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Oscillatory behaviour in functional differential systems of neutral type

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Abstract

In this paper are obtained several criteria for oscillations of a class of autonomous functional differential systems of neutral type. © 2002 Elsevier Science (USA). All rights reserved.

1. Introduction

The purpose of this paper is to investigate the oscillatory behaviour of the solutions of the linear functional differential system of neutral type

$$\frac{d}{dt}\left[x(t) - \int_{-1}^{0} d[\nu(\theta)]\,x(t-\tau(\theta))\right] = \int_{-1}^{0} d[\eta(\theta)]\,x(t-r(\theta)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $\nu(\theta)$ and $\eta(\theta)$ are real $n \times n$ matrix valued functions of bounded variation on $[-1,0]$, and $\tau(\theta)$ and $r(\theta)$ are real positive continuous functions on $[-1,0]$.

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Considering the value $R = \max\{\|\tau\|, \|r\|\}$, where

$$\|\tau\| = \max\{\tau(\theta)\colon -1 \leqslant \theta \leqslant 0\} \quad \text{and} \quad \|r\| = \max\{r(\theta)\colon -1 \leqslant \theta \leqslant 0\},$$

by a solution of (1) we mean a continuous function $x : [-R, \infty[\to \mathbb{R}$, such that

$$x(t) - \int_{-1}^{0} x(t - \tau(\theta))\, d[\nu(\theta)]$$

is differentiable and (1) is satisfied for every $t \geqslant 0$. A solution of (1), $x(t) = [x_1(t), \ldots, x_n(t)]^{\mathrm{T}}$, is said to be *oscillatory* if every component $x_i(t)$, $i = 1, \ldots, n$, has arbitrary large zeros; otherwise it is called *nonoscillatory*. Whenever all solutions of the system (1) are oscillatory, we will say that (1) is *totally oscillatory*. If (1) is totally oscillatory for every pair of delay functions $\tau(\theta), r(\theta)$, it will be called *totally oscillatory globally in the delays*.

According to [1], the analysis of the oscillatory behaviour of solutions of the system (1) can be based upon the existence or absence of real zeros of the characteristic equation

$$\det\left[\lambda\left(I - \int_{-1}^{0} \exp(-\lambda\tau(\theta))\, d[\nu(\theta)]\right) - \int_{-1}^{0} \exp(-\lambda r(\theta))\, d[\eta(\theta)]\right] = 0, \tag{2}$$

where by I we mean the $n \times n$ identity matrix. In fact, in this framework, one can conclude that (1) is totally oscillatory if and only if (2) has no real roots; nonoscillatory solutions will exist, whenever (2) has at least a real root.

We start by studying, in Section 2, the total oscillatory behavior of the scalar case of (1), that is the case where $n = 1$. Then, as an application, the general case is discussed in Section 3, in basis of the totally oscillatory behavior of the functional retarded differential system

$$\frac{d}{dt}x(t) = \int_{-1}^{0} d[\eta(\theta)]\, x(t - r(\theta)), \tag{3}$$

corresponding to have in (1), $\nu(\theta)$ constant on $[-1, 0]$.

Kirchner and Stroinski in [2] discuss the same problem for the system

$$\frac{d}{dt}\left[x(t) - \int_{-r}^{0} d[\alpha(\theta)]\, x(t + \theta)\right] = \int_{-r}^{0} d[\beta(\theta)]\, x(t + \theta), \tag{4}$$

where r is a positive real number and α and β are matrix valued functions of bounded variation on $[-r, 0]$, α atomic at zero. If we allowed in (1) $\tau(\theta)$ and $r(\theta)$ to be nonnegative, then with $r(\theta) = \tau(\theta) = -r\theta$ and $\alpha(\theta) = \nu(\theta/r)$ atomic at zero (notice that in (1), the restriction on $\tau(\theta)$ to be positive makes unnecessary any atomicity assumption on ν) and $\beta(\theta) = \eta(\theta/r)$, we obtain the class of systems (4). However, there will be some interest by considering (1) in order to understand the role of the delays on the oscillatory behavior of functional differential systems. Anyway, independently of the adopted formulation, the criteria obtained here are of different kind of those given in [2].

Both systems (1) and (4) include the differential-difference system

$$\frac{d}{dt}\left[x(t) - \sum_{j=1}^{m} B_j x(t-\tau_j)\right] = \sum_{j=1}^{m} A_j x(t-r_j), \tag{5}$$

where the A_j and B_j $(j = 1, \dots, m)$ are real $n \times n$ matrices and the τ_j and r_j $(j = 1, \dots, m)$ are positive real numbers. Regarding a system of this kind, in [2–6] several criteria are obtained on its total oscillatory behavior. We will develop here, for system (1), the method introduced in Section 2 of [4] with respect to (5). We will get, in particular, for this system, other different criteria, as well as a clearer picture on the oscillatory behavior of scalar equations.

By C^+ we will denote the subset of $C([-1,0],\mathbb{R})$ formed by all positive continuous functions on $[-1,0]$. With respect to a function $r \in C^+$, it will be often considered the value

$$m(r) = \min\{r(\theta):\ -1 \leqslant \theta \leqslant 0\}.$$

Denoting by $\mathbb{R}^{n\times n}$ the Banach space of all $n \times n$ real matrices, we take the space BV_n of all functions of bounded variation, $\eta : [-1,0] \to \mathbb{R}^{n\times n}$. For a given norm $\|\cdot\|$ in $\mathbb{R}^{n\times n}$, with $\eta \in BV_n$ and $\theta \in [-1,0]$, by $V_\eta(\theta)$ we mean the total variation of η on the interval $[-1,\theta]$. The total variation of η on $[-1,0]$, $V_\eta(0)$, will also be denoted by

$$\int_{-1}^{0} \big\|d[\eta(\theta)]\big\|.$$

For any $\eta \in BV_n$, we will consider the functions η_0 and η_1 of BV_n, which for $\theta \in [-1,0]$ are given, respectively, by

$$\eta_0(\theta) = \eta(0) - \eta(\theta), \qquad \eta_1(\theta) = \eta(\theta) - \eta(-1).$$

The space BV_1 of all real functions of bounded variation on $[-1,0]$ will be denoted simply by BV. For $\phi \in BV$, by

$$\int_{-1}^{0} |d\phi(\theta)|$$

we will mean the total variation of ϕ on $[-1,0]$, which in the case of ϕ to be monotonic corresponds to the value

$$\Delta_\phi = |\phi(0) - \phi(-1)|.$$

For ϕ monotonic, whenever ϕ is called increasing (or decreasing) we are implicitly excluding the possibility of ϕ be constant; in order to include this case, we will say that ϕ is nondecreasing (respectively, nonincreasing).

2. The scalar case

For a matter of convenience, the scalar case of (1) will be rewritten as

$$\frac{d}{dt}\left[x(t) - \int_{-1}^{0} x(t-\tau(\theta))\,dp(\theta)\right] = \int_{-1}^{0} x(t-r(\theta))\,dq(\theta) \tag{6}$$

for $p, q \in BV$ and $\tau, r \in C^+$. This equation will be totally oscillatory if and only if the function

$$F(\lambda) = \lambda - \lambda \int_{-1}^{0} \exp(-\lambda\tau(\theta))\,dp(\theta) - \int_{-1}^{0} \exp(-\lambda r(\theta))\,dq(\theta)$$

has no real zeros. Through the analysis of this property, the two theorems below, which will be important in the sequel, report some sufficient conditions for having (6) totally oscillatory.

In view of the method used in the next section, we will restrict ourselves to the case where $p(\theta)$ and $q(\theta)$ are monotonic functions on $[-1, 0]$. But, as for any $\phi \in BV$, $r \in C^+$ and $\lambda \geqslant 0$, one has

$$\left|\int_{-1}^{0} \exp(-\lambda r(\theta))\,d\phi(\theta)\right| \leqslant \exp(-\lambda m(r)) \int_{-1}^{0} |d\phi(\theta)|,$$

it holds that $F(\lambda) \to +\infty$ as $\lambda \to +\infty$. Therefore, in such framework, as $q(\theta)$ nondecreasing implies $F(0) \leqslant 0$, (6) will have then at least a nonoscillatory solution. So, under the assumption that $q(\theta)$ is monotonic, in order that (6) be totally oscillatory is necessary that $q(\theta)$ be decreasing on $[-1, 0]$, independently of the function $p(\theta)$.

Theorem 1. *Let* $p(\theta)$ *be nonincreasing and* $q(\theta)$ *decreasing on* $[-1, 0]$. *If* $\|r\| > \|\tau\|$, J *is a finite union of closed intervals such that* $r(\theta) > \|\tau\|$ *for* $\theta \in J$ *and*

$$1 + \Delta p \leqslant e \int_{J} (\|\tau\| - r(\theta))\,dq(\theta), \tag{7}$$

then (6) *is totally oscillatory.*

Proof. As $F(\lambda) > 0$ for every $\lambda \geqslant 0$, we only have to show that $F(\lambda)$ has no negative real zeros.

For $\lambda \neq 0$, the equation $F(\lambda) = 0$ can be reduced to the form $\Phi(\lambda) = \Psi(\lambda)$, where

$$\Phi(\lambda) = \exp(\lambda\|\tau\|) - \int_{-1}^{0} \exp\bigl(\lambda(\|\tau\| - \tau(\theta))\bigr)\, dp(\theta)$$

and

$$\Psi(\lambda) = \frac{1}{\lambda}\int_{-1}^{0} \exp\bigl(\lambda(\|\tau\| - r(\theta))\bigr)\, dq(\theta).$$

Notice that $\Phi(\lambda)$ is differentiable in $\mathbb{R}$ and has as derivative

$$\Phi'(\lambda) = \|\tau\|\exp(\lambda\|\tau\|) - \int_{-1}^{0} (\|\tau\| - \tau(\theta))\exp\bigl(\lambda(\|\tau\| - \tau(\theta))\bigr)\, dp(\theta).$$

As $\Phi'(\lambda) > 0$ for every real λ, it holds that $\Phi(\lambda)$ is strictly increasing.

On the other hand, since $q(\theta)$ is decreasing and $\exp(\lambda(\|\tau\| - \tau(\theta)))$ is positive, we have, for every real $\lambda < 0$,

$$\begin{aligned}
\Psi(\lambda) &\geqslant \frac{1}{\lambda}\int_{J} \exp\bigl(\lambda(\|\tau\| - r(\theta))\bigr)\, dq(\theta) \\
&\geqslant \int_{J} \left(\max\left\{\frac{\exp(\lambda(\|\tau\| - r(\theta)))}{\lambda}\colon\ \lambda < 0\right\}\right) dq(\theta) \\
&= e\int_{J} (\|\tau\| - r(\theta))\, dq(\theta).
\end{aligned}$$

Then assuming that $\Psi(\lambda_0) = \Phi(\lambda_0)$ for some real $\lambda_0 < 0$, we have that

$$e\int_{J} (\|\tau\| - r(\theta))\, dq(\theta) \leqslant \Psi(\lambda_0) = \Phi(\lambda_0) < \Phi(0) = 1 - (p(0) - p(-1)),$$

which is in contradiction with the condition (7). □

We illustrate the preceding theorem with the following example.

Example 2. For $\tau, r \in]0, +\infty[$, the equation

$$\frac{d}{dt}\left[x(t) + \int_{-1}^{0} x(t + \tau\theta - \tau)\exp(-\theta)\, d\theta\right] + \int_{-1}^{0} x(t + r\theta - r)\, d\theta = 0$$

corresponds to have in (6), $p(\theta) = \exp(-\theta)$, $\tau(\theta) = -\tau\theta + \tau$, $q(\theta) = -\theta$ and $r(\theta) = -r\theta + r$. Applying Theorem 1, one easily sees that this equation is totally oscillatory for every pair of real positive numbers, (τ, r), such that $0 < \tau \leqslant (3r-2)/4$.

When $p(\theta)$ and $q(\theta)$ are step functions with a finite number of jump points, one obtains the differential-difference equation

$$\frac{d}{dt}\left[x(t) - \sum_{j=1}^{m} p_j x(t-\tau_j)\right] = \sum_{j=1}^{m} q_j x(t-r_j), \tag{8}$$

where, for $j = 1, \ldots, m$, $p_j, q_j \in \mathbb{R}$ and $\tau_j, r_j \in \left]0, +\infty\right[$. Theorem 1 is formulated for this equation in the following corollary. The result obtained is of the same kind of the given in [5, Chapter 6] for the case $m = 1$.

Corollary 3. *If* $p_j, q_j \in \left]-\infty, 0\right]$ (*the* q_j *not all zero*), $\tau_1 < \cdots < \tau_m$ *and for some* $m_0 \in \{1, \ldots, m\}$, $0 < r_1 < \cdots < r_{m_0-1} \leqslant \tau_m < r_{m_0} < \cdots < r_m$, *then* (8) *will be totally oscillatory providing that*

$$1 - \sum_{j=1}^{m} p_j \leqslant e \sum_{j=m_0}^{m} (\tau_m - r_j) q_j.$$

Example 4. The equation

$$\frac{d}{dt}\left[x(t) + \frac{1}{2}x\left(t - \frac{1}{2}\right) + \frac{1}{2}x(t-1)\right] = ax(t-1) + bx(t-2)$$

is totally oscillatory if $a \leqslant 0$ and $b \leqslant -2e^{-1}$.

Remark 5. Theorem 1 and Corollary 3 have the disadvantage of excluding the cases, respectively, $r(\theta) = \tau(\theta)$ for every $\theta \in [-1, 0]$ and $\tau_j = r_j$ for all $j \in \{1, \ldots, m\}$. However, for example, with respect to Corollary 3, if $\tau_m \geqslant r_m$ and $p_m < 0$, then the corresponding function $F(\lambda)$ has at least a real zero (see [4, Remark 1]).

Theorem 6. *Let* $p(\theta)$ *be nondecreasing and* $q(\theta)$ *decreasing on* $[-1, 0]$. *Then*:

(i) *If* $\Delta p = 1$, *Eq.* (6) *is totally oscillatory globally in the delays.*
(ii) *Equation* (6) *is totally oscillatory if at least one of the following assumptions is satisfied*:

$$1 + e\int_{-1}^{0} r(\theta)\, dq(\theta) < \Delta p < 1, \tag{9}$$

$$1 < \Delta p < \frac{em(\tau)}{\|r\|} \log(\|r\| e \Delta_q), \tag{10}$$

$$1 < \Delta p \quad \textit{and}$$

$$\int_{-1}^{0} \frac{1}{e\tau(\theta)}\, dp(\theta) + \int_{-1}^{0} \frac{1}{\Delta_q r(\theta)} \log[\Delta_q r(\theta) e]\, dq(\theta) < 0. \tag{11}$$

This theorem is an immediate consequence of the following lemma.

Lemma 7. *Let $p(\theta)$ be nondecreasing and $q(\theta)$ decreasing on $[-1, 0]$. Then*:

(i) *$F(\lambda)$ has no real zeros if*

$$\Delta_p = 1. \tag{12}$$

(ii) *$F(\lambda)$ has no real negative zero if*

$$\Delta p > 1 + e \int_{-1}^{0} r(\theta)\, dq(\theta). \tag{13}$$

(iii) *$F(\lambda)$ has no real positive zero if at least one of the following assumptions is satisfied*:

$$\Delta_p \leqslant 1, \tag{14}$$

$$\Delta_p < \frac{em(\tau)}{\|r\|} \log(\|r\| e \Delta_q), \tag{15}$$

$$\int_{-1}^{0} \frac{1}{e\tau(\theta)}\, dp(\theta) + \int_{-1}^{0} \frac{1}{\Delta_q r(\theta)} \log[\Delta_q r(\theta) e]\, dq(\theta) < 0. \tag{16}$$

Proof. (i) Consider the functions

$$\varphi(\lambda) = 1 - \int_{-1}^{0} \exp(-\lambda \tau(\theta))\, dp(\theta)$$

and

$$\psi(\lambda) = \frac{1}{\lambda} \int_{-1}^{0} \exp(-\lambda r(\theta))\, dq(\theta);$$

as $F(0) = \Delta q > 0$, then $F(\lambda)$ has a real zero if and only if $\varphi(\lambda) = \psi(\lambda)$ for some real $\lambda \neq 0$.

As the function $\varphi(\lambda)$ is differentiable in $\mathbb{R}$ and

$$\varphi'(\lambda) = \int_{-1}^{0} \tau(\theta) \exp(-\lambda\tau(\theta))\, dp(\theta) \geqslant 0,$$

we have that $\varphi(\lambda)$ is nondecreasing.

On the other hand, analogously to the function $\Psi(\lambda)$ of the proof of Theorem 1, for every $\lambda \in]-\infty, 0[$ it holds that

$$\psi(\lambda) \geqslant -e \int_{-1}^{0} r(\theta)\, dq(\theta) \geqslant em(r)\Delta q > 0.$$

Moreover, for every $\lambda \in]0, +\infty[$ one has $\psi(\lambda) < 0$.

Thus $\varphi(0) = 1 - \Delta p = 0$, that is (12), implies that $\varphi(\lambda) \neq \psi(\lambda)$ for every real $\lambda \neq 0$; $\varphi(0) = 1 - \Delta p \geqslant 0$, that is (14), implies that $\varphi(\lambda) \neq \psi(\lambda)$ for every $\lambda > 0$. Analogously,

$$-e \int_{-1}^{0} r(\theta)\, dq(\theta) > \varphi(0) = 1 - \Delta p,$$

that is (13), implies that $\varphi(\lambda) \neq \psi(\lambda)$ for every $\lambda < 0$.

(ii) With respect to (15) and (16) let us suppose that $F(\lambda_0) = 0$ for some real $\lambda_0 > 0$ and consider the functions

$$f(\lambda) = \lambda - \int_{-1}^{0} \exp(-\lambda r(\theta))\, dq(\theta)$$

and

$$g(\lambda) = \lambda \int_{-1}^{0} \exp(-\lambda\tau(\theta))\, dp(\theta).$$

For $\lambda > 0$ we have

$$\begin{aligned} f(\lambda) &\geqslant \lambda + \exp(-\lambda\|r\|)\Delta_q \\ &\geqslant \min\bigl\{\lambda + \exp(-\lambda\|r\|)\Delta_q\colon \lambda \in \mathbb{R}\bigr\} \\ &= \frac{1}{\|r\|} \log(\|r\|\Delta_q e). \end{aligned}$$

On the other hand, for every $\lambda > 0$ it holds that

$$g(\lambda) \leqslant \int_{-1}^{0} \max\bigl\{\lambda \exp(-\lambda\tau(\theta))\colon \lambda > 0\bigr\}\, dp(\theta)$$

$$\leqslant \int_{-1}^{0} \frac{1}{e\tau(\theta)}\, dp(\theta) \leqslant \frac{\Delta_p}{em(\tau)}. \tag{17}$$

Hence $\lambda_0 > 0$ is such that $f(\lambda_0) = g(\lambda_0)$ and, by consequence, we have necessarily

$$\frac{1}{\|r\|} \log(\|r\| e \Delta_q) \leqslant f(\lambda_0) = g(\lambda_0) \leqslant \frac{\Delta_p}{em(\tau)},$$

which contradicts (15).

(iii) In the concerning to (16), notice that the function $f(\lambda)$ can be rewritten as

$$\begin{aligned}
f(\lambda) &= -\int_{-1}^{0} \left(\frac{\lambda}{\Delta_q} + \exp(-\lambda r(\theta)) \right) dq(\theta) \\
&\geqslant -\int_{-1}^{0} \min_{\lambda \in \mathbb{R}} \left(\frac{\lambda}{\Delta_q} + \exp(-\lambda r(\theta)) \right) dq(\theta) \\
&= -\int_{-1}^{0} \frac{1}{\Delta_q r(\theta)} \log(\Delta_q r(\theta) e)\, dq(\theta).
\end{aligned}$$

Thus, by (17), one has necessarily

$$-\int_{-1}^{0} \frac{1}{\Delta_q r(\theta)} \log[\Delta_q r(\theta) e]\, dq(\theta) \leqslant f(\lambda_0) = g(\lambda_0) \leqslant \int_{-1}^{0} \frac{1}{e\tau(\theta)}\, dp(\theta),$$

which contradicts (16). □

Remark 8. Condition (9) is included in [2, Theorem 3.3.].

Remark 9. Theorem 6 can be used to analyze the totally oscillatory behavior of the functional differential equation

$$\frac{d}{dt} x(t) = \int_{-1}^{0} x(t - r(\theta))\, dq(\theta), \tag{18}$$

corresponding to have, in (6), $p(\theta)$ constant on $[-1, 0]$. As $\Delta p = 0$, then with $q(\theta)$ decreasing, by (9) we can state that (18) is totally oscillatory if

$$\int_{-1}^{0} r(\theta)\, dq(\theta) < -\frac{1}{e}.$$

This result, for the case $r(\theta) = -r\theta$ $(r > 0,\ \theta \in [-1, 0])$, is included in several papers like [2,7,8].

Remark 10. Notice also that, even under the assumption of $q(\theta)$ be decreasing, Eq. (18) cannot be totally oscillatory globally in the delays. In fact, by considering the function $f(\lambda)$, introduced in the proof of the preceding lemma, the totally oscillatory behavior of (18) depends upon $f(\lambda)$ to be positive for every $\lambda \in \mathbb{R}$. But, for any $\lambda < 0$ it holds that

$$f(\lambda) \leqslant \lambda + \exp(-\lambda m(r))\Delta q.$$

Therefore, if $\lambda < -\Delta q$, one can have $f(\lambda) < 0$, providing that $r \in C^+$ be such that $m(r)$ be small enough. So, in order to have (18) totally oscillatory globally in the delays, we have to exclude that $q(\theta)$ be monotonic.

In Theorem 6, the second inequality of (11) is not very easy to handle in the applications. The same seems not happen to the remaining conditions, as one can see through the following examples.

Example 11. For any decreasing function $q(\theta)$ on $[-1, 0]$, by (i) of Theorem 6, the equation

$$\frac{d}{dt}\left[x(t) - \int_{-1}^{0} x(t - \tau(\theta))\,d\theta\right] = \int_{-1}^{0} x(t - r(\theta))\,dq(\theta)$$

is totally oscillatory globally in the delays.

Example 12. By making, in (6), $p(\theta) = \exp(\theta)$ and $q(\theta) = -\theta$, we obtain the equation

$$\frac{d}{dt}\left[x(t) - \int_{-1}^{0} x(t - \tau(\theta))\exp(\theta)\,d\theta\right] + \int_{-1}^{0} x(t - r(\theta))\,d\theta = 0.$$

As $\Delta p = 1 - e^{-1} < 1$, we have by (9) that, independently of the delay function $\tau \in C^+$, this equation is totally oscillatory for every $r \in C^+$ such that

$$e^{-2} < \int_{-1}^{0} r(\theta)\,d\theta$$

(for example, any $r \in C^+$ in manner that $m(r) > e^{-2}$).

Example 13. Now letting $p(\theta) = -e^{-\theta}$, $q(\theta) = -\theta$, and $\tau(\theta) = r(\theta) = -k\theta + k$, we have by (10) that the equation

$$\frac{d}{dt}\left[x(t) - \int_{-1}^{0} x(t + k\theta - k)e^{-\theta}\, d\theta\right] + \int_{-1}^{0} x(t + k\theta - k)\, d\theta = 0$$

is totally oscillatory, providing that $k \geqslant e/2$. In fact, notice that $1 < \Delta p = e - 1 < e$, $\|r\| = 2k$, $m(\tau) = k$, $\Delta q = 1$ and

$$2k \geqslant e \Leftrightarrow \frac{e}{2}\log(2ke) \geqslant e.$$

For the differential-difference equation

$$\frac{d}{dt}\left[x(t) - \sum_{j=1}^{m} p_j x(t - r_j)\right] = \sum_{j=1}^{m} q_j x(t - r_j), \tag{19}$$

one can apply also Theorem 6. The following corollary is then obtained. Several other different criteria can be seen in [3,5] and references therein.

Corollary 14. *Let for* $j = 1, \ldots, m$ *be* $p_j \in [0, +\infty[$, $q_j \in]-\infty, 0]$ (*not all zero*) *and* $r_j \in]0, +\infty[$. *Then*:

(i) (19) *is totally oscillatory globally in the delays if* $\sum_{j=1}^{m} p_j = 1$;
(ii) (19) *is totally oscillatory providing that at least one of the following assumptions be satisfied*:

$$1 + e\sum_{j=1}^{m} q_j r_j < \sum_{j=1}^{m} p_j < 1, \tag{20}$$

$$1 < \sum_{j=1}^{m} p_j < \frac{er_1}{r_m}\log\left(er_m\left|\sum_{j=1}^{m} q_j\right|\right) \quad (r_1 \leqslant \cdots \leqslant r_m), \tag{21}$$

$$\sum_{j=1}^{m} p_j > 1 \quad \textit{and}$$

$$\left|\sum_{j=1}^{m} q_j\right| \sum_{j=1}^{m} \frac{p_j}{r_j} < e\sum_{j=1}^{m} \frac{|q_j|}{r_j}\log\left(er_j\left|\sum_{j=1}^{m} q_j\right|\right). \tag{22}$$

These conditions are illustrated, respectively, in each one of the following examples.

Example 15. By (i) of Corollary 14 the equation

$$\frac{d}{dt}\left[x(t) - \frac{1}{2}x(t - r_1) - \frac{1}{2}x(t - r_2)\right] = ax(t - r_1) + bx(t - r_2)$$

for every $a, b \in]0, +\infty[$ is totally oscillatory globally in the delays.

Example 16. The condition (20) of Corollary 14 is obtained, in particular, in [8, Corollary 4.3]. This condition is satisfied if

$$\sum_{j=1}^{m} p_j < 1 \quad \text{and} \quad \sum_{j=1}^{m} q_j r_j < -\frac{1}{e}.$$

For example, the equation

$$\frac{d}{dt}\left[x(t) - \frac{1}{2}x(t-1) - \frac{1}{3}x(t-2)\right] = ax(t-1) + ax(t-2)$$

is totally oscillatory for every $a \in]-\infty, -1/3e[$.

Example 17. By (21), the equation

$$\frac{d}{dt}\left[x(t) - \frac{e}{2}x(t-1) - \frac{e}{2}x(t-2)\right] = ax(t-1) + bx(t-2)$$

with $a, b \in]-\infty, 0]$, is totally oscillatory if $a + b < -e/2$.

Example 18. Through (22), we can say that the equation

$$\frac{d}{dt}\left[x(t) - x(t-1) - x(t-2)\right] + \frac{1}{r}x(t-r) = 0$$

is totally oscillatory whenever $0 < r < 2e/3$.

3. Retarded and neutral systems

For $\eta, \nu \in BV_n$ and $\lambda \in \mathbb{R}$ the matrices

$$A(\lambda) = \lambda I - \int_{-1}^{0} \exp(-\lambda r(\theta))\, d[\eta(\theta)]$$

and

$$B(\lambda) = \lambda \int_{-1}^{0} \exp(-\lambda \tau(\theta))\, d[\nu(\theta)]$$

will play an important role, in order to obtain a sufficient condition for having system (1) totally oscillatory.

For that purpose, matrix measures, already introduced on this subject by other authors, will be used. For a matter of completeness we recall briefly its definition and the properties which will be used in the sequel.

For each induced norm $\|\cdot\|$ in $\mathbb{R}^{n\times n}$ we associate a matrix measure $\mu:\mathbb{R}^{n\times n}\to\mathbb{R}$, which is defined for any $C\in\mathbb{R}^{n\times n}$ as

$$\mu(C)=\lim_{\gamma\to 0+}\frac{\|I+\gamma C\|-1}{\gamma},$$

where by I we mean the identity matrix. Independently of the considered induced norm in $\mathbb{R}^{n\times n}$, a matrix measure has always the following properties (see [9]) for any $C\in\mathbb{R}^{n\times n}$:

(I) $-\|C\|\leqslant\mu(C)\leqslant\|C\|$.
(II) $\mu(C_1)-\mu(-C_2)\leqslant\mu(C_1+C_2)\leqslant\mu(C_1)+\mu(C_2)$ $(C_1,C_2\in\mathbb{R}^{n\times n})$.
(III) $\mu(\gamma C)=\gamma\mu(C)$ for every $\gamma\geqslant 0$.
(IV) $\mu(\gamma C)=|\gamma|\mu(-C)$ for every $\gamma\leqslant 0$.
(V) $-\mu(-C)\leqslant\|C^{-1}\|^{-1}$, if C is nonsingular.

Denoting by $\sigma(C)$ the spectrum of the matrix C and introducing the upper and lower bounds of the set $\operatorname{Re}\sigma(C)=\{\operatorname{Re}\lambda:\ \lambda\in\sigma(C)\}$, which are given by

$$s(C)=\max\{\operatorname{Re}z:\ z\in\sigma(C)\}\quad\text{and}\quad\ell(C)=\min\{\operatorname{Re}z:\ z\in\sigma(C)\},$$

we have

(VI) $-\mu(-C)\leqslant\ell(C)\leqslant s(C)\leqslant\mu(C)$.

If $\eta\in BV_n$ the continuity of μ on $\mathbb{R}^{n\times n}$ implies that $\mu\circ\eta\in BV$; in consequence, the following inequalities hold (see [2]):

(VII) If $\phi\in C([-1,0];\mathbb{R})$ is nonincreasing and positive, then

$$\mu\left(\int_{-1}^{0}\phi(\theta)\,d[\eta(\theta)]\right)\leqslant\int_{-1}^{0}\phi(\theta)\,d(\mu\circ\eta_1)(\theta).$$

(VIII) If $\phi\in C([-1,0];\mathbb{R})$ is nondecreasing and positive, then

$$\mu\left(\int_{-1}^{0}\phi(\theta)\,d[\eta(\theta)]\right)\leqslant-\int_{-1}^{0}\phi(\theta)\,d(\mu\circ\eta_0)(\theta).$$

Specifically, for any given induced norm $\|\cdot\|$ in $\mathbb{R}^{n\times n}$ and the corresponding matrix measure μ, the function $\eta\in BV_n$, which determines $A(\lambda)$, will be supposed in manner that the following assumption holds:

(A) $\mu\circ\eta_0$ is increasing and $\mu\circ\eta_1$ decreasing.

The following property of $A(\lambda)$ is then stated.

Theorem 19. *Under* (A), *let* $r \in C^+$ *be such that*:

(i) r *is decreasing and*

$$\int_{-1}^{0} r(\theta)\, d(\mu \circ \eta_1)(\theta) < -\frac{1}{e}; \tag{23}$$

(ii) r *is increasing and*

$$\int_{-1}^{0} r(\theta)\, d(\mu \circ \eta_0)(\theta) > \frac{1}{e}. \tag{24}$$

Then $\mu(-A(\lambda)) < 0$ *for every real* λ.

Proof. Since $\mu(\pm I) = \pm 1$ and $\mu(0) = 0$, by the properties (II) and (III) of the matrix measures we have, for every real λ,

$$\mu(-A(\lambda)) \leqslant -\lambda + \mu\left(\int_{-1}^{0} \exp(-\lambda r(\theta))\,[d\eta(\theta)]\right). \tag{25}$$

Assumption (A) implies the following inequalities:

$$\begin{aligned}\int_{-1}^{0} r(\theta)\, d(\mu \circ \eta_1)(\theta) &\geqslant \|r\|\big((\mu \circ \eta_1)(0) - (\mu \circ \eta_1)(-1)\big)\\ &= \|r\|\mu(\eta_1(0)),\end{aligned} \tag{26}$$

$$\int_{-1}^{0} r(\theta)\, d(\mu \circ \eta_0)(\theta) \leqslant -\|r\|\mu(\eta_0(-1)), \tag{27}$$

and for $\lambda \geqslant 0$,

$$\int_{-1}^{0} \exp(-\lambda r(\theta))\, d(\mu \circ \eta_0)(\theta) \geqslant -\exp(-\lambda\|r\|)\mu(\eta_0(-1)), \tag{28}$$

$$\int_{-1}^{0} \exp(-\lambda r(\theta))\, d(\mu \circ \eta_1)(\theta) \leqslant \exp(-\lambda\|r\|)\mu(\eta_1(0)). \tag{29}$$

Moreover, taking into account that $e^{u-1} \geqslant u$ for every real u, still holds that

$$\int_{-1}^{0} \exp(-\lambda r(\theta))\, d(\mu \circ \eta_1)(\theta) \leqslant -\lambda e \int_{-1}^{0} r(\theta)\, d(\mu \circ \eta_1)(\theta), \tag{30}$$

$$\int_{-1}^{0} \exp(-\lambda r(\theta))\, d(\mu \circ \eta_0)(\theta) \geqslant -\lambda e \int_{-1}^{0} r(\theta)\, d(\mu \circ \eta_0)(\theta) \tag{31}$$

for every real λ.

(1) Let then (i) be satisfied. By (23) and (26) we have $\mu(\eta_1(0)) = \mu(\eta_0(-1)) < 0$.

(1.1) Then letting $\lambda \geqslant 0$, by (28) we have

$$\int_{-1}^{0} \exp(-\lambda r(\theta))\, d(\mu \circ \eta_0)(\theta) > 0.$$

But as $r(\theta)$ is decreasing, $\exp(-\lambda r(\theta))$ is nondecreasing and positive. Therefore by property (VIII) of the matrix measures it holds that

$$\mu\left(\int_{-1}^{0} \exp(-\lambda r(\theta))\, [d\eta(\theta)]\right) \leqslant -\int_{-1}^{0} \exp(-\lambda r(\theta))\, d(\mu \circ \eta_0)(\theta) < 0.$$

Thus, by (25), $\mu(-A(\lambda)) < 0$ for every $\lambda \in [0, +\infty[$.

(1.2) For $\lambda < 0$, as $\exp(-\lambda r(\theta))$ is decreasing and positive, we have by (VII), (30) and (23) that

$$\begin{aligned}\mu\left(\int_{-1}^{0} \exp(-\lambda r(\theta))\, [d\eta(\theta)]\right) &\leqslant \int_{-1}^{0} \exp(-\lambda r(\theta))\, d(\mu \circ \eta_1)(\theta) \\ &\leqslant -\lambda e \int_{-1}^{0} r(\theta)\, d(\mu \circ \eta_1)(\theta) < \lambda.\end{aligned}$$

Hence, by (25), $\mu(-A(\lambda)) < 0$ also for every $\lambda \in]-\infty, 0[$.

(2) Assuming (ii), we have by (27) that $\mu(\eta_0(-1)) = \mu(\eta_1(0)) < 0$.

(2.1) Letting $\lambda \geqslant 0$, as $r(\theta)$ is increasing, we have $\exp(-\lambda r(\theta))$ nonincreasing and positive and by property (VII) of the matrix measures and (29) it holds that

$$\begin{aligned}\mu\left(\int_{-1}^{0} \exp(-\lambda r(\theta))\, [d\eta(\theta)]\right) &\leqslant \int_{-1}^{0} \exp(-\lambda r(\theta))\, d(\mu \circ \eta_1)(\theta) \\ &\leqslant \exp(-\lambda \|r\|)\mu(\eta_1(0)) < 0.\end{aligned}$$

Thus (25) implies that $\mu(-A(\lambda)) < 0$ for every $\lambda \in [0, +\infty[$.

(2.2) For $\lambda < 0$, as $\exp(-\lambda r(\theta))$ is increasing and positive, we have by (VIII), (31) and (24) that

$$\mu\left(\int_{-1}^{0} \exp(-\lambda r(\theta))\,[d\eta(\theta)]\right) \leqslant -\int_{-1}^{0} \exp(-\lambda r(\theta))\,d(\mu\circ\eta_0)(\theta)$$
$$\leqslant \lambda e \int_{-1}^{0} r(\theta)\,d(\mu\circ\eta_0)(\theta) < \lambda.$$

Hence, by (25), $\mu(-A(\lambda)) < 0$ also for every $\lambda \in]-\infty, 0[$. □

From the properties (V) and (VI) of the matrix measures the following corollary holds.

Corollary 20. *Under the assumptions of Theorem* 19*, the matrix $A(\lambda)$ is nonsingular and*

$$\left\|A(\lambda)^{-1}\right\| \leqslant -\frac{1}{\mu(-A(\lambda))} \tag{32}$$

for every real λ.

Remark 21. Since $\mu(-A(\lambda)) < 0$ implies $\det[A(\lambda)] \neq 0$, both cases in Theorem 19 express sufficient conditions to have the retarded functional differential system (3) totally oscillatory. This conclusion, for the case where $r(\theta) = -r\theta$ ($r > 0$, $\theta \in [-1, 0]$), is reported, in particular, by Kirchner and Stroinski in [5, Theorem 3.3].

Remark 22. Replacing the assumption (A) by the condition that for a given matrix measure μ the function $\eta \in BV_n$ is such that

$$\mu(\eta(\theta_1) - \eta(\theta_2)) \leqslant 0 \tag{33}$$

for every $\theta_1, \theta_2 \in [-1, 0]$, $\theta_1 < \theta_2$, then by use of similar arguments, one can also conclude that $\mu(-A(\lambda)) < 0$ for every real λ, providing that $r \in C^+$ be such that

$$\mu\left(\int_{-1}^{0} r(\theta)\,d[\eta(\theta)]\right) < -\frac{1}{e}. \tag{34}$$

This means that in such circumstances (3) is also totally oscillatory, a statement which is reported by Kong in [8] for the case where $r(\theta) = -r\theta$ ($r > 0$, $\theta \in [-1, 0]$). The relationship between conditions (33) and (34) and the assumptions of Theorem 19 is as follows. Condition (33) implies (A). In fact, if $\theta_1, \theta_2 \in [-1, 0]$ are such that $\theta_1 < \theta_2$, then by property (II) of the matrix measures it holds that

$$\mu(\eta_0(\theta_1)) - \mu(\eta_0(\theta_2)) \leqslant \mu(\eta_0(\theta_1) - \eta_0(\theta_2)) = \mu(\eta(\theta_2) - \eta(\theta_1)) \leqslant 0,$$
$$\mu(\eta_1(\theta_2)) - \mu(\eta_1(\theta_1)) \leqslant \mu(\eta_1(\theta_2) - \eta_1(\theta_1)) = \mu(\eta(\theta_2) - \eta(\theta_1)) \leqslant 0.$$

On the other hand, by property (VII) of the matrix measures, (23) implies (34) in the case where $r(\theta)$ is decreasing, and by (VIII), (24) implies (34) when $r(\theta)$ is increasing.

Theorem 19 and its corollary enable the following sufficient condition for having (1) totally oscillatory.

Theorem 23. *Under* (A), *let* $\tau, r \in C^+$ *be such that* r *is decreasing,* $\|r\| > \|\tau\|$ *and*

$$1 + \int_{-1}^{0} \|d[\nu(\theta)]\| \leqslant e \int_J (\|\tau\| - r(\theta))\, d(\mu \circ \eta_1)(\theta), \tag{35}$$

where J *is a finite union of closed intervals such that* $r(\theta) > \|\tau\|$ *for* $\theta \in J$. *Then* (1) *is totally oscillatory if at least one of the following assumptions is verified*:

$$\int_{-1}^{0} \|d[\nu(\theta)]\| \leqslant 1, \tag{36}$$

$$\int_{-1}^{0} \|d[\nu(\theta)]\| < \frac{em(\tau)}{\|r\|} \log\big[\|r\| e \big|\mu(\eta_0(-1))\big|\big], \tag{37}$$

$$\int_{-1}^{0} \frac{1}{e\tau(\theta)}\, dV_\nu(\theta) < \int_{-1}^{0} \frac{\log(|\mu(\eta_0(-1))| r(\theta) e)}{|\mu(\eta_0(-1))| r(\theta)}\, d(\mu \circ \eta_0)(\theta). \tag{38}$$

Proof. As, by (35),

$$\begin{aligned}
1 &< 1 + \int_{-1}^{0} \|d[\nu(\theta)]\| - e\|\tau\| \int_J d(\mu \circ \eta_1)(\theta) \\
&\leqslant e \int_J r(\theta)\, d(-(\mu \circ \eta_1))(\theta) \\
&\leqslant e \int_{-1}^{0} r(\theta)\, d(-(\mu \circ \eta_1))(\theta),
\end{aligned}$$

then

$$\int_{-1}^{0} r(\theta)\, d(\mu \circ \eta_1)(\theta) < -\frac{1}{e}.$$

So, by Theorem 19 we have $\mu(-A(\lambda)) < 0$ for every real λ. This implies that $A(\lambda)$ is nonsingular for every real λ, and so all solutions of (1) are oscillatory if and only if, for every $\lambda \in \mathbb{R}$,

$$\det[I - A^{-1}(\lambda)B(\lambda)] \neq 0.$$

But by properties (I) and (VI) of the matrix measures this occurs, in particular, whenever

$$\|A^{-1}(\lambda)B(\lambda)\| < 1, \tag{39}$$

for every real λ.

By Corollary 20 we obtain, for every real λ,

$$\|A^{-1}(\lambda)B(\lambda)\| \leqslant \|A^{-1}(\lambda)\| \cdot \|B(\lambda)\| \leqslant -\frac{\|B(\lambda)\|}{\mu(-A(\lambda))};$$

then if

$$\|B(\lambda)\| < -\mu(-A(\lambda)) \tag{40}$$

for every real λ, inequality (39) is satisfied.

But since for every real λ it holds that

$$\|B(\lambda)\| \leqslant |\lambda| \int_{-1}^{0} \exp(-\lambda\tau(\theta))\, dV_\nu(\theta)$$

and

$$\mu(-A(\lambda)) \leqslant -\lambda + \mu\left(\int_{-1}^{0} \exp(-\lambda r(\theta))\, [d\eta(\theta)]\right),$$

we have that if

$$|\lambda| \int_{-1}^{0} \exp(-\lambda\tau(\theta))\, dV_\nu(\theta) < \lambda - \mu\left(\int_{-1}^{0} \exp(-\lambda r(\theta))\, d[\eta(\theta)]\right) \tag{41}$$

for every real λ, then inequality (40) is satisfied.

(1) As, for $\lambda \leqslant 0$, $\exp(-\lambda r(\theta))$ is decreasing, we have

$$\mu\left(\int_{-1}^{0} \exp(-\lambda r(\theta))\, d[\eta(\theta)]\right) \leqslant \int_{-1}^{0} \exp(-\lambda r(\theta))\, d(\mu \circ \eta_1)(\theta).$$

Then taking the function

$$G(\lambda) = \lambda + \lambda \int_{-1}^{0} \exp(-\lambda\tau(\theta))\, dV_\nu(\theta) - \int_{-1}^{0} \exp(-\lambda r(\theta))\, d(\mu \circ \eta_1)(\theta),$$

(41) is satisfied for every $\lambda \leqslant 0$, if $G(\lambda) > 0$ for every $\lambda \in]-\infty, 0]$.

But $G(\lambda)$ is a specific case of the function $F(\lambda)$ introduced in Section 2, relatively to the nonincreasing function $p(\theta) = -V_\nu(\theta)$ and the decreasing function $q(\theta) = (\mu \circ \eta_1)(\theta)$ $(\theta \in [-1, 0])$. Hence, that $G(\lambda) > 0$ for every $\lambda \in]-\infty, 0]$, follows by Theorem 1 and its proof.

(2) For $\lambda > 0$, as $\exp(-\lambda r(\theta))$ is increasing, we have that

$$\mu\left(\int_{-1}^{0} \exp(-\lambda r(\theta))\, d[\nu(\theta)]\right) \leqslant -\int_{-1}^{0} \exp(-\lambda r(\theta))\, d(\mu \circ \eta_0)(\theta).$$

Therefore taking now

$$G(\lambda) = \lambda - \lambda \int_{-1}^{0} \exp(-\lambda\tau(\theta))\, dV_\nu(\theta) + \int_{-1}^{0} \exp(-\lambda r(\theta))\, d(\mu \circ \eta_0)(\theta),$$

we have analogously that (41) is satisfied for every real $\lambda > 0$, if $G(\lambda) > 0$ for every $\lambda \in]0, +\infty[$. But now $G(\lambda)$ is the function $F(\lambda)$ corresponding to $p(\theta) = V_\nu(\theta)$ and $q(\theta) = -(\mu \circ \eta_0)(\theta)$. Then by (iii) of Lemma 7 the theorem follows. $\square$

A similar theorem can be obtained, analogously, for the case where the delay function $r(\theta)$ is increasing.

Theorem 24. *Under* (A), *let $\tau, r \in C^+$ be such that r is increasing, $\|r\| > \|\tau\|$ and*

$$1 + \int_{-1}^{0} \|d[\nu(\theta)]\| \leqslant e \int_J (r(\theta) - \|\tau\|)\, d(\mu \circ \eta_0)(\theta),$$

where J is a finite union of closed intervals such that $r(\theta) > \|\tau\|$ when $\theta \in J$. Then (1) *is totally oscillatory if at least one of the following assumptions is satisfied*:

$$\int_{-1}^{0} \|d[\nu(\theta)]\| \leqslant 1,$$

$$\int_{-1}^{0} \|d[\nu(\theta)]\| < \frac{em(\tau)}{\|r\|} \log\big[\|r\|e\big|\mu(\eta_1(0))\big|\big],$$

$$\int_{-1}^{0} \frac{1}{e\tau(\theta)}\, dV_\nu(\theta) + \int_{-1}^{0} \frac{\log[|\mu(\eta_1(0))|r(\theta)e]}{|\mu(\eta_1(0))|r(\theta)}\, d(\mu\circ\eta_1)(\theta) < 0.$$

One can apply these theorems when μ is one of the following well known matrix measures of a matrix $C = [c_{ik}] \in \mathbb{R}^{n\times n}$,

$$\mu_1(C) = \max\left\{c_{kk} + \sum_{i\neq k} |c_{ik}|:\ k = 1,\dots,n\right\},$$

$$\mu_\infty(C) = \max\left\{c_{ii} + \sum_{k\neq i} |c_{ik}|:\ i = 1,\dots,n\right\},$$

which correspond, respectively, to the induced norms in $\mathbb{R}^{n\times n}$ given by

$$\|C\|_1 = \max\left\{\sum_{i=1}^{n} |c_{ik}|:\ k = 1,\dots,n\right\},$$

$$\|C\|_\infty = \max\left\{\sum_{k=1}^{n} |c_{ik}|:\ i = 1,\dots,n\right\}.$$

Going back to the scalar case, both Theorems 23 and 24 can be applied to Eq. (6) through the matrix measures μ_1 or μ_∞. For that purpose, observe that for any real number c one has $\mu_1(c) = \mu_\infty(c) = c$, and so the assumption (A) reduces to have $q(\theta)$ decreasing. Therefore, from those theorems, the following corollary is obtained.

Corollary 25. *Let $q(\theta)$ be decreasing on $[-1,0]$, $p \in BV$ and $\tau, r \in C^+$ be such that r is monotonic, $\|r\| > \|\tau\|$ and*

$$1 + \int_{-1}^{0} |dp(\theta)| \leqslant e \int_{J} (\|\tau\| - r(\theta))\, dq(\theta), \tag{42}$$

where J is a finite union of closed intervals such that $r(\theta) > \|\tau\|$ when $\theta \in J$. If at least one of the following assumptions is verified:

$$\int_{-1}^{0} |dp(\theta)| \leqslant 1, \tag{43}$$

$$\int_{-1}^{0} |dp(\theta)| \leqslant \frac{em(\tau)}{\|r\|} \log[\|r\| e\Delta_q], \tag{44}$$

$$\int_{-1}^{0} \frac{1}{e\tau(\theta)}\, dV_p(\theta) + \int_{-1}^{0} \frac{\log(\Delta_q r(\theta) e)}{\Delta_q r(\theta)}\, dq(\theta) < 0, \tag{45}$$

then Eq. (6) *is totally oscillatory.*

The main advantage of this corollary, with respect to the Theorems 1 and 6 of Section 2, is in the fact that the function $p(\theta)$ can be nonmonotonic. This situation is illustrated in the following example.

Example 26. For $\tau \in C^+$ and $k > 0$, let the equation

$$\frac{d}{dt}\left[x(t) - \int_{-1}^{0} \sin(2\pi\theta) x(t - \tau(\theta))\, d\theta\right] = \int_{-1}^{0} \theta x(t - k\theta^2 - 1)\, d\theta, \tag{46}$$

corresponding to

$$p(\theta) = -\frac{1}{2\pi}\cos(2\pi\theta), \qquad q(\theta) = \frac{\theta^2}{2}, \quad \text{and} \quad r(\theta) = k\theta^2 + 1.$$

Notice that $q(\theta)$ and $r(\theta)$ are decreasing, and $p(\theta)$ satisfies (43), since

$$\int_{-1}^{0} |dp(\theta)| = \frac{2}{\pi}.$$

As $\|r\| = k + 1$, taking $\tau(\theta)$ such that $\|\tau\| = 1$, we have $\|r\| > \|\tau\|$ and the right-hand term of (42) can be taken as

$$-e\int_{-1}^{0} k\theta^3\, d\theta = -ke\left[\frac{\theta^4}{4}\right]_{-1}^{0} = \frac{ke}{4}.$$

Thus, for any real $k > 0$ such that

$$1 + \frac{1}{2\pi} \leqslant \frac{ke}{4},$$

(42) is fulfilled and the corresponding equation (46) is totally oscillatory.

For the differential-difference system (5), specific versions of Theorems 23 and 24 can be obtained, such as the given in the following corollary.

Corollary 27. *For $j = 1, \dots, m$ let $\mu(A_j) \in\,]-\infty, 0]$ (not all zero), $\tau_1 < \cdots < \tau_m$, $0 < r_1 < \cdots < r_{m_0-1} \leqslant \tau_m < r_{m_0} < \cdots < r_m$ for some $m_0 \in \{1, \dots, m\}$, and*

$$1 + \sum_{j=1}^{m} \|B_j\| \leqslant e \sum_{j=m_0}^{m} (r_j - \tau_m)|\mu(A_j)|. \tag{47}$$

Then (5) *is totally oscillatory if at least one of the following assumptions is verified*:

$$\sum_{j=1}^{m} \|B_j\| \leqslant 1, \tag{48}$$

$$\sum_{j=1}^{m} \|B_j\| < \frac{e\tau_1}{r_m} \log\left(r_m e \left|\mu\left(\sum_{j=1}^{m} A_j\right)\right|\right), \tag{49}$$

$$\left|\mu\left(\sum_{j=1}^{m} A_j\right)\right| \sum_{j=1}^{m} \frac{\|B_j\|}{\tau_j} < e \sum_{j=1}^{m} \frac{|\mu(A_j)|}{r_j} \log\left(e r_j \left|\mu\left(\sum_{j=1}^{m} A_j\right)\right|\right). \tag{50}$$

Proof. As $\mu(A_1) \leqslant 0$, $\mu(A_m) \leqslant 0$, and by the property (II) of the matrix measures

$$\mu\left(\sum_{j=1}^{p} A_j\right) - \mu\left(\sum_{j=1}^{p-1} A_j\right) \leqslant \mu(A_p) \leqslant 0,$$

$$\mu\left(\sum_{j=q}^{m} A_j\right) - \mu\left(\sum_{j=q+1}^{m} A_j\right) \leqslant \mu(A_q) \leqslant 0$$

for $p = 2, \dots, m$, $q = 1, \dots, m-1$, the assumption (A) is fulfilled. On the other hand, for this case conditions (47) and (50) imply, for example, the conditions of Theorem 23 corresponding to (35) and (38), respectively. Conditions (48) and (49) are direct reformulations of (36) and (37), respectively. □

Remark 28. Excluding the conditions (49) and (50), a result of this kind is obtained in [4].

For the norms introduced above, we illustrate the preceding corollary in the following example.

Example 29. Consider the system

$$\frac{d}{dt}\big[x(t) - B_1 x(t-1) - B_2 x(t-2)\big] = A_1 x(t-1) + A_2 x(t-4), \tag{51}$$

where

$$B_1 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -2 & 0 \\ 1 & -2 \end{bmatrix}.$$

As $\|B_1\|_\infty = \|B_2\|_\infty = 2$, $\mu_\infty(A_1) = \mu_\infty(A_2) = -1$ and $\mu_\infty(A_1 + A_2) = -2$, conditions (47) and (50) of Corollary 27 are easily satisfied, and so (51) is totally oscillatory. Notice that since the matrix B_1 has a negative real eigenvalue, it is not possible to apply [2] to this system.

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