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Completeness of a relational calculus for program schemes

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To the memory of David Park, one of the first driving forces behind the μ-calculus

Abstract

The relational calculus $\mathscr{MU}$ was presented in Willem-Paul de Roever's dissertation as a framework for describing and proving properties of programs. $\mathscr{MU}$ is axiomatized by de Roever in stages. The next-to-last stage is the calculus $\mathscr{MU}_2$, namely $\mathscr{MU}$ without the recursive μ-operator. Its axioms include typed versions of Tarski's axioms for the calculus of relations, together with axioms for the projection functions. For $\mathscr{MU}$ there is, in addition, an axiom expressing the least-fixed-point property of terms containing the μ-operator, and Scott's induction rule. Thus $\mathscr{MU}_2$ is a calculus for nonrecursive program schemes. Around 1976 David Park conjectured that de Roever's axiomatization for $\mathscr{MU}_2$ is complete. In this paper, we confirm Park's conjecture.

Keywords: Relational calculus; Completeness; Recursive program schemes; Projection functions

1. Introduction

Relational frameworks can be used to provide semantic characterizations of programs. While operational semantics describes programs according to the way they operate on data, relational semantics captures the relationship that programs establish between input and output data. In his dissertation, Willem-Paul de Roever [11] defined two languages, $\mathscr{PL}$ and $\mathscr{MU}$, including syntax and semantics for both, and shows that there is a meaning-preserving translation from $\mathscr{PL}$ to $\mathscr{MU}$.

$\mathscr{PL}$ is a language for first-order recursive program schemes with call-by-value as parameter mechanism. Programs schemes in $\mathscr{PL}$ are abstractions of programs that may contain various kinds of commands, including conditional statements and calls to

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previously declared recursive procedures. The operational semantics for $\mathscr{PL}$ is defined in terms of a computation model.

$\mathscr{MU}$ is a language for binary relations over Cartesian products. It includes various constants and variables, to be interpreted as binary relations, with operators for forming terms that denote the union, intersection, relative product, and converse of relations. Types are required because a relation is conceived as a subset of a specified Cartesian product of two sets. There is an operator for forming the complement of a relation with respect to its specified product. The constants and variables are provided with source and target types. Types are either generators or products of generators. Types are to be interpreted as sets or Cartesian products of sets. Terms are restricted to those that obey certain natural typing laws. Let REL be the category of *nonempty* sets with binary relations as morphisms. Interpretations of $\mathscr{MU}_2$ are maps that send types to objects in REL and terms to morphisms in REL. De Roever showed that there is a translation from $\mathscr{PL}$ to $\mathscr{MU}$ preserving validity. Properties of program schemes may then be translated into $\mathscr{MU}$ and proved using the axioms and rules of $\mathscr{MU}$.

$\mathscr{MU}$ is axiomatized by de Roever in stages. The next-to-last stage is the calculus $\mathscr{MU}_2$, namely $\mathscr{MU}$ without the recursive μ-operator. Its axioms include typed versions of Tarski's axioms for the calculus of relations [13], together with axioms for the projection functions. For $\mathscr{MU}$ there is, in addition, an axiom expressing the least-fixed-point property of terms containing the μ-operator, and Scott's induction rule [12]. Thus, $\mathscr{MU}_2$ is a calculus for nonrecursive program schemes. David Park conjectured that de Roever's axiomatization for $\mathscr{MU}_2$ is already complete. (see, in [11], the summary of Chapter 7 on p. 5, the footnote on p. 49, and problem 3 on p. 87). In this paper we prove Park's conjecture.

Here is a sketch of the proof. Soundness is evident; completeness is a challenge. Start with a formula that cannot be proved. We wish to construct an interpretation in which the formula fails. First, by standard techniques, using certain equivalence classes of terms, we build an algebraic structure in which this formula does not hold. The axioms and rules of inference of $\mathscr{MU}_2$ are enough to insure that this algebraic structure is a Schröder category [6, 10] with direct products. Next, we wish to construct a relation algebra from the Schröder category. Jónsson's [6] construction does not apply directly because the category has infinitely many objects. There are two ways to proceed [3], [4]. The first way involves embedding the Schröder category into one whose Boolean algebras are all complete and atomic [3]. This method yields a complete atomic quasi-projective relation algebra. Quasi-projective relation algebras are representable by Tarski's Theorem [14, Theorem VII], [9, Corollary 8], [16, Theorem 8.4(iii)]. Once we know the relation algebra is representable, we know that the Schröder category is embeddable in REL, and we may use set-theoretical reasoning to complete the proof. The drawback of this first method is that it requires the addition of types to various results in the Jónsson–Tarski [7] theory of Boolean algebras with operators. This extra work is avoided here. We alter the construction of a relation algebra from a Schröder category, and get a relation algebra that is not necessarily complete and not necessarily atomic [4]. Its products exist only locally, so Tarski's Theorem does not apply. Instead, we use a stronger representation theorem [8, Theorem 9(2)] [9, Theorem 7], and again are able to reason as if we were in REL. The remainder of the proof is a set-theoretical construction of an interpretation involving true Cartesian products of sets. The uniqueness condition used at this late stage is not needed for the proofs of the representation theorems. Both theorems use the same underlying intuitive idea: to code up finite sequences. Unique codes are not needed, so both theorems have hypotheses involving only quasi-projection functions. Such functions behave like projection functions except for the uniqueness condition, which they need not satisfy.

If the axioms for projection functions are deleted from $\mathscr{MU}_2$, the resulting system is, in a certain sense, equivalent to the first-order logic of binary relations restricted to what can be said with three variables and proved with no more than four variables. There are first-order logically valid sentences that contain only three variables, but that nevertheless require arbitrarily large numbers of variables to prove. Tarski found the first example of this. He showed that the associative law for composition of binary relations can be expressed with only three variables, but requires four variables to prove. Lyndon found a law expressible in three variables that requires five variables to prove. Large numbers of variables can be handled if quasi-projection functions are available; the restriction to four variables becomes a "restriction" to four sequences of arbitrary length. An example of this coding procedure occurs at the end of de Roever's thesis, where he derive's Lyndon's law. In another example, quasi-projection functions are used to derive one of de Roever's axioms; see [16, Theorem 4.1(viii)]. Both de Roever's axiom and Lyndon's law need five variables to prove if quasi-projection functions are not available. For a general proof of completeness in the presence of quasi-projections, see [16, Chapter 4].

REL is an example of a distributive allegory [2] (also a division allegory), a Schröder category [6, 10], and a heterogeneous relation algebra [1]. These are, by definition, categories with additional structure. The class of morphisms between any two objects is required to be a distributive lattice in a distributive (or division) allegory, a Boolean algebra in a Schröder category, and a complete atomic Boolean algebra in a heterogeneous relation algebra. The algebras of formulas in $\mathscr{MU}_2$ are probably best viewed as Schröder categories, considering the close match between choice of operations and axioms. We want complementation, and need to consider algebras that are not complete and not atomic. Our construction of a relation algebra from an algebra of formulas extends Jónsson's [6] construction of a relation algebra from a finite Schröder category.

2. Types, basic symbols, terms, and formulas

In this and the next few sections we describe de Roever's system $\mathscr{MU}_2$ (but not the full system $\mathscr{MU}$, which includes the multi-variable least-fixed-point operator). We stay quite close to de Roever's notation and terminology.

Let G be a nonempty set, whose elements we will call "generators". (In [11, p. 35], G is assumed to be "the collection of possibly subscripted Greek letters".) Let Types be the universe of an absolutely free groupoid $\langle \mathsf{Types}, \otimes \rangle$ generated by G. What this means is that if $\langle F, \times \rangle$ is any groupoid (F is a nonempty set, $\times$ is a binary operation on F), then every map from G into F has a unique extension to a homomorphism from $\langle \mathsf{Types}, \otimes \rangle$ into $\langle F, \times \rangle$. The elements of Types are called **types**. Since G contains at least one element, Types is always an infinite set (countable if G is countable). The key elementary properties of types are

1. If $\eta, \xi \in \mathsf{Types}$, then $\eta \otimes \xi \in \mathsf{Types}$ and $\eta \otimes \xi \notin \mathsf{G}$.
2. If $\eta \otimes \xi = \mu \otimes \nu$ then $\eta = \mu$ and $\xi = \nu$.

From these two properties, plus the assumption that $\langle \mathsf{Types}, \otimes \rangle$ is generated by G, it can be shown that the groupoid $\langle \mathsf{Types}, \otimes \rangle$ is absolutely freely generated by G. In the next section, each type will be interpreted as a nonempty set. Each generator can be interpreted arbitrarily, with each product type interpreted as the direct product of the interpretations of its factors.

The class of **basic symbols** is the union of four classes, $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{X}$:

- $\mathscr{A}$ is the class of **individual relation constant** symbols:

$$\mathscr{A} = \{A_i^{\eta,\xi} : \eta, \xi \in \mathsf{Types}, i \in \omega\},$$

- $\mathscr{B}$ is the class of **Boolean relation constant** symbols:

$$\mathscr{B} = \bigcup_{\eta \in \mathsf{Types}, i \in \omega} \{p_i^{\eta,\eta}, {p'}_i^{\eta,\eta}\},$$

- $\mathscr{C}$ is the class of **logical relation constant** symbols:

$$\mathscr{C} = \bigcup_{\eta, \xi \in \mathsf{Types}} \{\Omega^{\eta,\xi}, U^{\eta,\xi}, E^{\eta,\eta}, \pi_1^{\eta\otimes\xi,\eta}, \pi_2^{\eta\otimes\xi,\xi}\},$$

- $\mathscr{X}$ is the class of **relation variable** symbols:

$$\mathscr{X} = \{X_i^{\eta,\xi} : \eta, \xi \in \mathsf{Types}, i \in \omega\}.$$

The class of basic symbols, namely $\mathscr{A} \cup \mathscr{B} \cup \mathscr{C} \cup \mathscr{X}$, generates a set of relation-algebraic terms according to the rule that if σ and τ are terms, then so are $\overline{\sigma}$, $\breve{\sigma}$, $\sigma;\tau$, $\sigma \cap \tau$, and $\sigma \cup \tau$. The terms have properties similar to types. For example, if $\sigma_0 \cup \tau_0 = \sigma_1 \cup \tau_1$ then $\sigma_0 = \sigma_1$ and $\tau_0 = \tau_1$. In fact, the set of terms is the universe of an algebra with two unary operations and three binary operations that is absolutely freely generated by $\mathscr{A} \cup \mathscr{B} \cup \mathscr{C} \cup \mathscr{X}$.

Each basic symbol has an ordered pair of types attached to it, as is indicated by de Roever's superscript notation. If $\langle \eta, \xi \rangle$ is the pair of types of a symbol, then that symbol is to be interpreted as a binary relation from the interpretation of type η to the interpretation of type ξ. This interpretation can be extended to terms that respect the types of the basic symbols they contain. We therefore single out a special class of terms, namely Terms, whose elements we call **typed terms**, by letting Terms be the domain of the type function $\mathsf{t}(\text{-})$. $\mathsf{t}(\text{-})$ maps each term to an ordered pair of types in

$\mathsf{Types} \times \mathsf{Types}$. $\mathsf{t}(\text{-})$ is the smallest function (the intersection of all functions) satisfying the following conditions for all $i \in \omega$, all $\sigma, \sigma_0, \sigma_1 \in \mathsf{Terms}$, and all $\eta, \xi, \mu \in \mathsf{Types}$:

1. $\mathsf{t}(A_i^{\eta,\xi}) = \mathsf{t}(X_i^{\eta,\xi}) = \langle \eta, \xi \rangle$,
2. $\mathsf{t}(\Omega^{\eta,\xi}) = \mathsf{t}(U^{\eta,\xi}) = \langle \eta, \xi \rangle$,
3. $\mathsf{t}(E^{\eta,\eta}) = \langle \eta, \eta \rangle$,
4. $\mathsf{t}(p_i^{\eta,\eta}) = \mathsf{t}({p'}_i^{\eta,\eta}) = \langle \eta, \eta \rangle$,
5. $\mathsf{t}(\pi_1^{\eta\otimes\xi,\eta}) = \langle \eta \otimes \xi, \eta \rangle$,
6. $\mathsf{t}(\pi_2^{\eta\otimes\xi,\xi}) = \langle \eta \otimes \xi, \xi \rangle$,
7. if $\mathsf{t}(\sigma) = \langle \eta, \xi \rangle$ then $\mathsf{t}(\breve{\sigma}) = \langle \xi, \eta \rangle$,
8. if $\mathsf{t}(\sigma) = \langle \eta, \xi \rangle$ then $\mathsf{t}(\overline{\sigma}) = \langle \eta, \xi \rangle$,
9. if $\mathsf{t}(\sigma_0) = \langle \eta, \xi \rangle = \mathsf{t}(\sigma_1)$ then $\mathsf{t}(\sigma_0 \cup \sigma_1) = \langle \eta, \xi \rangle = \mathsf{t}(\sigma_0 \cap \sigma_1)$,
10. if $\mathsf{t}(\sigma_0) = \langle \eta, \xi \rangle$ and $\mathsf{t}(\sigma_1) = \langle \xi, \mu \rangle$ then $\mathsf{t}(\sigma_0;\sigma_1) = \langle \eta, \mu \rangle$.

Many terms are not typed. For example, $A_0^{\eta,\xi} \cap A_0^{\xi,\eta} \notin \mathsf{Terms}$ whenever $\eta \neq \xi$. A function satisfying the conditions above may be undefined on an untyped term. Define functions $\mathsf{d}(\text{-})$ and $\mathsf{r}(\text{-})$ mapping Terms to Types by the condition that

$$\mathsf{t}(\sigma) = \langle \mathsf{d}(\sigma), \mathsf{r}(\sigma) \rangle$$

for all $\sigma \in \mathsf{Terms}$. We call $\mathsf{d}(\text{-})$ the **source** (or domain) type of σ and $\mathsf{r}(\text{-})$ the **target** (or range) type of σ. The three type functions $\mathsf{t}(\text{-})$, $\mathsf{d}(\text{-})$ and $\mathsf{r}(\text{-})$ are implicit in [11].

An **atomic formula** is an ordered pair $\langle \sigma, \tau \rangle$ of terms $\sigma, \tau \in \mathsf{Terms}$ such that $\mathsf{t}(\sigma) = \mathsf{t}(\tau)$. A **formula** is a set of atomic formulas. An **assertion** is an ordered pair of formulas, denoted "$\Phi \vdash \Psi$". (While preferring to use "$\vdash$" for the provability relation, we follow de Roever, who uses no special symbol to denote provability.) The interpretation of an assertion $\Phi \vdash \Psi$ is that the conjunction of the atomic formulas in Φ implies the conjunction of the atomic formulas in Ψ.

The atomic formula $\langle \sigma, \tau \rangle$ is written "$\sigma \subseteq \tau$", in order to reflect its semantic interpretation, that the relation assigned to σ is a subrelation of the relation assigned to τ. Here de Roever could have interpreted each atomic formula as an equality of relations rather than an inclusion, which would bring the system closer to equational logic. De Roever uses some standard notation shortcuts. "$\emptyset \vdash \Psi$" is shortened to "$\vdash \Psi$", and singleton formulas are written without braces, e.g., "$\sigma \subseteq \tau \vdash \tau \subseteq \rho$" replaces "$\{\sigma \subseteq \tau\} \vdash \{\tau \subseteq \rho\}$". "$\sigma = \tau$" denotes the formula $\{\sigma \subseteq \tau, \tau \subseteq \sigma\}$.

3. Interpretations of $\mathscr{MU}_2$

In the definition of interpretation we use some standard operations on sets and relations, namely,

the **direct product** of two sets:

$$X \times Y = \{\langle x, y \rangle : x \in X \text{ and } y \in Y\},$$

the **converse** of a binary relation:

$$R^{-1} = \{\langle y,x\rangle : \langle x,y\rangle \in R\},$$

the **difference** of two relations:

$$R \sim S = \{\langle x,y\rangle : \langle x,y\rangle \in R \text{ and } \langle x,y\rangle \notin S\},$$

the **relative product** of two relations:

$$R|S = \{\langle x,z\rangle : \exists_y(\langle x,y\rangle \in R \text{ and } \langle y,z\rangle \in S)\},$$

the **union** of two relations:

$$R \cup S = \{\langle x,y\rangle : \langle x,y\rangle \in R \text{ or } \langle x,y\rangle \in S\},$$

and the **intersection** of two relations:

$$R \cap S = \{\langle x,y\rangle : \langle x,y\rangle \in R \text{ and } \langle x,y\rangle \in S\}.$$

We will assume here that Types and Terms are disjoint, so that we may let an interpretation be a single function defined on their union $\mathsf{Types} \cup \mathsf{Terms}$. An **interpretation** is a function $\mathsf{m}(\text{-})$ that assigns each type $\eta \in \mathsf{Types}$ to a nonempty set $\mathsf{m}(\eta)$, assigns each term $\sigma \in \mathsf{Terms}$ to a binary relation $\mathsf{m}(\sigma)$, and satisfies the following conditions for all types $\eta, \xi \in \mathsf{Types}$ and all terms $\sigma, \tau \in \mathsf{Terms}$:

$$\begin{aligned}
&\mathsf{m}(\eta \otimes \xi) = \mathsf{m}(\eta) \times \mathsf{m}(\xi),\\
&\mathsf{m}(U^{\eta,\xi}) = \mathsf{m}(\eta) \times \mathsf{m}(\xi),\\
&\mathsf{m}(\Omega^{\eta,\xi}) = \emptyset,\\
&\mathsf{m}(E^{\eta,\eta}) = \{\langle x,x\rangle : x \in \mathsf{m}(\eta)\},\\
&\mathsf{m}(p_i^{\eta,\eta}) \cup \mathsf{m}({p'}_i^{\eta,\eta}) \subseteq \{\langle x,x\rangle : x \in \mathsf{m}(\eta)\},\\
&\mathsf{m}(p_i^{\eta,\eta}) \cap \mathsf{m}({p'}_i^{\eta,\eta}) = \emptyset,\\
&\mathsf{m}(\pi_1^{\eta\otimes\xi,\eta}) = \{\langle\langle x,y\rangle,x\rangle : x \in \mathsf{m}(\eta) \text{ and } y \in \mathsf{m}(\xi)\},\\
&\mathsf{m}(\pi_2^{\eta\otimes\xi,\xi}) = \{\langle\langle x,y\rangle,y\rangle : x \in \mathsf{m}(\eta) \text{ and } y \in \mathsf{m}(\xi)\},\\
&\mathsf{m}(\breve{\sigma}) = \mathsf{m}(\sigma)^{-1},\\
&\mathsf{m}(\overline{\sigma}) = (\mathsf{m}(\mathsf{d}(\sigma)) \times \mathsf{m}(\mathsf{r}(\sigma))) \sim \mathsf{m}(\sigma),\\
&\mathsf{m}(\sigma;\tau) = \mathsf{m}(\sigma)|\mathsf{m}(\tau),\\
&\mathsf{m}(\sigma \cup \tau) = \mathsf{m}(\sigma) \cup \mathsf{m}(\tau),\\
&\mathsf{m}(\sigma \cap \tau) = \mathsf{m}(\sigma) \cap \mathsf{m}(\tau).
\end{aligned}$$

It is perhaps worth remarking that de Roever only requires $p_i^{\eta,\eta}$ and ${p'}_i^{\eta,\eta}$ to be interpreted as mutually exclusive identity relations that are not necessarily exhaustive. The presence of the Boolean relation constant symbols and their interpretation is convenient for de Roever's later purposes, but is irrelevant here. The same is true for the individual relation constant symbols.

For each type $\eta \in \mathsf{Types}$ the nonempty set $\mathsf{m}(\eta)$ is called the **domain of type** η. A simple proof by induction shows that a term's interpretation is a relation that relates the domain of its source type to the domain of its target type, that is,

$$\mathsf{m}(\sigma) \subseteq \mathsf{m}(\mathsf{d}(\sigma)) \times \mathsf{m}(\mathsf{r}(\sigma)),$$

for all $\sigma \in \mathsf{Terms}$.

An interpretation $\mathsf{m}(\text{-})$ **satisfies** an atomic formula $\sigma \subseteq \tau$ if $\mathsf{m}(\sigma) \subseteq \mathsf{m}(\tau)$, and **satisfies** a formula Φ if $\mathsf{m}(\text{-})$ satisfies every atomic formula in Φ. Therefore $\mathsf{m}(\text{-})$ satisfies $\sigma = \tau$ if and only if $\mathsf{m}(\sigma) = \mathsf{m}(\tau)$. An assertion $\Phi \vdash \Psi$ is **valid** if every interpretation that satisfies Φ also satisfies Ψ.

4. Axioms of $\mathscr{MU}_2$

The axiomatization proposed by de Roever consists of the following five groups of rules and axioms: (1) "typed versions of axioms and rules for Boolean algebras (including axioms for Ω and U)", (2) axioms T_1–T_5, called "typed versions of Tarski's axioms for binary relations":

$$T_1 \vdash X_0^{\eta,\theta};(X_1^{\theta,\xi};X_2^{\xi,\nu}) = (X_0^{\eta,\theta};X_1^{\theta,\xi});X_2^{\xi,\nu},$$

$$T_2 \vdash ((X_0^{\eta,\xi})\breve{})\breve{} = X_0^{\eta,\xi},$$

$$T_3 \vdash (X_0^{\eta,\theta};(X_1^{\theta,\xi})\breve{} = (X_1^{\theta,\xi})\breve{};(X_0^{\eta,\theta})\breve{},$$

$$T_4 \vdash X_0^{\eta,\xi};E^{\xi,\xi} = X_0^{\eta,\xi},$$

$$T_5\ (X_0^{\eta,\theta};X_1^{\theta,\xi}) \cap X_2^{\eta,\xi} = \Omega^{\eta,\xi} \vdash (X_1^{\theta,\xi};(X_2^{\eta,\xi})\breve{}) \cap (X_0^{\eta,\theta})\breve{} = \Omega^{\theta,\eta},$$

(3) axiom U, for linking together the Boolean units:

$$U \vdash U^{\eta,\xi} \subseteq U^{\eta,\theta};U^{\theta,\xi},$$

(4) axioms P_1 and P_2, for the Boolean relation constants:

$$P_1 \vdash p_i^{\eta,\eta} \subseteq E^{\eta,\eta}, \quad \vdash p'^{\eta,\eta}_i \subseteq E^{\eta,\eta},$$

$$P_2 \vdash p_i^{\eta,\eta} \cap p'^{\eta,\eta}_i = \Omega^{\eta,\eta},$$

and (5) axioms C_1 and C_2, for the projection relation constants (restricted according to the suggestion on p. 52 of [11]):

$$C_1 \vdash [\pi_1^{\eta\otimes\xi,\eta};(\pi_1^{\eta\otimes\xi,\eta})\breve{}] \cap [\pi_2^{\eta\otimes\xi,\xi};(\pi_2^{\eta\otimes\xi,\xi})\breve{}] = E^{\eta\otimes\xi,\eta\otimes\xi},$$

$$C_2 \vdash [X_1^{\nu,\eta};X_2^{\eta,\mu}] \cap [X_3^{\nu,\xi};X_4^{\xi,\mu}]$$

$$= ([X_1^{\nu,\eta};(\pi_1^{\eta\otimes\xi,\eta})\breve{}] \cap [X_3^{\nu,\xi};(\pi_2^{\eta\otimes\xi,\xi})\breve{}]);([\pi_1^{\eta\otimes\xi,\eta};X_2^{\eta,\mu}] \cap [\pi_2^{\eta\otimes\xi,\xi};X_4^{\xi,\mu}]).$$

Axiom C_2 can be replaced by axiom C_2'.

$$C_2' \quad \vdash [(\pi_1^{\eta \otimes \xi, \eta})\breve{}\,;\pi_1^{\eta \otimes \xi, \eta}] \subseteq E^{\eta,\eta}, \quad \vdash [(\pi_2^{\eta \otimes \xi, \xi})\breve{}\,;\pi_2^{\eta \otimes \xi, \xi}] \subseteq E^{\xi,\xi},$$
$$\vdash (\pi_1^{\eta \otimes \xi, \eta})\breve{}\,;\pi_2^{\eta \otimes \xi, \xi} = U^{\eta,\xi}.$$

De Roever [11, Lemma 4.6] derives C_2' from C_2 (using the remaining axioms and rules). To get C_2 from C_2', see [16, Theorem 4.1(viii)].

5. Rules of inference for $\mathscr{MU}_2$

Given a finite index set J, a relation variable $X_{i_j}^{\eta_j,\xi_j} \in \mathscr{X}$ for every $j \in J$, and a function $\tau: J \to \mathsf{Terms}$ such that $\mathsf{t}(\tau_j) = \langle \eta_j, \xi_j \rangle$ for every $j \in J$, "$\sigma[\tau_j/X_{i_j}^{\eta_j,\xi_j}]_{j\in J}$" denotes the result of simultaneously replacing each occurrence of variable $X_{i_j}^{\eta_j,\xi_j}$ in σ with the term τ_j. This notation extends to formulas, so that

$$\Phi[\tau_j/X_{i_j}^{\eta_j,\xi_j}]_{j\in J} = \{\sigma[\tau_j/X_{i_j}^{\eta_j,\xi_j}]_{j\in J} \subseteq \rho[\tau_j/X_{i_j}^{\eta_j,\xi_j}]_{j\in J}\colon\ (\sigma \subseteq \rho) \in \Phi\}.$$

The only rule of inference explicitly mentioned by de Roever [11] is "the substitution rule":

$$\text{if } \Phi \vdash \Psi \quad \text{then } \Phi[\tau_j/X_{i_j}^{\eta_j,\xi_j}]_{j\in J} \vdash \Psi[\tau_j/X_{i_j}^{\eta_j,\xi_j}]_{j\in J}.$$

Item (1) in the axiomatization of $\mathscr{MU}_2$ is "typed versions of axioms and rules for Boolean algebras (including axioms for Ω and U)". De Roever does not make any more specific choice here, leaving it to the reader to supply a sufficient set. There are the specifically Boolean axioms, such as commutativity

$$\vdash X_0^{\eta,\xi} \cap X_1^{\eta,\xi} = X_1^{\eta,\xi} \cap X_0^{\eta,\xi}$$

and double negation

$$\vdash \overline{\overline{X_0^{\eta,\xi}}} = X_0^{\eta,\xi}$$

and structural rules for logical deduction, such as "tautology"

$$\Phi \vdash \Phi,$$

"cut"

$$\frac{\Phi \vdash \Psi \qquad \Psi \vdash \Theta}{\Phi \vdash \Theta}$$

and "weakening"

$$\frac{\Phi \vdash \Psi}{\Phi \cup \Theta \vdash \Psi}, \qquad \frac{\Phi \vdash \Psi}{\Phi \vdash \Psi \cap \Theta}.$$

The symmetry and transitivity axioms,

$$X_0^{\eta,\xi} = X_1^{\eta,\xi} \vdash X_1^{\eta,\xi} = X_0^{\eta,\xi},$$
$$\{X_0^{\eta,\xi} = X_1^{\eta,\xi}, X_1^{\eta,\xi} = X_2^{\eta,\xi}\} \vdash X_0^{\eta,\xi} = X_2^{\eta,\xi},$$

ensure that the relation $\equiv$, which relates σ to τ just in case $\vdash \sigma = \tau$ is provable, is an equivalence relation. Axioms that show $\equiv$ also a congruence relation in the algebra of typed terms are

$$X_0^{\eta,\xi} = X_1^{\eta,\xi} \vdash \overline{X_0^{\eta,\xi}} = \overline{X_1^{\eta,\xi}},$$
$$X_0^{\eta,\xi} = X_1^{\eta,\xi} \vdash (X_0^{\eta,\xi})\breve{} = (X_1^{\eta,\xi})\breve{},$$
$$X_0^{\eta,\xi} = X_1^{\eta,\xi} \vdash X_0^{\eta,\xi} \cup X_2^{\eta,\xi} = X_1^{\eta,\xi} \cup X_2^{\eta,\xi},$$
$$X_0^{\eta,\xi} = X_1^{\eta,\xi} \vdash X_0^{\eta,\xi} \cap X_2^{\eta,\xi} = X_1^{\eta,\xi} \cap X_2^{\eta,\xi},$$
$$X_0^{\eta,\xi} = X_1^{\eta,\xi} \vdash X_0^{\eta,\xi}; X_2^{\xi,\mu} = X_1^{\eta,\xi}; X_2^{\xi,\mu}.$$

We will henceforth assume that these and other axioms and rules are part of $\mathscr{MU}_2$, the exact choice being governed only by the need to remain sound and prove Lemma 1 below.

6. Soundness and completeness of $\mathscr{MU}_2$

Theorem 1 (Soundness). *Every $\mathscr{MU}_2$-provable assertion $\Phi \vdash \Psi$ is valid.*

Proof. It suffices to check that the $\mathscr{MU}_2$ axioms are valid and that the rules of inference preserve validity. $\square$

Theorem 2 (Completeness). *Every valid assertion is provable.*

Proof. It suffices to show that if an assertion in $\mathscr{MU}_2$ is not provable, then it fails in an interpretation $\mathsf{m}(\text{-})$ of $\mathscr{MU}_2$ and is therefore not valid. Let Φ and Ψ be formulas and suppose the assertion $\Phi \vdash \Psi$ is not provable in $\mathscr{MU}_2$. Define a binary relation $\equiv_\Phi$ on Terms:

$$\sigma \equiv_\Phi \tau \Leftrightarrow \Phi \vdash \sigma = \tau \text{ is } \mathscr{MU}_2\text{-provable.}$$

For every $\tau \in \mathsf{Terms}$ and all $\eta, \xi \in \mathsf{Types}$, let

$$|\sigma| = \{\tau\colon \tau \equiv_\Phi \sigma\},$$
$$B_{\eta\xi} = \{|\sigma|\colon \mathsf{t}(\sigma) = \langle \eta, \xi \rangle\}.$$

The following lemma says that we have constructed a Schröder category from Φ with Types as objects and $B_{\eta\xi}$ as the set of morphisms from object η to object ξ. We omit the easy but tedious details of the proof. The earlier parts of the lemma are useful in establishing the later parts.

Lemma 1.
1. $\equiv_\Phi$ *is an equivalence relation on* Terms.
2. *Equivalent terms have the same type*: *if* $\sigma \equiv_\Phi \tau$ *then* $\mathsf{t}(\sigma)=\mathsf{t}(\tau)$, $\mathsf{d}(\sigma)=\mathsf{d}(\tau)$, *and* $\mathsf{r}(\sigma)=\mathsf{r}(\tau)$.
3. $\equiv_\Phi$ *is a congruence relation with respect to the operations on terms*: *if* $\sigma \equiv_\Phi \tau$, *then*, *for every* $\rho \in \mathsf{Types}$, *we have*
   (a) $\overline{\sigma} \equiv_\Phi \overline{\tau}$,
   (b) $\breve{\sigma} \equiv_\Phi \breve{\tau}$,
   (c) *if* $\mathsf{t}(\rho)=\mathsf{t}(\sigma)$ *then* $\sigma\cup\rho \equiv_\Phi \tau\cup\rho$, $\rho\cup\sigma \equiv_\Phi \rho\cup\tau$, $\sigma\cap\rho \equiv_\Phi \tau\cap\rho$, *and* $\rho\cap\sigma \equiv_\Phi \rho\cap\tau$,
   (d) *if* $\mathsf{d}(\rho)=\mathsf{r}(\sigma)$ *then* $\sigma;\rho \equiv_\Phi \tau;\rho$,
   (e) *if* $\mathsf{r}(\rho)=\mathsf{d}(\sigma)$ *then* $\rho;\sigma \equiv_\Phi \rho;\tau$.
4. *If* $|\sigma|=|\tau|$ *then* $|\overline{\sigma}|=|\overline{\tau}|$ *and* $|\breve{\sigma}|=|\breve{\tau}|$.
5. *If* $\mathsf{t}(\sigma_0)=\mathsf{t}(\sigma_1)$, $|\sigma_0|=|\tau_0|$, *and* $|\sigma_1|=|\tau_1|$, *then* $|\sigma_0\cup\sigma_1|=|\tau_0\cup\tau_1|$ *and* $|\sigma_0\cap\sigma_1| = |\tau_0\cap\tau_1|$.
6. *If* $\mathsf{r}(\sigma_0)=\mathsf{d}(\sigma_1)$, $|\sigma_0|=|\tau_0|$, *and* $|\sigma_1|=|\tau_1|$, *then* $|\sigma_0;\sigma_1|=|\tau_0;\tau_1|$.
7. *For all* $\eta,\xi \in \mathsf{Types}$, $B_{\eta\xi}$ *is the universe of a Boolean algebra*

   $$\mathfrak{B}_{\eta\xi} = \langle B_{\eta\xi}, +_{\eta\xi}, \cdot_{\eta\xi}, {}^{-\eta\xi}, 0_{\eta\xi}, 1_{\eta\xi}\rangle,$$

   where

   $$0_{\eta\xi} = |\Omega^{\eta,\xi}|,$$
   $$1_{\eta\xi} = |U^{\eta,\xi}|,$$

   and, *for all* $\sigma,\tau \in \mathsf{Terms}$, *if* $\mathsf{t}(\sigma)=\mathsf{t}(\tau)=\langle\eta,\xi\rangle$ *then*

   $$|\sigma| +_{\eta\xi} |\tau| = |\sigma\cup\tau|,$$
   $$|\sigma|\cdot_{\eta\xi}|\tau| = |\sigma\cap\tau|,$$
   $$\overline{|\sigma|}^{\eta\xi} = |\overline{\sigma}|.$$

8. *There is a* $(\mathsf{Types}\times\mathsf{Types}\times\mathsf{Types})$*-indexed system of functions* $;^{\xi}_{\eta\mu} : B_{\eta\xi}\times B_{\xi\mu} \to B_{\eta\mu}$ *for* $\eta,\xi,\mu \in \mathsf{Types}$, *such that* $|\sigma|;^{\xi}_{\eta\mu}|\tau| = |\sigma;\tau|$ *whenever* $\sigma,\tau \in \mathsf{Terms}$, $\mathsf{t}(\sigma)=\langle\eta,\xi\rangle$, *and* $\mathsf{t}(\tau)=\langle\xi,\mu\rangle$.
9. *The* **unit law**

   $$1_{\eta\xi};^{\xi}_{\eta\mu}1_{\xi\mu} = 1_{\eta\mu},$$

   the **zero laws**

   $$0_{\eta\xi} = 0_{\eta\mu};^{\mu}_{\eta\xi}0_{\mu\xi} = 0_{\eta\mu};^{\mu}_{\eta\xi}1_{\mu\xi} = 1_{\eta\mu};^{\mu}_{\eta\xi}0_{\mu\xi},$$

and the **associative law**

$$(x;^{\xi}_{\eta\mu}y);^{\mu}_{\eta\nu}z = x;^{\xi}_{\eta\nu}(y;^{\mu}_{\xi\nu}z),$$

hold whenever $\eta,\xi,\mu,\nu \in \mathsf{Types}$, $x \in B_{\eta\xi}$, $y \in B_{\xi\mu}$, *and* $z \in B_{\mu\nu}$.

10. *There is a* $(\mathsf{Types} \times \mathsf{Types})$*-indexed system of functions* $\breve{}^{\eta\xi} : B_{\eta\xi} \to B_{\xi\eta}$ *for* $\eta,\xi \in \mathsf{Types}$, *such that* $|\breve{\sigma}|^{\eta\xi} = |\breve{\sigma}|$ *whenever* $\sigma \in \mathsf{Terms}$ *and* $\mathsf{t}(\sigma) = \langle \eta,\xi \rangle$.
11. *The* **cycle law** *holds*: *if* $\eta,\xi,\mu \in \mathsf{Types}$, $x \in B_{\eta\xi}$, $y \in B_{\xi\mu}$, *and* $z \in B_{\eta\mu}$, *then the following statements are equivalent*:

$$(x;^{\xi}_{\eta\mu}y)\cdot_{\eta\mu}z = 0_{\eta\mu}, \qquad (\breve{y};^{\xi}_{\mu\eta}\breve{x})\cdot_{\mu\eta}\breve{z} = 0_{\mu\eta},$$
$$(\breve{x};^{\eta}_{\xi\mu}z)\cdot_{\xi\mu}y = 0_{\xi\mu}, \qquad (\breve{z};^{\eta}_{\mu\xi}x)\cdot_{\mu\xi}\breve{y} = 0_{\mu\xi},$$
$$(z;^{\mu}_{\eta\xi}\breve{y})\cdot_{\eta\xi}x = 0_{\eta\xi}, \qquad (y;^{\mu}_{\xi\eta}\breve{z})\cdot_{\xi\eta}\breve{x} = 0_{\xi\eta}.$$

12. *The* **identity law** *holds*: *for all* $\eta,\xi \in \mathsf{Types}$ *and all* $x \in B_{\eta\xi}$,

$$|E^{\eta,\eta}|;^{\eta}_{\eta\xi}x = x = x;^{\xi}_{\eta\xi}|E^{\xi,\xi}|.$$

We have a Boolean algebra $\mathfrak{B}_{\eta\xi}$ for every $\langle \eta,\xi \rangle \in \mathsf{Types} \times \mathsf{Types}$. Let $\mathfrak{B}$ be the direct product of these algebras:

$$\mathfrak{B} = \prod_{\eta,\xi\in\mathsf{Types}} \mathfrak{B}_{\eta\xi}.$$

Then $\mathfrak{B}$ is a Boolean algebra. Given $s,s' \in B$, we wish to define relative multiplication by

$$(s;s')_{\eta\xi} = \sum_{\mu\in\mathsf{Types}}^{(\mathfrak{B}_{\eta\xi})} s_{\eta\mu};^{\mu}_{\eta\xi}s'_{\mu\xi} \tag{1}$$

for all $\eta,\xi \in \mathsf{Types}$, but one of the joins in (1) may not exist; the relevant Boolean algebra may not be complete. To get around this, we restrict our attention to S, the set of functions that are "eventually-0-or-1 off the diagonal":

$$\mathsf{S} = \left\{ s\colon s \in \prod_{\eta,\xi\in\mathsf{Types}} B_{\eta\xi},\ \{\langle \eta,\xi \rangle\colon \eta \neq \xi \text{ and } 0_{\eta\xi} \neq s_{\eta\xi} \neq 1_{\eta\xi}\} \text{ is finite} \right\}.$$

Clearly, S is closed under the operations of $\mathfrak{B}$. Let $\mathfrak{S}$ be the subalgebra of $\mathfrak{B}$ whose universe is S. For every $\tau \in \mathsf{Terms}$, let $\widetilde{\tau}$ be the sequence in S which is 0 everywhere except at $\langle \mathsf{d}(\tau),\mathsf{r}(\tau) \rangle$, where its value is $|\tau|$, that is, for all $\mu,\nu \in \mathsf{Types}$

$$(\widetilde{\tau})_{\mu\nu} = \begin{cases} |\tau| & \text{if } \langle \mu,\nu \rangle = \langle \mathsf{d}(\tau),\mathsf{r}(\tau) \rangle, \\ 0_{\mu\nu} & \text{if } \langle \mu,\nu \rangle \neq \langle \mathsf{d}(\tau),\mathsf{r}(\tau) \rangle. \end{cases}$$

Suppose $s,s' \in \mathsf{S}$ and $\eta,\xi \in \mathsf{Types}$. For almost all $\mu \in \mathsf{Types}$ (i.e., with only finitely many exceptions), we have $s_{\eta\mu} \in \{0_{\eta\mu},1_{\eta\mu}\}$ and $s'_{\mu\xi} \in \{0_{\mu\xi},1_{\mu\xi}\}$, hence $s_{\eta\mu};^{\mu}_{\eta\xi}s'_{\mu\xi} \in \{0_{\eta\xi},1_{\eta\xi}\}$ by the unit and zero laws in Lemma 1. It follows that $\{s_{\eta\mu};^{\mu}_{\eta\xi}s'_{\mu\xi} : \mu \in \mathsf{Types}\}$ is finite for all $\eta,\xi \in \mathsf{Types}$. We can therefore define $s;s'$ by (1). This defines a binary operation ; on S. Define a unary operation $\breve{}$ on S by $(\breve{s})_{\eta\xi} = (s_{\xi\eta})^{\breve{}\,\xi\eta}$ for all $\eta, \xi \in \mathsf{Types}$. Define an identity element $1'$ by

$$(1')_{\eta\xi} = \begin{cases} |E^{\eta,\eta}| & \text{if } \eta = \xi, \\ |0_{\eta\xi}| & \text{if } \eta \neq \xi \end{cases}$$

for all $\eta, \xi \in \mathsf{Types}$. Obviously $1' \in \mathsf{S}$, and it is easy to show that S is closed under the operations ; and $\breve{}$. This yields an algebra $\mathfrak{A} = \langle \mathfrak{S}, ;, \breve{}, 1' \rangle$. It follows from Lemma 1 that $\mathfrak{A}$ is a relation algebra.

Next we show that $\mathfrak{A}$ is a representable relation algebra. A relation algebra is tabular if every nonzero element has nonempty intersection with an element of the form $\breve{p};q$, where p and q are functional elements. We show that $\mathfrak{A}$ is tabular and use the fact that every tabular relation algebra is representable [9, Theorem 7]. (The term "tabular" is borrowed from a similar theorem in [2].) Suppose $0 \neq s \in \mathsf{S}$. Since s is not everywhere zero, there are types $\eta, \xi \in \mathsf{Types}$ such that

$$0_{\eta\xi} \neq s_{\eta\xi} \in B_{\eta\xi}.$$

Let

$$p = \widetilde{\pi_1^{\eta\otimes\xi,\eta}}, \qquad q = \widetilde{\pi_2^{\eta\otimes\xi,\xi}},$$

i.e., p and q are the sequences in S that are zero everywhere except for

$$p_{\eta\otimes\xi,\eta} = |\pi_1^{\eta\otimes\xi,\eta}|, \qquad q_{\eta\otimes\xi,\xi} = |\pi_2^{\eta\otimes\xi,\xi}|.$$

With the essential help of axiom C_2, it can be shown that $\breve{p};p \leqslant 1'$, $\breve{q};q \leqslant 1'$, and $1_{\eta\xi} = \breve{p};q$ [11, Lemma 4.6]. Hence p and q are functional elements such that $0 \neq s \cdot \breve{p};q$. This completes the proof that $\mathfrak{A}$ is tabular. Consequently, $\mathfrak{A}$ is representable. Since the assertion $\Phi \vdash \Psi$ is not provable, there must be some atomic formula $\psi \in \Psi$ such that $\Phi \vdash \psi$ is not provable. Suppose $\psi = \langle \sigma, \tau \rangle$. Then $\sigma \not\equiv_\Phi \tau$, so $|\sigma| \neq |\tau|$, hence $\widetilde{\sigma} \neq \widetilde{\tau}$. This inequality in $\mathfrak{A}$ will be inherited by one of the simple homomorphic images of $\mathfrak{A}$. Every relation algebra is a subdirect product of its simple homomorphic images [5, 7], so, if every homomorphism R from $\mathfrak{A}$ onto a simple relation algebra $\mathfrak{A}'$ agrees on two elements of $\mathfrak{A}$, then the elements must coincide. In particular, there must be a simple relation algebra $\mathfrak{A}'$ and a homomorphism $R : \mathfrak{A} \to \mathfrak{A}'$ from $\mathfrak{A}$ onto $\mathfrak{A}'$ such that $R(\widetilde{\sigma}) \neq R(\widetilde{\tau})$. The class of representable relation algebras is closed under the formation of homomorphic images [5, 15]. It follows from the representability of $\mathfrak{A}$ that $\mathfrak{A}'$ is also representable. This means that $\mathfrak{A}'$ is isomorphic to a proper relation algebra. We will simply assume that $\mathfrak{A}'$ actually *is* a proper relation algebra, so that its elements are actually binary relations, and its operations are the ordinary Boolean and relative operations on binary relations. Furthermore, since $\mathfrak{A}'$ is simple, we may also assume that its Boolean unit element 1 is a Cartesian square [7]. This means that there is some $U \neq \emptyset$ such that $1 = U \times U$ and $\mathfrak{A}'$ is a subalgebra of $\mathfrak{Re}(U)$. $\mathfrak{Re}(U)$ is the relation algebra of all binary relations on U. It may happen that U contains no ordered pairs. If we were to set $\mathsf{m}(\sigma) = R(\widetilde{\sigma})$ for every term, then every term would be interpreted as a binary relation, but, while $R(\widetilde{\pi_1^{\eta\otimes\xi,\eta}})$ and $R(\widetilde{\pi_2^{\eta\otimes\xi,\xi}})$ are functions, they may not be projection functions defined on a Cartesian product. To fix this we will replace the base set U with another base set V that is closed under pairing, i.e., $(\langle x,y\rangle \in V)$ for all $x,y\in V$. V and U share certain elements, namely those correlated with relations whose types are in the generating set of types G. Let

$$\begin{aligned} W &= \bigcup_{\gamma\in\mathsf{G}} \text{Field } (R(\widetilde{E^{\gamma,\gamma}})) \\ &= \{x\colon \langle x,x\rangle \in R(\widetilde{E^{\gamma,\gamma}}) \text{ and } \gamma\in\mathsf{G})\}. \end{aligned}$$

We will assume U has been chosen in such a way that W contains no ordered pairs. Let V be the closure of W under the formation of ordered pairs. Note that $W\subseteq V\cap U$, and it is not excluded that $V=U$. We can now define an interpretation $\mathsf{m}(\text{-})$ on the types by the recursive conditions:

$$\begin{aligned} &\mathsf{m}(\gamma) = \text{ Field } (R(\widetilde{E^{\gamma,\gamma}})) \quad \text{for every } \gamma\in\mathsf{G}, \\ &\mathsf{m}(\eta\otimes\xi) = \mathsf{m}(\eta)\times\mathsf{m}(\xi) \quad \text{for all } \eta,\xi\in\mathsf{Types}. \end{aligned}$$

Note that $\mathsf{m}(\eta)\subseteq W\subseteq V$ whenever $\eta\in\mathsf{G}$, and, by induction using the second condition, $\mathsf{m}(\eta)\subseteq V$ for every $\eta\in\mathsf{Types}$. Next we extend the interpretation from types to terms. Let P and Q be the true projection functions restricted to V, namely

$$\begin{aligned} P &= \{\langle\langle x,y\rangle,x\rangle\colon\ x,y\in V\}, \\ Q &= \{\langle\langle x,y\rangle,y\rangle\colon\ x,y\in V\}. \end{aligned}$$

Define F_η recursively: for all $\gamma\in\mathsf{G}$ and $\eta,\xi\in\mathsf{Types}$ let

$$\begin{aligned} &F_\gamma = R(\widetilde{E^{\gamma,\gamma}}), \\ &F_{\eta\otimes\xi} = (R(\widetilde{\pi_1^{\eta\otimes\xi,\eta}})|F_\eta|P^{-1})\cap(R(\widetilde{\pi_2^{\eta\otimes\xi,\xi}})|F_\xi|Q^{-1}). \end{aligned}$$

Using axiom C_2 and properties of ordered pairs, one can show by induction that these relations are all functions. The assumption that U contains no ordered pairs means that the ranges of these functions are disjoint. It follows from axiom C_1 that these functions are also all one-to-one. It is built into the structure of $\mathfrak{A}$ that the domains of these functions are disjoint. It follows that their union is a function. Let

$$F = \bigcup_{\eta\in\mathsf{Types}} F_\eta.$$

Then F is a bijection mapping U onto V. We transfer relations from U onto V via F, and thus define $\mathsf{m}(\text{-})$. If $\sigma\in\mathsf{Terms}$, $\eta,\xi\in\mathsf{Types}$, and $\mathsf{t}(\sigma)=\langle\eta,\xi\rangle$, then let

$$\mathsf{m}(\sigma) = (F_\eta)^{-1}|R(\tilde{\sigma})|F_\xi.$$

What remains is the routine verification that $\mathsf{m}(\text{-})$ is an interpretation of $\mathscr{MU}_2$ that satisfies Φ but not ψ. This shows that $\Phi\vdash\Psi$ is not valid. $\square$

References

[1] C. Brink, W. Kahl, G. Schmidt (Eds.), Relational Methods in Computer Science, Springer, Vienna-New York, 1997.

[2] P.J. Freyd, A. Scedrov, Categories, Allegories, North-Holland, Amsterdam, 1990.

[3] M.F. Frias, Fork Algebras in algebra, logic and computer science, Dissertation, Pont. Univ.-Rio, Rio de Janeiro, 1998.

[4] M.F. Frias, R.D. Maddux, Completeness of a relational calculus for program schemes, Proc. 13th Annual IEEE Symp. on Logic in Computer Science, June 21–24, Indianapolis, Indiana, IEEE Computer Society, Los Alamitos, 1988, pp. 127–134.

[5] B. Jónsson, Varieties of relation algebras Algebra Univ. 15 (1982) 273–298.

[6] B. Jónsson, Relation algebras and Schröder categories Discrete Math. 70 (1988) 27–45.

[7] B. Jónsson, A. Tarski, Boolean algebras with operators. Part II Amer. J. Math. 74 (1952) 127–162.

[8] R.D. Maddux, Topics in relation algebras, Dissertation, University of California, Berkeley, 1978.

[9] R.D. Maddux, Some sufficient conditions for the representability of relation algebras Algebra Univ. 8 (1978) 162–172.

[10] J.-P. Olivier, D.E. Serrato, Catégories de Dedekind. Morphismes dans les catégories de Schröder C. R. Acad. Sci. Paris 290 (1980) 939–941.

[11] W.-P. de Roever, Recursive program schemes: semantics and proof theory, Mathematical Centre Tracts 70, Mathematisch Centrum, Amsterdam, 1976.

[12] D.S. Scott, J.W. de Bakker, A theory of programs, unpublished seminar notes, IBM Seminar, Vienna, 1969, in: Klop, Meijer, Rutten (Eds.), J.W. de Bakker, 25 Jaar Semantiek. Liber Amicorum, Amsterdam, 1989.

[13] A. Tarski, On the calculus of relations J. Symbolic Logic 6 (1941) 73–89.

[14] A. Tarski, Some metalogical results concerning the calculus of relations J. Symbolic Logic 18 (1953) 188–189.

[15] A. Tarski, Contributions to the theory of models, III, Koninklijkle Nederlandse Akademie van Wetenschappen Proc. Series A. Mathematical Sciences (Indaga. Math., Vol. 17), Vol. 58, 1955, pp. 56–64.

[16] A. Tarski, S.R. Givant, A Formalization of Set Theory without Variables, Coll. Publ., Vol. 41, Amer. Math. Soc., Providence, RI, 1987.