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A Formal Power Series Operational Calculus for Quasi-Nilpotent Operators. II*

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1. INTRODUCTION

Throughout this paper we let T be a quasi-nilpotent, but not nilpotent, operator on a Banach space E. Suppose that B is the set of complex formal power series $f = \sum_{n=1}^{\infty} \lambda_n z^n$ for which the series $\bar{f}(T) = \sum_{n=1}^{\infty} \lambda_n T^n$ converges in some specified operator topology. If $\bar{f}(T)$ converges, we denote its sum by $f(T)$. In this paper we continue the study, begun in [3], of B and of the map $f \to f(T)$.

In Section 3, we will examine the structure of B and its ideals, under the same sort of hypotheses used in [3] to show that B is an algebra and $f \to f(T)$ is an algebra isomorphism. Under varying hypotheses we will describe certain properties of all non-zero ideals of $B^{\#}$, the algebra formed from B by adjunction of an identity, (Lemma (3.2)A), Theorems (3.8)A), (3.9), and (3.15)D), below). We are particularly interested in using these characterizations of ideals to compare the ranges, null-spaces, compactness, etc., of $f(T)$ and powers of T (Theorem (3.3), Corollary (3.17)). We also prove, under suitable hypotheses, that B is the only non-zero prime ideal in $B^{\#}$ (Theorems (3.8)B) and (3.15)E)) and that B has uncountably ascending and descending chains of ideals (Theorem (3.10)).

In Section 4, we show that, when properly interpreted, the main features of the operational calculus discussed in [3] still hold for suitable collections of power series f for which $\bar{f}(T)\phi$ is no longer required to converge for all ϕ; in this case $f(T)$ is no longer defined throughout B. For instance, the fact that the map $f \to f(T)$ is an algebra homomorphism for everywhere-defined $f(T)$ becomes $f(T)g(T) \subseteq fg(T)$ for suitable f and g (see Theorem (4.3) below).

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2. Basic Definitions

For convenience, we will now recall some of the basic definitions and a few of the results of [3]. We will also make a few related definitions we will need in this paper. We will naturally need to make liberal use of the results of [3] in the present paper.

A **proper** power series is a complex formal power series with zero constant term. A **proper algebra** is an algebra of proper power series which contains all proper polynomials. A **module** [3, Def. (2.5)] is a collection of proper series which contains all proper polynomials, and is a module over the ring of polynomials.

Definition (2.1). If f is a formal power series whose first non-zero term is $\lambda_n z^n$, then f is said to have **order** n (if $f = 0$, f has **order** ∞). If B is a module and n is a positive integer, $B^{(n)}$ is the set of all series in B whose order is greater than or equal to n. Also $B^{(0)} = B^{\#}$, the vector space sum of B and the constant series.

A proper algebra B will be called **analytically closed** if $h \circ f$ belongs to $B^{\#}$ whenever f belongs to B and h is a power series with positive radius of convergence. Similarly, if Φ is an algebra homomorphism from an analytically closed algebra to an algebra of operators, then Φ will be called an **analytic homomorphism** if $\Phi(h \circ f) = h(\Phi(f))$, for all f in B, and for all h with positive radius of convergence.

Definition (2.2). Suppose that T is a quasi-nilpotent, but not nilpotent, operator on a Banach space E and that B is the set of all proper power series f for which $f(T)$ converges in the strong operator topology. Then T is **formally representable** in the strong operator topology and B is a **representing algebra**, for T, in the strong operator topology if:

(A) B is an analytically closed algebra.

(B) The map $f \to f(T)$ is both an algebra isomorphism and an analytic homomorphism from B to the algebra of bounded linear operators on E.

In the same way, we define **formal representability** and **representing algebras** for the weak operator topology, for the uniform operator topology, and for absolute convergence in the uniform operator topology. In the next section it will usually be immaterial which of the above four types of convergence we are considering. We will therefore simply call B a **representing algebra** if it satisfies Definition (2.2) for a fixed but unspecified manner of convergence. We should mention that only minor modifications of a few proofs would be necessary to extend the results of [3] and of this paper to a large number of other types of convergence: such as, for instance, Cesaro convergence in the weak* operator topology [1, pp. 89–91 and 117–119].

The main goal of [3] was to find hypotheses on T which assured that it was formally representable for various types of convergence. This involved the study of the modules defined below, which play a central role in this paper.

DEFINITION (2.3). If B is a module and j is a non-negative integer, then $S_{-j}(B)$ is the module of all proper power series f for which fz^j belongs to B (in particular $S_{-0}(B) = B$). Also, $S_{-\infty}(B) = \bigcup_{m=1}^{\infty} S_{-m}(B)$. When the module B is clearly understood, we will often just write S_{-j} and $S_{-\infty}$.

We now review some results from [3] about formal representability.

Suppose that K is the collection of proper series f for which $\bar{f}(T)$ converges absolutely in the uniform operator topology. Then T is formally representable for absolute uniform convergence if and only if the map $f \rightarrow f(T)$ is injective from K to the bounded operators on E [3, Theorem (2.3)]. This map is injective if $S_{-\infty}(K)$ is a radical algebra [3, Lemma (2.7) and Theorem (2.9)]. $S_{-\infty}(K)$ is a radical algebra, in particular, whenever all $S_{-j}(K)$ are algebras [3, Lemma (2.8) and Theorem (2.9)]. If B is the set of proper f for which $\bar{f}(T)$ converges in the weak operator topology, then T is formally representable for weak, strong, and uniform operator topology convergence if all $S_{-j}(K)$ are algebras, and B is contained in some $S_{-m}(K)$, [3, Theorem (3.1)]. Finally, there are direct conditions on the sequence $\{\| T^n \|\}$ which guarantee that $S_{-\infty}(K)$ is radical, [3, Theorem (4.9)]; conditions under which all $S_{-j}(K)$ are algebras, [3, Theorems (2.10) and (4.7)]; and there is a simple condition guaranteeing that $B \subseteq S_{-m}(K)$, [3, Lemma (3.3)].

3. The Structure of Representing Algebras

Suppose that T is a quasi-nilpotent, but not nilpotent, operator on a Banach space E. In this section, we will be interested in two sorts of questions. First, we will assume that B is a representing algebra for T and we will try to describe the ideals of $B^{\#}$ under the assumptions that $S_{-\infty}(B)$ is radical or that some of the $S_{-j}(B)$ are algebras; and we will use this information about ideals to study the relation between the operators T and $f(T)$. Second, we will suppose that K is the set of proper f for which $\bar{f}(T)$ converges absolutely in the uniform operator topology, and that B is an algebra with $K \subseteq B \subseteq S_{-\infty}(K)$. Under this assumption we will try to get as many characterizations as possible of the property that $S_{-\infty}(K)$ is analytically closed (see Theorem (3.15)).

We will start with an abstract characterization of the inverse images of closed ideals of operators under the map $f \rightarrow f(T)$.

DEFINITION (3.1). If B is any proper algebra, a non-zero ideal J of $B^{\#}$ is called **formally closed** if $z^n \in J$ always implies $B^{(n)} \subseteq J$.

LEMMA (3.2). *Suppose that B and $S_{-\infty}(B)$ are proper radical algebras, then:*

(A) *Every non-zero ideal in $B^{\#}$ contains a power of the indeterminate z.*

(B) *The only formally closed ideals of $B^{\#}$ are the ideals $B^{(n)}$.*

Proof. Part (A) is just [3, Lemma (2.7)], so we prove (B). Suppose that J is a formally closed ideal, f is an element of J of order n, and $J \subsetneq B^{(n)}$ (i.e., n is the minimum of the order of elements of J). By (A), some z^m belongs to J. Let

$$f = z^n(\lambda + p) + g,$$

where λ is a non-zero scalar, p is a polynomial with zero constant term, and g belongs to $B^{(m)}$. Since J is formally closed, both g and $z^n(\lambda + p)$ belong to J. But B is a radical algebra, so $(\lambda + p)^{-1}$ belongs to $B^{\#}$. Hence z^n belongs to J. Thus $B^{(n)} \subseteq J$, and the proof is complete.

Since so many facts about algebras of operators or Banach algebras can be stated in terms of ideals, numerous applications of the above lemma suggest themselves. We list a few applications in the following theorem:

THEOREM (3.3). *Suppose that T is a quasi-nilpotent, but not nilpotent, operator on E; that B is a representing algebra for T; and that g is a non-zero series of order n in B. If $S_{-\infty}(B)$ is a radical algebra, then:*

(A) *Suppose ϕ belongs to E; then $g(T)\phi = 0$ if and only if $T^n\phi = 0$.*

(B) *$g(T)(E) \supseteq T^m(E)$ for some m.*

(C) $\mathrm{cl}(g(T)(E)) = \mathrm{cl}(T^n(E))$.

(D) *If $\{\phi_k\}$ is a sequence in E and $g(T)\phi_k \to 0$, then $\lim_k T^m\phi_k = 0$ for some m.*

(E) *If F_1 and F_2 are closed invariant subspaces for T, then $g(T)(F_1) \subseteq F_2$ if and only if $T^n(F_1) \subseteq F_2$.*

(F) *If $g(T)$ is compact, some T^m is compact.*

(G) *If $g(T)$ is strictly singular, then some T^m is strictly singular.*

(H) *Suppose S is a bounded linear transformation; then $Sg(T) = 0$ if and only if $ST^n = 0$.*

(I) *Suppose S and U are bounded linear transformations with $g(T) = SU$; then there is an integer m, and there are bounded transformations V and W, for which $T^m = SV = WU$.*

Proof. We need only apply Lemma (3.2) to suitable ideals for each of the properties (A) through (I). For (A) consider the formally closed ideal $\{f \in B: f(T)\phi = 0\}$. For (B) consider the ideal $\{f \in B: f(T)(E) \subseteq g(T)(E)\}$. For (C) consider the formally closed ideal $\{f \in B: f(T)(E) \subseteq \mathrm{cl}(g(T)(E))\}$. (D) is similar to (A). For (E) consider the formally closed ideal $\{f \in B: f(T)(F_1) \subseteq F_2\}$. For (F) consider the ideal $\{f \in B: f(T) \text{ is compact}\}$; this ideal is even formally closed if we are dealing with the uniform operator topology. The ideals for (G) and (H) are obvious. For (I) let C be the algebra of bounded operators on E and consider the ideals $\{f \in B: f(T) \in SC\}$ and $\{f \in B: f(T) \in CU\}$. This completes the proof.

Part (A) of the above theorem can be viewed as an extension of the fact that the map $f \to f(T)$ is an injection. If ϕ is a vector for which no $T^n\phi = 0$, then Part (A) shows that the map $f \to f(T)\phi$ is injective. It is well known that such ϕ exist (for a simple proof using formal power series, see [2, Theorem 5]).

The description of formally closed ideals given in Lemma (3.2) (B) can hardly be improved; but it would be useful to have more information about ideals which are not formally closed, such as the ideals used in the proofs of Theorem (3.3) (B), (F), (G), and (I). In Theorem (3.15) below, we see that simply adding the hypothesis that $K \subseteq B \subseteq S_{-\infty}(K)$, where K is that set of proper f for which $\bar{f}(T)$ converges absolutely in the uniform operator topology, implies that every non-zero ideal in $B^{\#}$ not only contains a power of z, but also contains a power of every element of B. Before proving this and other consequences of the added assumption $K \subseteq B \subseteq S_{-\infty}(K)$, we will show (Theorems (3.8) and (3.9)) that even more detailed information on the ideals of B can be obtained under the alternate assumption that some $S_{-n}(B)$ is an analytically closed algebra, or even just that $S_{-1}(B)$ is an algebra. (Recall that if $j < k$ and $S_{-k}(B)$ is an algebra, so is $S_{-j}(B)$ [3, Lemma (3.15)]; and if B is also an analytically closed algebra, so is S_{-j} [3, Lemma (3.16)].)

Theorems (3.8) and (3.9), describing the ideals of B when $S_{-1}(B)$ is an algebra or $S_{-n}(B)$ is an analytically closed algebra, will be simple consequences of the following two lemmas. We actually will prove somewhat more than we need to describe ideals of B. For, we will show that various properties of B are equivalent to S_{-n} being an algebra or a radical algebra, instead of just showing that these properties are consequences of S_{-n} being an algebra or radical algebra.

LEMMA (3.4). *Suppose that B is a proper algebra and that j, k, and n are non-negative integers with $n > 0$. Then the following are equivalent:*

(A) $S_{-n}(B)$ *is an algebra.*

(B) $(B^{(n)})^{j+1} = B^{(n)}z^{nj}$.

(C) $B^{(n+k)}B^{(n)} = B^{(n+k)}z^n$.

Proof. We first prove the equivalence of (A) and (B). Suppose that S_{-n} is an algebra. Let $f_1, f_2, \ldots, f_{j+1}$ belong to $B^{(n)}$. To prove (B), we must show that the product of the f's belongs to $B^{(n)}z^{nj}$. For each i between 1 and $j+1$, let $f_i = z^n g_i$. Then

$$f_1 f_2 \cdots f_{j+1} = z^n(g_1 g_2 \cdots g_{j+1})\, z^{nj}.$$

Each g_i belongs to $S^{\#}_{-n}$, which we are assuming to be an algebra. Hence the product of the f's belongs to $(S^{\#}_{-n} z^n)\, z^{nj} = B^{(n)}z^{nj}$. This proves (B).

Conversely, suppose that (B) holds and that f and g belong to S_{-n}. Then $(fz^n)(gz^n)(z^n)^{j-1}$ belongs to $(B^{(n)})^{j+1} = B^{(n)}z^{nj}$. Dividing by z^{nj}, we find that fgz^n belongs to $B^{(n)} \subseteq B$. Hence fg belongs to S_{-n}, which is therefore an algebra.

We complete the proof of the lemma by showing that (C) is equivalent to $B^{(n)}B^{(n)} = B^{(n)}z^n$, which is the formula of part (B) for $j = 1$. If $B^{(n)}B^{(n)} = B^{(n)}z^n$ then clearly every element of the smaller set $B^{(n+k)}B^{(n)}$ is divisible by z^n in B, so (C) holds. On the other hand, every element of $B^{(n)}$ is the sum of a polynomial of order n and an element of $B^{(n+k)}$. Hence (C) implies that all elements in $B^{(n)}B^{(n)}$ are divisible by z^n in B; and the lemma is proved.

Various other properties equivalent to S_{-n} being an algebra are given in [1, Theorem (3.3) and Corollaries (3.4) and (3.5)].

LEMMA (3.5). *Suppose that B is a proper algebra, n is a positive integer, and k is a non-negative integer. Then the following are equivalent:*

(A) *S_{-n} is a radical algebra.*

(B) *$B^{(n+k)}f = B^{(n+k)}B^{(n)}$ for all f of order n in B.*

Proof. Suppose that S_{-n} is a radical algebra. Since it is an algebra, Lemma (3.4) implies that $B^{(n+k)}B^{(n)} = B^{(n+k)}z^n$. Hence

$$B^{(n+k)}f \subseteq B^{(n+k)}z^n \tag{3.6}$$

for all f of order n in B. Let $f = z^n(\lambda + g) \in B$, where λ is a non-zero scalar and g belongs to the radical algebra S_{-n}. Then $(\lambda + g)^{-1}$ belongs to $S^{\#}_{-n}$ and $f' = z^n(\lambda + g)^{-1}$ is an element of order n of B. Thus formula (3.6) applies to f'; that is, $B^{(n+k)}f' \subseteq B^{(n+k)}z^n$. Multiplying this by f, we obtain

$$B^{(n+k)}z^{2n} = B^{(n+k)}f'f \subseteq B^{(n+k)}fz^n.$$

Dividing by z^n, we see that $B^{(n+k)}z^n \subseteq B^{(n+k)}f$. This, together with formula (3.6), proves (B).

Conversely, assume (B). Letting $f = z^n$ and applying Lemma (3.4), we see that S_{-n} is an algebra. Suppose g belongs to S_{-n}; to complete the proof we must show that $(1 + g)^{-1}$ belongs to $S^{\#}_{-n}$. Let $f = z^n(1 + g)$, which is then an element of order n in B. We wish to show that

$$z^{2n} \in B^{(n)}f. \tag{3.7}$$

For if (3.7) holds, $z^{2n} = f'f$ for some f' in $B^{(n)}$, but f' must equal $z^n(1 + g)^{-1}$; and $(1 + g)^{-1}$ would then be in S_{-n}. Formula (3.7) would of course be immediate if $k = 0$. For positive k, we have

$$B^{(n)} = \text{span}\{f, zf, \ldots, z^{k-1}f\} + B^{(n+k)}.$$

Multiplying this by z^n, we get

$$z^{2n} \in B^{(n)}z^n \subseteq B^{(n)}f + B^{(n+k)}z^n.$$

But, by hypothesis, $B^{(n+k)}z^n = B^{(n+k)}f \subseteq B^{(n)}f$. This proves Formula (3.7), and completes the proof of the theorem.

A careful examination of the above proof would show that S_{-n} is a radical algebra under the assumption, which at first glance appears weaker than (B), that all $B^{(n+k)}f \supseteq B^{(n+k)}z^n$. The reverse inclusions, $B^{(n+k)}f \subseteq B^{(n+k)}z^n$, for all f of order n, are easily shown to be equivalent to Lemma (3.5) (B) and hence to S_{-n} just being an algebra.

We now apply Lemmas (3.4) and (3.5) to describe the ideals of $B^{\#}$.

THEOREM (3.8). *Suppose that B and $S_{-1}(B)$ are proper algebras, and that $S_{-\infty}(B)$ is a radical algebra. Then:*

(A) *Every non-zero ideal of $B^{\#}$ contains a power of B.*

(B) *B is the only non-zero prime ideal in $B^{\#}$.*

(C) *B is a radical algebra.*

Proof. Suppose that J is a non-zero ideal of $B^{\#}$. Then, by Lemma (3.2), some z^j belongs to J. Applying Lemma (3.4) (B), with $n = 1$, we obtain

$$(B)^{j+1} = Bz^j \subseteq J.$$

This proves (A). (B) is an easy consequence of (A).

Since every maximal ideal is prime, (B) implies that B is the unique maximal ideal of $B^{\#}$. Hence B is the radical of $B^{\#}$, [5, p. 55], and the proof is complete.

THEOREM (3.9). *Suppose that B is a proper algebra, that $S_{-n}(B)$ is a radical algebra, and that f is an element of order n of B. Then every ideal of $B^{\#}$ that contains f contains all elements of $(B^{(n)})^2$.*

Proof. Suppose that f beongs to the ideal J. Then, applying Lemma (3.5), we see that $(B^{(n)})^2 = B^{(n)}f \subseteq J$. This completes the proof.

It would now be easy to apply Theorems (3.8) and (3.9) to the parts of Theorem (3.3) in which the ideals involved were not formally closed. We omit the obvious details, except to remark that if $S_{-n}(B)$ is taken to be analytically closed, then m can be taken as equal to $2n$ in Theorem (3.3) (B), (D), (F), (G), and (I). Also, Theorem (3.9) shows that if the g in Theorem (3.3) has order n, and $S_{-n}(B)$ is a radical algebra, then the assumption that $S_{-\infty}$ is radical can be dropped from the hypothesis of Theorem (3.3).

We now use Lemmas (3.4) and (3.5) to obtain some additional facts about the structure of representing algebras. First we show that representing algebras are far from being Noetherian or Artinian. In the remainder of this section, we let K be the set of all proper power series f for which $f(T)$ converges absolutely in the uniform norm.

THEOREM (3.10). *If B is a representing algebra for T, and if $S_{-1}(B)$ is an algebra, then B contains a chain of ideals which is both uncountably ascending and uncountably descending.*

Proof. We first notice that any vector space J for which $Bz \subseteq J \subseteq B$ is an ideal of B. For, $BJ \subseteq B^2 = Bz$, by Lemma (3.4). I claim that it will be enough to find an uncountable-dimensional vector space V, for which

$$\begin{aligned} &\text{(a)} \qquad V \subseteq S_{-1}(B) \\ &\text{(b)} \qquad V \cap B = \{0\}. \end{aligned} \tag{3.11}$$

If we find such a V, then $Vz \cap Bz = \{0\}$, and adding Bz to a subspace of Vz will give an ideal of B.

Let $\| T^n \| = c_n$, and choose a series $f = \sum_{n=1}^{\infty} \lambda_n z^n$ in $S_{-1}(K) \subseteq S_{-1}(B)$, for which the sequence $\{\lambda_n c_n\}$ is unbounded (such a series is given in [2, p. 151]). Since $\{\lambda_n c_n\}$ is unbounded, f cannot belong to B, because of the Banach-Steinhaus Theorem. Partition the positive integers in a collection of sets $\{G_j\}_{j=1}^{\infty}$, chosen such that each of the sequences of numbers $\{\lambda_n c_n \colon n \in G_j\}$ is unbounded. For each positive integer j, let f_j be the power series whose n'th coefficient is λ_n if $n \in G_j$, and is 0 otherwise. Let V be the space of all power series of the form $\sum_{j=1}^{\infty} \alpha_j f_j$, where $\{\alpha_j\}$ is a bounded sequence of complex numbers. V is easily seen to satisfy formulas (3.11), so the theorem is proved.

THEOREM (3.12). *Suppose that B and all $S_{-j}(B)$ are radical algebras. Let I be the collection of all ideals of $B^{\#}$ which can be written as a sum of a finite dimensional subspace of $B^{\#}$ and an ideal of the form $B^{(k)}z^n$. Then I is closed* *under addition, multiplication, and multiplication by elements of $B^\#$ (in particular, I contains all $B^{(k)}f$ for $k \geqslant 0$ and f in $B^\#$).*

Proof. The theorem is an easy consequence of Lemma (3.4) (C), Lemma (3.5) (B), and the fact that $B^{(m)}$ is of finite co-dimension in $B^{(k)}$ whenever $m \geqslant k$.

Actually the assumption that all $S_{-j}(B)$ are radical algebras is slightly redundant. If B is radical and all $S_{-j}(B)$ are algebras, then they can all be proved to be radical algebras by arguments nearly identical to [3, Lemma (3.7) (A) and Lemma (3.16)].

We now return to an examination of the implications of the assumption that $S_{-\infty}$ is a radical algebra, independently of assumption on S_{-n} for finite n. To obtain more information than we were able to obtain in Lemma (3.2), we add the assumption that $K \subseteq B \subseteq S_{-\infty}(K)$. This is a relatively natural assumption for collections of converging power series T [3, Lemma (3.3) and the following remark]. We do not exclude the important special case, $K = B$. In any case $S_{-\infty}(K) = S_{-\infty}(B)$.

Since f belongs to K if and only if $\bar{f}(T)$ converges absolutely, the assumption that $K \subseteq B \subseteq S_{-\infty}(K)$ will allow us to use techniques very similar to techniques often used in proving convergence of series of complex numbers. The notation in the following definition will make the arguments as painless as possible.

DEFINITION (3.13). If $f = \sum_{n=0}^{\infty} \lambda_n z^n$, then $|f| = \sum_{n=0}^{\infty} |\lambda_n| z^n$. $f \geqslant 0$ if and only if all $\lambda_n \geqslant 0$ (i.e., $f = |f|$). If f and g both have real coefficients, then $f \leqslant g$ if $g - f \geqslant 0$.

We also need the following simple computational lemma which we restate, without proof, from [3, Lemma (3.8)].

LEMMA (3.14). *If h is a proper power series with positive radius of convergence and k is a positive integer, then there exist power series $h_1, h_2, \ldots, h_k$, with positive radii of convergence, such that*

$$h \circ f = \sum_{i=1}^{k} (h_i \circ f^k) f^i$$

for all proper power series f.

We are now ready to prove a number of properties equivalent to $S_{-\infty}$ being radical.

THEOREM (3.15). *Suppose that T is a quasi-nilpotent, but not nilpotent, operator; that K is the collection of proper series for which $\bar{f}(T)$ converges* *absolutely in the uniform operator topology; and that B is any analytically closed algebra for which $K \subseteq B \subseteq S_{-\infty}(K)$. Then the following are equivalent:*

(A) *$S_{-\infty}$ is a radical algebra.*

(B) *$S_{-\infty}$ is analytically closed.*

(C) *All non-zero ideals of $B^{\#}$ contain a power of z.*

(D) *For all f in B; every non-zero ideal in $B^{\#}$ contains a power of f.*

(E) *B is the only non-zero prime ideal of $B^{\#}$.*

(F) *If $f \in S_{-\infty}$, some $f^k \in B$.*

(G) *If $f \in S_{-\infty}$, then there is an $m \geqslant 0$ for which $f^n \in B$ whenever $n \geqslant m$.*

Proof. It is immediately apparent that (B) $\Rightarrow$ (A), (D) $\Rightarrow$ (C), and (G) $\Rightarrow$ (F). That (A) $\Leftrightarrow$ (C) is proved in [3, Lemma (2.7)].

(D) $\Rightarrow$ (E) follows directly from the definition of prime ideal. Conversely, assume (E). Suppose that there was an f in B and a non-zero ideal J of $B^{\#}$ which contained no power of f. Using Zorn's Lemma, we could find a non-zero ideal Q which was maximal with respect to containing no power of f. Such Q would be a prime ideal [4, Theorem 1, p. 1], contradicting (E). Hence (E) $\Rightarrow$ (D).

Now we prove (F) $\Rightarrow$ (B). Suppose that f belongs to $S_{-\infty}$, and that h has positive radius of convergence. By assumption, some f^k belongs to the analytically closed algebra B. Let $h_1, h_2, \ldots, h_k$ be the series given by Lemma (3.14). Then $h_i \circ f^k \in B \subseteq S_{-\infty}$, for $i = 1, 2, \ldots, k$. Since $S_{-\infty}(K)$ is always an algebra, (B) follows from Lemma (3.14).

We complete the proof of the theorem by assuming (A), and hence (C), and proving (D) and (G). Suppose f belongs to $S_{-\infty}$; let $f = z^j(\lambda + g)$, where λ is a non-zero scalar and g is a proper series. g, and hence $| g |$, belong to $S_{-\infty}$. Since $S_{-\infty}$ is a radical algebra,

$$(1 - | g |)^{-1} = \sum_{k=0}^{\infty} | g |^k$$

belongs to $S^{\#}_{-\infty}$, and hence to some fixed $S_{-m}(K)^{\#}$. For all n

$$| g^n | \leqslant | g |^n \leqslant \sum | g |^k,$$

so all g^n belong to this $S_{-m}(K)$. Hence

$$f^n = z^{nj}(\lambda + g)^n \in (S_{-m}(K))^{\#} z^{nj}, \tag{3.16}$$

for all positive integers n. Therefore the above m satisfies condition (G) (in fact $f^n \in K \subseteq B$ whenever $n \geqslant m$). Finally, we suppose that J is a non-zero ideal of B, and show that some $f^n \in J$. Since we are assuming (C), some z^i belongs to J. Let $n = m + i$. Applying Formula (3.16), we obtain:

$$f^n \in Kz^i \subseteq Bz^i \subseteq J.$$

This completes the proof of the theorem.

Since condition (D) of the previous theorem gives more information about ideals in $B^{\#}$ than condition (C) gives, we could apply the above theorem to the sort of ideals used in Theorem (3.5). The following corollary, whose proof we omit, will serve as an illustration.

COROLLARY (3.17). *Suppose that B is a representing algebra for T with $K \subseteq B \subseteq S_{-\infty}(K)$. If $S_{-\infty}$ is a radical algebra and if f and g belong to B, then*

(A) *There are integers m and n such that $(f(T))^m (E) \subseteq g(T)(E)$ and $(g(T))^n (E) \subseteq f(T)(E)$.*

(B) *Some power of $f(T)$ is compact if and only if some power of $g(T)$ is compact.*

The only place the proof of Theorem (3.15) uses the fact that the algebra B is analytically closed is in the proof that (F) $\Rightarrow$ (B). If we replace (F) by

$$\text{(F)}' \qquad \text{if} \qquad f \in S_{-\infty}, \qquad \text{some} \qquad f^k \in K,$$

and similarly replace (G) by a statement (G)′ in which K replaces B; then the new conditions (A), (B), (C), (D), (E), (F)′, and (G)′ are equivalent. In fact, we can prove that B is automatically analytically closed under these assumptions. This would follow from Lemma (3.14), in exactly the same way that this lemma is used to prove that (F) $\Rightarrow$ (B) in Theorem (3.15). Thus we have proved:

COROLLARY (3.18). *Suppose that B is any algebra with $K \subseteq B \subseteq S_{-\infty}(K)$. If $S_{-\infty}$ is radical, then B is analytically closed.*

The equivalence of (A) and (B) in Theorem (3.15) explains an apparent anomaly in many of the results of this paper and of [3]. The only property of $S_{-\infty}(B)$ used in studying B is that $S_{-\infty}$ is radical (see for instance Lemma (3.2), Theorem (3.3), Theorem (3.8), and Theorem (3.15), as well as [3, Lemma (2.7)]); yet we were always able to prove the apparently stronger property that $S_{-\infty}$ is analytically closed [3, Theorems (2.9) and (2.10), and Theorem (4.9)]

The proof in Theorem (3.15) that $S_{-\infty}(K)$ is analytically closed whenever it is radical is given by the implications (A) $\Rightarrow$ (F) $\Rightarrow$ (C). We give a simpler, more direct proof by adapting classical arguments on power series majorization. The hypothesis in the following theorem is essentially the only property of K, $S_{-j}(K)$, and $S_{-\infty}(K)$ really used in Theorem (3.15).

THEOREM (3.19). *Suppose that R is a radical algebra of proper power series. Suppose also that:*

$$g \in R \qquad \text{and} \qquad |f| \leqslant |g| \qquad \text{together imply } f \in R.$$

Then R is analytically closed.

Proof. Suppose that f belongs to R, and that h is a series with positive radius of convergence. Then there exist positive numbers M and r such that

$$|h| \leqslant M(1 - rz)^{-1}.$$

Therefore

$$|h \circ f| \leqslant |h| \circ |f| \leqslant M(1 - r|f|)^{-1}.$$

Since $|f|$ belongs to R and R is radical, $(1 - r|f|)^{-1}$ belongs to R. This completes the proof.

We conclude this section with two results on the structure of $S_{-\infty}$.

THEOREM (3.20). *Suppose that B is an algebra with $K \subseteq B \subseteq S_{-\infty}$ and that $S_{-\infty}$ is analytically closed. If f and g are proper power series with g in $S_{-\infty}$, then the following are equivalent:*

(A) *f belongs to $S_{-\infty}$.*

(B) *fg belongs to $S_{-\infty}$.*

(C) *Some power of f belongs to $S_{-\infty}$.*

(D) *Some power of f belongs to B.*

Proof. (A) $\Rightarrow$ (B) since $S_{-\infty}$ is an algebra. (A) $\Rightarrow$ (D) is part of Theorem (3.15). (D) $\Rightarrow$ (C) is obvious. (B) $\Rightarrow$ (A) is a special case of [3, Lemma (4.13)].

We complete the proof by assuming (C) and proving (A). Without loss of generality we may assume the leading coefficient of f is 1. Suppose f^n belongs to $S_{-\infty}$ and let $f^n = z^{nk}(1 + g)$. Then g belongs to the analytically closed algebra $S_{-\infty}$. Hence $(1 + g)^{1/n}$, which is given by the binomial series, belongs to $S^{\#}_{-\infty}$. Therefore $f = z^k(1 + g)^{1/n}$ belongs to $S_{-\infty}$, and the proof is complete.

THEOREM (3.21). *If B and $S_{-\infty}(B)$ are radical algebras, then all non-zero ideals of $S^{\#}_{-\infty}$ are of the form $S^{\#}_{-\infty}z^n = S^{(n)}_{-\infty}$.*

Proof. $S_{-\infty}(S_{-\infty}(B)) = S_{-\infty}(B)$, which is radical. Moreover, it is clear that every ideal of $S^{\#}_{-\infty}$ is formally closed. So the theorem follows from Lemma (3.2). Alternately, this theorem can be viewed as a special case of [3, Lemma (4.13)].

4. Unbounded $f(T)$

As before, we consider a fixed quasi-nilpotent, but not nilpotent, operator T on a Banach space E. If $f = \sum_{n=0}^{\infty} \lambda_n z^n$ is an arbitrary complex formal power series, we define the operator $f(T)$ by

$$f(T)\phi = \sum_{n=0}^{\infty} \lambda_n T^n \phi,$$

whenever the series converges *weakly*. Such $f(T)$ would normally not be defined throughout E. We use weak convergence to maximize the domain of $f(T)$; all the results of this section would be completely unchanged by systematically substituting strong convergence for weak convergence. We will follow the standard practice of defining sums, products, inverses, and limits of operators on the largest possible domain; except that limits will always be in the weak topology on E. We will also assume the most basic elementary facts about operations on unbounded operators [6, pp. 297–299, 301–302].

Throughout this section we let K be the collection of proper f for which $\bar{f}(T)$ converges absolutely in the uniform operator topology; and we let B be the set of proper f for which $\bar{f}(T)$ converges in the weak operator topology. The main goal of this section is to examine in what sense the operational calculus of [3], particularly the properties of formal representability, can be extended to $S_{-\infty}(B)$ and certain special subalgebras, like $S_{-j}(B)$. Since we can hardly expect an adequate operational calculus for unbounded operators $f(T)$ under weaker conditions than we needed for bounded $f(T)$, we will from now on assume that T is an amenable operator (for convenience, Definition (4.1) below repeats the definition of amenable operator from [3]). In [3, Theorem (3.1)] we saw that amenable operators are formally representable in the weak, strong, and uniform operator topologies. Conditions on $\{\| T^n \|\}$ which guarantee that T is amenable are given in [3, Theorem (3.2) and (4.7)].

Definition (4.1). Suppose that T is a quasi-nilpotent, but not nilpotent operator, that B is the set of all proper f for which $\bar{f}(T)$ converges in the weak operator topology, and that K is the set of proper f for which $\bar{f}(T)$ converges absolutely in the uniform norm. Then T is *amenable* if

(A) $B \subseteq S_{-m}(K)$, for some $m \geqslant 0$.

(B) $S_{-j}(K)$ is an algebra for all $j \geqslant 0$.

Since we assume T to be amenable, $S_{-\infty}(B) = S_{-\infty}(K)$ is an analytically closed algebra. Thus the structure of $S_{-\infty}$ is rather completely described by

Theorems (3.20) and (3.21). It will also be fairly easy to describe the structure of $S_{-j}(K)$ and the other subalgebras of $S_{-\infty}$ that we consider (see Theorem (4.18) (A) and (E)).

Since the structure of $S_{-\infty}$ and its subalgebras is relatively easy to describe, the major question we consider in this section is: in what sense is the map $f \to f(T)$ an algebraic isomorphism and analytic homomorphism, when the f are permitted to range over $S_{-\infty}(B)$, or over some suitable subalgebra like $S_{-j}(B)$? The most naive examples show that we can no longer expect $f(T)\, g(T) = fg(T)$. For instance, suppose f belongs to $S_{-1}(B)$, but not to B. Then $f(T)\, T = fz(T)$ is defined throughout E, but $Tf(T)$ is defined only on the domain of $f(T)$. It is, however, true that

$$T^j g(T) \subseteq g(T)\, T^j = gz^j(T); \qquad \text{for all } g \in S_{-\infty} \text{ and } j > 0. \tag{4.2}$$

But if g does not belong to B, the inclusion cannot be reversed.

It would not be overly difficult to adapt the proof of [3, Lemma (3.12)] to prove $fg(T)\phi = f(T)\, g(T)\phi$ for ϕ in the domain of $f(T), g(T), f(T)\, g(T)$, and $fg(T)$. What makes some of the arguments we will give in this section so complicated is that the major results of this section describe the domains of the operators involved at the same time they give algebraic information about the map $f \to f(T)$.

We are concerned with two sorts of results about domains. In Theorem (4.3), which is the major result of this section, we consider the largest possible domains of the operators involved. In particular, we show that $f(T)\, g(T) \subseteq fg(T)$ and that the domain of $f(T)\, g(T)$ contains

$$\operatorname{Dom} f(T) \cap \operatorname{Dom} g(T),$$

whenever f and g belong to $S_{-\infty}$.

In Theorem (4.18) we show that, for certain natural analytically closed subalgebras R of $S_{-\infty}$ (these will include in particular all $S_{-j}(B)$), there are natural associated subspaces F of E for which $f(T) F \subseteq F$ for all f in R, and the map which takes f into the restriction of $f(T)$ to F is an algebraic and analytic isomorphism.

In Theorem (4.22) we also show that if T is injective and f belongs to $S_{-\infty}$, then $f(T)$ has a natural extension to a closed operator $[f(T)]$. Moreover, the map $f \to [f(T)]$ behaves essentially in the same way as the map $f \to f(T)$.

THEOREM (4.3). *Suppose that T is an amenable operator on E, that f and g are non-zero series in $S_{-\infty}(K) = S_{-\infty}(B)$, and that h is a non-constant series with positive radius of convergence.*

Then

(A) $g(T) \not\subseteq 0$.

(B) $f(T) + g(T) \subseteq (f + g)(T)$.

(C) (1) $f(T)g(T) \subseteq fg(T)$.

(2) $\mathrm{Dom}\, f(T) \cap \mathrm{Dom}\, g(T) \subseteq \mathrm{Dom}\, f(T)g(T)$.

(3) $\mathrm{Dom}\, f(T)g(T) = \mathrm{Dom}\, g(T) \cap \mathrm{Dom}\, fg(T)$.

(D) (1) $h(f(T)) \subseteq h \circ f(T)$.

(2) $\mathrm{Dom}\, h(f(T)) = \mathrm{Dom}\, f(T)$.

Proof. (A) is easy. Choose some j such that gz^j belongs to B. Since T is amenable, $gz^j(T) = g(T)\, T^j \neq 0$, so $g(T) \not\subseteq 0$. We omit the obvious proof of (B).

We now give a rather involved proof of (C). Choose a positive integer m for which both f and g belong to $S_{-m}(K)$. Let

$$f = p + f' \qquad \text{and} \qquad g = q + g'$$

where p and q are polynomials and f' and g' are power series of order at least m. Then

$$fg = pq + pg' + f'q + f'g'. \tag{4.4}$$

We will need to prove the following formulas:

$$p(T)\, q(T) = pq(T). \tag{4.5}$$

(a) $$p(T)\, g'(T) \subseteq pg'(T). \tag{4.6}$$

(b) $$\mathrm{Dom}\, p(T)\, g'(T) = \mathrm{Dom}\, g(T).$$

$$f'(T)\, q(T) = f'q(T). \tag{4.7}$$

(a) $$f'(T)\, g'(T) \subseteq f'g'(T). \tag{4.8}$$

(b) $$\mathrm{Dom}\, f'g'(T) = E.$$

(c) $$\mathrm{Dom}\, f'(T)\, g'(T) = \mathrm{Dom}\, g(T).$$

Assume, for now, that Formulas (4.5) through (4.8) have been proved. Under this assumption we will show that all of Part (C) of the theorem is true. After doing this, we will then prove Formulas (4.5) through (4.8).

First notice that, since $p(T)$ and $q(T)$ are defined throughout E and since $\mathrm{Dom}\, f'(T)\, g'(T) = \mathrm{Dom}\, g(T) = \mathrm{Dom}\, g'(T)$ (by Formula (4.8) (c)), we can conclude that:

$$f(T)\, g(T) = p(T)\, q(T) + p(T)\, g'(T) + f'(T)\, q(T) + f'(T)\, g'(T). \tag{4.9}$$

We can now use Part (B) of the theorem, and Formulas (4.5) through (4.8), to compare the summands in (4.4) with those in (4.9). This yields $f(T)g(T) \subseteq fg(T)$. Further, suppose that ϕ belongs to $\operatorname{Dom} f(T) \cap \operatorname{Dom} g(T)$. Certainly ϕ belongs to the domains of $p(T)\,q(T)$ and $p(T)\,g'(T)$. Since $q(T)$ is a bounded operator which commutes with $f'(T)$, ϕ also belongs to $\operatorname{Dom} f'(T)\,q(T) \supseteq \operatorname{Dom} f(T)$. By (4.8) (c), ϕ belongs to the domain of $f'(T)g'(T)$. Hence ϕ belongs to the domain of $f(T)g(T)$, and Part (C) (2) of the theorem is true.

Since $f(T)\,g(T) \subseteq fg(T)$, it is clear that

$$\operatorname{Dom} f(T)\,g(T) \subseteq \operatorname{Dom} fg(T) \cap \operatorname{Dom} g(T).$$

We will prove the reverse inclusion. Suppose ϕ belongs to the domains of $fg(T)$ and $g(T)$. As before, it is clear that ϕ belongs to the domains of $p(T)\,q(T)$ and $p(T)\,g'(T)$; and it follows from (4.8) (c) that ϕ belongs to the domain of $f'(T)\,g'(T)$. Thus, to prove (C) (3), we need only show that ϕ belongs to the domain of $f'(T)\,q(T)$ which, by (4.7), is the same as the domain of $f'q(T)$. ϕ belongs to the domain of $fg(T)$, by assumption. It obviously belongs to the domain of $pq(T)$. It belongs to the domains of $pg'(T)$ and $f'(T)\,g'(T)$ by Formulas (4.6) (a) and (4.8) (b), respectively. Thus if we solve Formula (4.4) for $f'q$ and apply Part (B) of the theorem, we find that ϕ belongs to the domain of $f'q(T)$ and hence to the domain of $f(T)\,g(T)$. We have thus reduced the proof of Part (C) of the theorem to proving Formulas (4.5) through (4.8).

Formula (4.5) is obvious. Formula (4.6) is an easy consequence of Formula (4.2).

We now prove (4.7). Let $q = z^n(\lambda + q')$, where λ is a non-zero scalar and q' is a polynomial with zero constant term. Then $q'(T)$ is quasi-nilpotent, so $\lambda + q'(T)$ and $(\lambda + q'(T))^{-1}$ are both bounded operators which commute with $f'(T)$ [6, pp. 301–302]. Hence

$$\begin{aligned} f'(T)\,q(T) &= (\lambda + q'(T))\,(\lambda + q'(T))^{-1} f'(T)\,q(T) \\ &\subseteq (\lambda + q'(T))\,f'(T)\,(\lambda + q'(T))^{-1}\,q(T) \\ &= (\lambda + q'(T))\,f'(T)\,T^n \subseteq f'(T)\,q(T) \end{aligned}$$

so all the above operators are equal and have the same domains. In particular,

$$\operatorname{Dom} f'(T)\,q(T) = \operatorname{Dom} f'(T)\,T^n = \operatorname{Dom} f'z^n(T).$$

Moreover, if we apply Formula (4.6) with $f'z^n$ and $\lambda + q'$ in place of g' and p, respectively, we obtain

$$f'(T)\,q(T) = (\lambda + q'(T))\,f'z^n(T) \subseteq f'q(T).$$

Thus to complete the proof of (4.7), we need only prove

$$\operatorname{Dom} f'q(T) = \operatorname{Dom} f'z^n(T). \tag{4.10}$$

For each ϕ in E define

$$R(\phi) = \{s \in S_{-m}(K) : \phi \in \operatorname{Dom} s(T)\}. \tag{4.11}$$

Then clearly $K \subseteq R(\phi) \subsetneq S_{-m}(K)$ and $R(\phi)$ is a module. Hence by [3, Lemma (3.6)] $R(\phi)$ is an algebra. Since $\lambda + q'$ and $(\lambda + q')^{-1}$ both belong to $K^{\#} \subsetneq R(\phi)^{\#}$, we conclude that $f'q = f'z^n(\lambda + q')$ belongs to $R(\phi)$ if and only if $f'z^n = f'q(\lambda + q')^{-1}$ belongs to $R(\phi)$. This proves (4.10), and hence proves (4.7).

We now give a proof of Formula (4.8) which, except for the care required to prove (4.8) (c), is almost identical to the proof of [3, Lemma (3.12)]. Let $f' = \sum a_n z^n$ and $g' = \sum b_n z^n$, and recall that f' and g' belong to $S_{-m}(K)^{(m)}$. Let $|f'| = z^m f''$ and $|g'| = z^m g''$. Then both f'' and g'' belong to the algebra $S_{-2m}(K)^{\#}$. Therefore $|f'|\,|g'|$ and $f'g'$ belong to K. This proves (4.8) (b), and also implies that the iterated series

$$\sum_{n=2m}^{\infty} \sum_{k=m}^{n-m} |a_k|\,|b_{n-k}|\,\|T^n\phi\|$$

converges for all ϕ in E. Hence the double series

$$\sum_{n,k=m}^{\infty} a_n b_k T^{n+k}\phi \tag{4.12}$$

converges absolutely in the strong topology on E. One rearrangement of (4.12) is the series for $fg(T)\,\phi$. Another rearrangement is

$$\sum_n a_n \sum_k T^n(b_k T^k \phi). \tag{4.13}$$

But, *provided* ϕ belongs to $\operatorname{Dom} g(T) = \operatorname{Dom} g'(T)$, the series $\sum_k b_k T^k \phi$ converges weakly. Since all T^n are weakly continuous, the sum of (4.13) is $f'(T)\,g'(T)\,\phi$. This proves (4.8) (a) and (c), and completes the proof of Part (C) of the theorem.

We now prove (D) of the theorem. Using Theorem (3.15) (F), or [3, Formula (3.11)], we see that there is an integer k for which f^k belongs to K. We may suppose h has zero constant term. Let $h_1, h_2, \ldots, h_k$ be the series given by Lemma (3.14). Applying Part (C) of the current theorem and the fact that K is a representing algebra (Definition (2.2)), we obtain:

$$\begin{aligned}(h_i \circ f^k)\, f^i(T) &\supseteq (h_i \circ f^k)\,(T)\,(f(T))^i \\ &= h_i(f^k(T))(f(T))^i \supseteq h_i(f(T)^k)\, f(T)^i, \qquad i = 1, 2, \ldots, k;\end{aligned}$$

and also $\operatorname{Dom} f(T) = \operatorname{Dom}(f(T))^j = \operatorname{Dom} h_i(f(T)^k)f(T)^i$ for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots$. Applying Lemma (3.14) and Part (B) of the current theorem we see that $h \circ f(T) \supseteq \sum_{i=1}^{n} (h_i \circ f^k) f^i(T)$. Using similar reasoning, plus the fact that $\operatorname{Dom} h(f(T)) \subseteq \operatorname{Dom} f(T)$, we get $h(f(T)) = \sum_{i=1}^{n} h_i(f(T)^k)f(T)^i$. This completes the proof of the theorem.

Corollary (4.14). *Suppose that T is an amenable operator on E, I is the identity on E, and f belongs to $S_{-\infty}$. Then*

(A) $I - f(T)$ *is injective.*

(B) $\operatorname{Dom}(I - f(T)) = \operatorname{Range}(I - f(T))$.

(C) $(I - f(T))^{-1} = \sum_{n=0}^{\infty} (f(T))^n = (1 - f)^{-1}(T)$.

Proof. Theorem (4.3) (D) shows that $\sum_{n=0}^{\infty} (f(T))^n \subseteq (1 - f)^{-1}(T)$, with equality if $\operatorname{Dom}(1 - f)^{-1}(T) \subseteq \operatorname{Dom} f(T)$. Let $h = -\sum_{n=1}^{\infty} z^n$; then $(1 - f)^{-1} = 1 - h \circ f$, so $\operatorname{Dom}(1 - f)^{-1}(T) = \operatorname{Dom} h \circ f(T)$. But $f = h \circ (h \circ f)$, so, by Theorem (4.3) (D) again, $\operatorname{Dom} h \circ f(T) \subseteq \operatorname{Dom} f(T)$. Applying Theorem (4.3) (C), we find that both $(1 - f)(T)(1 - f)^{-1}(T)$ and $(1 - f)^{-1}(T)(1 - f)(T)$ equal the restriction of the identity I to $\operatorname{Dom} f(T) = \operatorname{Dom}(1 - f)(T) = \operatorname{Dom}(1 - f)^{-1}(T)$. This completes the proof of the corollary.

Notice that the argument in the above theorem shows that

$$\operatorname{Dom} h \circ f(T) = \operatorname{Dom} h(f(T)) = \operatorname{Dom} f(T),$$

whenever h is a power series of order one and positive radius of convergence. In particular, a similar theorem could be proved equating $(1 + f)^{1/n}(T)$ with $(1 + f(T))^{1/n}$; where the n'th roots are calculated from the binomial series.

We now consider the actions of certain subalgebras of $S_{-\infty}(B)$ on certain natural subspaces of E. Explicitly we are concerned with the situation of the following definition.

Definition (4.14). Suppose F is a non-void subspace of E and R is a non-void subset of $S_{-\infty}(B)$. The module $A(F)$ and the subspace $D(R)$ are defined by:

$$A(F) = \{f \in S_{-\infty}(B) : F \subseteq \operatorname{Dom} f(T)\}$$

$$D(R) = \bigcap_{f \in R} \operatorname{Dom} f(T).$$

Sets of the form $A(F)$ and $D(R)$ will be called **standard modules** and **standard subspaces**, respectively.

The elemntary lattice properties of the maps $F \to A(F)$ and $R \to D(R)$ are fairly obvious. We should note in particular that R is a standard module if and only if $R = AD(R)$, and that F is a standard subspace if and only if $F = DA(F)$. Thus the restriction of the maps $F \to A(F)$ and $R \to D(R)$ to the sets of standard subspaces and standard modules, respectively, are inverse lattice anti-isomorphisms.

LEMMA (4.15). *If R is a standard module, $F = D(R)$, and k is a non-negative integer, then $S_{-k}(R) = A(T^k(F))$.*

Proof. We simply unravel definitions.

$$f \in S_{-k}(R) \Leftrightarrow fz^k \in R$$
$$= A(F) \Leftrightarrow F \subseteq \operatorname{Dom} f(T)\, T^k \Leftrightarrow T^k(F) \subseteq \operatorname{Dom} f(T) \Leftrightarrow f \in A(T^k(F)).$$

If R is a standard module, $F = D(R)$, and k is a non-negative integer, we define

$$F_k = D(S_{-k}(R)). \tag{4.16}$$

Since Lemma (4.15) shows that $S_{-k}(R)$ is a standard module, we can conclude

$$S_{-k}(R) = A(F_k) = A(T^k(F)). \tag{4.17}$$

Probably the most interesting application of (4.15), (4.16), (4.17) is to the standard module $B = A(E)$ and the standard subspace $E = D(B)$. In this case, each $S_{-(k+1)}(B)$ properly contains $S_{-k}(B)$ and hence each E_k properly contains E_{k+1} [2].

So far our discussion of standard modules and subspaces has not used the fact that T is amenable, and would apply to arbitrary quasi-nilpotent T. Theorem (4.18) below, our major result on standard modules and subspaces, will require that T be amenable. Essentially Theorem (4.18) says that if $R = A(F)$ and $F = D(R)$, then the set R, and the map which takes f in R to the restriction of $f(T)$ to F, have essentially the same properties as B and the map $f \to f(T)$ from B to the bounded linear operators on E.

THEOREM (4.18). *Suppose that T is an amenable operator on E, $R = A(F)$, and $F = D(R)$. Define the map e, with domain R, by letting $e(f)$ be the restriction of $f(T)$ to F. Then:*

(A) *R is an analytically closed algebra.*

(B) *If f belongs to R, $f(T)(F) \subseteq F$.*

(C) *e is an algebraic and analytic homomorphism from R to the algebra of linear operators with domain F and range in F.*

(D) *If no $T^n(F) = \{0\}$, then e is injective.*

(E) *$S_{-i}(R)$ is an analytically closed algebra for $0 \leqslant i \leqslant \infty$.*

(F) *If f belongs to R, $(I + f(T))(F) = F$.*

(G) *If g belongs to $S_{-\infty}(B)$ and ϕ belongs to $F \cap \operatorname{Dom} g(T)$, then $g(T)\phi$ belongs to F.*

Proof. We first prove (A). For each positive integer k, let

$$A(k) = A \cap S_{-k}(B).$$

Then $A(k)$ is a module and $B \subseteq A(k) \subseteq S_{-\infty}(B)$. Therefore $A(k)$ is an analytically closed algebra, [3, Lemmas (3.6) and (3.7)]. So, $A = \cup A(k)$ is also an analytically closed algebra. Since $S_{-j}(R)$ is a standard algebra, by Lemma (4.15); (E) is a special case of (A).

(G) follows from Theorem (4.3) (C) (3); and (B) is a special case of (G). (C) follows from (B) and Theorem (4.3).

Since $S_{-\infty}(R) = S_{-\infty}(B)$ is analytically closed, all non-zero ideals of R contain a power of z. Hence if no $T^n(F) = \{0\}$, the kernel of e must be the zero ideal. This proves (D).

Finally, $(1 + f)^{-1}$ belongs to R and $(I + f(T))^{-1} = (1 + f)^{-1}(T)$, by Corollary (4.14). So (F) follows by applying Part (B) to $(1 + f)^{-1}(T)$. This completes the proof of the theorem.

Because both R and all $S_{-j}(R)$ are analytically closed algebras, whenever R is a standard module, the main feature of the structure theory of the previous section applies to R. In particular, Lemmas (3.2), (3.4), and (3.5) and Theorems (3.8), (3.9), and (3.12), all hold with R in place of B. As an illustration of how to use these structure theorems, we prove the following generalization of Theorem (3.3) (A).

COROLLARY (4.19). *Suppose that g is a series in $S_{-\infty}(B)$ of order n and that ϕ belongs to* Dom $g(T)$. *Then $g(T)\phi = 0$ if and only if $T^n\phi = 0$.*

Proof. Let $R = A\{\phi\}$ and

$$J = \{f \in R: f(T)\phi = 0\}.$$

Then J is a formally closed ideal in R; so the corollary follows from Lemma (3.2) and Theorem (4.18).

Next we show that the degree of the power series f in $S_{-\infty}$ can be determined by the mapping properties of $f(T)$.

COROLLARY (4.20). *Suppose that F is a standard subspace, k and n are non-negative integers, f is a series of order n in $S_{-\infty}$, and ϕ belongs to* Dom $f(T)$. *Then ϕ belongs to F_k if and only if $f(T)\phi$ belongs to F_{k+n}.*

Proof. Since $F_{k+n} = (F_k)_n$, there is no loss of generality in letting $k = 0$.

Let $f = z^n g$. If ϕ belongs to F, $T^n\phi$ belongs to F_n. Hence $f(T)\phi = g(T)(T^n\phi)$ belongs to F_n, by Theorem (4.18) (G).

Conversely, suppose that $f(T)\phi = g(T)(T^n\phi)$ belongs to F_n. Since $T^n\phi \in \mathrm{Dom}\, g(T) \cap \mathrm{Dom}\, f(T)$, $f(T)\phi \in \mathrm{Dom}\, g(T) = \mathrm{Dom}(g(T))^{-1}$. Therefore $g^{-1}(T)f(T)\phi = (g(T))^{-1}f(T)\phi = T^n\phi$ belongs to F_n. Hence ϕ belongs to F, and the proof is complete.

If R is a standard module and $F = D(R)$ contains some ϕ for which $T^n\phi$ is never zero, then F has uncountable Hamel dimension; since the map $f \to f(T)\phi$ is then injective. Thus, for instance, $E_k \supseteq T^k(E)$ has uncountable dimension. On the other hand, $S_{-\infty}$ is a standard module for which $D(S_{-\infty}) = \cap E_k$ can, depending on T, be anything from a dense proper subspace of E to $\{0\}$, [2, p. 151]. Hence Theorem (4.18) always gives nontrivial information for $R = S_{-k}(B)$, k finite; but may fail to say anything for $R = S_{-\infty}$. Of course Theorem (4.3) is applicable for all amenable T.

The relation between $S_{-\infty}$ and $\cap T^k(E)$ is in some ways more natural than that between $S_{-\infty}$ and $D(S_{-\infty}) = \cap E_k$.

THEOREM (4.21). *Suppose that T is an amenable operator and*

$$T^\infty(E) = \cap T^k(E) \neq \{0\}.$$

Then:

(A) *$f(T)(T^\infty(E)) = T^\infty(E)$ for all f in $S^\#_{-\infty}$.*

(B) *The map which takes f in $S_{-\infty}$ to the restriction of $f(T)$ to $T^\infty(E)$ is an algebraic and analytic isomorphism.*

Proof. (B) will follow from (A) and Theorem (4.3), so we prove (A). Since each T^n commutes with $f(T)$, it is clear that $f(T)(T^\infty(E)) \subseteq T^\infty(E)$. Let $f = z^n(\lambda + g)$; then

$$f(T)(T^\infty(E)) = (\lambda + g)(T)\,T^n(T^\infty(E)) = (\lambda + g)(T)(T^\infty(E)).$$

But $(\lambda + g)^{-1}$ belongs to $S^\#_{-\infty}$ and $(\lambda + g)^{-1}(T) = (\lambda + g(T))^{-1}$. So $(\lambda + g(T))^{-1}(T^\infty(E)) \subseteq T^\infty(E)$, and the proof is complete.

We have thus far had nothing to say about the continuity or closure of the operators $f(T)$. If T is injective and $f \in S_{-k}$, it is easy to see that the operator

$$[f(T)] = T^{-k}f(T)\,T^k$$

is a closed extension of $f(T)$. Notice that the definition of $[f(T)]$ given above is independent of the integer k, *provided* $f \in S_{-k}$. In fact it is not hard to show that $[f(T)] = g(T)^{-1}f(T)g(T)$ for any g in B of order at least k. Our final result, Theorem (4.22) below, shows that the map $f \to [f(T)]$ shares the most important features of the map $f \to f(T)$.

THEOREM (4.22). *Suppose that T is an injective amenable operator on E, that f and g are non-zero series in $S_{-\infty}$, and that h is a non-constant power series with positive radius of convergence. Then*

(A) $[g(T)]$ *is injective.*

(B) $[f(T)] + [g(T)] \subseteq [(f+g)(T)]$.

(C) (1) $[f(T)]\,[g(T)] \subseteq [fg(T)]$.

(2) $\operatorname{Dom}[f(T)] \cap \operatorname{Dom}[g(T)] \subseteq \operatorname{Dom}[f(T)]\,[g(T)]$.

(3) $\operatorname{Dom}[f(T)]\,[g(T)] = \operatorname{Dom}[g(T)] \cap \operatorname{Dom}[fg(T)]$.

(D) (1) $h([f(T)]) \subseteq [h \circ f(T)]$.

(2) $\operatorname{Dom} h([f(T)]) = \operatorname{Dom}[f(T)]$.

Proof. (A) follows from Corollary (4.19), and (B) is easy. Choose a positive integer m for which f and g both belong to $S_{-m}(K)$. We obtain (C) (1) of this theorem from Theorem (4.3) (C) (1), as follows:

$$\begin{aligned}[f(T)]\,[g(T)] &= T^{-m}f(T)\,T^mT^{-m}g(T)\,T^m \\ &\subseteq T^{-m}f(T)\,g(T)\,T^m \subseteq T^{-m}fg(T)\,T^m \\ &= [fg(T)].\end{aligned}$$

The easiest way to prove the rest of Part (C) is to define p, f', q, and g' as in Theorem (4.3) and prove analogues of Formulas (4.5) through (4.8), by replacing $f(T)$, $pg'(T)$, etc. by $[f(T)]$, $[pg'(T)]$, etc. The analogue of (4.5) is clear, and the analogue of (4.6) follows from

$$T^j[g'(T)] \subseteq [g'(T)]\,T^j = [g'z^j(T)].$$

The proof of (4.7) is essentially unchanged, provided we notice that every bounded linear operator which commutes with T also commutes with $[f'(T)]$.

Since we have already proved (C) (1) of this theorem, the analogue of (4.8) (a) holds. The analogue of (4.8) (b) holds since, by (4.8) (b), $f'g'(T) = [f'g'(T)]$. We complete the proof of (C) by proving the following analogue of (4.8) (c):

$$\operatorname{Dom}[f'(T)]\,[g'(T)] = \operatorname{Dom}[g(T)]. \tag{4.23}$$

Suppose that ϕ belongs to the domain of $[g(T)]$, which is the same as the domain of $[g'(T)] = T^{-m}g'(T)\,T^m$. Then $g'(T)\,T^m\phi$ belongs to E_{2m}, by Corollary (4.20). Hence

$$[g'(T)]\,\phi \in E_m \subseteq \operatorname{Dom} f'(T) \subseteq \operatorname{Dom}[f'(T)].$$

This proves (4.23), and hence Part (C). The proof of Theorem (4.22) (D) is a direct modification of the proof of (4.3) (D), so the proof of Theorem (4.22) is complete.

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