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Note

The multichromatic numbers of some Kneser graphs

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Abstract

The Kneser graph $K(m,n)$ has the n-subsets of $\{1,2,\dots,m\}$ as its vertices, two such vertices being adjacent whenever they are disjoint. The kth multichromatic number of the graph G is the least integer t such that the vertices of G can be assigned k-subsets of $\{1,2,\dots,t\}$, so that adjacent vertices of G receive disjoint sets. The values of $\chi_k(K(m,n))$ are computed for $n = 2$, 3 and bounded for $n \geqslant 4$.

Keywords: Graph; Kneser; multichromatic number

1. Introduction

For any positive integers t and k, a *k-tuple coloring* of a graph G with t colors assigns to each vertex of G, a k-subset of $I_t = \{1,2,\dots,t\}$ so that adjacent vertices receive disjoint sets. Equivalently, such a coloring is a family $\{C_\lambda\}_{\lambda=1}^{t}$ of independent sets of vertices of G such that each vertex belongs to k of the C_λ's. The kth *(multi)chromatic number* of G, denoted by $\chi_k(G)$, is the least integer t such that G has a k-tuple coloring with t colors. These colorings were first studied in the early 1970s and the reader is referred to [10] for more information.

For positive integers $m \geqslant 2n$, the *Kneser graph* $K(m,n)$ has all the n-subsets of I_m as its vertices, two such vertices being adjacent whenever they are disjoint as sets. It is natural to ask for the kth chromatic number of the Kneser graph $K(m,n)$. The following conjecture was made in [11] and also discussed in [3, 6].

Conjecture 1. If $k = qn - r$ where $q \geqslant 1$ and $0 \leqslant r < n$, then $\chi_k(K(m,n)) = qm - 2r$.

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When $k = 1$, this conjecture asserts that $\chi(K(m, n)) = \chi_1(K(m, n)) = m - 2n + 2$ which is equivalent to the Kneser conjecture proved in [1, 9]. When $k = qn$, Conjecture 1 asserts that $\chi_{qn}(K(m, n)) = qm$. This is proved in the corollary to [11, Theorem 9]. The following proposition was proved in [11].

Proposition 2. *Let* k *be any positive integer and let G be any graph with at least one edge. Then*

$$\chi_{k+1}(G) \geqslant 2 + \chi_k(G).$$

It follows from this proposition that for each $K(m, n)$ the conjecture is valid for $k = 1, 2, \ldots, n$. The conjecture has also been proved for $K(2n + 1, n)$ for all k and n [11, Theorem 7], and for $k = 4$, $n = 3$ and $m \geqslant 6$ in [7]. The conjecture is trivially true for $n = 1$, and it will be proved below that it is also true for $n = 2, 3$. It was shown in [4] that for each $n \geqslant 2$, there exists a constant $c = c(n)$ such that $\chi_{n+1}(K(m, n)) \geqslant 2m - c$ for all sufficiently large m. This result will also be sharpened below.

2. The main result

A set of vertices of $K(m, n)$ is independent if and only if its vertices are pairwise nondisjoint as sets. Such a set is said to be of *type* 1 if the intersection of all of its vertices is nonempty, and of *type* 2 otherwise. If C is a type 1 independent set of vertices of $K(m, n)$ and i is an element of each of its vertices, then C will be said to be *anchored* at i. The following facts are well known:

Erdős et al. [5]. *Every independent set of vertices of* $K(m, n)$ *contains at most* $\binom{m-1}{n-1}$ *vertices.*

Hilton and Milner [8]. *Every type* 2 *independent set of vertices of* $K(m, n)$ *contains at most*

$$1 + \binom{m-1}{n-1} - \binom{m-n-1}{n-1}$$

vertices.

The proof of Proposition 4 below relies on the following technical lemma.

Lemma 3. *Let m and n denote positive integers and set*

$$g(m, n) = m - 2n + 1 - \frac{m}{n} - (m - 2n + 2)\left[\binom{m-n-1}{n-1} - 1\right]\binom{m-1}{n-1}^{-1}.$$

Then

$$g(m, n) < 0 \qquad \text{for } n = 1, 2, 3 \text{ and } m > 2n,$$

$$g(m, n) < n^2 - 3n + 4 \quad \text{for } n \geqslant 4 \text{ and } m > 2n.$$

Proof. Elementary computations can be used to show that $g(n) < 0$ for $n = 1, 2, 3$. As for $n \geqslant 4$, it follows from well-known inequalities (see [2, p. 5]) that for $m > 2n$,

$$\binom{m-n-1}{n-1}\binom{m-1}{n-1}^{-1} \geqslant \left(\frac{m-2n}{m-n-1}\right)^n$$
$$= \left(1 - \frac{n-1}{m-n-1}\right)^n \geqslant 1 - \frac{n(n-1)}{m-n-1}.$$

Consequently,

$$m - 2\text{n} + 1 - \frac{m}{n} - (m-2n+2)\left[\binom{m-n-1}{n-1} - 1\right]\binom{m-1}{n-1}^{-1}$$
$$\leqslant m - 2n + 2 - \frac{m}{n} - (m-2n+2)\left(1 - \frac{n(n-1)}{m-n-1}\right)$$
$$= \frac{m-2n+2}{m-n-1}n(n-1) - \frac{m}{n} < n^2 - 3n + 4,$$

the last inequality being verifiable by the methods of elementary calculus. □

Proposition 4. *For any positive integers $m \geqslant 2n$, k, and q,*

$$\chi_{2q-1}(K(m,2)) = qm - 2, \qquad \chi_{2q}(K(m,2)) = qm,$$
$$\chi_{3q-2}(K(m,3)) = qm - 4, \qquad \chi_{3q-1}(K(m,3)) = qm - 2, \qquad \chi_{3q}(K(m,3)) = qm$$
$$qm - 2r \geqslant \chi_k(K(m,n)) \geqslant qm - 2r - (n^2 - 3n + 4),$$

where $k = qn - r$, $0 \leqslant r < n$.

Proof. As noted above, $\chi_{qn}(K(m,n)) = qm$ was already proved in [11]. It therefore follows from Proposition 2 that $\chi_{qn-r}(K(m,n)) \leqslant qm - 2r$. Since $K(2n,n)$ is bipartite it follows that $\chi_k(K(2n,n)) = 2k$ for all k. Thus, the proposition holds for $m = 2n$, and we now assume that $m > 2n$. Set

$$f(n) = 0 \quad \text{for } n = 1, 2, 3,$$
$$f(n) = n^2 - 3n + 4 \quad \text{for } n \geqslant 4.$$

It follows from Proposition 2 that it suffices to prove the present proposition for the case $r = n - 1$. Fix n and suppose by the way of contradiction that m is the least positive integer that is at least $2n + 1$ and for which there exist a positive integer q such that

$$\chi_{qn-(n-1)}(K(m,n)) \leqslant qm - 2(n-1) - f(n) - 1.$$

Set $\Lambda = qm - 2(n-1) - f(n) - 1$ and let $\{C_\lambda\}_{\lambda=1}^{\Lambda}$ be a family of independent sets such that every vertex of $K(m,n)$ belongs to $qn - (n-1)$ of the C_λ's.

Case 1: There exists an integer $i \in \{1, 2, \ldots, m\}$ such that at least q of the C_λ's are anchored at i. It may be assumed, without loss of generality, that $i = m$ and that for each λ such that

$$\Lambda - q < \lambda \leqslant \Lambda,$$

C_λ is anchored at m. Let M denote the set of vertices of $K(m, n)$ that contain m and set $\bar{C}_\lambda = C_\lambda - M$ for each λ. Then $\{\bar{C}_\lambda\}_{\lambda=1}^{\Lambda-q}$ is a family of independent sets of vertices of $K(m-1, n)$ such that each vertex of $K(m-1, n)$ is contained in $qn - (n-1)$ of the $\bar{C}_\lambda$'s. It follows from the minimality of m that

$$q(m-1) - 2(n-1) - f(n) \leqslant \chi_{qn-(n-1)}(K(m-1, n))$$
$$\leqslant \Lambda - q = qm - 2(n-1) - f(n) - 1 - q,$$

which is impossible.

Case 2: For each integer $i \in \{1, 2, \ldots, m\}$ at most $q-1$ of the C_λ's are anchored at i. Since every type 1 C_λ that contains the arbitrary vertex $\{i_1, i_2, \ldots, i_n\}$ of $K(m, n)$ must be anchored at some $i_j, j \in \{1, 2, \ldots, n\}$, it follows that each such vertex is contained in at most $n(q-1)$ type 1 C_λ's. Hence, each vertex is contained in at least one C_λ of type 2. Thus the type 2 C_λ's constitute a (1-tuple) coloring of $K(m, n)$. Consequently, if d denotes the number of type 2 C_λ's, then

$$d \geqslant \chi(K(m, n)) = m - 2n + 2. \tag{1}$$

For each λ, set $d_\lambda = \binom{m-1}{n-1} - |C_\lambda|$. It follows from the Erdős–Ko–Rado theorem that $d_\lambda \geqslant 0$. Since $\{C_\lambda\}_{\lambda=1}^{\Lambda}$ is a $[qn - (n-1)]$-tuple coloring of $K(m, n)$, we have

$$\sum_{\lambda=1}^{\Lambda} d_\lambda = \Lambda\binom{m-1}{n-1} - \sum_{\lambda=1}^{\Lambda} |C|_\lambda = \Lambda\binom{m-1}{n-1} - [qn - (n-1)]\binom{m}{n}$$
$$= \binom{m-1}{n-1}\left(m - 2n + 1 - \frac{m}{n} - f(n)\right). \tag{2}$$

By the Hilton–Milner theorem $d_\lambda \geqslant \binom{m-n-1}{n-1} - 1$ whenever C_λ is of type 2 and so

$$d\left[\binom{m-n-1}{n-1} - 1\right] \leqslant \binom{m-1}{n-1}\left(m - 2n + 1 - \frac{m}{n} - f(n)\right). \tag{3}$$

Combining (1) and (3) we get

$$(m - 2n + 2)\left[\binom{m-n-1}{n-1} - 1\right] \leqslant \binom{m-1}{n-1}\left(m - 2n + 1 - \frac{m}{n} - f(n)\right)$$

or

$$f(n) \leqslant m - 2n + 1 - \frac{m}{n} - (m - 2n + 2)\left[\binom{m-n-1}{n-1} - 1\right]\binom{m-1}{n-1}^{-1}.$$

It follows from the lemma that $f(n) < 0$ for $n = 1, 2, 3$, and that $f(n) < n^2 - 3n + 4$ for $n \geqslant 4$. In either case this contradicts the definition of $f(n)$. $\square$

3. Conclusions

A *homomorphism* of the graph G into the graph H is a function $f: V(G) \to V(H)$ that maps adjacent vertices to adjacent vertices. It is immediate that

$\chi_n(G) \leqslant m$ if and only if there exists a homomorphism of G into $K(m, n)$.

In view of this, the computation of the n-tuple multichromatic numbers would resolve the issue of just when there exists a homomorphism $\eta: K(s, t) \to K(m, n)$. Ref. [12] contains an application of such a homomorphism.

A graph G is said to be *(k, t)-colorable* if there exists a k-tuple coloring of G with t colors. The computation of the multichromatic numbers of the Kneser graphs would also provide an answer to the following question:

> *For which ordered 4-tuples of integers (k, t, m, n) does (k, t)-colorability imply (m, n)-colorability?*

It may be of interest to note that $K(m, 1)$, $K(m, 2)$, $K(m, 3)$ and $K(2n + 1, n)$ were the only Kneser graphs whose chromatic numbers were known prior to Lovász's proof of the Kneser conjecture. They are also the only Kneser graphs for which Conjecture 1 has been verified for all values of k.

Added in proof: The multichromatic numbers of $K(m, 2)$ (i.e. case $k = 2$ of Proposition 4) were computed previously and independently by Claude Tardif. The author is indebted to both Claude Tradif and Sandi Klavžar for reviving his interest in this topic.

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