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INFINITE FAMILIES OF BI-EMBEDDINGS

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A graph G is said to be (p_1, p_2) be-embeddable if there exist two subgraphs H_1 and H_2 of G with $H_1 \cup H_2 = G$ so that H_1 is embedded on S_{p_1} and H_2 is embedded on S_{p_2} where S_{p_1} and S_{p_2} are orientable surfaces of genera p_1 and p_2 respectively. Several infinite families of bi-embeddings of complete graphs are exhibited. Results from a computer search for additional bi-embeddings are also included. Finally, an unsolved problem is presented.

1. Basic definitions

Using the definition in [6], a graph G is said to be *embedded* on a surface S if there exists a set of distinct points of S which correspond to vertices of G and a set of distinct curves on S which are pairwise disjoint except for endpoints and which correspond to the edges of G. Intuitively, G is embedded on S if G can be drawn on S so that no two edges of G intersect. If all of the regions which are bounded by these curves on S are triangles, the embedding is called a *triangular embedding*.

If v is a vertex of graph G, define a *rotation* of v as a cyclic permutation of all edges adjacent to v. A *rotation* π of the graph G is a rotation for each vertex.

If graph G has a rotation π, construct a walk $v_0e_0v_1e_1v_2 \cdots v_te_tv_{t+1}$ such that e_1 follows e_0 in the rotation of v_1 and in general e_i follows e_{i-1} in the rotation of v_i. This walk is called the *circuit* generated by v_0 and e_0 and induced by rotation π. In this paper, • denotes a clockwise rotation and ∘ a counterclockwise rotation of a vertex.

If every circuit induced by a rotation π is a triangle, π is called a *triangular rotation*.

A current graph is one in which a number and an orientation are assigned to each edge such that if $e = (v_i, v_j)$ is an edge with current a, the current for (v_j, v_i) is $-a$. Current graphs are described in detail in [9]. A current graph must satisfy the following properties:

1. The rotation induces one single circuit.
2. Each vertex has valence 3, 2, or 1.

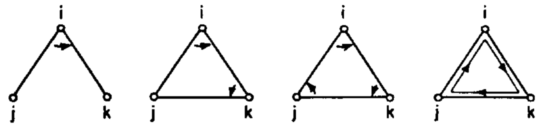


Fig. 1. The triangular rotation.

3. If there are $(n-\epsilon)/2$ edges, the elements $1, 2, \ldots, (n-\epsilon)/2$ of Z_n appear on each edge exactly once, where $\epsilon = 1$ if n is odd and $\epsilon = 0$ if n is even.

4. At each vertex of valence 3 the sum of the currents is zero (Kirchhoff's Current Law).

5. Each vertex identified by a letter, such as z, is of valence 1 and the current flowing into the vertex generates the group $\mathbb{Z}_n$. Such a vertex is called a *vortex*.

6. Each element of order 2 in the group appears on a dead-end edge.

A current graph which satisfies Properties 1–4 will always yield a triangular rotation for K_n. The current graph produces a rotation for the 0 vertex of K_n. The rotation is found by following the circuit. The rotation then yields a scheme for row 0 and the Addition Rule [9] gives the scheme for all other vertices.

If row i is of the form

$$i.\ \cdots\cdots j\ k\ \cdots\cdots$$

then the kth row must have

$$k.\ \cdots\cdots i\ j\ \cdots\cdots.$$

Fig. 1 illustrates the triangular rotation obtained.

To see how a current graph yields the correct kth row, first obtain the 0 row from the ith row:

$$0.\ \cdots\cdots j-i\ \ k-i\ \cdots\cdots i-k\ \ h\ \cdots\cdots.$$

Fig. 2 shows the local picture on the current graph. Adding k to each element, we obtain the kth row:

$$k.\ \cdots\cdots i\ \ h+k\ \cdots\cdots.$$

By Kirchhoff's Current Law, $h = (j-i) + (i-k) = j-k$ to obtain

$$k.\ \cdots\cdots i\ j\ \cdots\cdots.$$

Thus the rotation is triangular. This produces a triangular embedding of K_n since all regions are triangles.

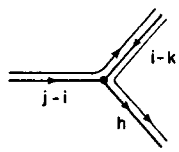


Fig. 2.

A graph G is said to be (p_1, p_2) bi-embeddable if there exist two subgraphs H_1 and H_2 of G with $H_1 \cup H_2 = G$ such that H_1 is embedded on S_{p_1} and H_2 is embedded on S_{p_2}, where S_p is the orientable surface of genus p. If $G = K_n$ and H_1 has n vertices and $H_2 = \bar{H}_1$, then K_n is said to be bi-embedded on the two surfaces.

$N(p_1, p_2)$ is an integer such that $K_{N(p_1,p_2)}$ cannot be edge-partitioned into two graphs bi-embeddable on S_{p_1} and S_{p_2} but $K_{N(p_1,p_2)-1}$ can be. The best-possible upper bound for $N(p_1, p_2)$ has been found [2]:

$$N(p_1, p_2) \leqslant \tfrac{1}{2}(15 + \sqrt{73 + 48(p_1 + p_2)}). \tag{1}$$

The proof, which is found in [2], is based on Euler's formula and properties of the complete graph.

A few cases of equality were known. Tutte [10] and Battle et al. [4] independently showed that $N(0, 0) = 9$. Ringel [8] and Beineke [5] showed that $N(1, 1) = 14$, and Beineke's work also showed $N(2, 2) = 15$. In [3] Anderson and White found seven new cases using current graphs, and in [13] Anderson found some infinite families of bi-embeddings using previously known embeddings of $K_{s(n)}$ and sK_n for $s = 3$ and 4.

2. Results

The following theorems present some infinite families of bi-embeddings:

Theorem 1. *Let* $p_1 = 12sk + k + 1$ *and* $p_2 = 12s^2 - 11s - 12sk - k$ *for* $k = 0, 1, 2, \ldots$ *and* $s \geqslant 3k + 2$. *Then,* $N(p_1, p_2) = 12s + 2$.

Proof. The current graphs in *Fig. 3*, with currents in Z_{12s+1}, give bi-embeddings of K_{12s+1} on S_{p_1} and S_{p_2} so that $N(p_1, p_2) \geqslant 12s + 2$. (1) shows that $N(p_1, p_2) \leqslant 12s + 2$.

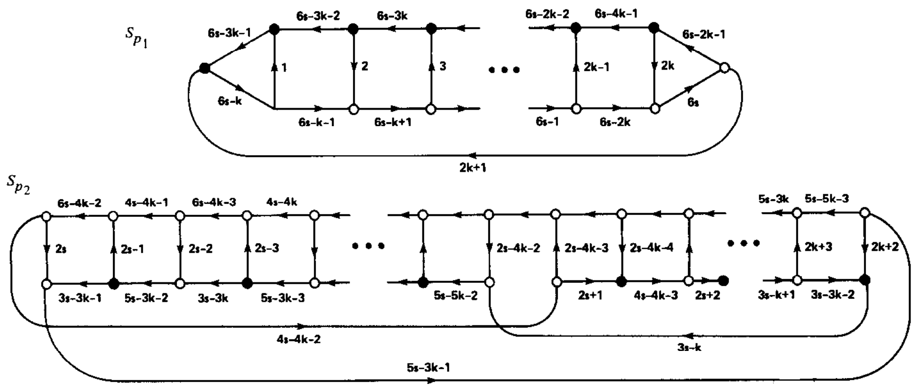
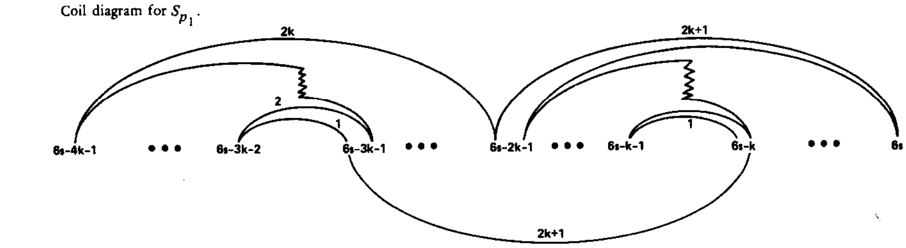


Fig. 3. Current graphs which yield on embedding of K_{12s+1} on S_{p_1} and S_{p_2} for $p_1 = 12sk + k + 1$ and $p_2 = 12s^2 - 11s - 12sk - k$.

Coil diagram for S_{p_1}.



Coil diagram for S_{p_2}.

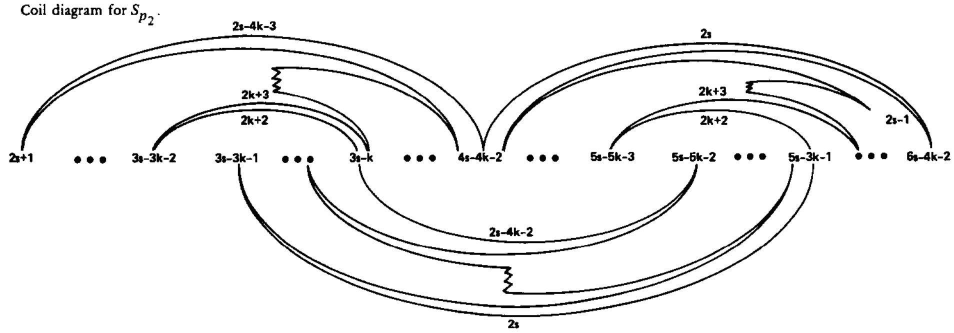
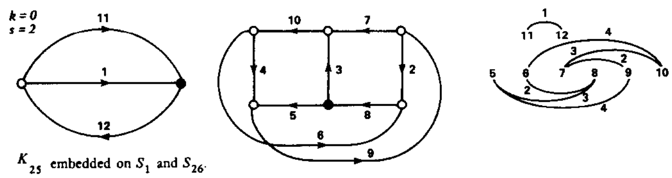
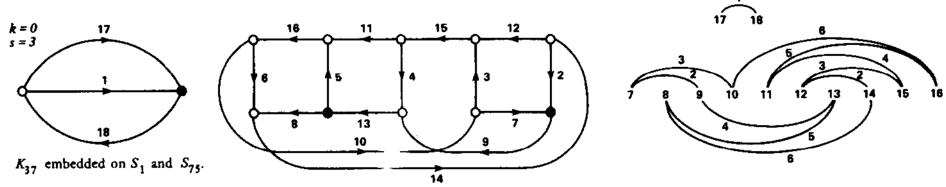


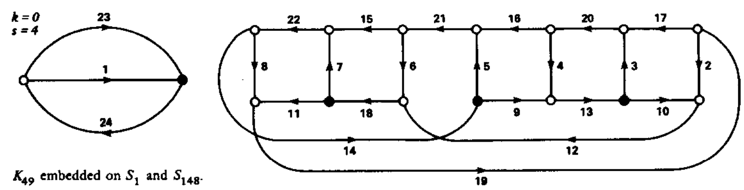
Fig. 4.



K_{25} embedded on S_1 and S_{26}.



K_{37} embedded on S_1 and S_{75}.



K_{49} embedded on S_1 and S_{148}.

Fig. 5.

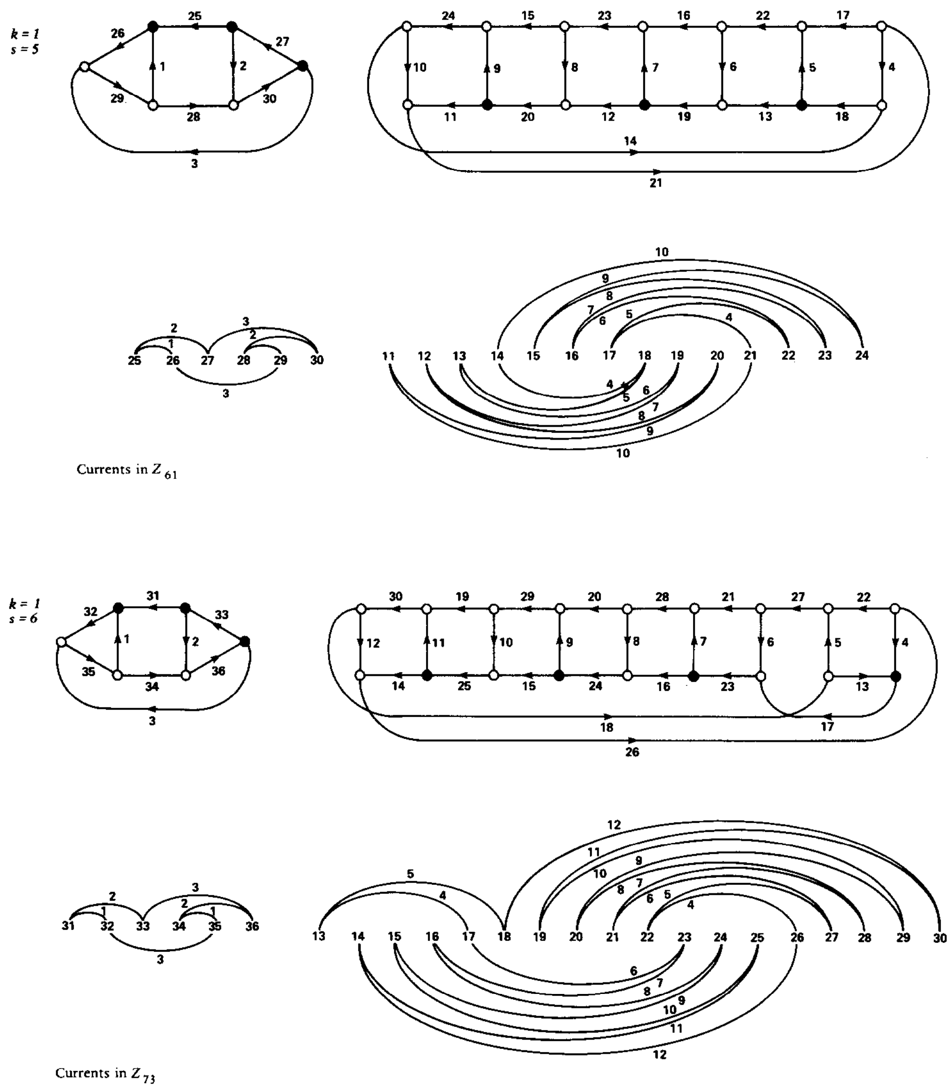


Fig. 6.

The patterns for the horizontal rungs on the current graphs for S_{p_1} and S_{p_2} are derived from the coil diagrams illustrated in *Fig. 4*. (Coil diagrams are explained in detail in [9].) □

Fig. 5 illustrates the first three cases of Theorem 1 for $k = 0$ and $s = 2$, 3, and 4 with currents in Z_{25}, Z_{37} and Z_{49} respectively. The bi-embedding of K_{25} on S_1 and S_{26} was first found by Anderson and White in [3]. *Fig. 6* illustrates the first two cases for $k = 1$ and $s = 5$ and 6 with currents in Z_{61} and Z_{73} respectively. *Fig. 7* illustrates the case for $k = 2$ and $s = 8$ with currents in Z_{97}.

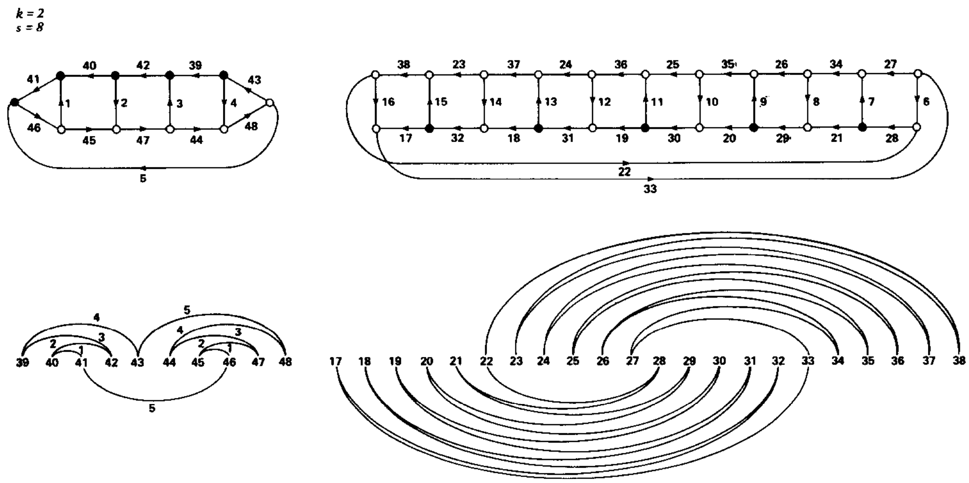


Fig. 7. Currents in Z_{97}.

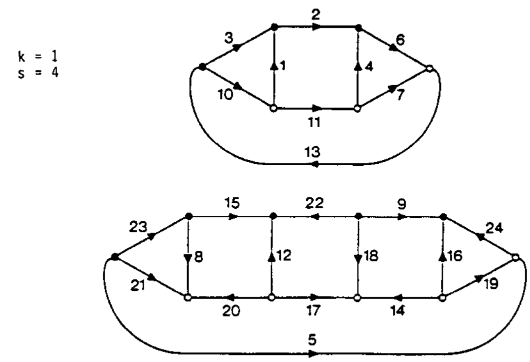


Fig. 8. Currents in Z_{49}.

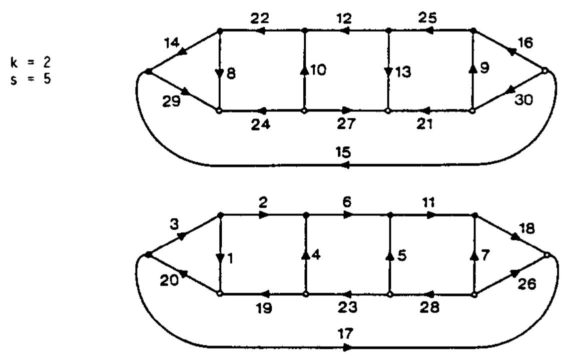


Fig. 9. Currents in Z_{61}.

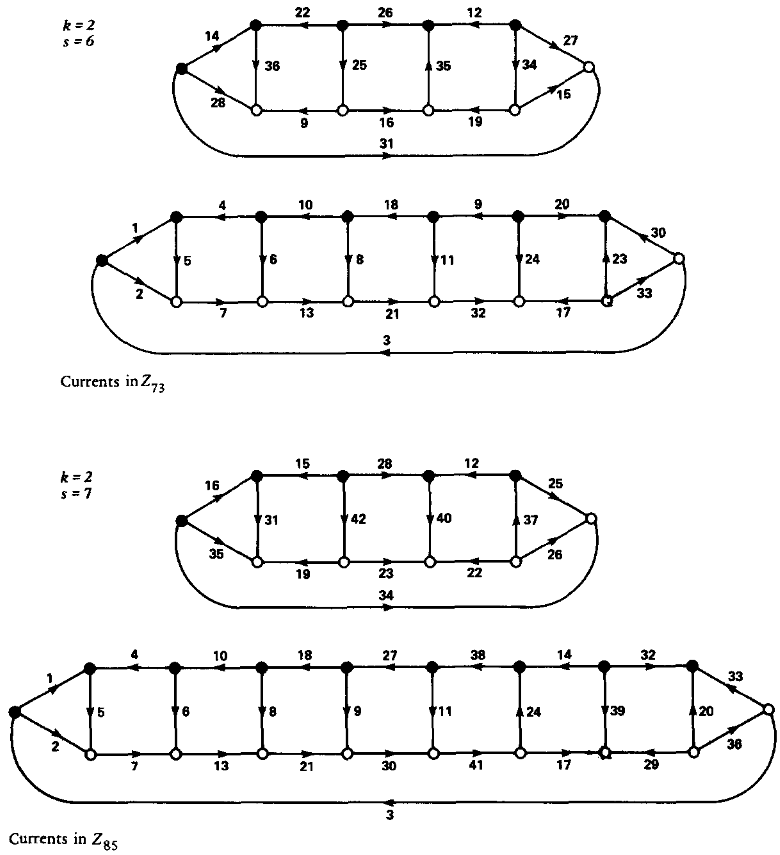


Fig. 10.

Some examples like those in Theorem 1 except with $s < 3k + 2$ have also been determined. The bi-embedding for the case of $k = 0$ and $s = 1$ is simply $N(1, 1) = 13$. The bi-embedding of K_{37} on S_{38} and S_{38} for the case of $k = 1$ and $s = 3$ was done in [3]. The current graphs for the case $k = 1$ and $s = 4$ are presented in *Fig. 8* with currents in Z_{49}. The current graphs for the cases of $k = 2$ and $s = 5$, 6, and 7 are illustrated in *Figs 9* and *10* with currents in Z_{61}, Z_{73} and Z_{85} respectively. These current graphs were found using DEC VAX 11/780 and 11/750 computers [7].

Corollary. *Let* $p_1' = 3s$ *and* $p_2' = 12s^2 - 12s + 1$ *for* $s \geq 2$. *Then,* $N(p_1', p_2') = 12s + 3$.

Proof. Fixing $k = 0$ in Theorem 1, modify the current graph for the first surface so that the current $6s$ is on a dead-end edge and the current 1 and $6s - 1$ are on edges incident with vortices. Since $\gcd(12s, 1) = 1$ and $\gcd(12s, 6s - 1) = 1$, these

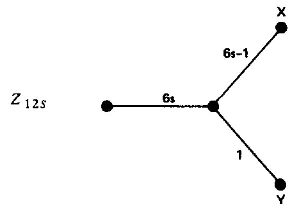


Fig. 11. Current graph for S_{3s}, $s \geqslant 2$.

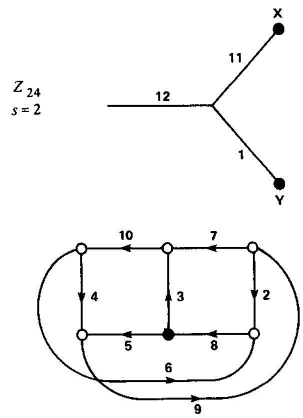


Fig. 12. $K_{26} - K_2$ embedding on S_6 and S_{25}.

currents generate the group Z_{12s}. *Fig. 11* illustrates the new current graph for $S_{p_1'}$ with currents in Z_{12s}. The second surface, $S_{p_2'}$, has the same current graph as in Theorem 1 for S_{p_2} but with currents in Z_{12s} this time. Thus, $K_{12s+2} - K_2$ is bi-embedded on $S_{p_1'}$ and $S_{p_2'}$. The missing K_2 can be placed inside one of the faces of the triangulation on $S_{p_2'}$. Thus, $N(p_1', p_2') \geqslant 12s + 3$. By (1), $N(p_1', p_2') \leqslant 12s + 3$. □

Fig. 12 illustrates the Corollary for the case $s = 2$ with currents in Z_{24}.

Theorem 2. *Let* $p_1 = 48s^2 + 36s + 1$ *and* $p_2 = 144s^2 + 216s + 81$ *for* $s \geqslant 0$. *Then* $N(p_1, p_2 + u) = 48s + 39$ *with* $0 \leqslant u \leqslant 4 + 8s$ *if s is even and* $4 \leqslant u \leqslant 4 + 8s$ *if s is odd.*

Proof. The current graphs in *Fig. 13* with currents in Z_{48s+36} given a bi-embedding of $K_{48s+38} - K_2$ on S_{p_1} and S_{p_2} for s even. The missing K_2 can be placed inside one of the faces of the triangulation on S_{p_1}. Note that the currents on the vortex edges generate the group Z_{48s+36} since $\gcd(48s + 36, 15s + 11) = 1$ and $\gcd(48s + 36, 9s + 7) = 1$ for s even. Thus $N(p_1, p_2 + u) \geqslant 48s + 39$. (1) shows that $N(p_1, p_2+u) \leq 48s+39$. *Fig. 14* illustrates the current graphs for the case $s=0$.

For s odd, however, let $\gcd(48s+36, 15s+11) = i$ and $\gcd(48s+36, 9s+7) = j$. Then $i=2$ and $j=4$ if $s \equiv 1 \pmod 4$, and $i=4$ and $j=2$ if $s \equiv 3 \pmod 4$. In the case where the gcd $=2$, the vortex corresponds to two disjoint vertices, say x_0 and

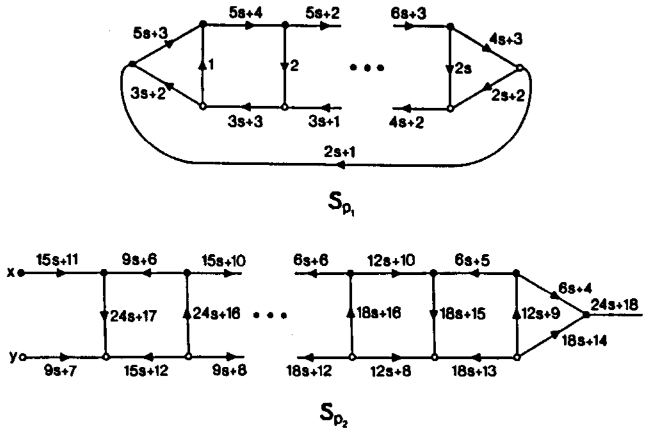


Fig. 13. Current graphs which yield an embedding of $K_{48s+36} - K_2$ on S_{p_1} and S_{p_2}, for $p_1 = 48s^2 + 36s + 1$ and $p_2 = 144s^2 + 216s + 81$.

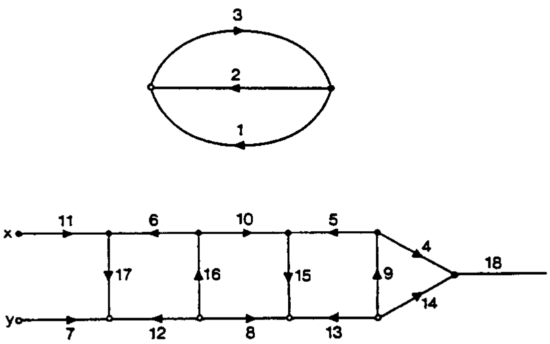


Fig. 14. The current graphs for $s=0$.

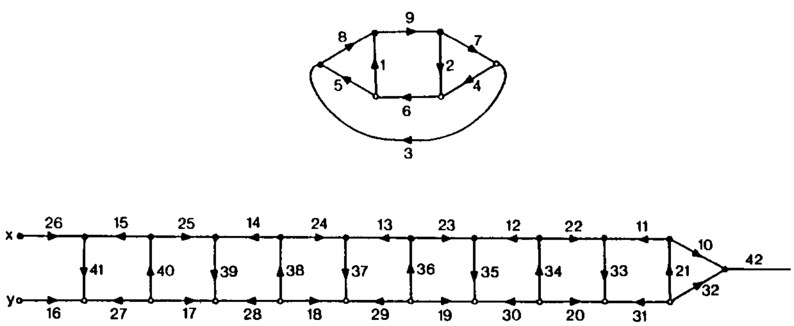


Fig. 15. The current graphs for $s=1$.

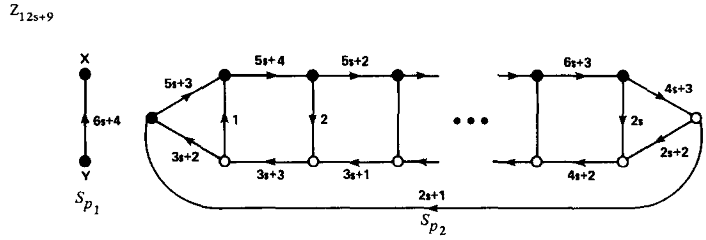


Fig. 16. Current graphs which yield an embedding of $K_{12s+11} - K_2$ on S_{p_1} and S_{p_2} where $p_1 = 0$ and $p_2 = 12s^2 + 9s + 1$.

x_1. One vertex, say x_0, is adjacent to all even-numbered vertices and the other, x_1, is adjacent to all odd-numbered vertices. Thus one handle must be added to the surface S_{p_2} so that x_0 and x_1 are adjacent. This handle can then be labeled x (or y as appropriate).

In the case where the gcd is 4, the vortex corresponds to four disjoint vertices: x_0 and x_1 adjacent to two disjoint sets of even-numbered vertices and x_2 and x_3 adjacent to two disjoint sets of odd-numbered vertices. Thus one handle must be added to S_{p_2} to connect x_0 and x_1 label it t_0. Another handle must be added to connect x_2 and x_3, label it t_1. And then a third handle must be added to connect t_0 and t_1, label it x (or y as appropriate).

Thus a total of four handles must be added to S_{p_2}, when s is odd. *Fig. 15* illustrates the current graph for $s = 1$. □

Theorem 3. *Let* $p_1 = 0$ *and* $p_2 = 12s^2 + 9s + 1$ *for* $s \geq 0$. *Then* $N(p_1, p_2) = 12s + 12$.

Proof. The current graphs in *Fig. 16*, with currents in Z_{12s+9}, give bi-embeddings of $K_{12s+11} - K_2$ on S_{p_1} and S_{p_2}. Again, the missing K_2 can be placed inside one of the faces of the triangulation on S_{p_2}. Thus, $N(p_1, p_2) \geq 12s + 12$. (1) shows that $N(p_1, p_2) \leq 12s + 12$. □

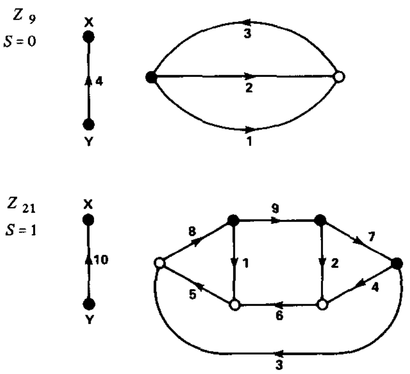


Fig. 17. (Top) K_{11} embedding on S_0 and (Bottom) K_{23} embedding on S_0 and S_{22}.

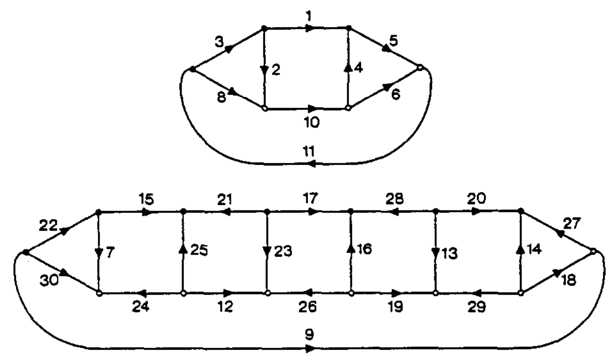


Fig. 18. K_{61} embedding on S_{62} and S_{184} so that $N(62, 184) = 62$.

Fig. 17 illustrates the cases for $s = 0$ and $s = 1$ for Theorem 3 with currents in Z_9 and Z_{21} respectively.

The following are more new values for $N(p_1, p_2)$ which were found with a computer [7].

1. *Fig. 18* illustrates a different bipartition of K_{61}, this time showing K_{61} embedded on S_{62} and S_{184}. Again using (1), $N(62, 184) = 62$.
2. The current graphs in *Fig. 19*, with currents in Z_{85}, yield an embedding of K_{85} on S_{256} and S_{256}. Thus $N(256, 256) = 86$.
3. The current graphs in *Fig. 20*, with currents in Z_{37}, yield an embedding of K_{37} on S_1 and S_{75}. Thus, $N(1, 75) = 38$.
4. The current graphs in *Fig. 21*, with currents in Z_{24}, yield an embedding of $K_{26} - K_2$ on S_1 and S_{30}. The missing K_2 can be placed inside one of the faces of the triangular embedding on S_1. Thus, $N(1, 30) = 27$.
5. The current graphs in *Fig. 22*, with currents in Z_{48}, yield an embedding of $K_{50} - K_2$ on S_1 and S_{156}. Thus, $N(1, 156) = 51$.

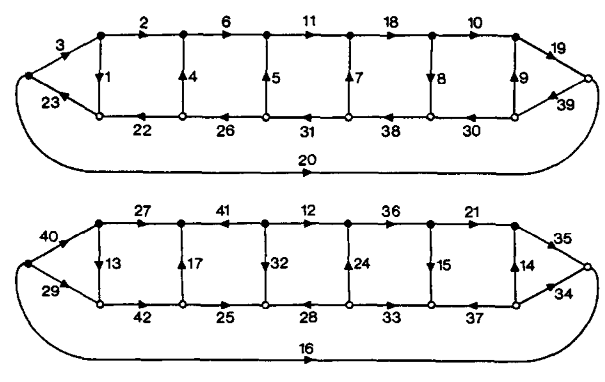


Fig. 19. K_{85} embedding on S_{256} and S_{256} so that $N(256, 256) = 86$.

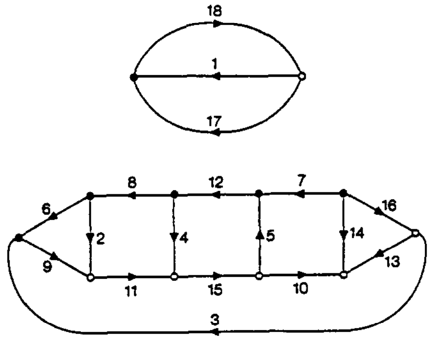


Fig. 20. K_{37} embedding on S_1 and S_{75} so that $N(1, 75) = 38$.

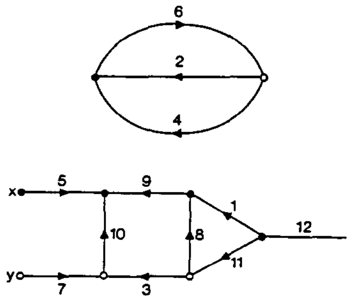


Fig. 21. $K_{26} - K_2$ embedding on S_1 and S_{30} so that $N(1, 30) = 27$.

6. The current graphs in *Fig. 23*, with currents in Z_{24}, yield a different bi-embedding of $K_{26} - K_2$ than that in *Fig. 21*. Here it is embedding on S_6 and S_{25}, so that $N(6, 25) = 27$.
7. The current graphs in *Fig. 24*, with currents in Z_{36}, yield a different bi-embedding of $K_{38} - K_2$ than that in *Fig. 14*. Here is is embedding on S_9 and S_{73}, so that $N(9, 73) = 39$.

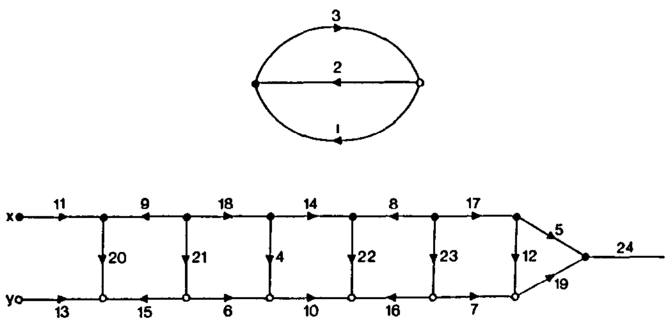


Fig. 22. $K_{50} - K_2$ embedding on S_1 and S_{156} so that $N(1, 156) = 51$.

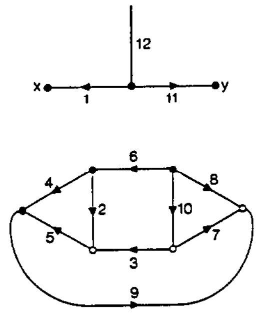


Fig. 23. $K_{26} - K_2$ embedding on S_6 and S_{25} so that $N(6, 25) = 27$.

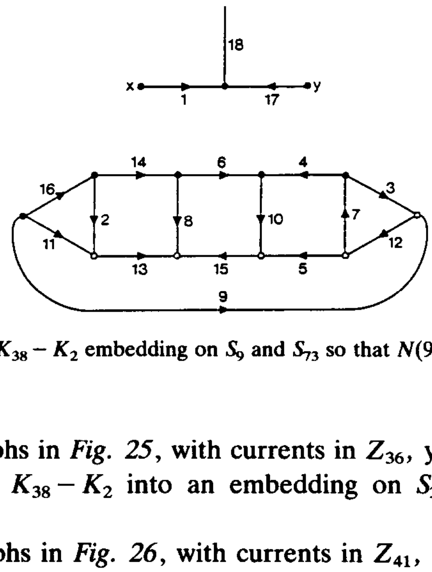


Fig. 24. $K_{38} - K_2$ embedding on S_9 and S_{73} so that $N(9, 73) = 39$.

8. The current graphs in *Fig. 25*, with currents in Z_{36}, yield a third bipartition of the edges of $K_{38} - K_2$ into an embedding on S_{37} and S_{45}. This gives $N(37, 45) = 39$.
9. The current graphs in *Fig. 26*, with currents in Z_{41}, yield an embedding of $K_{45} - K_4$ on S_{42} and S_{81}. The missing K_4 can be placed inside one of the faces of the triangular embedding on S_{42}. Thus, $N(42, 81) = 46$.

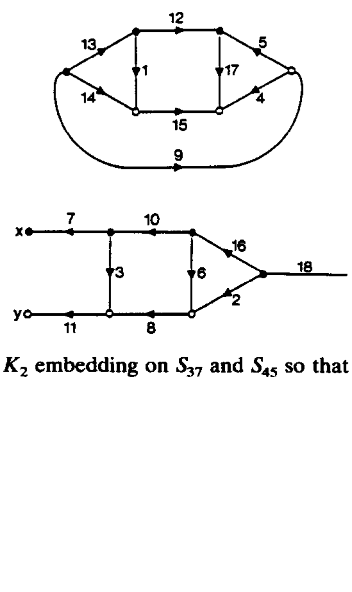


Fig. 25. $K_{38} - K_2$ embedding on S_{37} and S_{45} so that $N(37, 45) = 39$.

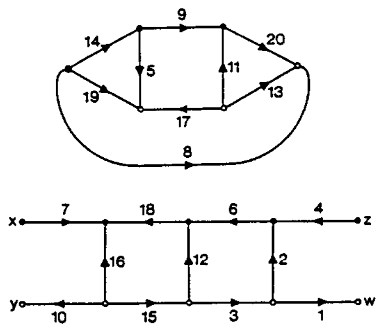


Fig. 26. $K_{45} - K_4$ embedding on S_{42} and S_{81} so that $N(42, 81) = 46$.

3. Unsolved problem

In [2] it was determined that $N(p_1, p_2) \leq (\frac{1}{2}(15 + \sqrt{73 + 48(p_1 + p_2)})$. All examples found to date indicate that this inequality is indeed an equality, but except for the examples presented here and in [3], this is a conjecture. In particular $N(p, p)$ is known in only a few cases.

For a given genus p, what is the largest n such that K_n can be bi-embedding on S_p and S_p? From [2] it is known that $n \leq (\frac{1}{2}(13 + \sqrt{73 + 96p}))$.

In general, define the *bigenus* $\beta(G)$ of a graph G as the minimal genus of an orientable surface on which G can be bi-embedded. That is, $\beta(G) = p$ is the smallest p for which G can be bipartitioned into two subgraphs H and $\bar{H}$ so that H and $\bar{H}$ are each embeddable on S_p.

For example, $\beta(K_9) = 0$, $\beta(K_{13}) = 1$, $\beta(K_{14}) = 2$, $\beta(K_{37}) = 38$, $\beta(K_{61}) = 123$ and $\beta(K_{85}) = 256$.

Conjecture. $\beta(K_n) = \{(n^2 - 13n + 24)/24\}$ *where* $\{x\}$ *denotes the smallest integer greater than or equal to* x.

Let $\beta(K_n) = p$. From [1], $n - 1 \leq 12 + 12(2p - 2)/n$. Hence, $p \geq (n^2 - 13n + 24)/24$. Note that equality is attained in the six known examples of $\beta(K_n)$.

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