

Consequently,

$$\chi(AX) \le rQ\chi(TX) \le rQa\chi(X)$$

(cf. Section 2).

Observe that the assumptions of our theorem yield $rQa = r_0Qa < 1$, which implies that A is a contraction with respect to χ on the ball $B(\theta, r)$. Thus, applying Theorem 1, we infer that there exists a function $x \in B(\theta, r)$ that is a solution of Eq. (4). This completes the proof.

Now we provide a few examples illustrating the applicability of Theorem 4.

EXAMPLE 1. Consider Eq. (1), which we write here in the following form:

$$x(t) = 1 + x(t)\int_0^1 \frac{t}{t+s}\varphi(s)x(s)\,ds.$$

Obviously this equation is a particular case of Eq. (1), where $k(t,s) = t/(t+s)$ and $T(x) = x$.

It is easy to check that in this situation the assumptions of Theorem 4 are satisfies for $a = 1$ and $b = 1$. Thus case 1° of Theorem 4 is fulfilled. Moreover, it may be verified that

$$Q = \sup\left\{\int_0^1 \frac{t|\varphi(s)|}{t+s}ds : t \in I\right\} \le \|\varphi\| \ln 2.$$

Thus, to ensure that the inequality $bQ = Q < 1/4$ will be valid, it is enough to take $\varphi \in C(I)$ such that $\|\varphi\| < (1/4)\ln 2$.

The above calculations show that a result from [3] is an easy corollary of Theorem 4.

EXAMPLE 2. Let us take the following integral equation:

$$x(t) = 1 + \frac{t^2x(t)}{1+|x(t)|}\int_0^1 \frac{1}{(t+s)^2}\exp(-1/(t+s))\varphi(s)x(s)\,ds. \quad (5)$$

Put (according to Theorem 4)

$$(Tx)(t) = \frac{tx(t)}{1+|x(t)|}, \qquad k(t,s) = \frac{t}{(t+s)^2}\exp(-1/(t+s)).$$

Then it is easy to check that the assumptions of Theorem 4 are satisfied for $b = 1$ and $a = 2$. Indeed, taking an arbitrary function $x \in C(I)$, we have

$$\begin{aligned} |(Tx)(t_2) - (Tx)(t_1)| &= \left| \frac{t_2 x(t_2)}{1 + |x(t_2)|} - \frac{t_1 x(t_1)}{1 + |x(t_1)|} \right| \\ &\leq |t_2 - t_1| + 2|x(t_2) - x(t_1)|. \end{aligned}$$

This yields,

$$\omega(Tx, \varepsilon) \leq \varepsilon + 2\omega(x, \varepsilon),$$

and consequently,

$$\omega_0(TX) \leq 2\omega_0(X),$$

for any bounded subset X of the space C.

Now observe that in view of $a = 2 \geq 2b = 2$, case 2° of Theorem 4 comes into play. Thus the function $\varphi \in C$ has to be taken in such a way that $bQ = Q < 1/4$. This implies immediately that the second part of case 2°, i.e., $a - b \geq a^2 Q$, is satisfied.

Let us notice that the existence result for Eq. (5) deduced with the help of Theorem 4 cannot be obtained as a consequence of results given in [3].

4. FINAL REMARKS

In this section we summarize the paper with a few comments and remarks.

At first let us observe that the existence result contained in Theorem 4 can be obtained with help of the fixed-point principle formulated in Theorem 3. A similar approach has been realized in paper [3], but it seems that the method determined by Theorem 1 is easier and more natural. On the other hand, it is clear that this method is based on the fact that the space $C(I)$ is a Banach algebra.

Second, let us mention that we can investigate equations with a more complicated form than Eq. (4). For example, the method of Theorem 4 (or Theorem 3) can be adopted for the study of the Chandrasekhar integral equation with nonlinear perturbation of the form

$$x(t) = 1 + x(t)\int_0^1 \frac{t}{t+s}\varphi(s)x(s)\,ds + \int_0^1 F(t, s, x(t), x(s))\,ds. \quad (6)$$

This equation was considered in [6], for example.

Based on Example 1, we will assume that $\varphi \in C = C(I)$ and that φ satisfies the following condition:

(α) $\|\varphi\| \leq 1/4 \ln 2$.

Furthermore, we assume the following hypotheses:

(β) $F\colon I^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function such that

$$|F(t,s,x,u) - F(t,s,y,u)| \leq k|x-y|$$

and

$$|F(t,s,0,u)| \leq m|u|,$$

where k and m are some constants,

(γ) $k + m < 1$ and $(1-k-m)^2 \geq 1/\ln 2$.

Under the above assumptions, we can easily prove the theorem on the existence of solutions of Eq. (6). We omit the details.

Let us pay attention to the fact that the existence result for Eq. (6) was proved in paper [6] under more complicated and more restrictive assumptions.

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