Oscillations of a System of Delay Logistic Equations

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We obtained sufficient conditions for the oscillation of all positive solutions of the system

\[ \dot{N}_i(t) = N_i(t) \left( a_i - \sum_{j=1}^{m} b_{ij} N_j(t-\tau) \right), \quad i = 1, 2, \ldots, m \]

about its steady state. We also obtained sufficient conditions for the existence of a nonoscillatory solution of this system. † 1990 Academic Press, Inc.

1. INTRODUCTION

Our aim in this paper is to obtain sufficient conditions for the oscillation of all positive solutions of the system of delay logistic equations

\[ \dot{N}_i(t) = N_i(t) \left( a_i - \sum_{j=1}^{m} b_{ij} N_j(t-\tau) \right), \quad i = 1, 2, \ldots, m, \quad (1) \]

where

\[ \tau \in (0, \infty) \quad \text{and} \quad a_i, b_{ij} \in \mathbb{R} \quad \text{for} \quad i, j = 1, 2, \ldots, m \quad (2) \]

about its steady state

\[ N^* = [N_1^*, N_2^*, \ldots, N_m^*]^T \]

that is about the solution of the system

\[ \sum_{j=1}^{m} b_{ij} N_j^* = a_i \quad \text{for} \quad i = 1, 2, \ldots, m. \quad (3) \]
We will also obtain sufficient conditions for the existence of a non-oscillatory solution of (1).

Together with the system (1) we assume initial conditions of the form

\[ N_i(t) = \varphi_i(t), \quad -\tau \leq t \leq 0 \]

where \( \varphi_i \in C\left([-\tau, 0], \mathbb{R}^+\right) \) and \( \varphi_i(0) > 0 \) (4)

for \( i = 1, 2, ..., m \). Then (1) and (4) has a unique solution \( N(t) = [N_1(t), N_2(t), ..., N_m(t)]^T \) valid for all \( t \geq 0 \) and such that

\[ N_i(t) > 0 \quad \text{for} \quad t \geq 0 \quad \text{and} \quad i = 1, 2, ..., m. \]

Throughout this paper we assume that the system (3) has a solution \( N^* \) with positive components.

Set

\[ N_i(t) = N_i^* e^{\nu_i(t)} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad i = 1, 2, ..., m. \]

Then the functions \( x_i(t) \) satisfy the system of delay equations

\[ \dot{x}_i(t) + \sum_{j=1}^{m} p_{ij} \left[ e^{\nu_i(t-\tau_j)} - 1 \right] = 0, \quad i = 1, 2, ..., m, \] (5)

where

\[ p_{ij} = b_{ij} N_i^* \quad \text{for} \quad i, j = 1, 2, ..., m. \]

We will say that a solution \( N(t) = [N_1(t), ..., N_m(t)]^T \) of (1) oscillates about \( N^* = [N_1^*, ..., N_m^*]^T \) if for some \( i = 1, 2, ..., m \) the function \( N_i(t) - N_i^* \) has arbitrarily large zeros. If, on the other hand, each of the functions \( N_i(t) - N_i^* \) for \( i = 1, 2, ..., m \) is eventually different from zero, then we will say that the solution \( N(t) \) is nonoscillatory about \( N^* \).

It is clear that for every \( \varphi \) satisfying (4) the solution of (1) and (4) oscillates about \( N^* \) if and only if every solution of (5) oscillates about \( [0, 0, ..., 0]^T \).

For some recent work concerning the oscillation of systems we refer to [2] and [3] and the references cited therein.

2. SUFFICIENT CONDITIONS FOR OSCILLATION

The main result in this section is the following:

**Theorem 1.** Consider the system (1) and assume that (2) is satisfied and that (3) has a solution \( N^* = [N_1^*, N_2^*, ..., N_m^*]^T \) with positive components.
Set
\[ \mu = \min_{1 \leq i \leq m} \left[ N_i^* \left( b_{i_i} - \sum_{i=1, i \neq i}^m |b_{i_j}| \right) \right] \quad (6) \]
and suppose that
\[ \mu \tau e > 1. \quad (7) \]

Then for every \( \phi \) satisfying (4) the solution of (1) and (4) oscillates about \( N^* \).

The proof of Theorem 1 makes use of Barbalat's lemma [1] which for convenience we state and prove below.

**Lemma 1 (Barbalat).** Let \( y \in C^1([t_0, \infty), \mathbb{R}) \) be such that
\[ \lim_{t \to \infty} y(t) \text{ exists and is finite} \]
and
\[ \dot{y}(t) \text{ is uniformly continuous on } [t_0, \infty). \]

Then
\[ \lim_{t \to \infty} \dot{y}(t) = 0. \]

**Proof.** Otherwise there exists a sequence \( \{t_n\} \) such that
\[ \lim_{n \to \infty} t_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \dot{y}(t_n) = l \neq 0. \]

The limit \( l \) is finite or infinite. We will assume that \( l > 0 \) or \( l = \infty \). The case where \( l < 0 \) or \( l = -\infty \) is similar and will be omitted. Let \( l \in (0, l) \). Then for \( \varepsilon - l/2 \) there exists a \( \delta = \delta(\varepsilon) > 0 \) such that
\[ t_n \leq t \leq t_n + \frac{\delta}{3} \Rightarrow \dot{y}(t) \geq \frac{l}{3}. \]

By integrating we obtain
\[ y \left( t_n + \frac{\delta}{2} \right) - y(t_n) \geq \frac{l\delta}{6} \]
which as \( n \to \infty \) leads to a contradiction.
3. Proof of Theorem 1

Assume, for the sake of contradiction, that (1) and (4) has a solution \( N(t) \) which is nonoscillatory about \( N^* \). Then (5) has a solution \( x(t) = [x_1(t), x_2(t), ..., x_m(t)]^T \) which is nonoscillatory about \([0, 0, ..., 0]^T\). First, we claim that

\[
\lim_{t \to \infty} x_i(t) = 0, \quad i = 1, 2, ..., m. \tag{8}
\]

To this end, set for \( t \) sufficiently large

\[
\delta_i = \text{sgn } x_i(t) \quad \text{for } i = 1, 2, ..., m
\]

\[
z_i(t) = \delta_i x_i(t) \quad \text{for } i = 1, 2, ..., m
\]

and

\[
V(t) = \sum_{i=1}^{m} z_i(t).
\]

Then

\[
z_i(t) > 0 \quad \text{for } i = 1, 2, ..., m
\]

and

\[
\dot{z}_i(t) + \sum_{j=1}^{m} p_{ij} \frac{\delta_j e^{\delta_j z_j(t-\tau)} - 1}{\delta_j} = 0, \quad i = 1, 2, ..., m. \tag{9}
\]

Hence

\[
\dot{z}_i(t) + p_{ii} \frac{e^{\delta_i z_i(t-\tau)} - 1}{\delta_i} - \sum_{j=1, j \neq i}^{m} |p_{ij}| \frac{e^{\delta_j z_j(t-\tau)} - 1}{\delta_j} \leq 0, \quad i = 1, ..., m. \tag{10}
\]

By summing up (vertically) the terms in (10) for \( i = 1, 2, ..., m \) and using (6), we find

\[
\dot{\nu}(t) + \mu \sum_{i=1}^{m} \frac{e^{\delta_i z_i(t-\tau)} - 1}{\delta_i} \leq 0. \tag{11}
\]

As \( \mu > 0 \) and

\[
\frac{e^{\delta_i z_i(t-\tau)} - 1}{\delta_i} > 0 \quad \text{for } i = 1, 2, ..., m
\]

it follows that

\[
\dot{\nu}(t) < 0
\]
and so

\[ L = \lim_{t \to \infty} v(t) \]  

exists and is nonnegative. From (12) we see that \( z_i(t) \) is bounded for each \( i = 1, 2, \ldots, m \). Hence, from (9), \( \dot{z}_i(t) \) and \( \ddot{z}_i(t) \) are also bounded for each \( i = 1, 2, \ldots, m \). Thus, \( \dot{v}(t) \) is uniformly continuous and by Barbata's lemma,

\[ \lim_{t \to \infty} \dot{v}(t) = 0. \]

It follows from (11) that \( \lim_{t \to \infty} z_i(t) = 0 \) for each \( i = 1, 2, \ldots, m \). This completes the proof of (8).

Next, we rewrite (5) in the form

\[ \dot{x}_i(t) + \sum_{j=1}^{m} P_{ij}(t) x_j(t-\tau) = 0, \quad i = 1, 2, \ldots, m, \]  

that is,

\[ \dot{x}_i(t) + \sum_{j=1}^{m} P_{ij}(t) x_j(t-\tau) = 0, \quad i = 1, 2, \ldots, m, \]  

(13)

where

\[ P_{ij}(t) = p_{ij} \frac{e^{\gamma_j(t-\tau)} - 1}{x_j(t-\tau)} \quad \text{for} \quad i, j = 1, 2, \ldots, m \]  

and

\[ \lim_{t \to \infty} P_{ij}(t) = p_{ij} \quad \text{for} \quad i, j = 1, 2, \ldots, m. \]  

(14)

From (13) we see that

\[ \dot{z}_i(t) + P_{ii}(t) z_i(t-\tau) - \sum_{j=1, j \neq i}^{m} |P_{ij}(t)| z_j(t-\tau) \leq 0, \quad i = 1, 2, \ldots, m, \]  

(15)

where, as before,

\[ \delta_i = \text{sgn} \, x_i(t) \quad \text{and} \quad z_i(t) = \delta_i x_i(t) \quad \text{for} \quad i = 1, 2, \ldots, m. \]

Let \( \varepsilon > 0 \) be chosen in such a way that

\[ (\mu - \varepsilon) \tau e > 1 \]  

(16)
which is possible in view of (7). Now, summing up (15) (vertically) for \( i = 1, 2, ..., m \) we obtain

\[
\dot{v}(t) + \sum_{j=1}^{m} \left[ P_{ij}(t) - \sum_{i=1, i \neq j}^{m} |P_{ij}(t)| \right] \dot{z}_j(t - \tau) \leq 0
\]

which in view of (14), (6), and (7) implies that

\[
\dot{v}(t) + (\mu - \varepsilon) v(t - \tau) \leq 0. \tag{17}
\]

It follows from [4] that because of (16), inequality (17) cannot have an eventually positive solution. This is a contradiction and the proof is complete.

4. EXISTENCE OF A NONOSCILLATORY SOLUTION

The main result in this section is the following:

**Theorem 2.** Consider the system (1) and assume that \( \tau > 0, b_{ij} > 0 \) for \( i, j = 1, 2, ..., m \) and that (3) has a solution \( N^* = [N^*_1, N^*_2, ..., N^*_m] \) with positive components. Suppose that

\[
\rho \leq 1,
\]

where \( \rho \) denotes the spectral radius of the \( m \times m \) matrix with \( (i, j) \) components equal to \( b_{ij} N^*_j \). Then (1) has a solution which is nonoscillatory about the steady state \( N^* \).

The proof of Theorem 2 will be facilitated by the next four lemmas which are interesting in their own right.

In the sequel, inequalities and equalities about matrices and vectors are assumed to hold componentwise.

Consider the systems of equations

\[
\dot{x}(t) + P(t) f(x(t - \tau)) = 0, \quad t \geq t_0 \tag{19}
\]

and

\[
\dot{y}(t) + Qy(t - \tau) = 0, \quad t \geq t_0 \tag{20}
\]

and the inequality

\[
\dot{z}(t) + P(t) f(z(t - \tau)) \leq 0, \quad t \geq t_0, \tag{21}
\]

where \( \tau > 0, P(t) \) is an \( m \times m \) matrix with positive and continuous components, \( Q \) is an \( m \times m \) matrix with positive and constant components,
and \( f = [f_1, f_2, \ldots, f_m]^T \) is nondecreasing and such that \( f \in C[R^m, R^m] \) and for any \( u = [u_1, \ldots, u_m]^T \) with \( u \neq 0 \),
\[
u_i f_i(u_1, \ldots, u_m) > 0 \quad \text{for } i = 1, \ldots, m.
\]

**Lemma 2.** Assume that (22) holds, \( \tau > 0 \), \( P(t) > 0 \) and continuous and that inequality (21) has an eventually positive solution. Then the corresponding Eq. (19) also has an eventually positive solution.

**Proof.** Let \( z(t) \) be an eventually positive solution of (21) and let \( T \geq t_0 \) be such that
\[
z(t) > 0 \quad \text{for } t \geq T - \tau.
\]

Then
\[
\dot{z}(t) < 0 \quad \text{for } t \geq T
\]
and so
\[
\lim_{t \to \infty} z(t) = L
\]
exists and is a finite and nonnegative vector. Thus, integrating (21) from \( T \) to \( \infty \) we obtain
\[
L + \int_T^\infty P(s) f(z(s-\tau)) \, ds \leq z(t), \quad t \geq T.
\]

Let \( W \) denote the set of nonnegative and nonincreasing functions \( w \) on \([T, \infty)\) such that
\[
L \leq w(t) \leq z(t) \quad \text{for } t \geq T.
\]

For every \( w \in W \), set
\[
\tilde{w}(t) = \begin{cases} w(t), & t \geq T \\ w(T) + z(t) - z(T), & T - \tau \leq t < T. \end{cases}
\]

Define the mapping \( S \) on \( W \) by
\[
(Sw)(t) = L + \int_t^\infty P(s) f(\tilde{w}(s-\tau)) \, ds, \quad t \geq T.
\]

In view of (23), \( S \) maps \( W \) into \( W \) and all the hypotheses of the Knaster–Tarski fixed point theorem, see [5], are satisfied. Hence, there exists a point \( x \in W \) such that \( Sx = x \). Clearly, \( x \) satisfies Eq. (19) and so the proof
will be complete if we show that eventually, $x(t) > 0$. For $T - \tau \leq t < T$, we have

$$x(t) = x(T) + z(t) - z(T) > 0.$$  

Now assume, for the sake of contradiction, that there exists a $t^* \geq T$ such that

$$x(t) > 0 \quad \text{for} \quad T - \tau \leq t < t^*$$

while

$$x(t^*) = 0.$$

Then by (19) and (22), $\dot{x}(t^*) = -P(t^*) f(x(t^* - \tau)) < 0$ which contradicts the fact that $x \in S$ and, consequently, that $x(t) \geq 0$ for $t \geq T$. The proof is complete.

**Lemma 3.** Let $q, \tau \in (0, \infty)$ be such that

$$q \tau e < 1.$$

Then the equation

$$\mu + q e^{-\mu \tau} = 0$$

has a negative root.

**Proof.** Set

$$F(\mu) = \mu - q e^{-\mu \tau}$$

and observe that

$$F(0) F\left( -\frac{1}{\tau} \right) = q \left( -\frac{1}{\tau} + q e \right) = q \frac{q \tau e - 1}{\tau} < 0$$

from which the result follows.

**Lemma 4.** Let $A$ be an $m \times m$ matrix and let $\tau, \lambda_0, \mu_0 \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ be such that

$$A \xi = \lambda_0 \xi$$

and

$$\mu_0 + \lambda_0 e^{-\mu_0 \tau} = 0.$$
Then

\[ z(t) = e^{r(t)} \xi \]

is a solution of

\[ \dot{z}(t) + Az(t - \tau) = 0. \]

**Proof.**

\[
\dot{z}(t) + A z(t - \tau) = \mu_0 e^{i\mu t} \xi + A e^{i\mu(t - \tau)} \xi = e^{i\mu t} [A + \mu_0 e^{\mu \tau}] \xi
\]

\[
= e^{i\mu t} (A - \lambda_0 I) \xi = 0.
\]

The next lemma gives sufficient conditions for the system (19) to have a nonoscillatory solution.

**Lemma 5.** Consider the system (19) and assume that \( P(t) \) is a continuous \( m \times m \) matrix such that

\[
\lim_{t \to \infty} P(t) = Q,
\]

where the constant matrix \( Q \) has positive components and is such that

\[
\rho(Q) \tau e < 1, \tag{24}
\]

where \( \rho(Q) \) is the spectral radius of \( Q \) and \( \tau > 0 \). Suppose that \( f \) satisfies (22) and that there exists a \( \delta > 0 \) such that either

\[
f(u) \leq u \quad \text{for} \quad 0 < u \leq \delta \tag{25}
\]

or

\[
f(u) \geq u \quad \text{for} \quad -\delta \leq u < 0. \tag{26}
\]

Then the system (19) has a nonoscillatory solution.

**Proof.** We will assume that (25) holds and we will prove that (19) has an eventually positive solution. When (26) holds, one can set \( v(t) = -x(t) \) and show, by a similar argument, that (26) has an eventually negative solution. To this end, choose \( \varepsilon > 0 \) and \( T \geq t_0 \) such that

\[
\rho(Q(\varepsilon)) \tau e < 1
\]

and

\[
0 \leq P(t) \leq Q(\varepsilon) \quad \text{for} \quad t \geq T,
\]

where

\[
Q(\varepsilon) = \begin{pmatrix}
\lambda_0 & \mu_0 e^{\mu \tau} \\
0 & \lambda_0
\end{pmatrix}
\]
where $Q(\varepsilon)$ is the $m \times m$ matrix with $ij$-components equal to $q_{ij} + \varepsilon$ and $ho(Q(\varepsilon))$ denotes the spectral radius of $Q(\varepsilon)$. Let $\xi$ be a positive eigenvector of $Q(\varepsilon)$ associated with the eigenvalue $\rho(Q(\varepsilon))$. Such an eigenvector exists because $Q(\varepsilon)$ is a positive matrix and $\rho(Q(\varepsilon))$ is the largest eigenvalue of $Q(\varepsilon)$.

Let $\mu_0(\varepsilon)$ be a negative root of the equation

$$
\mu + \rho(Q(\varepsilon)) e^{-\mu t} = 0
$$

as is guaranteed by Lemma 3 and let $z(t) = e^{\mu(t)} e^{\varepsilon t} \xi$ be the positive solution of the equation

$$
\dot{z}(t) + Q(\varepsilon) z(t - \tau) = 0
$$

which is guaranteed by Lemma 4. Then

$$
0 = \dot{z}(t) + Q(\varepsilon) z(t - \tau) \geq \dot{z}(t) + P(t) z(t - \tau) \geq \dot{z}(t) + P(t) f(z(t - \tau)).
$$

Hence, by Lemma 2, the system (19) has also an eventually positive solution. The proof is complete.

**Proof of Theorem 2.** As we saw in the introduction, the transformation

$$
N_i(t) = N_* e^{\lambda_i t} \quad \text{for} \quad i = 1, 2, \ldots, m \quad \text{and} \quad t \geq 0
$$

reduces system (1) to system (5). On the other hand, $N(t)$ is nonoscillatory about $N^*$ if and only if $x(t)$ is nonoscillatory about $[0, 0, \ldots, 0]^T$. Thus it suffices to show that system (5) has a nonoscillatory solution. For each $u = [u_1, u_2, \ldots, u_m]^T$, set

$$
f(u) = [e^{u_1} - 1, e^{u_2} - 1, \ldots, e^{u_m} - 1]^T
$$

and observe that $f$ is nondecreasing and

$$
(e^{u_i} - 1) u_i > 0 \quad \text{for} \quad u_i \neq 0
$$

and

$$
e^{u_i} - 1 \geq u_i \quad \text{for} \quad u < 0
$$

which implies that $f$ satisfies conditions (22) and (26). By applying Lemma 5 to Eq. (5) we see that (5) has a nonoscillatory solution. The proof is complete.
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