JOURNAL OF MULTIVARIATE ANALYSIS 2, 135–144 (1972)

Some Approximation Problems in the Theory of Stationary Processes

YU. A. ROZANOV

Steklov Mathematical Institute, Academy of Sciences of the U.S.S.R., Moscow, U.S.S.R.

In this paper, necessary and sufficient conditions for the regularity of a general (multivariate) stationary process are obtained. These subsume all the known criteria of regularity for such processes.

Let $\{\xi_\alpha(t)\}$ be an arbitrary family of univariate stationary processes with discrete time $t = 0, \pm 1, \ldots$. If we consider every component $\xi_\alpha(t)$, $-\infty < t < \infty$, as a function of t in some Hilbert space H, then the stationarity of the family means that the inner product $(\xi_\alpha(s), \xi_\beta(t))$ of $\xi_\alpha(s)$ and $\xi_\beta(t)$ in H depends only on $s - t$.

It is well known that each component $\xi_\alpha(t)$, $-\infty < t < \infty$, can be represented in the form $\xi_\alpha(t) = U^t a$, where $a = \xi_\alpha(0)$ and U is an unitary operator in a space H, generated by all elements $\xi_\alpha(t)$, which we shall denote as

$$H = \bigvee_{\alpha,t} \xi_\alpha(t).$$

For our purposes, we may consider only those components $\xi_\alpha(t)$, which are linearly independent. Following our paper [1], we shall consider a general stationary process as the pair $\{\mathscr{A}, U^t\}$ of some separable subspace $\mathscr{A}$ and a group of unitary operators U^t, $-\infty < t < \infty$, in the Hilbert space H. (In fact, it is a family of univariate stationary processes $\xi_a(t) = U^t a$, $-\infty < t < \infty$, where $a \in \mathscr{A}$ is a new parameter.)

The classical approximation problems, e.g. extrapolation or interpolation

Received October 14, 1971.

AMS 1970 subject classification: Primary 60G10, 60G25; Secondary.

Key words and phrases: Multidimensional stationary processes; spectral density; regularity; factorization of spectral density; Kolmogorov condition; Zasuhin condition.

(see, for example, [2]), raise the following general question. Let B be some subspace in

$$H = \bigvee_{-\infty < t - < \infty} U^t \mathscr{A}$$

and

$$L = \bigvee_{-\infty < t < \infty} U^t B$$

be the minimal invariant subspace containing B. When does $L = H$?

We study this question here and we obtain, as a corollary, new conditions for the regularity[1]

$$\bigcap_n H(-\infty, n) = \{0\} \tag{1}$$

where

$$H(-\infty, n) = \bigvee_{t<n} U^t \mathscr{A}.$$

We obtain also some other corollaries which can be useful in the extrapolation and interpolation problems.[2]

1. We assume that there is a so-called spectral density f_λ, $-\pi \leqslant \lambda \leqslant \pi$, which is a measurable positive operator-valued function in the Hilbert space $\mathscr{A}$ such that

$$(U^t a_1, a_2) = \int_{-\pi}^{\pi} e^{i\lambda t} (f_\lambda a_1, a_2)\, d\lambda$$

for all a_1, $a_2 \in \mathscr{A}$ and $-\infty < t < \infty$.

It is convenient to deal with $f_\lambda^{1/2}$ instead of f_λ, where $f_\lambda^{1/2}$ is an arbitrary measurable self-adjoint operator-valued function in $\mathscr{A}$,

$$[f_\lambda^{1/2}]^2 = f_\lambda, \qquad -\pi \leqslant \lambda \leqslant \pi,$$

and to consider the Hilbert space $\mathscr{L}^2(\mathscr{A})$ of all measurable $\mathscr{A}$-valued functions $a(\lambda)$, $\int_\pi^\pi \| a(\lambda) \|^2 d\lambda < \infty$, with the inner product of any elements $a_1(\lambda), a_2(\lambda) \in \mathscr{L}^2(\mathscr{A})$ determined by

$$\int_{-\pi}^{\pi} (a_1(\lambda), a_2(\lambda))\, d\lambda,$$

[1] In other words, it means that the stationary process $\{\mathscr{A}, U^t\}$ is purely nondeterministic.

[2] Some similar results were presented at the Symposium on "Linear operators and approximations" at Oberwolfach, August, 1971.

namely, our stationary process $\{\mathscr{A}, U^t\}$ with the spectral density f_λ is ***isometric*** to the pair $\{f_\lambda^{1/2}\mathscr{A}, e^{i\lambda t}\}$ of the subspace $f_\lambda^{1/2}\mathscr{A} \subset \mathscr{L}^2(\mathscr{A})$ and the group of unitary operators of multiplication by the scalar function $e^{i\lambda t}$

$$(U^t a_1, a_2) = \int_{-\pi}^{\pi} (e^{i\lambda t} f_\lambda^{1/2} a_1, f_\lambda^{1/2} a_2)\, d\lambda.$$

The corresponding subspace H in $\mathscr{L}^2(\mathscr{A})$ will be

$$H = \bigvee_{-\infty<t<\infty} e^{i\lambda t} f_\lambda^{1/2} \mathscr{A}. \tag{2}$$

Let B be some subspace in $\mathscr{L}^2(\mathscr{A})$ and $S = \{\mathscr{C}_1(\lambda), \mathscr{C}_2(\lambda), \ldots\}$ be a complete system of functions in B. We denote by $B_S(\lambda)$ the subspace in the Hilbert space $\mathscr{A}$ generated by all values $\mathscr{C}_1(\lambda)$, $\mathscr{C}_2(\lambda)$, ... Obviously, the closure $\overline{B(\lambda)} = \overline{B_S(\lambda)}$ does not depend on S in the sense that $\overline{B_{S_1}(\lambda)} = \overline{B_{S_2}(\lambda)}$ for almost all λ, $-\pi \leqslant \lambda \leqslant \pi$, if S_1, S_2 are any two complete systems in B. We call $\overline{B(\lambda)}$, $-\pi \leqslant \lambda \leqslant \pi$, a space-function, generated by the space $B \subseteq \mathscr{L}^2(\mathscr{A})$.

LEMMA 1. *The subspace*

$$L = \bigvee_{-\infty<t<\infty} e^{i\lambda t} B \tag{3}$$

consists of all functions $\mathscr{C}(\lambda) \in \mathscr{L}^2(\mathscr{A})$ *such that*

$$\mathscr{C}(\lambda) \in \overline{B(\lambda)} \qquad \textit{for} \quad \text{a.a. } \lambda. \tag{4}$$

Proof. Every function $\mathscr{C}(\lambda)$ from the subspace L is a limit for a.a. λ of some linear form $\sum_k e^{i\lambda k} \mathscr{C}_k(\lambda)$, where the functions $\mathscr{C}_k(\lambda)$ belong to B, so that $\mathscr{C}(\lambda)$ satisfies the condition (4). Note that for any $\mathscr{C}(\lambda)$ from B and a scalar measurable function $c(\lambda)$, $\int_\pi^\pi |c(\lambda)|^2 \| \mathscr{C}(\lambda)\|^2\, d\lambda < \infty$, the product $c(\lambda)\mathscr{C}(\lambda)$ being a function from L. Besides, for any $\mathscr{C}_1(\lambda)$, $\mathscr{C}_2(\lambda)$, the inner product $(\mathscr{C}_1(\lambda), \mathscr{C}_2(\lambda))$ is a measurable function of λ, $-\pi \leqslant \lambda \leqslant \pi$. Therefore under the condition (4), the projection $\mathscr{C}^{(n)}(\lambda) = \sum_{k=1}^n C_k(\lambda)\, \mathscr{C}_k(\lambda)$ of the value $\mathscr{C}(\lambda) \in \mathscr{A}$ on the linear manifold of $\mathscr{C}_1(\lambda)$, $\mathscr{C}_2(\lambda)$, ..., $\mathscr{C}_n(\lambda) \in \mathscr{A}$ as a function of λ belongs to L, and for a complete system $\mathscr{C}_1(\lambda)$, $\mathscr{C}_2(\lambda)$, ... in B we have

$$\lim_{n\to\infty} \| \mathscr{C}^{(n)}(\lambda) - \mathscr{C}(\lambda)\|^2 = 0 \qquad \text{for} \quad \text{a.a. } \lambda,$$

where $\| \mathscr{C}^{(n)}(\lambda) - \mathscr{C}(\lambda)\|^2 \leqslant \| \mathscr{C}(\lambda)\|^2$. So

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \| \mathscr{C}^{(n)}(\lambda) - \mathscr{C}(\lambda)\|^2\, d\lambda = 0 \qquad \text{and} \qquad \mathscr{C}(\lambda) \in L.$$

As a corollary of this lemma we obtain the following result: *the relation* $L = H$ *is true if and only if*

$$\overline{B(\lambda)} = \overline{f_\lambda^{1/2}\mathscr{A}} \qquad \text{for} \quad \text{a.a. } \lambda. \tag{5}$$

2. We shall be interested in the structure of subspaces

$$B = H(T) \ominus H(S),$$

where T, S are some sets of integers and for any set T,

$$H(T) = \bigvee_{t \in T} e^{i\lambda t} f_\lambda^{1/2} \mathscr{A}.$$

One can say that $H(T)$ is the subspace, generated by all values $\xi_a(t)$, $t \in T$, of univariate components of the stationary process under consideration, and B is "the innovation" in comparison with $H(S)$.

Let us consider a linear space L_T of all $\mathscr{A}$-valued integrable functions $\varphi(\lambda)$, $-\pi \leqslant \lambda \leqslant \pi$, with the Fourier decomposition of the form

$$\varphi(\lambda) \sim \sum_{k \in T} a_k e^{i\lambda k} \tag{6}$$

such that

$$\varphi(\lambda) \in f_\lambda^{1/2} \mathscr{A} \qquad \text{for} \quad \text{a.a. } \lambda \tag{7}$$

and

$$\int_{-\pi}^{\pi} \| f_\lambda^{-1/2} \varphi(\lambda) \|^2 \, d\lambda < \infty \tag{8}$$

where $f_\lambda^{-1/2}$ means the inverse operator from $f_\lambda^{1/2}\mathscr{A}$ to $\overline{f_\lambda^{1/2}\mathscr{A}}$ that uniquely determines $f_\lambda^{-1/2}$ for the self-adjoint operator $f_\lambda^{1/2}$ in the Hilbert space $\mathscr{A}$.

LEMMA 2. *Let T be the complement of a set S, in the above notation, i.e., TUS is the set of all integers. Then the subspace $B = H(T) \ominus H(S)$ in $\mathscr{L}^2(\mathscr{A})$ can be described as*

$$B = f_\lambda^{-1/2} L_T . \tag{9}$$

Proof. If $\mathscr{C}(\lambda) \in B$ then for all $a \in \mathscr{A}$

$$\int_{-\pi}^{\pi} e^{-i\lambda s} (\mathscr{C}(\lambda), f_\lambda^{1/2} a) \, d\lambda = \int_{-\pi}^{\pi} e^{-i\lambda s} (f_\lambda^{1/2} \mathscr{C}(\lambda), a) \, d\lambda = 0, \qquad s \in S, \tag{10}$$

so the integrable function $\varphi(\lambda) = f_\lambda^{1/2}\mathscr{C}(\lambda)$ satisfies the conditions (6)–(8). For any $a \in \mathscr{A}$ and a value $\varphi(\lambda) \in f_\lambda^{1/2}\mathscr{A}$ we have

$$(f_\lambda^{-1/2}\varphi(\lambda), f_\lambda^{1/2}a) = (\varphi(\lambda), a),$$

and if the conditions (6)–(8) hold true then by Lemma 1 the corresponding function $\mathscr{C}(\lambda) = f_\lambda^{-1/2}\varphi(\lambda)$ belongs to H and satisfies the relation (10), i.e. $\mathscr{C}(\lambda) \in B$.

As a corollary of our Lemma 2 we obtain the following result:

COROLLARY. *The relation*

$$\bigvee_{-\infty<t<\infty} e^{i\lambda t}B = H \tag{11}$$

holds true if and only if

$$\overline{f_\lambda^{1/2}\mathscr{A}} = \overline{L_T(\lambda)} \qquad for \quad \text{a.a. } \lambda \tag{12}$$

where $\overline{L_T(\lambda)}$ is the space function, generated by the subspace $L_T \subseteq \mathscr{L}^1(\mathscr{A})$ of all functions $\varphi(\lambda)$ which satisfy the conditions (6)–(8).

Note that this result can be useful for the extrapolation problem if we take $S = (-\infty, 0)$ and $T = [0, \infty)$. In this case the corresponding subspace L_T is the "Hardy class" of functions

$$\varphi(\lambda) \sim \sum_{k=0}^{\infty} a_k e^{i\lambda k} \tag{13}$$

which are boundary values of analytical functions $\Gamma(z) = \sum_{k=0}^{\infty} a_k z^k$, $|z| < 1$ (more exactly, for any $a \in \mathscr{A}$ a scalar function $(\varphi(\lambda), a)$ is a boundary value of the analytical function $(\Gamma(z), a)$ from the well-known Hardy class H^1). It is worth mentioning that for $S = (-\infty, 0)$ and $T = [0, \infty)$ *the relation* (11) *is equivalent to the regularity* of the corresponding stationary process. From our condition (12) it is very easy to see that the dimension $\dim f_\lambda^{1/2}\mathscr{A}$, which is equal to the dimension $\dim \overline{L_T(\lambda)}$ of the "analytical" space-function $\overline{L_T(\lambda)}$, is *a constant*

$$\dim f_\lambda^{1/2}\mathscr{A} = N \qquad \text{for} \quad \text{a.a. } \lambda. \tag{14}$$

(obviously $\dim f_\lambda^{1/2}\mathscr{A}$ is the so-called rank of our stationary process). We shall return once more to the condition (12) of the regularity in Section 2.

When considering the interpolation problem we have to take $S =$

$(-\infty, 0) \cup (0, \infty)$ and $T = \{0\}$. The corresponding subspace L_T will be the subspace of all constant functions $\varphi(\lambda) = a$, $a \in \mathscr{A}$, for which

$$\int_{-\pi}^{\pi} \| f_\lambda^{-1/2} a \|^2 \, d\lambda < \infty. \tag{15}$$

Recall that the relation (11) in this case means that our stationary process is "fully minimal"—see [3]. As we have seen above, *it is true if and only if the condition* (12) *satisfies:*

$$f_\lambda^{1/2} \mathscr{A} = \mathscr{A} \qquad \textit{for} \quad \text{a.a. } \lambda \tag{16}$$

and the relation (15) *holds for all* $a \in \mathscr{A}$.

In connection with this, note that if we take a sequence $\xi_k(t) = U^t a_k (k = 1, 2, \ldots, -\infty < t < \infty)$ where $a_1, a_2, \ldots$ is a *basis* in $\mathscr{A}$, then the condition for the stationary process $\{A, U^t\}$ to be "fully minimal" is equivalent to the following: *the sequence* $\{\xi_k(t)\}$ *is minimal in the sense that any element* $\xi_k(t)$ *does not belong to the closed linear manifold of all other elements* $\xi_j(s)$ (*cf.* [1]).

As a simple corollary of the condition (16) we obtain also the following result for a "fully minimal" stationary process of rank N, which is given as a family of univariate components $\{\xi_\alpha(t)\}$.

COROLLARY. *There are N components* $\xi_{\alpha_1}(t), \ldots, \xi_{\alpha_N}(t)$ *such that each component* $\xi_\alpha(t)$ *can be represented in the form*

$$\xi_\alpha(t) = \sum_{k=1}^{N} c_{\alpha k} \xi_{\alpha_k}(t), \tag{17}$$

where $\mathscr{C}_{\alpha k}$; $k = 1, \ldots, N$, *are some constants.*

[Indeed as it follows from (16), the space $\mathscr{A} = H(0)$ has the dimension N and for any basis $\xi_{\alpha_1}(0), \ldots, \xi_{\alpha_N}(0)$ in $\mathscr{A}$ we have the representation (17).]

3. As we have noted above, the relation (11) for "the innovation"

$$B = H[0, \infty] \ominus H(-\infty, 0)$$

is equivalent to the regularity which means that

$$\bigcap_n H(-\infty, n) = \{0\}.$$

The question on the regularity was posed and completely solved by Kolmogorov (1939) for univariate stationary processes. The new developments of general (in particular, multivariate) stationary processes started in 1957 and the regularity was considered by many authors but in the general case the question on the regularity has not been solved yet.[3]

From the very beginning of the study of a general stationary process $\{\mathscr{A}, U^t\}$ it was clear that *the regularity is equivalent to the factorization*

$$f_\lambda = \varphi_\lambda \cdot \varphi_\lambda^* \tag{18}$$

of the spectral density f_λ by an "analytical" operator-valued function φ_λ in the Hilbert space $\mathscr{A}$ with the Fourier decomposition

$$\varphi_\lambda = \sum_{k=0}^{\infty} \Phi_k e^{i\lambda k}. \tag{19}$$

More exactly, for all $a_1, a_2 \in \mathscr{A}$ scalar functions $(\varphi_\lambda a_1, a_2)$ are boundary values of analytical functions $\Gamma(z) = \sum_{k=0}^{\infty}(\Phi_k a_1, a_2)z^k$ from the well-known Hardy class H^1 [under the condition (18) $\Gamma(z)$ belongs to H^2].

Let us consider the following condition: *For some "analytical" operator-valued function φ_λ of the type* (19) *such that*

$$\varphi_\lambda \mathscr{A} \subseteq f_\lambda^{1/2}\mathscr{A}, \qquad \overline{\varphi_\lambda \mathscr{A}} = \overline{f_\lambda^{1/2}\mathscr{A}} \qquad \text{a.e. } \lambda \tag{20}$$

the relation

$$\int_{-\pi}^{\pi} \| f_\lambda^{-1/2}\varphi_\lambda a \|^2 \, d\lambda < \infty \tag{21}$$

is satisfied (remember that $f_\lambda^{-1/2}$ is the inverse operator from $f_\lambda^{1/2}\mathscr{A}$ into the closure $\overline{f_\lambda^{1/2}\mathscr{A}}$).

In the case of univariate stationary processes this condition means that for some scalar function φ_λ which is a boundary value of an analytical function of the Hardy class H^1 (shortly: $\varphi_\lambda \in H^1$) we have

$$\int_{-\pi}^{\pi} \frac{|\varphi_\lambda|^2}{f_\lambda} \, d\lambda < \infty. \tag{22}$$

From this relation, it follows that

$$\int_{-\pi}^{\pi} \log \frac{|\varphi_\lambda|^2}{f_\lambda} \, d\lambda = \int_{-\pi}^{\pi} \log |\varphi_\lambda|^2 \, d\lambda - \int_{-\pi}^{\pi} \log f_\lambda \, d\lambda < \infty$$

[3] The relevant references can be found, for example, in the books [4, 5] and the recent paper [6].

and since $\log|\varphi_\lambda|^2$ is integrable for any $\varphi_\lambda \in H^1$ it is equivalent to the well-known *Kolmogorov condition* of the regularity

$$\int_{-\pi}^{\pi} \log f_\lambda \, d\lambda > -\infty.$$

In the case of multivariate stationary processes with nondegenerated matrix-valued spectral density f_λ the conditions (20)–(21) mean that for some "analytical" matrix-valued function φ_λ with components from the Hardy class H^1

$$\int_{-\pi}^{\pi} \operatorname{Tr}[\varphi_\lambda{}^* f_\lambda^{-1} \varphi_\lambda] \, d\lambda < \infty. \tag{23}$$

From this relation it follows that

$$\int_{-\pi}^{\pi} \log[\det \varphi_\lambda{}^* f_\lambda^{-1} \varphi_\lambda] \, d\lambda =$$
$$\int_{-\pi}^{\pi} \log|\det \varphi_\lambda|^2 \, d\lambda - \int_{-\pi}^{\pi} \log[\det f_\lambda] \, d\lambda < \infty$$

and since $\det \varphi_\lambda \in H^{1/n}$, $n = \dim \mathscr{A}$, it is equivalent to the well-known *Zasuhin condition* of the regularity

$$\int_{-\pi}^{\pi} \log[\det f_\lambda] \, d\lambda > -\infty.$$

Let us consider another well-known condition which is *sufficient* for the regularity of a general stationary process with a nondegenerate spectral density f_λ:

$$\int_{-\pi}^{\pi} \log \| f_\lambda^{-1} \|^{-1} \, d\lambda > -\infty. \tag{24}$$

Under this relation the scalar function $\| f_\lambda^{-1} \|^{-1}$ can be represented as $\| f_\lambda^{-1} \|^{-1} = |\theta(\lambda)|^2$, where $\theta(\lambda) \in H^1$. It is very easy to see that the operator-valued function

$$\varphi_\lambda = \theta(\lambda) I, \tag{25}$$

where I is the identity operator in $\mathscr{A}$, satisfies our conditions (20)–(21).

By a similar argument we can easily show that the conditions (20)–(21) give us all known criteria of regularity.

THEOREM. *For the regularity of a general stationary process* $\{A, U^t\}$ *with a spectral density* f_λ *the conditions* (20)–(21) *are necessary and sufficient.*

Proof. We shall go by the following path: the regularity $\Rightarrow$ the factorization $\Rightarrow$ the conditions (20)–(21) $\Rightarrow$ the regularity.

Let a stationary process $\{\mathscr{A}, U^t\}$ with a spectral density f_λ be regular (nondeterministic) and $\{B, U^t\}$ be a corresponding "innovation" process from Wold decomposition

$$H(-\infty, t] = \sum_{s\leqslant t} \oplus U^s B$$

(see, for example, [2]). We can assume that $H = \mathscr{L}^2(B)$ and $\mathscr{A}$ is a subspace in

$$H(-\infty, 0] = \sum_{s\leqslant 0} \oplus e^{i\lambda s} B,$$

consisting of some functions $a(\lambda) \in \mathscr{L}^2(B)$

$$a(\lambda) \sim \sum_{k=0}^{\infty} \mathscr{C}_k e^{-i\lambda k},$$

where the Fourier coefficients

$$\mathscr{C}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda k} a(\lambda)\, d\lambda = \Psi_k a, \qquad a \in \mathscr{A},$$

determine certain linear operators Ψ_k; $k = 0, 1,...$ from $\mathscr{A}$ into B, so that for the corresponding operator-valued function

$$\psi(\lambda) = \sum_{k=0}^{\infty} \Psi_k e^{-i\lambda k}, \qquad -\pi \leqslant \lambda \leqslant \pi,$$

we have $a(\lambda) = \psi(\lambda) a$, $a \in \mathscr{A}$. Obviously

$$\begin{aligned}(U^t a_1, a_2) &= \int_{-\pi}^{\pi} e^{i\lambda t} (\psi_\lambda a_1, \psi_\lambda a_2)\, d\lambda \\ &= \int_{-\pi}^{\pi} e^{i\lambda t} (\psi_\lambda^* \psi_\lambda a_1, a_2)\, d\lambda = \int_{-\pi}^{\pi} e^{i\lambda t} (f_\lambda a_1, a_2)\, d\lambda\end{aligned}$$

for any elements $a_1, a_2 \in \mathscr{A}$ and

$$f_\lambda = \psi_\lambda^* \cdot \psi_\lambda \text{ for a.a. } \lambda.$$

Let V be an isometric operator from B to $\mathscr{A}$. Then the operator-valued function

$$\varphi_\lambda = \psi_\lambda^* V^* = \sum_{k=0}^{\infty} \Psi_k^* V^* e^{i\lambda k}$$

will give us the factorization (18).

Now, let we have the factorization (18). Obviously

$$\| f_\lambda^{1/2} a \|^2 = \| \varphi_\lambda{}^* a \|^2, \qquad a \in \mathscr{A}. \tag{26}$$

Since $\varphi_\lambda{}^* a = 0$ for all elements a, which are orthogonal to the subspace $\varphi_\lambda \mathscr{A}$, the relation (26) shows us that $\overline{\varphi_\lambda \mathscr{A}} = \overline{f_\lambda^{1/2} \mathscr{A}}$. Without loss of generality, we can consider $f_\lambda^{1/2}$ and $\varphi_\lambda{}^*$ only on the subspace $\overline{f^{1/2} \mathscr{A}}$. If we determine the unitary operator V_λ from $\overline{f_\lambda^{1/2} \mathscr{A}}$ into $\varphi^* \mathscr{A}$ by the relation $V_\lambda f_\lambda^{1/2} a = \varphi^* a$, $a \in \mathscr{A}$, then one can see that the operator-valued function $\varphi_\lambda = f_\lambda^{1/2} V_\lambda{}^*$ satisfies all the conditions (20)–(21).

Let us consider now any operator-valued function φ_λ, which satisfies these conditions. Obviously, functions $\mathscr{C}(\lambda) = \varphi_\lambda a$, $a \in \mathscr{A}$, belong to the corresponding space L_T, $T = [0, \infty)$,—see (6)–(8)—and under conditions (20)–(21) we have the relation (12), but we have shown above that this relation is equivalent to regularity.

The proof of our theorem is thus complete.

References

[1] Rozanov, Yu. A. (1957). On the linear interpolation of stationary processes with discrete time. *Dokl. Akad. Nauk SSSR* **116** 923–926.

[2] Rozanov, Yu. A. (1958). Spectral theory of multidimensional stationary random processes with discrete time. *Uspehi Mat. Nauk* **XIII**, 2(80) 93–142 (in Russian).

[3] Masani, P. (1960). The prediction theory of multivariate stochastic processes, III. *Acta Math.* **104** 142–162.

[4] Rozanov, Yu. A. (1963). *Stationary Random Processes.* Nauka, Moscow (English translation; Holden-Day, San Franciso, 1967).

[5] Sz.-Nagy, B. and Foiaş, C. (1967). *Analyse Harmonique des Opérateurs de l'espace Hilbert.* Budapest. Masson et C^{ie}, Académiai Kiado.

[6] Kallianpur, G. and Mandrekar, V. (1971). Spectral theory of stationary $\mathscr{H}$-valued processes. *J. Multivariate Analysis* **1** 1–16.