Strength of convergence in the orbit space of a groupoid ☆

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A B S T R A C T

Let G be a second-countable locally-compact Hausdorff groupoid with a Haar system, and let $\{x_n\}$ be a sequence in the unit space $G^{(0)}$ of G. We show that the notions of strength of convergence of $\{x_n\}$ in the orbit space $G^{(0)}/G$ and measure-theoretic accumulation along the orbits are equivalent ways of realising multiplicity numbers associated to a sequence of induced representation of the groupoid C^*-algebra.

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1. Introduction

Suppose H is a locally-compact Hausdorff group acting freely and continuously on a locally-compact Hausdorff space X, so that (H, X) is a free transformation group. In [13, pp. 95–96] Green gives an example of a free non-proper action of $H = \mathbb{R}$ on a subset X of $\mathbb{R}^3$; the non-properness comes down to the existence of $z \in X$, $\{x_n\} \subset X$, and two sequences $\{s_n\}$ and $\{t_n\}$ in H such that

(i) $s_n^{-1} \cdot x_n \to z$ and $t_n^{-1} \cdot x_n \to z$; and
(ii) $t_n s_n^{-1} \to \infty$ as $n \to \infty$, in the sense that $\{t_n s_n^{-1}\}$ has no convergent subsequence.

In [2, Definition 2.2], and subsequently in [3, p. 2], the sequence $\{x_n\}$ is said to converge 2-times in the orbit space to $z \in X$. Each orbit $H \cdot x$ gives an induced representation $\mathrm{Ind}\,\epsilon_x$ of the associated transformation-group C^*-algebra $C_0(X) \rtimes H$ which is irreducible, and the k-times convergence of $\{x_n\}$ in the orbit space to $z \in X$ translates into statements about various multiplicity numbers associated to $\mathrm{Ind}\,\epsilon_z$ in the spectrum of $C_0(X) \rtimes H$, as in [2, Theorem 2.5], [3, Theorem 1.1] and [4, Theorem 2.1].

Upper and lower multiplicity numbers associated to irreducible representations π of a C^*-algebra A were introduced by Archbold [1] and extended to multiplicity numbers relative to a net of irreducible representations by Archbold and Spielberg [9]. The upper multiplicity $M_U(\pi)$ of π, for example, counts 'the number of nets of orthogonal equivalent pure states which can converge to a common pure state associated to π' [6, p. 26]. The definition of k-times convergence and [2,

Theorem 2.5] were very much motivated by a notion of k-times convergence in the dual space of a nilpotent Lie group [16] and its connection with relative multiplicity numbers (see, for example, [6, Theorem 2.4] and [7, Theorem 5.8]).

Theorem 1.1 of [3] shows that the topological property of a sequence $\{x_n\}$ converging k-times in the orbit space to $z \in X$ is equivalent to (1) a measure theoretic accumulation along the orbits $G \cdot x_n$ and (2) that the lower multiplicity of Ind ϵ_z relative to the sequence $\{\text{Ind } \epsilon_{x_n}\}$ is at least k. In this paper we prove that the results of [3] generalise to principal groupoids. In our main arguments we have tried to preserve as much as possible the structure of those in [3], although the arguments presented here are often more complicated in order to cope with the partially defined product in a groupoid and the set of measures that is a Haar system compared to the fixed Haar measure used in the transformation-group case. Our theorems have led us to a new class of examples exhibiting k-times convergence in groupoids that are not based on transformation groups, thus justifying our level of generality. Given a row-finite directed graph E, Kumjian, Pask, Raeburn and Renault in [15] used the set of all infinite paths in E to construct an r-discrete groupoid G_E, called a *path groupoid*. We prove that G_E is principal if and only if E contains no cycles (Proposition 8.1). We then exhibit principal G_E with Hausdorff and non-Hausdorff orbits space, respectively, both with a k-times converging sequence in the orbit space. In particular, our examples can be used to find a groupoid G_E whose C^*-algebra has non-Hausdorff spectrum and distinct upper and lower multiplicity counts among its irreducible representations.

2. Preliminaries

We denote the unit space of a groupoid G by $G^{(0)}$. For $x \in G^{(0)}$ we call the set $r(s^{-1}(\{x\})) = s(r^{-1}(\{x\}))$ the *orbit* of x and denote it by $[x]$. For a subset U of $G^{(0)}$ we define $G_U := s^{-1}(U)$, $G^U := r^{-1}(U)$, and $G|_U := s^{-1}(U) \cap r^{-1}(U)$. We denote the set of all positive integers by $\mathbb{P}$ and the set of all non-negative integers by $\mathbb{N}$. We write $\lfloor r \rfloor$ for the integer part of a non-negative real number r.

Definition 2.1. A *right Haar system* on a groupoid G is a set $\{\lambda_x : x \in G^{(0)}\}$ of non-negative Radon measures on G such that

(i) supp $\lambda_x = G_x$ $(= s^{-1}(\{x\}))$ for all $x \in G^{(0)}$;
(ii) for $f \in C_c(G)$, the function $x \mapsto \int f \, d\lambda_x$ on $G^{(0)}$ is in $C_c(G^{(0)})$; and
(iii) for $f \in C_c(G)$ and $\gamma \in G$,

$$\int f(\alpha\gamma) \, d\lambda_{r(\gamma)}(\alpha) = \int f(\alpha) \, d\lambda_{s(\gamma)}(\alpha).$$

We will refer to (ii) as the *continuity of the Haar system* and to (iii) as *Haar-system invariance*. The collection $\{\lambda^x : x \in G^{(0)}\}$ of measures where $\lambda^x(E) := \lambda_x(E^{-1})$ is a *left Haar system*, which is a system of measures such that supp $\lambda^x = G^x$ and, for $f \in C_c(G)$, $x \mapsto \int f \, d\lambda^x$ is continuous and $\int f(\gamma\alpha) \, d\lambda^{s(\gamma)}(\alpha) = \int f(\alpha) \, d\lambda^{r(\gamma)}(\alpha)$. Given that we can easily convert a right Haar system $\{\lambda_x\}$ into a left Haar system $\{\lambda^x\}$ and vice versa, we will simply refer to a *Haar system* λ and use subscripts to refer to elements of the right Haar system $\{\lambda_x\}$ and superscripts to refer to elements of the left Haar system $\{\lambda^x\}$.

The following lemma follows from the invariance of the Haar system and the Dominated Convergence Theorem; we omit the proof.

Lemma 2.2 *(Haar-system invariance). Suppose G is a locally-compact Hausdorff groupoid with Haar system λ. If $K \subset G$ is compact and $\gamma \in G$ with $s(\gamma) = x$ and $r(\gamma) = y$, then $\lambda_x(K\gamma) = \lambda_y(K)$ and $\lambda^x(\gamma^{-1}K) = \lambda^y(K)$.*

Definition 2.3 below is Definition 2.45 in the unpublished book [17]. Alternative descriptions of the induced representation may be found in [19, p. 234] and [23, pp. 81–82].

Definition 2.3. Suppose G is a second-countable locally-compact Hausdorff groupoid with Haar system λ and let μ be a Radon measure on $G^{(0)}$.

(i) We write $\nu = \mu \circ \lambda = \int \lambda^x \, d\mu$ for the measure on G defined for every Borel-measurable function $f : G \to \mathbb{C}$ by $\int_G f(\gamma) \, d\nu(\gamma) = \int_{G^{(0)}} \int_G f(\gamma) \, d\lambda^x(\gamma) \, d\mu(x)$. We call ν the measure induced by μ, and we write ν^{-1} for the image of ν under the homeomorphism $\gamma \mapsto \gamma^{-1}$.
(ii) For $f \in C_c(G)$, Ind $\mu(f)$ is the operator on $L^2(G, \nu^{-1})$ defined by the formula

$$\big(\text{Ind } \mu(f)\xi\big)(\gamma) = \int_G f(\alpha)\xi\big(\alpha^{-1}\gamma\big) \, d\lambda^{r(\gamma)}(\alpha) = \int_G f(\gamma\alpha)\xi\big(\alpha^{-1}\big) \, d\lambda^{s(\gamma)}(\alpha).$$

In this paper we are interested in representations that are induced by point-mass measures δ_x on $G^{(0)}$. We denote Ind δ_x by L^x for all $x \in G^{(0)}$ as in [19] and [11].

Remark 2.4. It follows from the definition of the induced measure that for $x \in G^{(0)}$, the measure ν induced by δ_x is equal to λ^x. In particular we have $\nu^{-1} = \lambda_x$, so L^x acts on $L^2(G, \lambda_x)$. The operator L^x is then given by

$$\left(L^x(f)\xi\right)(\gamma) = \int_G f\left(\gamma \alpha^{-1}\right)\xi(\alpha)\, d\lambda_x(\alpha)$$

for all $\xi \in L^2(G, \lambda_x)$ and all $\gamma \in G$. There is a close relationship between the convolution on $C_c(G)$ and these induced representations: recall that for $f, g \in C_c(G)$, the convolution $f * g \in C_c(G)$ is given by

$$f * g(\gamma) = \int_G f\left(\gamma \alpha^{-1}\right) g(\alpha)\, d\lambda_{s(\gamma)}(\alpha) \quad \text{for all } \gamma \in G,$$

so that

$$\left(L^x(f)g\right)(\gamma) = f * g(\gamma) \quad \text{for any } x \in G^{(0)} \text{ and } \gamma \in G_x.$$

We denote the norm in $L^2(G, \lambda_x)$ by $\|\cdot\|_x$. Finally note that when G is a second-countable locally-compact principal groupoid that admits a Haar system, each L^x is irreducible by [19, Lemma 2.4].

Remark 2.5. If $G = (H, X)$ is a second-countable free transformation group, then the representations L^x defined above are unitarily equivalent to the representations $\text{Ind}\,\epsilon_x$ used in [3]. Specifically, let ν be a choice of right Haar measure on H and Δ the associated modular function. The map $\iota : C_c(H \times X) \to C_c(H \times X)$ defined by $\iota(f)(t, x) = f(t, x)\Delta(t)^{1/2}$ extends to an isomorphism ι of the groupoid C^*-algebra $C^*(H \times X)$ onto the transformation-group C^*-algebra $C_0(X) \rtimes H$ [23, p. 58]. Fix $x \in X$. Then there is a unitary $U_x : L^2(H, \nu) \to L^2(H \times X, \lambda_x)$, characterised by $U(\xi)(h, y) = \xi(h)\delta_x(h^{-1} \cdot y)$ for $\xi \in C_c(H)$, and $U(\text{Ind}\,\epsilon_x(\iota(f)))U^* = L^x(f)$ for $f \in C^*(H \times X)$.

Let A be a C^*-algebra. We write θ for the canonical surjection from the space $P(A)$ of pure states of A to the spectrum $\hat{A}$ of A. We frequently identify an irreducible representation π with its equivalence class in $\hat{A}$ and we write $\mathcal{H}_\pi$ for the Hilbert space on which $\pi(A)$ acts.

Let $\pi \in \hat{A}$ and let $\{\pi_\alpha\}$ be a net in $\hat{A}$. We now recall the definitions of *upper* and *lower multiplicity* $M_U(\pi)$ and $M_L(\pi)$ from [1], and *relative upper* and *relative lower multiplicity* $M_U(\pi, \{\pi_\alpha\})$ and $M_L(\pi, \{\pi_\alpha\})$ from [9]. Let $\mathcal{N}$ be the weak*-neighborhood base at zero in the dual A^* of A consisting of all open sets of the form

$$N = \left\{\psi \in A^*: \left|\psi(a_i)\right| < \epsilon,\ 1 \leqslant i \leqslant n\right\},$$

where $\epsilon > 0$ and $a_1, a_2, \ldots, a_n \in A$. Suppose ϕ is a pure state of A associated with π and let $N \in \mathcal{N}$. Define

$$V(\phi, N) = \theta\left((\phi + N) \cap P(A)\right),$$

an open neighborhood of π in $\hat{A}$. For $\sigma \in \hat{A}$ let

$$\text{Vec}(\sigma, \phi, N) = \left\{\eta \in \mathcal{H}_\sigma: \|\eta\| = 1,\ \left(\sigma(\cdot)\eta \mid \eta\right) \in \phi + N\right\}.$$

Note that $\text{Vec}(\sigma, \phi, N)$ is non-empty if and only if $\sigma \in V(\phi, N)$. For any $\sigma \in V(\phi, N)$ define $d(\sigma, \phi, N)$ to be the supremum in $\mathbb{P} \cup \{\infty\}$ of the cardinalities of finite orthonormal subsets of $\text{Vec}(\sigma, \phi, N)$. Write $d(\sigma, \phi, N) = 0$ when $\text{Vec}(\sigma, \phi, N)$ is empty.

Define

$$M_U(\phi, N) = \sup_{\sigma \in V(\phi, N)} d(\sigma, \phi, N) \in \mathbb{P} \cup \{\infty\}.$$

Note that if $N' \in \mathcal{N}$ and $N \subset N$, then $M_U(\phi, N') \leqslant M_U(\phi, N)$. Now define

$$M_U(\phi) = \inf_{N \in \mathcal{N}} M_U(\phi, N) \in \mathbb{P} \cup \{\infty\}.$$

By [1, Lemma 2.1], the value of $M_U(\phi)$ is independent of the pure state ϕ associated to π and so $M_U(\pi) := M_U(\phi)$ is well defined. For lower multiplicity, assume that $\{\pi\}$ is not open in $\hat{A}$, and using [1, Lemma 2.1] again, define

$$M_L(\pi) := \inf_{N \in \mathcal{N}}\left(\liminf_{\sigma \to \pi,\, \sigma \neq \pi} d(\sigma, \phi, N)\right) \in \mathbb{P} \cup \{\infty\}.$$

Now suppose that $\{\pi_\alpha\}_{\alpha \in \Lambda}$ is a net in $\hat{A}$. For $N \in \mathcal{N}$ let

$$M_U\left(\phi, N, \{\pi_\alpha\}\right) = \limsup_{\alpha \in \Lambda} d(\pi_\alpha, \phi, N) \in \mathbb{N} \cup \{\infty\}.$$

Note that if $N' \in \mathcal{N}$ and $N' \subset N$ then $M_U(\phi, N', \{\pi_\alpha\}) \leqslant M_U(\phi, N, \{\pi_\alpha\})$. Then

$$M_U\left(\pi, \{\pi_\alpha\}\right) := \inf_{N \in \mathcal{N}} M_U\left(\phi, N, \{\pi_\alpha\}\right) \in \mathbb{N} \cup \{\infty\},$$

is well defined because the right-hand side is independent of the choice of ϕ by an argument similar to the proof of [1, Lemma 2.1]. Similarly define

$$M_L\big(\phi, N, \{\pi_\alpha\}\big) := \liminf_{\alpha \in \Lambda} d(\pi_\alpha, \phi, N) \in \mathbb{N} \cup \{\infty\},$$

and let

$$M_L\big(\pi, \{\pi_\alpha\}\big) = \inf_{N \in \mathcal{N}} M_L\big(\phi, N, \{\pi_\alpha\}\big) \in \mathbb{N} \cup \{\infty\}.$$

It follows that for any irreducible representation π and any net $\{\pi_\alpha\}_{\alpha \in \Lambda}$ of irreducible representations,

$$M_L\big(\pi, \{\pi_\alpha\}\big) \leqslant M_U\big(\pi, \{\pi_\alpha\}\big) \leqslant M_U(\pi)$$

and, if $\{\pi_\alpha\}$ converges to π with $\pi_\alpha \neq \pi$ eventually, $M_L(\pi) \leqslant M_L(\pi, \{\pi_\alpha\})$. Finally, if $\{\pi_\beta\}$ is a subnet of $\{\pi_\alpha\}$, then

$$M_L\big(\pi, \{\pi_\alpha\}\big) \leqslant M_L\big(\pi, \{\pi_\beta\}\big) \leqslant M_U\big(\pi, \{\pi_\beta\}\big) \leqslant M_U\big(\pi, \{\pi_\alpha\}\big).$$

3. Lower multiplicity and k-times convergence I

A key goal for this paper is to describe the relationship between multiplicities of induced representations and strength of convergence in the orbit space. We start this section by recalling the definition of k-times convergence in a groupoid from [11]. We then show that if a sequence converges k-times in the orbit space of a principal groupoid, then the lower multiplicity of the associated sequence of representations is at least k; the converse will be shown in Section 6.

Recall that a sequence $\{\gamma_n\} \subset G$ tends to infinity if it admits no convergent subsequence.

Definition 3.1. Let $k \in \mathbb{P}$. A sequence $\{x_n\}$ in $G^{(0)}$ is k-times convergent in $G^{(0)}/G$ to $z \in G^{(0)}$ if there exist k sequences $\{\gamma_n^{(1)}\}, \{\gamma_n^{(2)}\}, \ldots, \{\gamma_n^{(k)}\} \subset G$ such that

(i) $s(\gamma_n^{(i)}) = x_n$ for all n and $1 \leqslant i \leqslant k$;
(ii) $r(\gamma_n^{(i)}) \to z$ as $n \to \infty$ for $1 \leqslant i \leqslant k$; and
(iii) if $1 \leqslant i < j \leqslant k$ then $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \to \infty$ as $n \to \infty$.

The proof of the following proposition is based on [2, Theorem 2.3] and a part of [3, Theorem 1.1].

Proposition 3.2. *Suppose G is a second-countable locally-compact Hausdorff principal groupoid with Haar system λ. Let $z \in G^{(0)}$ and suppose that $\{x_n\}$ is a sequence in $G^{(0)}$ that converges k-times to z in $G^{(0)}/G$. Then $M_L(L^z, \{L^{x_n}\}) \geqslant k$.*

Proof. We will use a contradiction argument. Suppose that $M_L(L^z, \{L^{x_n}\}) = r < k$. Fix a real-valued $g \in C_c(G)$ so that $\|g\|_z > 0$. Define $\eta \in L^2(G, \lambda_z)$ by $\eta(\alpha) = \|g\|_z^{-1} g(\alpha)$ for all $\alpha \in G$. Then

$$\|\eta\|_z^2 = \|g\|_z^{-2} \int g(\alpha)^2 \, d\lambda_z(\alpha) = \|g\|_z^{-2} \|g\|_z^2 = 1,$$

so η is a unit vector in $L^2(G, \lambda_z)$ and the GNS construction of $\phi := (L^z(\cdot)\eta \mid \eta)$ is unitarily equivalent to L^z. By the definition of lower multiplicity we now have

$$M_L\big(L^z, \{L^{x_n}\}\big) = \inf_{N \in \mathcal{N}} M_L\big(\phi, N, \{L^{x_n}\}\big) = r,$$

so there exists $N \in \mathcal{N}$ such that

$$M_L\big(\phi, N, \{L^{x_n}\}\big) = \liminf_n d\big(L^{x_n}, \phi, N\big) = r,$$

and consequently there exists a subsequence $\{y_m\}$ of $\{x_n\}$ such that

$$d\big(L^{y_m}, \phi, N\big) = r \quad \text{for all } m. \tag{3.1}$$

Since any subsequence of a sequence that is k-times convergent is also k-times convergent, we know that $\{y_m\}$ converges k-times to z in $G^{(0)}/G$.

We will now use the k-times convergence of $\{y_m\}$ to construct k sequences of unit vectors with sufficient properties to establish our contradiction. By the k-times convergence of $\{y_m\}$ there exist k sequences

$$\big\{\gamma_m^{(1)}\big\}, \big\{\gamma_m^{(2)}\big\}, \ldots, \big\{\gamma_m^{(k)}\big\} \subset G$$

satisfying

(i) $s(\gamma_m^{(i)}) = y_m$ for all m and $1 \leqslant i \leqslant k$;

(ii) $r(\gamma_m^{(i)}) \to z$ as $m \to \infty$ for $1 \leqslant i \leqslant k$; and

(iii) if $1 \leqslant i < j \leqslant k$ then $\gamma_m^{(j)}(\gamma_m^{(i)})^{-1} \to \infty$ as $m \to \infty$.

For each $1 \leqslant i \leqslant k$ and $m \geqslant 1$, define $\eta_m^{(i)}$ by

$$\eta_m^{(i)}(\alpha) = \|g\|_{r(\gamma_m^{(i)})}^{-1} g\big(\alpha(\gamma_m^{(i)})^{-1}\big) \quad \text{for all } \alpha \in G.$$

It follows from Haar-system invariance that

$$\begin{aligned}
\big\|\eta_m^{(i)}\big\|_{y_m}^2 &= \|g\|_{r(\gamma_m^{(i)})}^{-2} \int g\big(\alpha(\gamma_m^{(i)})^{-1}\big)^2 \, d\lambda_{y_m}(\alpha) \\
&= \|g\|_{r(\gamma_m^{(i)})}^{-2} \int g(\alpha)^2 \, d\lambda_{r(\gamma_m^{(i)})}(\alpha) \\
&= \|g\|_{r(\gamma_m^{(i)})}^{-2} \|g\|_{r(\gamma_m^{(i)})}^2 = 1,
\end{aligned}$$

so the $\eta_m^{(i)}$ are unit vectors in $L^2(G, \lambda_{y_m})$.

Now suppose that $1 \leqslant i < j \leqslant k$. Then

$$\big(\eta_m^{(i)} \mid \eta_m^{(j)}\big)_{y_m} = \|g\|_{r(\gamma_m^{(i)})}^{-1} \|g\|_{r(\gamma_m^{(j)})}^{-1} \int g\big(\alpha(\gamma_m^{(i)})^{-1}\big) g\big(\alpha(\gamma_m^{(j)})^{-1}\big) \, d\lambda_{y_m}(\alpha). \tag{3.2}$$

Since $\gamma_m^{(i)}(\gamma_m^{(j)})^{-1} \to \infty$, $\gamma_m^{(i)}(\gamma_m^{(j)})^{-1}$ is eventually not in the compact set $(\operatorname{supp} g)^{-1}(\operatorname{supp} g)$, and so there exists m_0 such that if $m \geqslant m_0$, then

$$(\operatorname{supp} g)\gamma_m^{(i)} \cap (\operatorname{supp} g)\gamma_m^{(j)} = \emptyset.$$

(To see this claim, note that if $(\operatorname{supp} g)\gamma_m^{(i)} \cap (\operatorname{supp} g)\gamma_m^{(j)} \neq \emptyset$ then there exist $\alpha, \beta \in \operatorname{supp} g$ such that $\alpha\gamma_m^{(i)} = \beta\gamma_m^{(j)}$, and so $\gamma_m^{(i)}(\gamma_m^{(j)})^{-1} = \alpha^{-1}\beta \in (\operatorname{supp} g)^{-1}(\operatorname{supp} g)$.) For the integrand of (3.2) to be non-zero, both $\alpha(\gamma_m^{(i)})^{-1}$ and $\alpha(\gamma_m^{(j)})^{-1}$ must be in $\operatorname{supp} g$, so α must be in $(\operatorname{supp} g)\gamma_m^{(i)} \cap (\operatorname{supp} g)\gamma_m^{(j)}$. But this is not possible if $m \geqslant m_0$. Thus, for any distinct i, j, we will eventually have $\eta_m^{(i)} \perp \eta_m^{(j)}$.

For the last main component of this proof we will establish that

$$\big(\mathrm{L}^{y_m}(\cdot)\eta_m^{(i)} \mid \eta_m^{(i)}\big) \to \big(\mathrm{L}^z(\cdot)\eta \mid \eta\big) = \phi \quad \text{as } m \to \infty$$

in the dual of $C^*(G)$ with the weak* topology for each i. Fix $f \in C_c(G)$. We have

$$\begin{aligned}
\big(\mathrm{L}^z(f)\eta \mid \eta\big) &= \int_G \big(\mathrm{L}^z(f)\eta\big)(\alpha)\eta(\alpha) \, d\lambda_z(\alpha) \\
&= \int_G \int_G f\big(\alpha\beta^{-1}\big)\eta(\beta)\eta(\alpha) \, d\lambda_z(\beta) \, d\lambda_z(\alpha) \\
&= \|g\|_z^{-2} \int_G \int_G f\big(\alpha\beta^{-1}\big)g(\beta)g(\alpha) \, d\lambda_z(\beta) \, d\lambda_z(\alpha). \tag{3.3}
\end{aligned}$$

Now fix $1 \leqslant i \leqslant k$. By the invariance of the Haar system we have

$$\begin{aligned}
\big(\mathrm{L}^{y_m}(f)\eta_m^{(i)} \mid \eta_m^{(i)}\big) &= \int_G \int_G f\big(\alpha\beta^{-1}\big)\eta_m^{(i)}(\beta)\eta_m^{(i)}(\alpha) \, d\lambda_{y_m}(\beta) \, d\lambda_{y_m}(\alpha) \\
&= \|g\|_{r(\gamma_m^{(i)})}^{-2} \int_G \int_G f\big(\alpha\beta^{-1}\big)g\big(\alpha(\gamma_m^{(i)})^{-1}\big)g\big(\beta(\gamma_m^{(i)})^{-1}\big) \, d\lambda_{y_m}(\beta) \, d\lambda_{y_m}(\alpha) \\
&= \|g\|_{r(\gamma_m^{(i)})}^{-2} \int_G \int_G f\big(\alpha\beta^{-1}\big)g(\alpha)g(\beta) \, d\lambda_{r(\gamma_m^{(i)})}(\beta) \, d\lambda_{r(\gamma_m^{(i)})}(\alpha) \\
&= \|g\|_{r(\gamma_m^{(i)})}^{-2} \int_G f * g(\alpha)g(\alpha) \, d\lambda_{r(\gamma_m^{(i)})}(\alpha). \tag{3.4}
\end{aligned}$$

We know that $r(\gamma_m^{(i)}) \to z$ as $m \to \infty$ so, by the continuity of the Haar system, $\|g\|_{r(\gamma_m^{(i)})} \to \|g\|_z$ as $m \to \infty$. Since $f * g \in C_c(G)$ we can apply the continuity of the Haar system with (3.3) and (3.4) to see that

$$\big(\mathrm{L}^{y_m}(f)\eta_m^{(i)} \mid \eta_m^{(i)}\big) = \|g\|_{r(\gamma_m^{(i)})}^{-2} \int_G f * g(\alpha)g(\alpha) \, d\lambda_{r(\gamma_m^{(i)})}(\alpha) \to \|g\|_z^{-2} \int_G f * g(\alpha)g(\alpha) \, d\lambda_z(\alpha) = \big(\mathrm{L}^z(f)\eta \mid \eta\big)$$

as $m \to \infty$.

We have thus shown that, for each i,

$$\big(\mathrm{L}^{y_m}(\cdot)\eta_m^{(i)} \mid \eta_m^{(i)}\big) \to \big(\mathrm{L}^z(\cdot)\eta \mid \eta\big) = \phi$$

in the dual of $C^*(G)$ equipped with the weak* topology. Thus there exists m_1 such that for any $m \geqslant m_1$ and any $1 \leqslant i \leqslant k$, the pure state $(\mathrm{L}^{y_m}(\cdot)\eta_m^{(i)} \mid \eta_m^{(i)})$ is in $\phi + N$. We have now established that every $\eta_m^{(i)}$ with $m \geqslant \max\{m_0, m_1\}$ is in $\mathrm{Vec}(\mathrm{L}^{y_m}, \phi, N)$ with $\eta_m^{(i)} \perp \eta_m^{(j)}$ for $i \neq j$, so $d(\mathrm{L}^{y_m}, \phi, N) \geqslant k$ for all $m \geqslant \max\{m_0, m_1\}$, contradicting our choice of $\{y_m\}$ that in (3.1) had $d(\mathrm{L}^{y_m}, \phi, N) = r < k$ for all m. $\square$

4. Measure ratios and k-times convergence

In this section we show that lower bounds on measure ratios along orbits give strength of convergence in the orbit space. We begin by generalising [3, Proposition 4.1]. A subset S of a topological space X is *locally closed* if there exist an open set U of X and a closed set V of X such that $S = U \cap V$; this is equivalent to S being open in the closure of S with the subspace topology by, for example, [24, Lemma 1.25].

Proposition 4.1. *Let G be a second-countable locally-compact Hausdorff principal groupoid with Haar system λ. Let $k \in \mathbb{P}$ and $z \in G^{(0)}$ with $[z]$ locally closed in $G^{(0)}$. Assume that $\{x_n\}$ is a sequence in $G^{(0)}$ such that $[x_n] \to [z]$ uniquely in $G^{(0)}/G$. Suppose $\{W_m\}$ is a basic decreasing sequence of compact neighborhoods of z such that each m satisfies*

$$\liminf_n \lambda_{x_n}\big(G^{W_m}\big) > (k-1)\lambda_z\big(G^{W_m}\big).$$

Then $\{x_n\}$ converges k-times in $G^{(0)}/G$ to z.

Proof. Let $\{K_m\}$ be an increasing sequence of compact subsets of G such that $G = \bigcup_{m \geqslant 1} \mathrm{Int}\, K_m$. By the regularity of λ_z, for each $m \geqslant 1$ there exist $\delta_m > 0$ and an open neighborhood U_m of $G_z^{W_m}$ such that

$$\liminf_n \lambda_{x_n}\big(G^{W_m}\big) > (k-1)\lambda_z(U_m) + \delta_m. \tag{4.1}$$

We will construct, by induction, a strictly increasing sequence of positive integers $\{n_m\}$ such that, for all $n \geqslant n_m$,

$$\lambda_{x_n}\big(K_m\alpha \cap G^{W_m}\big) < \lambda_z(U_m) + \delta_m/k \quad \text{for all } \alpha \in G_{x_n}^{W_m}, \quad \text{and} \tag{4.2}$$

$$\lambda_{x_n}\big(G^{W_m}\big) > (k-1)\lambda_z(U_m) + \delta_m. \tag{4.3}$$

By applying Lemma 5.5 with $\delta = \lambda_z(U_1) - \lambda_z(G^{W_1}) + \delta_1/k$ there exists n_1 such that $n \geqslant n_1$ implies

$$\lambda_{x_n}\big(K_1\alpha \cap G^{W_1}\big) < \lambda_z(U_1) + \delta_1/k \quad \text{for all } \alpha \in G_{x_n}^{W_1},$$

establishing (4.2) for $m = 1$. If necessary we can increase n_1 to ensure (4.3) holds for $m = 1$ by considering (4.1). Assuming that we have constructed $n_1 < n_2 < \cdots < n_{m-1}$, we apply Lemma 5.5 with $\delta = \lambda_z(U_m) - \lambda_z(G^{W_m}) + \delta_m/k$ to obtain $n_m > n_{m-1}$ such that (4.2) holds, and again, if necessary, increase n_m to obtain (4.3).

If $n_1 > 1$ then, for each $1 \leqslant n < n_1$ and $1 \leqslant i \leqslant k$, let $\gamma_n^{(i)} = x_n$. For each $n \geqslant n_1$ there is a unique m such that $n_m \leqslant n < n_{m+1}$. For every such n and m choose $\gamma_n^{(1)} \in G_{x_n}^{W_m}$ (which is always non-empty by (4.3)). Using (4.2) and (4.3) we have

$$\begin{aligned}
\lambda_{x_n}\big(G^{W_m} \backslash K_m\gamma_n^{(1)}\big) &= \lambda_{x_n}\big(G^{W_m}\big) - \lambda_{x_n}\big(G^{W_m} \cap K_m\gamma_n^{(1)}\big) \\
&> \big((k-1)\lambda_z(U_m) + \delta_m\big) - \big(\lambda_z(U_m) + \delta_m/k\big) \\
&= (k-2)\lambda_z(U_m) + \frac{(k-1)}{k}\delta_m.
\end{aligned}$$

So for each $n \geqslant n_1$ and its associated m we can choose $\gamma_n^{(2)} \in G_{x_n}^{W_m} \backslash K_m\gamma_n^{(1)}$. We now have

$$\lambda_{x_n}\big(G^{W_m}\backslash(K_m\gamma_n^{(1)}\cup K_m\gamma_n^{(2)})\big) = \lambda_{x_n}\big(G^{W_m}\backslash K_m\gamma_n^{(1)}\big) - \lambda_{x_n}\big((G^{W_m}\backslash K_m\gamma_n^{(1)})\cap K_m\gamma_n^{(2)}\big)$$
$$\geqslant \lambda_{x_n}\big(G^{W_m}\backslash K_m\gamma_n^{(1)}\big) - \lambda_{x_n}\big(G^{W_m}\cap K_m\gamma_n^{(2)}\big)$$
$$> \bigg((k-2)\lambda_z(U_m) + \frac{(k-1)}{k}\delta_m\bigg) - \big(\lambda_z(U_m) + \delta_m/k\big)$$
$$= (k-3)\lambda_z(U_m) + \frac{(k-2)}{k}\delta_m,$$

enabling us to choose $\gamma_n^{(3)} \in G_{x_n}^{W_m}\backslash(K_m\gamma_n^{(1)}\cup K_m\gamma_n^{(2)})$. By continuing this process, for each $j = 3,\ldots,k$ and each $n \geqslant n_1$ we have

$$\lambda_{x_n}\Bigg(G^{W_m}\backslash\Bigg(\bigcup_{i=1}^{j-1}K_m\gamma_n^{(i)}\Bigg)\Bigg) > (k-j)\lambda_z(U_m) + \frac{(k-j-1)\delta_m}{k},$$

enabling us to choose

$$\gamma_n^{(j)} \in G_{x_n}^{W_m}\backslash\Bigg(\bigcup_{i=1}^{j-1}K_m\gamma_n^{(i)}\Bigg). \tag{4.4}$$

Note that for $n_m \leqslant n < n_{m+1}$ we have $\gamma_n^{(j)} \notin K_m\gamma_n^{(i)}$ for $1 \leqslant i < j \leqslant k$.

We will now establish that x_n converges k-times to z in $G^{(0)}/G$ by considering the $\gamma_n^{(i)}$. Note that $s(\gamma_n^{(i)}) = x_n$ for all n and i by our choice of the $\gamma_n^{(i)}$. To see that $r(\gamma_n^{(i)}) \to z$ as $n \to \infty$ for $1 \leqslant i \leqslant k$, first fix i and let V be an open neighborhood of z. Since $W_m \to \{z\}$ there exists m_0 such that $m \geqslant m_0$ implies $W_m \subset V$. For each $n \geqslant n_{m_0}$ there exists a $m \geqslant m_0$ such that $n_m \leqslant n < n_{m+1}$, and so $r(\gamma_n^{(i)}) \in W_m \subset V$.

Finally we claim that $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \to \infty$ as $n \to \infty$ for $1 \leqslant i < j \leqslant k$. Fix $i < j$ and let K be a compact subset of G. There exists m_0 such that $K \subset K_m$ for all $m \geqslant m_0$. Then for $n \geqslant n_{m_0}$ there exists $m \geqslant m_0$ such that $n_m \leqslant n < n_{m+1}$. By (4.4) we know

$$\gamma_n^{(j)} \in G_{x_n}^{W_m}\backslash(K_m\gamma_n^{(i)}) = \big(G_{x_n}^{W_m}(\gamma_n^{(i)})^{-1}\gamma_n^{(i)}\big)\backslash(K_m\gamma_n^{(i)}) = \big((G_{x_n}^{W_m}(\gamma_n^{(i)})^{-1})\backslash K_m\big)\gamma_n^{(i)},$$

and so $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \in (G_{x_n}^{W_m}(\gamma_n^{(i)})^{-1})\setminus K_m \subset G\setminus K_m \subset G\setminus K$, enabling us to conclude that $\{x_n\}$ converges k-times in $G^{(0)}/G$ to z. $\quad\square$

In Proposition 4.4 below we prove a generalisation of a part of [3, Proposition 4.2]; to do this we need the following two lemmas.

Lemma 4.2. *Suppose G is a second-countable groupoid with Haar system λ and let K be a compact subset of G. If $\{x_n\} \subset G^{(0)}$ is a sequence that converges to $z \in G^{(0)}$, then*

$$\limsup_n \lambda_{x_n}(K) \leqslant \lambda_z(K).$$

Proof. Fix $\epsilon > 0$. By the outer regularity of λ_z, there exists an open neighborhood U of K such that

$$\lambda_z(K) \leqslant \lambda_z(U) < \lambda_z(K) + \epsilon/2.$$

By Urysohn's Lemma there exists $f \in C_c(G)$ with $0 \leqslant f \leqslant 1$ such that f is identically one on K and zero off U. In particular we have

$$\lambda_z(K) \leqslant \int f\,d\lambda_z < \lambda_z(K) + \epsilon/2. \tag{4.5}$$

The continuity of the Haar system implies $\int f\,d\lambda_{x_n} \to \int f\,d\lambda_z$, so there exists n_0 such that $n \geqslant n_0$ implies

$$\int f\,d\lambda_z - \epsilon/2 < \int f\,d\lambda_{x_n} < \int f\,d\lambda_z + \epsilon/2.$$

By our choice of f we have $\lambda_{x_n}(K) \leqslant \int f\,d\lambda_{x_n}$, so

$$\lambda_{x_n}(K) \leqslant \int f\,d\lambda_{x_n} < \int f\,d\lambda_z + \epsilon/2.$$

Combining this with (4.5) enables us to observe that for $n \geqslant n_0$, $\lambda_{x_n}(K) < \lambda_z(K) + \epsilon$, completing the proof. $\quad\square$

Lemma 4.3. *Suppose G is a second-countable groupoid with Haar system λ and let K be a compact subset of G. For every $\epsilon > 0$ and $z \in G^{(0)}$ there exists a neighborhood U of z in $G^{(0)}$ such that $x \in U$ implies $\lambda_x(K) < \lambda_z(K) + \epsilon$.*

Proof. Fix $\epsilon > 0$ and $z \in G^{(0)}$. Let $\{U_n\}$ be a decreasing neighborhood basis for z in $G^{(0)}$. If our claim is false, then each U_n contains an element x_n such that $\lambda_{x_n}(K) \geqslant \lambda_z(K) + \epsilon$. But since each $x_n \in U_n$, $x_n \to z$, and so by Lemma 4.2 there exists n_0 such that $n \geqslant n_0$ implies $\lambda_{x_n}(K) < \lambda_z(K) + \epsilon$, a contradiction. □

Proposition 4.4. *Suppose G is a second-countable locally-compact Hausdorff groupoid with Haar system λ. Suppose that $z \in G^{(0)}$ with $[z]$ locally closed in $G^{(0)}$ and suppose $\{x_n\}$ is a sequence in $G^{(0)}$. Assume that for every open neighborhood V of z in $G^{(0)}$ such that G_z^V is relatively compact, $\lambda_{x_n}(G^V) \to \infty$ as $n \to \infty$. Then, for every $k \geqslant 1$, the sequence $\{x_n\}$ converges k-times in $G^{(0)}/G$ to z.*

Proof. Let $\{K_m\}$ be an increasing sequence of compact subsets of G such that $G = \bigcup_{m \geqslant 1} \operatorname{Int} K_m$. By Lemma 4.3, for each K_m there exists an open neighborhood V_m of z such that $x \in V_m$ implies $\lambda_x(K_m) < \lambda_z(K_m) + 1$. Since $[z]$ is locally closed, by Lemma 4.1(1) in [12] we can crop V_1 if necessary to ensure that $G_z^{V_1}$ is relatively compact. By further cropping each V_m we may assume that $\{V_m\}$ is a decreasing neighborhood basis of z. By our hypothesis, for each m there exists n_m such that

$$n \geqslant n_m \quad \text{implies} \quad \lambda_{x_n}(G^{V_m}) > k(\lambda_z(K_m) + 1). \tag{4.6}$$

Note that for any $\gamma \in G_{x_n}^{V_m}$ with $n \geqslant n_m$, we have $r(\gamma) \in V_m$, and so $\lambda_{r(\gamma)}(K_m) < \lambda_z(K_m) + 1$. By Haar-system invariance we know that $\lambda_{r(\gamma)}(K_m) = \lambda_{x_n}(K_m \gamma)$, which shows us that

$$\lambda_{x_n}(K_m \gamma) < \lambda_z(K_m) + 1. \tag{4.7}$$

If necessary we can increase the elements of $\{n_m\}$ so that it is a strictly increasing sequence.

We now proceed as in the proof of Proposition 4.1. For all $n < n_1$ and $1 \leqslant i \leqslant k$ let $\gamma_n^{(i)} = x_n$. For each $n \geqslant n_1$ there exists a unique number $m(n)$ such that $n_{m(n)} \leqslant n < n_{m(n)+1}$. For the remainder of this proof we will write m instead of $m(n)$ because the specific n will be clear from the context. For each $n \geqslant n_1$ choose $\gamma_n^{(1)} \in G_{x_n}^{V_m}$. Then by (4.6) and (4.7) we have

$$\begin{aligned}
\lambda_{x_n}(G^{V_m} \setminus K_m \gamma_n^{(1)}) &= \lambda_{x_n}(G^{V_m}) - \lambda_{x_n}(G^{V_m} \cap K_m \gamma_n^{(1)}) \\
&\geqslant \lambda_{x_n}(G^{V_m}) - \lambda_{x_n}(K_m \gamma_n^{(1)}) \\
&> k(\lambda_z(K_m) + 1) - (\lambda_z(K_m) + 1) \\
&= (k-1)(\lambda_z(K_m) + 1).
\end{aligned}$$

We can thus choose $\gamma_n^{(2)} \in G_{x_x}^{V_m} \setminus K_m \gamma_n^{(1)}$ for each $n \geqslant n_1$. This now gives us

$$\begin{aligned}
\lambda_{x_n}(G^{V_m} \setminus (K_m \gamma_n^{(1)} \cup K_m \gamma_n^{(2)})) &= \lambda_{x_n}(G^{V_m} \setminus K_m \gamma_n^{(1)}) - \lambda_{x_n}((G^{V_m} \setminus K_m \gamma_n^{(1)}) \cap K_m \gamma_n^{(1)}) \\
&\geqslant \lambda_{x_n}(G^{V_m} \setminus K_m \gamma_n^{(1)}) - \lambda_{x_n}(K_m \gamma_n^{(2)}) \\
&> (k-1)(\lambda_z(K_m) + 1) - (\lambda_z(K_m) + 1) \\
&= (k-2)(\lambda_z(K_m) + 1).
\end{aligned}$$

Continuing in this manner we can choose

$$\gamma_n^{(j)} \in G_{x_n}^{V_m} \setminus \left(\bigcup_{i=1}^{j-1} K_m \gamma_n^{(i)} \right)$$

for every $n \geqslant n_1$ and $j = 3, \dots, k$. The tail of the proof of Proposition 4.1 establishes our desired result. □

5. Measure ratios and bounds on lower multiplicity

In this section we show that upper bounds on measure ratios along orbits give upper bounds on multiplicities.

Lemma 5.1. *Suppose G is a second-countable locally-compact Hausdorff groupoid. Suppose $z \in G^{(0)}$ and $[z]$ is locally closed. Then the restriction of r to $G_z/(G|_{\{z\}})$ is a homeomorphism onto $[z]$. If in addition G is principal, then the restriction of r to G_z is a homeomorphism onto $[z]$.*

Proof. We consider the transitive groupoid $G|_{[z]}$. Since $[z]$ is locally closed, $G|_{[z]}$ is a second-countable locally-compact Hausdorff groupoid. Thus $G|_{[z]}$ is Polish by, for example, [24, Lemma 6.5]. Now [22, Theorem 2.1] applies to give the result. □

Theorem 5.2 is based on [3, Theorem 3.1]; it is only an intermediary result which will be used to prove a sharper bound in Theorem 5.8.

Theorem 5.2. *Suppose G is a second-countable locally-compact Hausdorff principal groupoid with Haar system λ. Let $M \in \mathbb{R}$ with $M \geqslant 1$, suppose $z \in G^{(0)}$ such that $[z]$ is locally closed and let $\{x_n\}$ be a sequence in $G^{(0)}$. Suppose there exists an open neighborhood V of z in $G^{(0)}$ such that G_z^V is relatively compact and*

$$\lambda_{x_n}(G^V) \leqslant M\lambda_z(G^V)$$

frequently (in the sense that there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lambda_{x_{n_i}}(G^V) \leqslant M\lambda_z(G^V)$ for all i). Then $\mathrm{M_L}(\mathrm{L}^z, \{\mathrm{L}^{x_n}\}) \leqslant \lfloor M^2 \rfloor$.

Proof. Fix $\epsilon > 0$ such that $M^2(1+\epsilon)^2 < \lfloor M^2 \rfloor + 1$. We will build a function $D \in C_c(G)$ such that $\mathrm{L}^z(D^* * D)$ is a rank-one projection and

$$\mathrm{Tr}\big(\mathrm{L}^{x_n}(D^* * D)\big) < M^2(1+\epsilon)^2 < \lfloor M^2 \rfloor + 1$$

frequently. By the generalised lower semi-continuity result of [9, Theorem 4.3] we will have

$$\liminf \mathrm{Tr}\big(\mathrm{L}^{x_n}(D^* * D)\big) \geqslant \mathrm{M_L}\big(\mathrm{L}^z, \{\mathrm{L}^{x_n}\}\big)\mathrm{Tr}\big(\mathrm{L}^z(D^* * D)\big) = \mathrm{M_L}\big(\mathrm{L}^z, \{\mathrm{L}^{x_n}\}\big),$$

and the result will follow.

For the next few paragraphs we will be working with G_z equipped with the subspace topology. Note that λ_z can be thought of as a Radon measure on G_z with $\lambda_z(S \cap G_z) = \lambda_z(S)$ for any λ_z-measurable subset S of G. Fix $\delta > 0$ such that

$$\delta < \frac{\epsilon\lambda_z(G^V)}{1+\epsilon} < \lambda_z(G^V).$$

Since λ_z is inner regular on open sets and G_z^V is G_z-open, there exists a G_z-compact subset W of G_z^V such that

$$0 < \lambda_z(G_z^V) - \delta < \lambda_z(W).$$

Since W is G_z-compact there exists a G_z-compact neighborhood W_1 of W that is contained in G_z^V and there exists a continuous function $g : G_z \to [0, 1]$ that is identically one on W and zero off the interior of W_1. We have

$$\lambda_z(G^V) - \delta = \lambda_z(G_z^V) - \delta < \lambda_z(W) \leqslant \int_{G_z} g(t)^2 \, d\lambda_z(t) = \|g\|_z^2,$$

and hence

$$\frac{\lambda_z(G^V)}{\|g\|_z^2} < 1 + \frac{\delta}{\|g\|_z^2} < 1 + \frac{\delta}{\lambda_z(G^V) - \delta} < 1 + \epsilon. \tag{5.1}$$

By Lemma 5.1 the restriction $\tilde{r}$ of r to G_z is a homeomorphism onto $[z]$. So there exists a continuous function $g_1 : \tilde{r}(W_1) \to [0, 1]$ such that $g_1(\tilde{r}(\gamma)) = g(\gamma)$ for all $\gamma \in W_1$. Thus $\tilde{r}(W_1)$ is $[z]$-compact, which implies that $\tilde{r}(W_1)$ is $G^{(0)}$-compact. Since we know that $G^{(0)}$ is second countable and Hausdorff, Tietze's Extension Theorem can be applied to extend g_1 to a continuous map $g_2 : G^{(0)} \to [0, 1]$. Because $\tilde{r}(W_1)$ is a compact subset of the open set V, there exist a compact neighborhood P of $\tilde{r}(W_1)$ contained in V and a continuous function $h : G^{(0)} \to [0, 1]$ that is identically one on $\tilde{r}(W_1)$ and zero off the interior of P. Note that h has compact support that is contained in P.

We set $f(x) = h(x)g_2(x)$. Then $f \in C_c(G^{(0)})$ with $0 \leqslant f \leqslant 1$ and

$$\mathrm{supp}\, f \subset \mathrm{supp}\, h \subset P \subset V. \tag{5.2}$$

Note that

$$\|f \circ r\|_z^2 = \int_{G_z} f\big(\tilde{r}(\gamma)\big)^2 \, d\lambda_z(\gamma)$$

$$= \int_{G_z} h\big(\tilde{r}(\gamma)\big)^2 g_2\big(\tilde{r}(\gamma)\big)^2 \, d\lambda_z(\gamma)$$

$$\geqslant \int_{W_1} h\big(\tilde{r}(\gamma)\big)^2 g(\gamma)^2 \, d\lambda_z(\gamma)$$

$$= \int_{W_1} g(\gamma)^2 \, d\lambda_z(\gamma)$$

$$= \|g\|_z^2 \tag{5.3}$$

since $\operatorname{supp} g \subset W_1$ and h is identically one on $\tilde{r}(W_1)$. We now define $F \in C_c(G^{(0)})$ by

$$F(x) = \frac{f(x)}{\|f \circ r\|_z}. \tag{5.4}$$

Then $\|F \circ r\|_z = 1$ and

$$F \circ r(\gamma) \neq 0 \quad \Longrightarrow \quad h(r(\gamma)) \neq 0 \quad \Longrightarrow \quad r(\gamma) \in V \quad \Longrightarrow \quad \gamma \in G^V. \tag{5.5}$$

Let $N = \operatorname{supp} F$ so that $N = \operatorname{supp} f \subset V$ by (5.2) and (5.4). Since G_z^V is relatively compact by our hypothesis, the set $\overline{G_z^N}$ is compact. Let $b \in C_c(G)$ be a function that is identically one on $(\overline{G_z^N})(\overline{G_z^N})^{-1}$ and has range contained in $[0, 1]$. We can assume that b is self-adjoint by considering $\frac{1}{2}(b + b^*)$ if necessary. Define $D \in C_c(G)$ by

$$D(\gamma) := F(r(\gamma))F(s(\gamma))b(\gamma).$$

For $\xi \in L^2(G, \lambda_u)$ and $\gamma \in G$ we have

$$
\begin{aligned}
(L^u(D)\xi)(\gamma) &= \int_G D(\gamma\alpha^{-1})\xi(\alpha)\,d\lambda_u(\alpha) \\
&= \int_G F(r(\gamma))F(s(\alpha^{-1}))b(\gamma\alpha^{-1})\xi(\alpha)\,d\lambda_u(\alpha) \\
&= F(r(\gamma))\int_G F(r(\alpha))b(\gamma\alpha^{-1})\xi(\alpha)\,d\lambda_u(\alpha).
\end{aligned}
$$

In the case where $u = z$, if $\alpha, \gamma \in \operatorname{supp} F \circ r \cap s^{-1}(z)$, then $r(\alpha), r(\gamma) \in \operatorname{supp} F = N$ and $\gamma, \alpha \in G_z^N$. This implies $b(\gamma\alpha^{-1}) = 1$, so

$$(L^z(D)\xi)(\gamma) = \int_G F(r(\alpha))\xi(\alpha)\,d\lambda_z(\alpha) = (\xi \mid F \circ r)_z F \circ r(\gamma),$$

and $L^z(D)$ is a rank-one projection.

By the hypothesis on V there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lambda_{x_{n_i}}(G^V) \leqslant M\lambda_z(G^V)$$

for all $i \geqslant 1$. If we define $E := \{\gamma \in G \colon F(r(\gamma)) \neq 0\}$ then E is open with

$$\lambda_{x_{n_i}}(E) \leqslant \lambda_{x_{n_i}}(G^V) \leqslant M\lambda_z(G^V) \tag{5.6}$$

by (5.5) and

$$\int_G (F \circ r(\gamma))^2\,d\lambda_{x_{n_i}}(\gamma) \leqslant \frac{\lambda_{x_{n_i}}(E)}{\|f \circ r\|_z^2} \leqslant \frac{M\lambda_z(G^V)}{\|g\|_z^2} \tag{5.7}$$

by (5.3). Consider the continuous function $T(\alpha, \beta) := F(r(\alpha))F(r(\beta))b(\alpha\beta^{-1})$. Note that

$$
\begin{aligned}
\int_G T(\alpha, \beta)^2\,d(\lambda_{x_{n_i}} \times \lambda_{x_{n_i}})(\alpha, \beta) &= \int_G F(r(\alpha))^2 F(r(\beta))^2 b(\alpha\beta^{-1})^2\,d(\lambda_{x_{n_i}} \times \lambda_{x_{n_i}})(\alpha, \beta) \\
&\leqslant \|F\|_\infty^4 \int_G \chi_{E \times E}(\alpha, \beta)\,d(\lambda_{x_{n_i}} \times \lambda_{x_{n_i}})(\alpha, \beta) \\
&= \|F\|_\infty^4 \lambda_{x_{n_i}}(E)^2,
\end{aligned}
$$

which is finite by (5.6). Thus

$$T \in L^2(G \times G, \lambda_{x_{n_i}} \times \lambda_{x_{n_i}}),$$

and since T is conjugate symmetric, [20, Proposition 3.4.16] implies that $L^{x_{n_i}}(D)$ is the self-adjoint Hilbert–Schmidt operator on $L^2(G, \lambda_{x_{n_i}})$ with kernel T. It follows that $L^{x_{n_i}}(D^* * D)$ is a trace-class operator, and since we equip the Hilbert–Schmidt operators with the trace norm, we have

$$\operatorname{Tr} L^{x_{n_i}}(D^* * D) = \|T\|_{L^2(\lambda_{x_{n_i}} \times \lambda_{x_{n_i}})}^2.$$

Applying Fubini's Theorem to T now gives

$$
\begin{aligned}
\operatorname{Tr} L^{x_{n_i}}\big(D^* * D\big) &= \int_G \int_G F\big(r(\alpha)\big)^2 F\big(r(\beta)\big)^2 b\big(\alpha\beta^{-1}\big)^2 \, d\lambda_{x_{n_i}}(\alpha)\, d\lambda_{x_{n_i}}(\beta) \\
&\leqslant \left(\int_G F\big(r(\alpha)\big)^2 \, d\lambda_{x_{n_i}}(\alpha) \right)^2 \\
&\leqslant \frac{M^2 \lambda_z(G^V)^2}{\|g\|_z^4} \quad \text{(using (5.7))} \\
&< M^2(1+\epsilon)^2 \quad \text{(using (5.1))}.
\end{aligned}
\tag{5.8}
$$

Now

$$
\operatorname{M}_L\big(L^z, \{L^{x_n}\}\big) \leqslant \liminf_n \operatorname{Tr}\big(L^{x_n}\big(D^* * D\big)\big) \leqslant M^2(1+\epsilon)^2 < \lfloor M^2 \rfloor + 1,
$$

and hence $\operatorname{M}_L(L^z, \{L^{x_n}\}) \leqslant \lfloor M^2 \rfloor$, completing the proof. $\quad\square$

The following proposition is an immediate consequence of Theorem 5.2 and Proposition 4.4. This result will be strengthened later in Corollary 6.5, where we will show that these three items are in fact equivalent.

Proposition 5.3. *Suppose G is a second-countable locally-compact Hausdorff principal groupoid with Haar system λ. Let $z \in G^{(0)}$ and let $\{x_n\}$ be a sequence in $G^{(0)}$. Assume that $[z]$ is locally closed in $G^{(0)}$. Consider the following properties.*

(1) $\operatorname{M}_L(L^z, \{L^{x_n}\}) = \infty$.
(2) *For every open neighborhood V of z such that G_z^V is relatively compact, $\lambda_{x_n}(G^V) \to \infty$ as $n \to \infty$.*
(3) *For each $k \geqslant 1$, the sequence $\{x_n\}$ converges k-times in $G^{(0)}/G$ to z.*

Then (1) *implies* (2) *and* (2) *implies* (3).

Our next goal is to sharpen the $\lfloor M^2 \rfloor$ bound in Theorem 5.2. This strengthened theorem appears later on as Theorem 5.8. We will first establish several results to assist in strengthening this bound.

Lemma 5.4. *Suppose G is a second-countable groupoid and $x, y \in G^{(0)}$. If $\overline{[x]} = \overline{[y]}$ and $[x]$ is locally closed, then $[x] = [y]$.*

Proof. We have $x \in \overline{[y]}$, so there exists $\{\gamma_n\} \subset G$ such that $s(\gamma_n) = y$ and $r(\gamma_n) \to x$. Since $[x]$ is locally closed, there exists an open subset U of G such that $[x] = U \cap \overline{[x]}$. Then $r(\gamma_n)$ is eventually in U, so eventually $r(\gamma_n) \in U \cap \overline{[y]} = U \cap \overline{[x]} = [x]$. Thus there exists $\gamma \in G$ with $s(\gamma) = y$ and $r(\gamma) \in [x]$, as required. $\quad\square$

Lemma 5.5. *Suppose G is a second-countable groupoid with Haar system λ. Let W be a compact neighborhood of $z \in G^{(0)}$ and let K be a compact subset of G. Let $\{x_n\}$ be a sequence in $G^{(0)}$ such that $[x_n] \to [z]$ uniquely in $G^{(0)}/G$. Then for every $\delta > 0$ there exists n_0 such that, for every $n \geqslant n_0$ and every $\gamma \in G_{x_n}^W$,*

$$
\lambda_{x_n}\big(K\gamma \cap G^W\big) < \lambda_z\big(G^W\big) + \delta.
$$

Proof. Suppose not. Then, by passing to a subsequence if necessary, for each n there exists $\gamma_n \in G_{x_n}^W$ such that

$$
\lambda_{x_n}\big(K\gamma_n \cap G^W\big) \geqslant \lambda_z\big(G^W\big) + \delta.
\tag{5.9}
$$

Since each $r(\gamma_n)$ is in the compact set W, we can pass to a subsequence so that $r(\gamma_n) \to y$ for some $y \in G^{(0)}$. This implies $[r(\gamma_n)] \to [y]$, but $[r(\gamma_n)] = [s(\gamma_n)] = [x_n]$ and $[x_n] \to [z]$ uniquely, so $[y] = [z]$. Choose $\psi \in G$ with $s(\psi) = z$ and $r(\psi) = y$. By Haar-system invariance

$$
\lambda_{x_n}\big(K\gamma_n \cap G^W\big) = \lambda_{r(\gamma_n)}\big(K \cap G^W\big),
$$

so by applying Lemma 4.2 with the compact space $K \cap G^W$ and $\{r(\gamma_n)\}$ converging to y,

$$
\begin{aligned}
\limsup_n \lambda_{x_n}\big(K\gamma_n \cap G^W\big) &= \limsup_n \lambda_{r(\gamma_n)}\big(K \cap G^W\big) \\
&\leqslant \lambda_y\big(K \cap G^W\big) \quad \text{(by Lemma 4.2)} \\
&= \lambda_z\big(K\psi \cap G^W\big) \quad \text{(Haar-system invariance)} \\
&\leqslant \lambda_z\big(G^W\big).
\end{aligned}
$$

This contradicts our assertion (5.9). $\quad\square$

The following is a generalisation of [3, Lemma 3.3].

Lemma 5.6. *Suppose G is a groupoid with Haar system λ. Fix $\epsilon > 0$, $z \in G^{(0)}$ and let V be an open neighborhood of $z \in G^{(0)}$ such that $\lambda_z(G^V) < \infty$. Then there exists an open relatively-compact neighborhood V_1 of z such that $\overline{V_1} \subset V$ and*

$$\lambda_z(G^V) - \epsilon < \lambda_z(G^{V_1}) \leqslant \lambda_z(G^{\overline{V_1}}) \leqslant \lambda_z(G^V) < \lambda_z(G^{V_1}) + \epsilon.$$

Proof. We use G_z equipped with the subspace topology to find a compact subset λ_z-estimate of V. This estimate is then used to obtain the required open set V_1. Since G_z^V is G_z-open, by the regularity of λ_z there exists a compact subset W of G_z^V such that $\lambda_z(W) > \lambda_z(G_z^V) - \epsilon$. Then $r(W)$ is compact and contained in V, so there exists an open relatively-compact neighborhood V_1 of $r(W)$ such that $\overline{V_1} \subset V$. Then

$$\lambda_z(G^V) - \epsilon < \lambda_z(W) \leqslant \lambda_z(G^{V_1}) \leqslant \lambda_z(G^{\overline{V_1}}) \leqslant \lambda_z(G^V) < \lambda_z(W) + \epsilon \leqslant \lambda_z(G^{V_1}) + \epsilon,$$

as required.   □

The following lemma is equivalent to the claim in [10, Proposition 3.6] that $[x] \mapsto [L^x]$ from $G^{(0)}/G$ to the spectrum of $C^*(G)$ is open.

Lemma 5.7. *Suppose G is a second-countable locally-compact Hausdorff groupoid with Haar system λ. If $\{x_n\}$ is a sequence in $G^{(0)}$ with $L^{x_n} \to L^z$, then $[x_n] \to [z]$.*

Proof. We prove the contrapositive. Suppose $[x_n] \nrightarrow [z]$. Then there exists an open neighborhood U_0 of $[z]$ in $G^{(0)}/G$ such that $[x_n]$ is frequently not in U_0. Let $q : G^{(0)} \to G^{(0)}/G$ be the quotient map $x \mapsto [x]$. Then $U_1 := q^{-1}(U_0)$ is an open invariant neighborhood of z and $x_n \notin U_1$ frequently. Note that $C^*(G|_{U_1})$ is isomorphic to a closed two-sided ideal I of $C^*(G)$ (see [18, Lemma 2.10]).

We now claim that $I \subset \ker L^{x_n}$ whenever $x_n \notin U_1$. Suppose $x_n \notin U_1$ and recall from Remark 2.4 that L^{x_n} acts on $L^2(G, \lambda_{x_n})$. Fix $f \in C_c(G)$ such that $f(\gamma) = 0$ whenever $\gamma \notin G|_{U_1}$ and fix $\xi \in L^2(G, \lambda_{x_n})$. Then by Remark 2.4 we have

$$\|L^{x_n}(f)\xi\|_{x_n}^2 = \int_G \left(\int_G f(\gamma\alpha^{-1})\xi(\alpha) \, d\lambda_{x_n}(\alpha) \right)^2 d\lambda_{x_n}(\gamma).$$

When evaluating the inner integrand, we have $s(\alpha) = s(\gamma) = x_n$, so $\gamma\alpha^{-1} \in G|_{[x_n]}$. Since U_1 is invariant with $x_n \notin U_1$, it follows that $\gamma\alpha^{-1} \notin G|_{U_1}$, and so $f(\gamma\alpha^{-1}) = 0$. Thus $\|L^{x_n}(f)\xi\|_{x_n} = 0$, and since ξ was fixed arbitrarily, $L^{x_n}(f) = 0$. This implies that $I \subset \ker L^{x_n}$.

We now conclude by observing that since $I \subset \ker L^{x_n}$ frequently, $L^{x_n} \notin \hat{I}$ frequently. But $\hat{I}$ is an open neighborhood of L^z, so $L^{x_n} \nrightarrow L^z$.   □

We may now proceed to strengthening the $\lfloor M^2 \rfloor$ bound in Theorem 5.2. This theorem is a generalisation of [3, Theorem 3.5].

Theorem 5.8. *Suppose G is a second-countable locally-compact Hausdorff principal groupoid with Haar system λ. Let $M \in \mathbb{R}$ with $M \geqslant 1$, suppose $z \in G^{(0)}$ such that $[z]$ is locally closed and let $\{x_n\}$ be a sequence in $G^{(0)}$. Suppose there exists an open neighborhood V of z in $G^{(0)}$ such that G_z^V is relatively compact and*

$$\lambda_{x_n}(G^V) \leqslant M\lambda_z(G^V)$$

frequently. Then $M_L(L^z, \{L^{x_n}\}) \leqslant \lfloor M \rfloor$.

Proof. If L^{x_n} does not converge to L^z, then $M_L(L^z, \{L^{x_n}\}) = 0 < \lfloor M \rfloor$. So we assume from now on that $L^{x_n} \to L^z$. Lemma 5.7 now shows that $[x_n] \to [z]$. Next we claim that we may assume, without loss of generality, that $[z]$ is the unique limit of $\{[x_n]\}$ in $G^{(0)}/G$. To see this, note that $M_L(L^z, \{L^{x_n}\}) \leqslant \lfloor M^2 \rfloor < \infty$ by Theorem 5.2. Hence, by [3, Proposition 3.4], $\{L^z\}$ is open in the set of limits of $\{L^{x_n}\}$. So there exists an open neighborhood U_2 of L^z in $C^*(G)^\wedge$ such that L^z is the unique limit of $\{L^{x_n}\}$ in U_2. By [19, Proposition 2.5] there is a continuous function $L : G^{(0)}/G \to C^*(G)^\wedge$ such that $[x] \mapsto L^x$ for all $x \in G^{(0)}$. Define $p : G^{(0)} \to G^{(0)}/G$ by $p(x) = [x]$ for all $x \in G^{(0)}$. Then p is continuous, and

$$Y := (L \circ p)^{-1}(U_2)$$

is an open G-saturated neighborhood of z in $G^{(0)}$. Note that $x_n \in Y$ eventually.

Now suppose that, for some $y \in Y$, $[x_n] \to [y]$ in Y/G and hence in $G^{(0)}/G$. Then $L^{x_n} \to L^y$ by [19, Proposition 2.5], and $L^y \in U_2$ since $y \in (L \circ p)^{-1}(U_2)$. But $\{L^{x_n}\}$ has the unique limit L^z in U_2, so $L^z = L^y$ and hence $\overline{[z]} = \overline{[y]}$. Since $[z]$ is locally closed, Lemma 5.4 shows that $[z] = [y]$ in $G^{(0)}$ and hence in Y.

We know Y is an open saturated subset of $G^{(0)}$, so $C^*(G|_Y)$ is isomorphic to a closed two-sided ideal J of $C^*(G)$. We can apply [8, Proposition 5.3] with the C^*-subalgebra J to see that $M_L(L^z, \{L^{x_n}\})$ is the same whether we compute it in the

ideal J or in $C^*(G)$. Since Y is G-invariant, $G_z^V = G_z^{V \cap Y}$ and eventually $G_{x_n}^V = G_{x_n}^{V \cap Y}$. We may thus consider $G|_Y$ instead of G and therefore assume that $[z]$ is the unique limit of $[x_n]$ in $G^{(0)}/G$ as claimed.

As in [3], the idea for the rest of the proof is the same as in Theorem 5.2, although more precise estimates are used. Fix $\epsilon > 0$ such that $M(1+\epsilon)^2 < \lfloor M \rfloor + 1$ and choose $\kappa > 0$ such that

$$\kappa < \frac{\epsilon \lambda_z(G^V)}{1+\epsilon} < \lambda_z(G^V). \tag{5.10}$$

By Lemma 5.6 there exists an open relatively compact neighborhood V_1 of z such that $\overline{V_1} \subset V$ and

$$0 < \lambda_z(G^V) - \kappa < \lambda_z(G^{V_1}) \leqslant \lambda_z(G^{\overline{V_1}}) \leqslant \lambda_z(G^V) < \lambda_z(G^{V_1}) + \kappa.$$

Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lambda_{x_{n_i}}(G^V) \leqslant M\lambda_z(G^V)$$

for all $i \geqslant 1$. Then

$$
\begin{aligned}
\lambda_{x_{n_i}}(G^{V_1}) &\leqslant \lambda_{x_{n_i}}(G^V) \\
&\leqslant M\lambda_z(G^V) \\
&< M(\lambda_z(G^{V_1}) + \kappa) \\
&< M\lambda_z(G^{V_1}) + M\epsilon(\lambda_z(G^V) - \kappa) \quad \text{(by (5.10))} \\
&< M\lambda_z(G^{V_1}) + M\epsilon\lambda_z(G^{V_1}) \\
&= M(1+\epsilon)\lambda_z(G^{V_1})
\end{aligned}
\tag{5.11}
$$

for all i. Since

$$\frac{\lambda_z(G^{V_1})(\lambda_z(G^{V_1}) + \kappa + 1/j)}{(\lambda_z(G^{V_1}) - 1/j)^2} \to 1 + \frac{\kappa}{\lambda_z(G^{V_1})} < 1 + \epsilon$$

as $j \to \infty$, there exists $\delta > 0$ such that $\delta < \lambda_z(G^{V_1})$ and

$$\frac{\lambda_z(G^{V_1})(\lambda_z(G^{\overline{V_1}}) + \delta)}{(\lambda_z(G^{V_1}) - \delta)^2} < \frac{\lambda_z(G^{V_1})(\lambda_z(G^{V_1}) + \kappa + \delta)}{(\lambda_z(G^{V_1}) - \delta)^2} < 1 + \epsilon. \tag{5.12}$$

We will now construct a function $F \in C_c(G^{(0)})$ with support inside V_1. Since λ_z is inner regular on open sets and $G_z^{V_1}$ is G_z-open, there exists a G_z-compact subset W of $G_z^{V_1}$ such that

$$0 < \lambda_z(G_z^{V_1}) - \delta < \lambda_z(W).$$

Since W is G_z-compact there exists a G_z-compact neighborhood W_1 of W that is contained in $G_z^{V_1}$ and there exists a continuous function $g : G_z \to [0, 1]$ that is identically one on W and zero off the interior of W_1. We have

$$\lambda_z(G^{V_1}) - \delta < \lambda_z(W) \leqslant \int_{G_z} g(t)^2 \, d\lambda_z(t) = \|g\|_z^2. \tag{5.13}$$

By Lemma 5.1 the restriction $\tilde{r}$ of r to G_z is a homeomorphism onto $[z]$. So there exists a continuous function $g_1 : \tilde{r}(W_1) \to [0, 1]$ such that $g_1(\tilde{r}(\gamma)) = g(\gamma)$ for all $\gamma \in W_1$. Thus $\tilde{r}(W_1)$ is $[z]$-compact, which implies that $\tilde{r}(W_1)$ is $G^{(0)}$-compact. Since we know that $G^{(0)}$ is second countable and Hausdorff, Tietze's Extension Theorem can be applied to show that g_1 can be extended to a continuous map $g_2 : G^{(0)} \to [0, 1]$. Because $\tilde{r}(W_1)$ is a compact subset of the open set V_1, there exist a compact neighborhood P of $\tilde{r}(W_1)$ contained in V_1 and a continuous function $h : G^{(0)} \to [0, 1]$ that is identically one on $\tilde{r}(W_1)$ and zero off the interior of P. Note that h has compact support that is contained in P.

We set $f(x) = h(x)g_2(x)$. Then $f \in C_c(G^{(0)})$ with $0 \leqslant f \leqslant 1$ and

$$\operatorname{supp} f \subset \operatorname{supp} h \subset P \subset V_1. \tag{5.14}$$

Note that

$$
\begin{aligned}
\|f \circ r\|_z^2 &= \int_{G_z} f(\tilde{r}(\gamma))^2 \, d\lambda_z(\gamma) \\
&= \int_{G_z} h(\tilde{r}(\gamma))^2 g_2(\tilde{r}(\gamma))^2 \, d\lambda_z(\gamma)
\end{aligned}
$$

$$\geqslant \int\limits_{W_1} h\big(\tilde{r}(\gamma)\big)^2 g(\gamma)^2 \, d\lambda_z(\gamma)$$

$$= \int\limits_{W_1} g(\gamma)^2 \, d\lambda_z(\gamma)$$

$$= \|g\|_z^2 \tag{5.15}$$

since supp $g \subset W_1$ and h is identically one on $\tilde{r}(W_1)$. We now define $F \in C_c(G^{(0)})$ by

$$F(x) = \frac{f(x)}{\|f \circ r\|_z}. \tag{5.16}$$

Then $\|F \circ r\|_z = 1$ and

$$F \circ r(\gamma) \neq 0 \implies h\big(r(\gamma)\big) \neq 0 \implies r(\gamma) \in V_1 \implies \gamma \in G^{V_1}. \tag{5.17}$$

Let $N = \operatorname{supp} F$. Suppose K is an open relatively compact symmetric neighborhood of $(\overline{G_z^N})(\overline{G_z^N})^{-1}$ in G and choose $b \in C_c(G)$ such that b is identically one on $(\overline{G_z^N})(\overline{G_z^N})^{-1}$ and identically zero off K. As in Theorem 5.2 we may assume that b is self-adjoint by considering $\frac{1}{2}(b + b^*)$. Define $D \in C_c(G)$ by $D(\gamma) := F(r(\gamma))F(s(\gamma))b(\gamma)$. By the same argument as in Theorem 5.2, $L^z(D)$, and hence $L^z(D^* * D)$, is the rank one projection determined by the unit vector $F \circ r \in L^2(G, \lambda_z)$. From (5.8) we have

$$\operatorname{Tr}\big(L^{x_{n_i}}\big(D^* * D\big)\big) = \int\limits_G F\big(r(\beta)\big)^2 \left(\int\limits_G F\big(r(\alpha)\big)^2 b\big(\alpha\beta^{-1}\big)^2 \, d\lambda_{x_{n_i}}(\alpha) \right) d\lambda_{x_{n_i}}(\beta).$$

Since b is identically zero off K, the inner integrand is zero unless $\alpha\beta^{-1} \in K$. Combining this with (5.14) and the fact that $\operatorname{supp} \lambda_{x_{n_i}} \subset G_{x_{n_i}}$ enables us to see that this inner integrand is zero unless $\alpha \in G_{x_{n_i}}^{V_1} \cap K\beta$. Thus

$$\operatorname{Tr}\big(L^{x_{n_i}}\big(D^* * D\big)\big) \leqslant \int\limits_{\beta \in G_{x_{n_i}}^{V_1}} F\big(r(\beta)\big)^2 \left(\int\limits_{\alpha \in G_{x_{n_i}}^{V_1} \cap K\beta} F\big(r(\alpha)\big)^2 \, d\lambda_{x_{n_i}}(\alpha) \right) d\lambda_{x_{n_i}}(\beta).$$

$$\leqslant \frac{1}{\|f \circ r\|_z^4} \int\limits_{\beta \in G_{x_{n_i}}^{V_1}} 1 \left(\int\limits_{\alpha \in G_{x_{n_i}}^{V_1} \cap K\beta} 1 \, d\lambda_{x_{n_i}}(\alpha) \right) d\lambda_{x_{n_i}}(\beta).$$

Since $\overline{V_1}$ and $\overline{K}$ are compact, by Lemma 5.5 there exists i_0 such that for every $i \geqslant i_0$ and any $\beta \in G_{x_{n_i}}^{\overline{V_1}}$,

$$\lambda_{x_{n_i}}\big(K\beta \cap G^{\overline{V_1}}\big) < \lambda_z\big(G^{\overline{V_1}}\big) + \delta.$$

So, provided $i \geqslant i_0$,

$$\operatorname{Tr}\big(L^{x_{n_i}}\big(D^* * D\big)\big) \leqslant \frac{1}{\|f \circ r\|_z^4} \int\limits_{\beta \in G_{x_{n_i}}^{V_1}} \lambda_{x_{n_i}}\big(K\beta \cap G_{x_{n_i}}^{V_1}\big) \, d\lambda_{x_{n_i}}(\beta)$$

$$\leqslant \frac{1}{\|f \circ r\|_z^4} \int\limits_{\beta \in G_{x_{n_i}}^{V_1}} \big(\lambda_z\big(G_z^{\overline{V_1}}\big) + \delta\big) \, d\lambda_{x_{n_i}}(\beta)$$

$$< \frac{(\lambda_z(G^{\overline{V_1}}) + \delta)\lambda_{x_{n_i}}(G^{V_1})}{\|f \circ r\|_z^4}$$

$$< \frac{M(1 + \epsilon)(\lambda_z(G^{\overline{V_1}}) + \delta)\lambda_z(G^{V_1})}{\|g\|_z^4} \quad (\text{by (5.11) and (5.15)})$$

$$< \frac{M(1 + \epsilon)(\lambda_z(G^{\overline{V_1}}) + \delta)\lambda_z(G^{V_1})}{(\lambda_z(G^{V_1}) - \delta)^2} \quad (\text{by (5.13)})$$

$$< M(1 + \epsilon)^2 \quad (\text{by (5.12)}).$$

We can now make our conclusion as in [3, Theorem 3.5]: by generalised lower semi-continuity [9, Theorem 4.3],

$$\liminf_n \operatorname{Tr}\bigl(L^{x_n}\bigl(D^* * D\bigr)\bigr) \geqslant M_L\bigl(L^z, \bigl\{L^{x_n}\bigr\}\bigr) \operatorname{Tr}\bigl(L^z\bigl(D^* * D\bigr)\bigr) = M_L\bigl(L^z, \bigl\{L^{x_n}\bigr\}\bigr).$$

We now have

$$M_L\bigl(L^z, \bigl\{L^{x_n}\bigr\}\bigr) \leqslant \liminf_n \operatorname{Tr}\bigl(L^{x_n}\bigl(D^* * D\bigr)\bigr) \leqslant M(1+\epsilon)^2 < \lfloor M \rfloor + 1,$$

and so $M_L(L^z, \{L^{x_n}\}) \leqslant \lfloor M \rfloor$, as required. $\quad\square$

6. Lower multiplicity and *k*-times convergence II

We proved in Proposition 3.2 that if a sequence converges *k*-times in the orbit space of a principal groupoid, then the lower multiplicity of the associated sequence of representations is at least *k*. In this section we will prove the converse. The first result in this section generalises [3, Lemma 5.1]; with the exception of notation changes, the proof is the same as the proof in [3].

Lemma 6.1. *Suppose G is a second-countable locally-compact Hausdorff principal groupoid. Let $k \in \mathbb{P}$, $z \in G^{(0)}$, and $\{x_n\}$ be a sequence in $G^{(0)}$. Assume that $[z]$ is locally closed in $G^{(0)}$ and that there exists $R > k - 1$ such that for every open neighborhood U of z with G_z^U relatively compact we have*

$$\liminf_n \lambda_{x_n}\bigl(G^U\bigr) \geqslant R\lambda_z\bigl(G^U\bigr).$$

Given an open neighborhood V of z such that G_z^V is relatively compact, there exists a compact neighborhood N of z with $N \subset V$ such that

$$\liminf_n \lambda_{x_n}\bigl(G^N\bigr) > (k-1)\lambda_z\bigl(G^N\bigr).$$

Proof. Apply Lemma 5.6 to V with $0 < \epsilon < \frac{R-k+1}{R}\lambda_z(G^V)$ to get an open relatively-compact neighborhood V_1 of z with $\overline{V_1} \subset V$ and

$$\lambda_z\bigl(G^V\bigr) - \epsilon < \lambda_z\bigl(G^{V_1}\bigr) \leqslant \lambda_z\bigl(G^{\overline{V_1}}\bigr) \leqslant \lambda_z\bigl(G^V\bigr) < \lambda_z\bigl(G^{V_1}\bigr) + \epsilon.$$

Since $G_z^{V_1}$ is relatively compact we have

$$
\begin{aligned}
\liminf_n \lambda_{x_n}\bigl(G^{\overline{V_1}}\bigr) &\geqslant \liminf_n \lambda_{x_n}\bigl(G^{V_1}\bigr) \\
&\geqslant R\lambda_z\bigl(G^{V_1}\bigr) \quad \text{(by hypothesis)} \\
&> R\bigl(\lambda_z\bigl(G^V\bigr) - \epsilon\bigr) \\
&> (k-1)\lambda_z\bigl(G^V\bigr) \quad \text{(by our choice of } \epsilon\text{)} \\
&\geqslant (k-1)\lambda_z\bigl(G^{\overline{V_1}}\bigr).
\end{aligned}
$$

So we may take $N = \overline{V_1}$. $\quad\square$

Remark 6.2. The preceding lemma also holds when lim inf is replaced by lim sup. No modification of the proof is needed beyond replacing the two occurrences of lim inf with lim sup.

We may now proceed to our main theorem.

Theorem 6.3. *Suppose G is a second-countable locally-compact Hausdorff principal groupoid that admits a Haar system λ. Let k be a positive integer, let $z \in G^{(0)}$ and let $\{x_n\}$ be a sequence in $G^{(0)}$. Assume that $[z]$ is locally closed in $G^{(0)}$. Then the following are equivalent:*

(1) *the sequence $\{x_n\}$ converges k-times in $G^{(0)}/G$ to z;*
(2) $M_L(L^z, \{L^{x_n}\}) \geqslant k$;
(3) *for every open neighborhood V of z in $G^{(0)}$ such that G_z^V is relatively compact we have*

$$\liminf_n \lambda_{x_n}\bigl(G^V\bigr) \geqslant k\lambda_z\bigl(G^V\bigr);$$

(4) *there exists a real number $R > k - 1$ such that for every open neighborhood V of z in $G^{(0)}$ with G_z^V relatively compact we have*

$$\liminf_n \lambda_{x_n}(G^V) \geqslant R\lambda_z(G^V); \quad and$$

(5) *there exists a basic decreasing sequence of compact neighborhoods $\{W_m\}$ of z in $G^{(0)}$ such that, for each $m \geqslant 1$,*

$$\liminf_n \lambda_{x_n}(G^{W_m}) > (k-1)\lambda_z(G^{W_m}).$$

Proof. We know that (1) implies (2) by Proposition 3.2.

Suppose (2). If $M_L(L^z, \{L^{x_n}\}) \geqslant k$, then $M_L(L^z, \{L^{x_n}\}) > \lfloor k - \epsilon \rfloor$ for all $\epsilon > 0$. By Theorem 5.8, for every $G^{(0)}$-open neighborhood V of z such that G_z^V is relatively compact, $\lambda_{x_n}(G^V) > (k - \epsilon)\lambda_z(G^V)$ eventually, and hence (3) holds.

It is immediately true that (3) implies (4).

Suppose (4). We will construct the sequence $\{W_m\}$ of compact neighborhoods inductively. Let $\{V_j\}$ be a basic decreasing sequence of open neighborhoods of z such that $G_z^{V_1}$ is relatively compact (such neighborhoods exist by [12, Lemma 4.1(1)]). By Lemma 6.1 there exists a compact neighborhood W_1 of z such that $W_1 \subset V_1$ and $\lambda_{x_n}(G^{W_1}) > (k-1)\lambda_z(G^{W_1})$. Now assume there are compact neighborhoods $W_1, W_2, \ldots, W_m$ of z with $W_1 \supset W_2 \supset \cdots \supset W_m$ such that

$$W_i \subset V_i \quad \text{and} \quad \lambda_{x_n}(G^{W_i}) > (k-1)\lambda_z(G^{W_i}) \tag{6.1}$$

for all $1 \leqslant i \leqslant m$. Apply Lemma 6.1 to $(\text{Int } m) \cap V_{m+1}$ to obtain a compact neighborhood W_{m+1} of z such that $W_{m+1} \subset (\text{Int } W_m) \cap V_{m+1}$ and (6.1) holds for $i = m+1$, establishing (5).

Suppose (5). We begin by showing that $[x_n] \to [z]$ in $G^{(0)}/G$. Let $q : G^{(0)} \to G^{(0)}/G$ be the quotient map. Let U be a neighborhood of $[z]$ in $G^{(0)}/G$ and $V = q^{-1}(U)$. There exists m such that $W_m \subset V$. Since $\liminf_n \lambda_{x_n}(G^{W_m}) > 0$ there exists n_0 such that $G_{x_n}^{W_m} \neq \emptyset$ for all $n \geqslant n_0$. Thus, for $n \geqslant n_0$, $[x_n] = q(x_n) \in q(W_m) \subset q(V) = U$, so $[x_n]$ is eventually in every neighborhood of $[z]$ in $G^{(0)}/G$.

Now suppose that $M_L(L^z, \{L^{x_n}\}) < \infty$. Then, as in the proof of Theorem 5.8, we may localise to an open invariant neighborhood Y of z such that $[z]$ is the unique limit in Y/G of $[x_n]$. Eventually $W_m \subset Y$, and so the sequence $\{x_n\}$ converges k-times in $Y/(G|_Y) = Y/G$ to z by Proposition 4.1 applied to the groupoid $G|_Y$. This implies that the sequence $\{x_n\}$ converges k-times in $G^{(0)}/G$.

Finally, if $M_L(L^z, \{L^{x_n}\}) = \infty$, then $\{x_n\}$ converges k-times in $G^{(0)}/G$ to z by Proposition 5.3, establishing (1) and completing the proof. $\quad\square$

Corollary 6.4. *Suppose that G is a second-countable locally-compact Hausdorff principal groupoid such that all the orbits are locally closed. Let $k \in \mathbb{P}$ and let $z \in G^{(0)}$ such that $[z]$ is not open in $G^{(0)}$. Then the following are equivalent:*

(1) *whenever $\{x_n\}$ is a sequence in $G^{(0)}$ which converges to z with $[x_n] \neq [z]$ eventually, then $\{x_n\}$ is k-times convergent in $G^{(0)}/G$ to z;*
(2) $M_L(L^z) \geqslant k$.

Proof. Assume (1). We must first establish that $\{L^z\}$ is not open in $C^*(G)^\wedge$. If this is not the case, then $\{L^z\}$ is open and we can apply [10, Proposition 3.6] to see that $\{[z]\}$ is open in $G^{(0)}/G$, and so $[z]$ is open in $G^{(0)}$, contradicting our assumption. Since $\{L^z\}$ is not open in $C^*(G)^\wedge$, we can apply [3, Lemma A.2] to see that there exists a sequence $\{\pi_i\}$ of irreducible representations of $C^*(G)$ such that each π_i is not unitarily equivalent to L^z, $\pi_i \to L^z$ in $C^*(G)^\wedge$, and

$$M_L(L^z) = M_L(L^z, \{\pi_i\}) = M_U(L^z, \{\pi_i\}). \tag{6.2}$$

Since the orbits are locally closed, the map $G^{(0)}/G \to C^*(G)^\wedge$ such that $[x] \mapsto L^x$ is a homeomorphism by [10, Proposition 5.1].[1] It follows that the mapping $G^{(0)} \to C^*(G)^\wedge$ such that $x \mapsto L^x$ is an open surjection, so by [24, Proposition 1.15] there is a sequence $\{x_n\}$ in $G^{(0)}$ such that $x_n \to z$ and $\{L^{x_n}\}$ is unitarily equivalent to a subsequence of $\{\pi_i\}$. By (6.2),

$$M_L(L^z) = M_U(L^z, \{\pi_i\}) \geqslant M_U(L^z, \{L^{x_n}\}) \geqslant M_L(L^z, \{L^{x_n}\}).$$

We know by (1) that $\{x_n\}$ converges k-times to z in $G^{(0)}/G$, so it follows from Theorem 6.3 that $M_L(L^z) \geqslant M_L(L^z, \{L^{x_n}\}) \geqslant k$.

Assume (2). If $\{x_n\}$ is a sequence in $G^{(0)}$ which converges to z such that $[x_n] \neq [z]$ eventually, then

$$M_L(L^z, \{L^{x_n}\}) \geqslant M_L(L^z) \geqslant k.$$

By Theorem 6.3, $\{x_n\}$ is k-times convergent to z in $G^{(0)}/G$. $\quad\square$

The next corollary improves Proposition 5.3 and is an immediate consequence of Proposition 5.3 and Theorem 6.3.

---

[1] Proposition 5.1 in [10] states that if a principal groupoid has locally closed orbits, then the map from $G^{(0)}/G$ to $C^*(G)^\wedge$ where $[x] \mapsto L^x$ is a 'homeomorphism from $G^{(0)}/G$ into $C^*(G)^\wedge$'. The proof explicitely shows that this map is a surjection.

Corollary 6.5. *Suppose that G is a second-countable locally-compact Hausdorff principal groupoid with Haar system λ. Let $z \in G^{(0)}$ and let $\{x_n\}$ be a sequence in $G^{(0)}$. Assume that $[z]$ is locally closed. Then the following are equivalent*:

(1) $M_L(L^z, \{L^{x_n}\}) = \infty$.
(2) *For every open neighborhood V of z such that G_z^V is relatively compact, $\lambda_{x_n}(G^V) \to \infty$ as $n \to \infty$.*
(3) *For each $k \geqslant 1$, the sequence $\{x_n\}$ converges k-times in $G^{(0)}/G$ to z.*

7. Upper multiplicity and *k*-times convergence

The results in this section are corollaries of Theorems 5.8 and 6.3: they relate k-times convergence, measure ratios and upper multiplicity numbers, generalising all the upper-multiplicity results of [3]. We begin with the upper-multiplicity analogue of Theorem 5.8.

Theorem 7.1. *Suppose that G is a second-countable locally-compact Hausdorff principal groupoid with Haar system λ. Let $M \in \mathbb{R}$ with $M \geqslant 1$, let $z \in G^{(0)}$ and let $\{x_n\}$ be a sequence in $G^{(0)}$. Assume that $[z]$ is locally closed. Suppose that there exists an open neighborhood V of z in $G^{(0)}$ such that G_z^V is relatively compact and*

$$\lambda_{x_n}(G^V) \leqslant M\lambda_z(G^V) < \infty$$

eventually. Then $M_U(L^z, \{L^{x_n}\}) \leqslant \lfloor M \rfloor$.

Proof. Since G is second countable, $C^*(G)$ is separable. By [3, Lemma A.1] there exists a sequence $\{L^{x_{n_i}}\}$ such that

$$M_U(L^z, \{L^{x_n}\}) = M_U(L^z, \{L^{x_{n_i}}\}) = M_L(L^z, \{L^{x_{n_i}}\}).$$

By Theorem 5.8, $M_L(L^z, \{L^{x_{n_i}}\}) \leqslant \lfloor M \rfloor$, so $M_U(L^z, \{L^{x_n}\}) \leqslant \lfloor M \rfloor$. □

Corollary 7.2. *Suppose that G is a second-countable locally-compact Hausdorff principal groupoid with Haar system λ such that all the orbits are locally closed. Let $M \in \mathbb{R}$ with $M \geqslant 1$ and let $z \in G^{(0)}$. If for every sequence $\{x_n\}$ in $G^{(0)}$ which converges to z there exists an open neighborhood V of z in $G^{(0)}$ such that G_z^V is relatively compact and*

$$\lambda_{x_n}(G^V) \leqslant M\lambda_z(G^V) < \infty$$

frequently, then $M_U(L^z) \leqslant \lfloor M \rfloor$.

Proof. Since G is second countable, $C^*(G)$ is separable, and so we can apply [5, Lemma 1.2] to see that there exists a sequence $\{\pi_n\}$ in $C^*(G)^\wedge$ that converges to L^z such that

$$M_L(L^z, \{\pi_n\}) = M_U(L^z, \{\pi_n\}) = M_U(L^z).$$

Since the orbits are locally closed, the map $G^{(0)}/G \to C^*(G)^\wedge$ such that $[x] \mapsto L^x$ is a homeomorphism by [10, Proposition 5.1]. In particular, the mapping $G^{(0)} \to C^*(G)^\wedge$ such that $x \mapsto L^x$ is an open surjection, so by [24, Proposition 1.15] there exists a sequence $\{x_i\}$ in $G^{(0)}$ converging to z such that $\{[L^{x_i}]\}$ is a subsequence of $\{[\pi_n]\}$. By Theorem 5.8, $M_L(L^z, \{L^{x_n}\}) \leqslant \lfloor M \rfloor$. Since

$$M_U(L^z) = M_L(L^z, \{\pi_n\}) \leqslant M_L(L^z, \{L^{x_i}\}) \leqslant M_U(L^z, \{L^{x_i}\}) \leqslant M_U(L^z, \{\pi_n\}) = M_U(L^z),$$

we obtain $M_U(L^z) \leqslant \lfloor M \rfloor$, as required. □

In Proposition 4.1 we generalised the first part of [3, Proposition 4.1]. We will now generalise the second part. The argument we use is similar to that used in Proposition 4.1.

Proposition 7.3. *Let G be a second-countable locally-compact Hausdorff principal groupoid with Haar system λ. Let $k \in \mathbb{P}$ and $z \in G^{(0)}$ with $[z]$ locally closed in $G^{(0)}$. Assume that $\{x_n\}$ is a sequence in $G^{(0)}$ such that $[x_n] \to [z]$ uniquely in $G^{(0)}/G$. Suppose $\{W_m\}$ is a basic decreasing sequence of compact neighborhoods of z such that each m satisfies*

$$\limsup_n \lambda_{x_n}(G^{W_m}) > (k-1)\lambda_z(G^{W_m}).$$

Then there exists a subsequence of $\{x_n\}$ which converges k-times in $G^{(0)}/G$ to z.

Proof. Let $\{K_m\}$ be an increasing sequence of compact subsets of G such that $G = \bigcup_{m \geqslant 1} \mathrm{Int}\, K_m$. By the regularity of λ_z, for each $m \geqslant 1$ there exist $\delta_m > 0$ and an open neighborhood U_m of $G_z^{W_m}$ such that

$$\limsup_n \lambda_{x_n}(G^{W_m}) > (k-1)\lambda_z(U_m) + \delta_m. \tag{7.1}$$

We will construct, by induction, a strictly increasing sequence of positive integers $\{i_m\}$ such that, for all m,

$$\lambda_{x_{i_m}}\big(K_m\alpha \cap G^{W_m}\big) < \lambda_z(U_m) + \delta_m/k \quad \text{for all } \alpha \in G^{W_m}_{x_{i_m}}, \quad \text{and} \tag{7.2}$$

$$\lambda_{x_{i_m}}\big(G^{W_m}\big) > (k-1)\lambda_z(U_m) + \delta_m. \tag{7.3}$$

By Lemma 5.5 with $\delta = \lambda_z(U_1) - \lambda_z(G^{W_1}) + \delta_1/k$, there exists n_1 such that $n \geqslant n_1$ implies

$$\lambda_{x_n}\big(K_1\alpha \cap G^{W_1}\big) < \lambda_z(U_1) + \delta_1/k \quad \text{for all } \alpha \in G^{W_m}_{x_n}.$$

By considering (7.1) with $m = 1$ we can choose $i_1 \geqslant n_1$ such that

$$\lambda_{x_{i_1}}\big(G^{W_1}\big) > (k-1)\lambda_z(U_1) + \delta_1.$$

Assuming that $i_1 < i_2 < \cdots < i_{m-1}$ have been chosen, we can apply Lemma 5.5 with $\delta = \lambda_z(U_m) - \lambda_z(G^{W_m}) + \delta_m/k$ to obtain $n_m > i_{m-1}$ such that

$$n \geqslant n_m \quad \text{implies} \quad \lambda_{x_n}\big(K_m\alpha \cap G^{W_m}\big) < \lambda_z(U_m) + \delta_m/k \quad \text{for all } \alpha \in G^{W_m}_{x_n},$$

and then by (7.1) we can choose $i_m \geqslant n_m$ such that

$$\lambda_{x_{i_m}}\big(G^{W_m}\big) > (k-1)\lambda_z(U_m) + \delta_m.$$

For each $m \in \mathbb{P}$ choose $\gamma^{(1)}_{i_m} \in G^{W_m}_{x_{i_m}}$ (which is non-empty by (7.3)). By (7.2) and (7.3) we have

$$\lambda_{x_{i_m}}\big(G^{W_m} \setminus K_m\gamma^{(1)}_{i_m}\big) = \lambda_{x_{i_m}}\big(G^{W_m}\big) - \lambda_{x_{i_m}}\big(G^{W_m} \cap K_m\gamma^{(1)}_{i_m}\big)$$
$$> (k-1)\lambda_z(U_m) + \delta_m - \big(\lambda_z(U_m) + \delta_m/k\big)$$
$$= (k-2)\lambda_z(U_m) + \frac{k-1}{k}\delta_m.$$

So we can choose $\gamma^{(2)}_{i_m} \in G^{W_m}_{x_{i_m}} \setminus K_m\gamma^{(1)}_{i_m}$. This implies, as in the proof of Proposition 4.1, that

$$\lambda_{x_{i_m}}\big(G^{W_m} \setminus \big(K_m\gamma^{(1)}_{i_m} \cup K_m\gamma^{(2)}_{i_m}\big)\big) > (k-3)\lambda_z(U_m) + \frac{(k-2)}{k}\delta_m,$$

enabling us to choose $\gamma^{(3)}_{i_m} \in G^{W_m}_{x_{i_m}} \setminus (K_m\gamma^{(1)}_{i_m} \cap K_m\gamma^{(2)}_{i_m})$. Continuing in this way for $j = 3, \ldots, k$, for each i_m we choose

$$\gamma^{(j)}_{i_m} \in G^{W_m}_{x_{i_m}} \setminus \left(\bigcup_{l=1}^{j-1} K_m\gamma^{(l)}_{i_m}\right). \tag{7.4}$$

Note that $\gamma^{(j)}_{i_m} \notin K_m\gamma^{(l)}_{i_m}$ for $1 \leqslant l < j \leqslant k$.

We claim that $r(\gamma^{(l)}_{i_m}) \to z$ as $m \to \infty$ for $1 \leqslant l \leqslant k$. To see this, fix l and let V be an open neighborhood of z. Since $\{W_m\}$ is a decreasing neighborhood basis for z there exists m_0 such that $m \geqslant m_0$ implies $W_m \subset V$, and so $r(\gamma^{(l)}_{i_m}) \in W_m \subset V$.

Finally we claim that $\gamma^{(j)}_{i_m}(\gamma^{(l)}_{i_m})^{-1} \to \infty$ as $m \to \infty$ for $1 \leqslant l < j \leqslant k$. Fix $l < j$ and let K be a compact subset of G. There exists m_0 such that $K \subset K_m$ for all $m \geqslant m_0$. By (7.4) we know

$$\gamma^{(j)}_{i_m} \in G^{W_m}_{x_{i_m}} \setminus \big(K_m\gamma^{(l)}_{i_m}\big) = \big(G^{W_m}_{x_{i_m}}\big(\gamma^{(l)}_{i_m}\big)^{-1}\gamma^{(l)}_{i_m}\big) \setminus \big(K_m\gamma^{(l)}_{i_m}\big) = \big(\big(G^{W_m}_{x_{i_m}}\big(\gamma^{(l)}_{i_m}\big)^{-1}\big) \setminus K_m\big)\gamma^{(l)}_{i_m}.$$

So provided $m \geqslant m_0$, $\gamma^{(j)}_{i_m}(\gamma^{(l)}_{i_m})^{-1} \in (G^{W_m}_{x_{i_m}}(\gamma^{(l)}_{i_m})^{-1}) \setminus K_m \subset G \setminus K_m \subset G \setminus K$, enabling us to conclude that $\{x_{i_m}\}$ converges k-times in $G^{(0)}/G$ to z. $\quad\square$

Theorem 7.4. *Suppose that G is a second-countable locally-compact Hausdorff principal groupoid with Haar system λ. Let $k \in \mathbb{P}$, let $z \in G^{(0)}$, and let $\{x_n\}$ be a sequence in $G^{(0)}$ such that $[x_n]$ converges to $[z]$ in $G^{(0)}/G$. Assume that $[z]$ is locally closed. Then the following are equivalent*:

(1) *there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges k-times in $G^{(0)}/G$ to z;*
(2) $M_U(L^z, \{L^{x_n}\}) \geqslant k$;
(3) *for every open neighborhood V of z such that G^V_z is relatively compact we have*

$$\limsup_n \lambda_{x_n}\big(G^V\big) \geqslant k\lambda_z\big(G^V\big);$$

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(4) *there exists a real number $R > k - 1$ such that for every open neighborhood V of z in $G^{(0)}$ with G_z^V relatively compact we have*

$$\limsup_n \lambda_{x_n}(G^V) \geqslant R\lambda_z(G^V); \quad and$$

(5) *there exists a basic decreasing sequence of compact neighborhoods $\{W_m\}$ of z in $G^{(0)}$ such that, for each $m \geqslant 1$,*

$$\limsup_n \lambda_{x_n}(G^{W_m}) > (k-1)\lambda_z(G^{W_m}).$$

Proof. If (1) holds then $M_L(L^z, \{L^{x_{n_i}}\}) \geqslant k$ by Theorem 6.3, and so

$$M_U(L^z, \{L^{x_n}\}) \geqslant M_U(L^z, \{L^{x_{n_i}}\}) \geqslant M_L(L^z, \{L^{x_{n_i}}\}) \geqslant k.$$

If (2) holds then by [3, Lemma A.1] there is a subsequence $\{x_{n_r}\}$ such that $M_L(L^z, \{L^{x_{n_r}}\}) = M_U(L^z, \{L^{x_n}\})$ so that $M_L(L^z, \{L^{x_{n_r}}\}) \geqslant k$. Let V be any open neighborhood of z in $G^{(0)}$ such that G_z^V is relatively compact. Then

$$\limsup_n \lambda_{x_n}(G^V) \geqslant \limsup_r \lambda_{x_{n_r}}(G^V) \geqslant \liminf_r \lambda_{x_{n_r}}(G^V) \geqslant k\lambda_z(G^V),$$

using Theorem 6.3 for the last step.

That (3) implies (4) is immediate.

That (4) implies (5) follows by making references to Remark 6.2 rather than Lemma 6.1 in the (4) implies (5) component of the proof of Theorem 6.3.

Assume (5). First suppose that $M_L(L^z, \{L^{x_n}\}) < \infty$. Since $[x_n] \to [z]$, we can use an argument found at the beginning of the proof of Theorem 5.8 to obtain an open G-invariant neighborhood Y of z in $G^{(0)}$ so that if we define $H := G|_Y$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $[x_{n_i}] \to [z]$ uniquely in $H^{(0)}/H$. Proposition 7.3 now shows us that there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ that converges k-times in $H^{(0)}/H$ to z. It follows that $\{x_{n_{i_j}}\}$ converges k-times in $G^{(0)}/G$ to z.

When $M_L(L^z, \{L^{x_n}\}) = \infty$, $\{x_n\}$ converges k-times in $G^{(0)}/G$ to z by Corollary 6.5, establishing (1). $\quad\square$

Corollary 7.5. *Suppose that G is a second-countable locally-compact Hausdorff principal groupoid such that all the orbits are locally closed. Let $k \in \mathbb{P}$ and let $z \in G^{(0)}$. Then the following are equivalent*:

(1) *there exists a sequence $\{x_n\}$ in $G^{(0)}$ which is k-times convergent in $G^{(0)}/G$ to z*;
(2) $M_U(L^z) \geqslant k$.

Proof. Assume (1). By the definitions of upper and lower multiplicity,

$$M_U(L^z) \geqslant M_U(L^z, \{L^{x_n}\}) \geqslant M_L(L^z, \{L^{x_n}\}).$$

By Theorem 6.3 we know that $M_L(L^z, \{L^{x_n}\}) \geqslant k$, establishing (2).

Assume (2). By [5, Lemma 1.2] there exists a sequence $\{\pi_n\}$ converging to L^z such that $M_L(L^z, \{\pi_n\}) = M_U(L^z, \{\pi_n\}) = M_U(L^z)$. Since the orbits are locally closed, by [10, Proposition 5.1] the mapping $G^{(0)} \to C^*(G)^\wedge : x \mapsto L^x$ is a surjection. So there is a sequence $\{L^{x_n}\}$ in $C^*(G)^\wedge$ such that L^{x_n} is unitarily equivalent to π_n for each n. Then

$$M_L(L^z, \{L^{x_n}\}) \geqslant M_L(L^z, \{\pi_n\}) = M_U(L^z) \geqslant k,$$

and it follows from Theorem 6.3 that $\{x_n\}$ is k-times convergent in $G^{(0)}/G$ to z. $\quad\square$

Corollary 7.6. *Suppose that G is a second-countable locally-compact Hausdorff principal groupoid with Haar system λ. Let $z \in G^{(0)}$ and let $\{x_n\} \subset G^{(0)}$ be a sequence converging to z. Assume that $[z]$ is locally closed. Then the following are equivalent*:

(1) *there exists an open neighborhood V of z such that G_z^V is relatively compact and*

$$\limsup_n \lambda_{x_n}(G^V) < \infty;$$

(2) $M_U(L^z, \{L^{x_n}\}) < \infty$.

Proof. Suppose that (1) holds. Since $C^*(G)$ is separable, it follows from [3, Lemma A.1] that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$M_L(L^z, \{L^{x_{n_j}}\}) = M_U(L^z, \{L^{x_{n_j}}\}) = M_U(L^z, \{L^{x_n}\}).$$

By (1) and Corollary 6.5, $M_L(L^z, \{L^{x_n}\}) < \infty$. Hence $M_U(L^z, \{L^{x_n}\}) < \infty$, as required.

Suppose that (1) fails. Let $\{V_i\}$ be a basic decreasing sequence of open neighborhoods of z such that $G_z^{V_1}$ is relatively compact (such neighborhoods exist by [12, Lemma 4.1(1)]). Then

$$\limsup_n \lambda_{x_n}(G^{V_i}) = \infty \quad \text{for each } i$$

and we may choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lambda_{x_{n_i}}(G^{V_i}) \to \infty$ as $i \to \infty$.

Let V be any open neighborhood of z such that G_z^V is relatively compact. There exists i_0 such that $V_i \subset V$ for all $i \geqslant i_0$. Then, for $i \geqslant i_0$,

$$\lambda_{x_{n_i}}\big(G^{V_i}\big) \leqslant \lambda_{x_{n_i}}\big(G^V\big).$$

Thus $\lambda_{x_{n_i}}(G^V) \to \infty$ as $i \to \infty$. By Corollary 6.5, $M_L(L^z, \{L^{x_n}\}) = \infty$. Hence $M_U(L^z, \{L^{x_n}\}) = \infty$, that is (2) fails. $\square$

Corollary 7.7. *Suppose G is a second-countable locally-compact Hausdorff principal groupoid with Haar system λ such that all the orbits are locally closed. Let $z \in G^{(0)}$. Then the following are equivalent*:

(1) $M_U(L^z) < \infty$;
(2) *there exists an open neighborhood V of z such that G_z^V is relatively compact and*

$$\sup_{x \in V} \lambda_x\big(G^V\big) < \infty.$$

Proof. If (2) holds then (1) holds by Corollary 7.2.

Let $\{V_i\}$ be a basic decreasing sequence of open neighborhoods of z such that $G_z^{V_1}$ is relatively compact. If (2) fails then $\sup_{x \in V_i}\{\lambda_x(G^{V_i})\} = \infty$ for each i and we may choose a sequence $\{x_i\}$ such that $x_i \in V_i$ for all i and $\lambda_{x_i}(G^{V_i}) \to \infty$. Since $\{V_i\}$ is a basic decreasing sequence, $x_i \to z$.

Let V be an open neighborhood of z such that G_z^V is relatively compact. There exists i_0 such that $V_i \subset V$ for all $i \geqslant i_0$. Then, for $i \geqslant i_0$,

$$\lambda_{x_i}\big(G^{V_i}\big) \leqslant \lambda_{x_i}\big(G^V\big).$$

Thus $\lambda_{x_i}(G^V) \to \infty$. By Corollary 7.6, $M_U(L^z, \{L^{x_i}\}) = \infty$. Hence $M_U(L^z) = \infty$, and so (1) fails. $\square$

8. Graph algebra examples

We begin this section by introducing the notion of a directed graph as well as some related concepts as in the expository book [21], although some notation is also taken from [15]. A *directed graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and functions $r, s : E^1 \to E^0$. The elements of E^0 and E^1 are called *vertices* and *edges* respectively. For each edge e, call $s(e)$ the *source* of e and $r(e)$ the *range* of e. A directed graph E is *row finite* if $r^{-1}(v)$ is finite for every $v \in E^0$.

A *finite path* in a directed graph E is a finite sequence $\alpha = \alpha_1 \alpha_2 \cdots \alpha_k$ of edges α_i with $s(\alpha_j) = r(\alpha_{j+1})$ for $1 \leqslant j \leqslant k-1$; write $s(\alpha) = s(\alpha_k)$ and $r(\alpha) = r(\alpha_1)$, and call $|\alpha| := k$ the *length* of α. An *infinite path* $x = x_1 x_2 \cdots$ is defined similarly, although $s(x)$ remains undefined. Let E^* and E^∞ denote the set of all finite paths and infinite paths in E respectively. If $\alpha = \alpha_1 \cdots \alpha_k$ and $\beta = \beta_1 \cdots \beta_j$ are finite paths then, provided $s(\alpha) = r(\beta)$, let $\alpha\beta$ be the path $\alpha_1 \cdots \alpha_k \beta_1 \cdots \beta_j$. When $x \in E^\infty$ with $s(\alpha) = r(x)$ define αx similarly. A *cycle* is a finite path α of non-zero length such that $s(\alpha) = r(\alpha)$.

When v is a vertex, f is an edge, and there is exactly one infinite path with range v that includes the edge f, then we denote this infinite path by $[v, f]^\infty$. When there is exactly one finite path α with $r(\alpha) = v$ and $\alpha_{|\alpha|} = f$, we denote α by $[v, f]^*$. In [15] two paths $x, y \in E^\infty$ are defined to be *shift equivalent* with lag $k \in \mathbb{Z}$ (written $x \sim_k y$) if there exists $N \in \mathbb{N}$ such that $x_i = y_{i+k}$ for all $i \geqslant N$.

Suppose E is a row-finite directed graph. We refer to the groupoid constructed from E by Kumjian, Pask, Raeburn and Renault in [15] as the *path groupoid*. Before describing this construction we caution that we are using the now standard notation for directed graphs which has the range and source swapped from the notation used in [15]. This new convention is due to the development of the higher-rank graphs, where edges become morphisms in a category and the new convention ensures that "composition of morphisms is compatible with multiplication of operators in $B(\mathcal{H})$" [21, p. 2]. The path groupoid $G = G_E$ constructed from E is defined as follows:

$$G := \big\{(x, k, y) \in E^\infty \times \mathbb{Z} \times E^\infty : x \sim_k y\big\}.$$

For elements of

$$G^{(2)} := \big\{\big((x, k, y), (y, l, z)\big) : (x, k, y), (y, l, z) \in G\big\},$$

Kumjian, Pask, Raeburn, and Renault defined

$$(x, k, y) \cdot (y, l, z) := (x, k+l, z),$$

and for arbitrary $(x, k, y) \in G$, defined

$$(x, k, y)^{-1} := (y, -k, x).$$

For each $\alpha, \beta \in E^*$ with $s(\alpha) = s(\beta)$, let $Z(\alpha, \beta)$ be the set

$$\big\{(x, k, y) : x \in Z(\alpha), \ y \in Z(\beta), \ k = |\beta| - |\alpha|, \ x_i = y_{i+k} \text{ for } i > |\alpha|\big\}.$$

By [15, Proposition 2.6], the collection of sets

$$\big\{Z(\alpha,\beta)\colon \alpha,\beta \in E^*,\ s(\alpha) = s(\beta)\big\}$$

is a basis of compact open sets for a second-countable locally-compact Hausdorff topology on G that makes G r-discrete. Kumjian, Pask, Raeburn and Renault equipped G with the Haar system consisting of counting measures, which they observe is possible by first showing that a Haar system exists for the groupoid with [23, Proposition I.2.8], and then using [23, Lemma I.2.7] to show that they can choose the system of counting measures.

By [15, Corollary 2.2], the cylinder sets

$$Z(\alpha) := \big\{x \in E^\infty \colon x_1 = \alpha_1, \ldots, x_{|\alpha|} = \alpha_{|\alpha|}\big\}$$

parameterised by $\alpha \in E^*$ form a basis of compact open sets for a locally-compact σ-compact totally-disconnected Hausdorff topology on E^∞. After identifying each $(x, 0, x) \in G^{(0)}$ with $x \in E^\infty$, [15, Proposition 2.6] tells us that the topology on $G^{(0)}$ is identical to the topology on E^∞.

For a row-finite directed graph E, Kumjian, Pask, Raeburn and Renault use the path groupoid G to construct the usual groupoid C^*-algebra $C^*(G)$, and show how a collection of partial isometries subject to some relations derived from E generate $C^*(G)$. More recently, a C^*-algebra $C^*(E)$ is constructed from a collection of partial isometries subject to slightly weakened relations derived from E. The slightly weakened relations permit non-zero partial isometries to be related to sources in the graph, and as a result $C^*(E)$ is isomorphic to $C^*(G)$ only when E contains no sources. It turns out that $C^*(E)$ and $C^*(G)$ can be substantially different: an example in [14] describes a graph with sources where $C^*(G)$ has continuous trace while $C^*(E)$ does not. In this paper we are only interested in groupoid C^*-algebras, so we will make no further mention of the graph algebra $C^*(E)$.

Since we wish to apply Theorem 6.3 to path groupoids, we must be able to show that the path groupoids we consider are principal.

Proposition 8.1. *Suppose E is a row-finite directed graph. The path groupoid G constructed from E is principal if and only if E contains no cycles.*

Proof. We first show that if E contains no cycles then G is principal. Suppose G is not principal. Then there exist $x, y \in E^\infty$ and distinct $\gamma, \delta \in G$ such that $r(\gamma) = r(\delta) = x$ and $s(\gamma) = s(\delta) = y$. It follows that there exist $a, b \in \mathbb{Z}$ such that $\gamma = (x, a, y)$ and $\delta = (x, b, y)$. Notice that since $\gamma \neq \delta$, $a \neq b$. We may assume without loss of generality that $a > b$.

Now $\gamma = (x, a, y)$ implies $x \sim_a y$ and $\delta = (x, b, y)$ implies $x \sim_b y$, so there exists N such that

$$n \geqslant N \quad \Longrightarrow \quad x_n = y_{n+a} = y_{n+b},$$

and so $x_n = y_{n+a} = y_{n+a-b+b} = x_{n+a-b}$. Thus E contains a cycle of length at most $a - b$.

We now show that if G is principal then E contains no cycles. Suppose E contains the cycle $\alpha = \alpha_1 \alpha_2 \cdots \alpha_k$. Then $x := \alpha\alpha \cdots$ is in E^∞ with $x \sim_k x$, so both $(x, 0, x)$ and (x, k, x) are in G. It follows that G is not principal. $\quad\square$

Example 8.2 (*2-times convergence in a path groupoid*). Let E be the graph
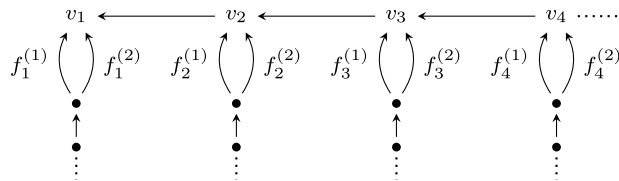
and let G be the path groupoid. For each $n \geqslant 1$ define $x^{(n)} := [v_1, f_n^{(1)}]^\infty$ and let z be the infinite path with range v_1 that passes through each v_n. Then $\{x^{(n)}\}$ converges 2-times in $G^{(0)}/G$ to z.

Proof. We will describe two sequences in G as in Definition 3.1. For each $n \geqslant 1$ define $\gamma_n^{(1)} := (x^{(n)}, 0, x^{(n)})$ and $\gamma_n^{(2)} := ([v_1, f_n^{(2)}]^\infty, 0, x^{(n)})$. It follows immediately that $s(\gamma_n^{(1)}) = x^{(n)} = s(\gamma_n^{(2)})$ for all n and that both $r(\gamma_n^{(1)})$ and $r(\gamma_n^{(2)})$ converge to z as $n \to \infty$. It remains to show that $\gamma_n^{(2)}(\gamma_n^{(1)})^{-1} \to \infty$ as $n \to \infty$.

Let K be a compact subset of G. Our goal is to show that $\gamma_n^{(2)}(\gamma_n^{(1)})^{-1} = \gamma_n^{(2)}$ is eventually not in K. Since sets of the form $Z(\alpha, \beta)$ for some $\alpha, \beta \in E^*$ form a basis for the topology on the path groupoid, for each $\gamma \in K$ there exist $\alpha^{(\gamma)}, \beta^{(\gamma)} \in E^*$ with $s(\alpha^{(\gamma)}) = s(\beta^{(\gamma)})$ so that $Z(\alpha^{(\gamma)}, \beta^{(\gamma)})$ is an open neighborhood of γ in G. Thus $\bigcup_{\gamma \in K} Z(\alpha^{(\gamma)}, \beta^{(\gamma)})$ is an open cover of the compact set K, and so admits a finite subcover $\bigcup_{i=1}^I Z(\alpha^{(i)}, \beta^{(i)})$.

We now claim that for any fixed $n \in \mathbb{P}$, if there exists i with $1 \leqslant i \leqslant I$ such that $\gamma_n^{(2)} \in Z(\alpha^{(i)}, \beta^{(i)})$, then $|[v_1, f_n^{(2)}]^*| \leqslant |\alpha^{(i)}|$. Temporarily fix $n \in \mathbb{P}$ and suppose there exists i with $1 \leqslant i \leqslant I$ such that $\gamma_n^{(2)} \in Z(\alpha^{(i)}, \beta^{(i)})$. Suppose the converse:

that $|\alpha^{(i)}| < |[v_1, f_n^{(2)}]^*|$. Since $\gamma_n^{(2)} \in Z(\alpha^{(i)}, \beta^{(i)})$, it follows that $r(\gamma_n^{(2)}) = [v_1, f_n^{(2)}]^\infty \in Z(\alpha^{(i)})$, and so $\alpha_p^{(i)} = [v_1, f_n^{(2)}]_p^\infty$ for every $1 \leqslant p \leqslant |\alpha^{(i)}|$. By examining the graph we can see that $s([v_1, f_n^{(2)}]_p^\infty) = v_{p+1}$ for all $1 \leqslant p < |[v_1, f_n^{(2)}]^*|$. Since we also know that $|\alpha^{(i)}| < |[v_1, f_n^{(2)}]^*|$, we can deduce that $s(\alpha^{(i)}) = v_j$ for some j. Furthermore since $s(\alpha^{(i)}) = s(\beta^{(i)})$, $s(\beta^{(i)}) = v_j$. There is only one path with source v_j and range v_1, so $\alpha^{(i)} = \beta^{(i)}$. Note that when $k = |\beta^{(i)}| - |\alpha^{(i)}|$, the set $Z(\alpha^{(i)}, \beta^{(i)})$ is by definition equal to

$$\big\{ (x, k, y) \colon x \in Z(\alpha^{(i)}), \ y \in Z(\beta^{(i)}), \ x_p = y_{p+k} \text{ for } p > |\alpha^{(i)}| \big\},$$

so since $\gamma_n^{(2)} \in Z(\alpha^{(i)}, \beta^{(i)})$ and $\alpha^{(i)} = \beta^{(i)}$, we can see that $s(\gamma_n^{(2)})_p = r(\gamma_n^{(2)})_p$ for all $p > |\alpha^{(i)}|$. We know $s(\gamma_n^{(2)}) = [v_1, f_n^{(1)}]^\infty$ and $r(\gamma_n^{(2)}) = [v_1, f_n^{(2)}]^\infty$, so $[v_1, f_n^{(2)}]_p^\infty = [v_1, f_n^{(1)}]_p^\infty$ for all $p > |\alpha^{(i)}|$. In particular, since we assumed that $|[v_1, f_n^{(2)}]^*| > |\alpha^{(i)}|$, we have

$$\big[v_1, f_n^{(2)} \big]_{|[v_1, f_n^{(2)}]^*|}^\infty = \big[v_1, f_n^{(1)} \big]_{|[v_1, f_n^{(2)}]^*|}^\infty,$$

so that $f_n^{(2)} = f_n^{(1)}$. But $f_n^{(1)}$ and $f_n^{(2)}$ are distinct, so we have found a contradiction, and we must have $|[v_1, f_n^{(2)}]^*| \leqslant |\alpha^{(i)}|$.

Our next goal is to show that each $Z(\alpha^{(i)}, \beta^{(i)})$ contains at most one $\gamma_n^{(2)}$. Fix $n, m \in \mathbb{P}$ and suppose that both $\gamma_n^{(2)}$ and $\gamma_m^{(2)}$ are in $Z(\alpha^{(i)}, \beta^{(i)})$ for some i. We will show that $n = m$. Since $\gamma_n^{(2)} \in Z(\alpha^{(i)} \beta^{(i)})$, $r(\gamma_n^{(2)}) = [v_1, f_n^{(2)}]^\infty \in Z(\alpha^{(i)})$. Thus there exists $x \in E^\infty$ such that $[v_1, f_n^{(2)}]^* x \in Z(\alpha^{(i)})$ and, since $|[v_1, f_n^{(2)}]^*| \leqslant |\alpha^{(i)}|$, we can crop x to form a finite $\epsilon \in E^*$ such that $[v_1, f_n^{(2)}]^* \epsilon = \alpha^{(i)}$. Similarly there exists $\delta \in E^*$ such that $[v_1, f_m^{(2)}]^* \delta = \alpha^{(i)}$. Then

$$\big[v_1, f_n^{(2)} \big]^* \epsilon = \alpha^{(i)} = \big[v_1, f_m^{(2)} \big]^* \delta,$$

which we can see by looking at the graph is only possible if $n = m$. We have thus shown that if $\gamma_n^{(2)}$ and $\gamma_m^{(2)}$ are in $Z(\alpha^{(i)}, \beta^{(i)})$, then $\gamma_n^{(2)} = \gamma_m^{(2)}$.

Let $S = \{n \in \mathbb{P} \colon \gamma_n^{(2)} \in K\}$. Since $K \subset \bigcup_{i=1}^l Z(\alpha^{(i)}, \beta^{(i)})$ and since $\gamma_n^{(2)}, \gamma_m^{(2)} \in Z(\alpha^{(i)}, \beta^{(i)})$ implies $n = m$, S can contain at most I elements. Then S has a maximal element n_0 and $\gamma_n^{(2)} \notin K$ provided $n > n_0$. Thus $\gamma_n \to \infty$ as $n \to \infty$, and we have shown that $x^{(n)}$ converges 2-times to z in $G^{(0)}/G$. $\square$

Example 8.3 *(k-times convergence in a path groupoid).* For any fixed positive integer k, let E be the graph
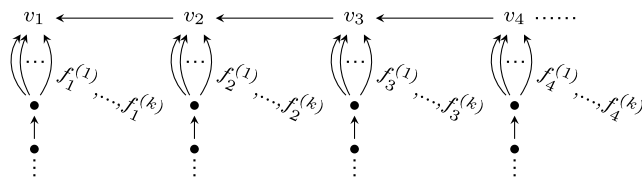
and let G be the path groupoid. For each $n \geqslant 1$ define $x^{(n)} := [v_1, f_n^{(1)}]^\infty$ and let z be the infinite path that passes through each v_n. Then the sequence $\{x^{(n)}\}$ converges k-times in $G^{(0)}/G$ to z.

Proof. After defining $\gamma_n^{(i)} := ([v_1, f_n^{(i)}]^\infty, 0, x^{(n)})$ for each $1 \leqslant i \leqslant k$, an argument similar to that in Example 8.2 establishes the k-times convergence. $\square$

Example 8.4 *(Lower multiplicity 2 and upper multiplicity 3).* Consider the graph E described by
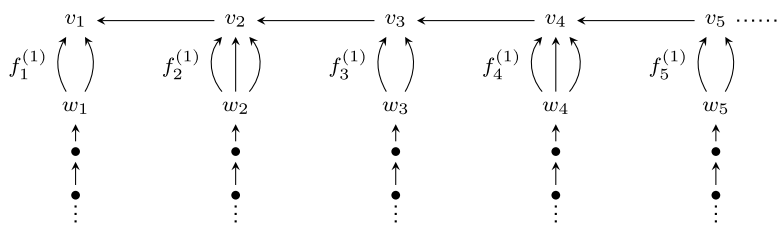
where for each odd $n \geqslant 1$ there are exactly two paths $f_n^{(1)}, f_n^{(2)}$ with source w_n and range v_n, and for each even $n \geqslant 2$ there are exactly three paths $f_n^{(1)}, f_n^{(2)}, f_n^{(3)}$ with source w_n and range v_n. Let G be the path groupoid, define $x^{(n)} := [v_1, f_n^{(1)}]^\infty$ for every $n \geqslant 1$, and let z be the infinite path that meets every vertex v_n (so z has range v_1). Then

$$\mathsf{M}_\mathsf{L}\big(\mathsf{L}^z, \{ \mathsf{L}^{x^{(n)}} \} \big) = 2 \quad \text{and} \quad \mathsf{M}_\mathsf{U}\big(\mathsf{L}^z, \{ \mathsf{L}^{x^{(n)}} \} \big) = 3.$$

Proof. We know that $\{x^{(n)}\}$ converges 2-times to z in $G^{(0)}/G$ by the argument in Example 8.2, so we can apply Theorem 6.3 to see that $M_L(L^z, \{L^{x^{(n)}}\}) \geqslant 2$. We can see that the subsequence $\{x^{(2n)}\}$ of $\{x^{(n)}\}$ converges 3-times to z in $G^{(0)}/G$ by Example 8.3. Theorem 7.4 now tells us that $M_U(L^z, \{L^{x^{(n)}}\}) \geqslant 3$.

Now suppose $M_L(L^z, \{L^{x^{(n)}}\}) \geqslant 3$. Then by Theorem 6.3, $\{x^{(n)}\}$ converges 3-times to z in $G^{(0)}/G$, so there must exist three sequences $\{\gamma_n^{(1)}\}$, $\{\gamma_n^{(2)}\}$, and $\{\gamma_n^{(3)}\}$ as in the definition of k-times convergence (Definition 3.1). For each odd n, there are only two elements in G with source $x^{(n)}$, so there must exist $1 \leqslant i < j \leqslant 3$ such that $\gamma_n^{(i)} = \gamma_n^{(j)}$ frequently. Then $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} = r(\gamma_n^{(i)})$ frequently and, since $r(\gamma_n^{(i)}) \to z$, $\{\gamma_n^{(j)}(\gamma_n^{(i)})^{-1}\}$ admits a convergent subsequence. Thus $\gamma_n^{(j)}(\gamma_n^{(i)})^{-1} \nrightarrow \infty$, contradicting the definition of k-times convergence.

If $M_U(L^z, \{L^{x^{(n)}}\}) \geqslant 4$, then by Theorem 7.4 there is a subsequence of $\{x^{(n)}\}$ that converges 4-times to z in $G^{(0)}/G$. A similar argument to that in the preceding paragraph shows that this is not possible since there are at most 3 edges between any v_n and v_m. It follows that $M_L(L^z, \{L^{x^{(n)}}\}) = 2$ and $M_U(L^z, \{L^{x^{(n)}}\}) = 3$. $\square$

Lemma 8.5. *In Example* 8.2,

$$M_L\big(L^z, \big\{L^{x^{(n)}}\big\}\big) = M_U\big(L^z, \big\{L^{x^{(n)}}\big\}\big) = 2;$$

and in Example 8.3,

$$M_L\big(L^z, \big\{L^{x^{(n)}}\big\}\big) = M_U\big(L^z, \big\{L^{x^{(n)}}\big\}\big) = k.$$

Proof. The same argument as that found in Example 8.4 can be used to demonstrate this lemma. The explicit proof was given for Example 8.4 since it covers the case where the upper and lower multiplicities are distinct. $\square$

In the next example we will add some structure to the graph from Example 8.2 to create a path groupoid G with non-Hausdorff orbit space that continues to exhibit 2-times convergence.

Example 8.6. Let E be the directed graph
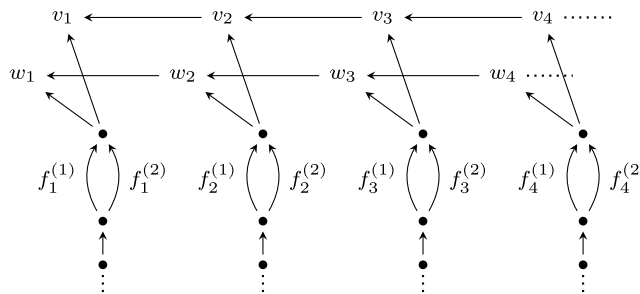
and let G be the path groupoid. For every $n \geqslant 1$ let $x^{(n)}$ be the infinite path $[v_1, f_n^{(1)}]^\infty$. Let x be the infinite path with range v_1 that passes through each v_n and let y be the infinite path with range w_1 that passes through each w_n. Then the orbit space $G^{(0)}/G$ is not Hausdorff and $\{x^{(n)}\}$ converges 2-times in $G^{(0)}/G$ to both x and y.

Proof. To see that $\{x^{(n)}\}$ converges 2-times to x in $G^{(0)}/G$, consider the sequences $\{([v_1, f_n^{(2)}]^\infty, 0, x^{(n)})\}$ and $\{(x^{(n)}, 0, x^{(n)})\}$ and follow the argument as in Example 8.2. To see that $\{x^{(n)}\}$ converges 2-times to y in $G^{(0)}/G$, consider the sequences $\{([w_1, f_n^{(1)}]^\infty, 0, x^{(n)})\}$ and $\{([w_1, f_n^{(2)}]^\infty, 0, x^{(n)})\}$. While it is tempting to think that this example exhibits 4-times convergence (or even 3-times convergence), this is not the case (see Example 8.4 for an argument demonstrating this). We know $x^{(n)}$ converges k-times to x in $G^{(0)}/G$, so $[x^{(n)}] \to [x]$ in $G^{(0)}/G$, and similarly $[x^{(n)}] \to [y]$ in $G^{(0)}/G$. It follows that $G^{(0)}/G$ is not Hausdorff since $[x] \neq [y]$. $\square$

In all of the examples above, the orbits in $G^{(0)}$ are closed and hence $C^*(G)^\wedge$ and $G^{(0)}/G$ are homeomorphic by [10, Proposition 5.1]. By combining the features of the graphs in Examples 8.4 and 8.6 we obtain a principal groupoid whose C^*-algebra has non-Hausdorff spectrum and distinct upper and lower multiplicities among its irreducible representations.

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