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An asymptotic result for Laguerre–Sobolev orthogonal polynomials

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Abstract

Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to the Sobolev inner product

$$(f,g)_S = \int_0^{+\infty} f(x)g(x)x^\alpha e^{-x}\,dx + \lambda \int_0^{+\infty} f'(x)g'(x)x^\alpha e^{-x}\,dx,$$

where $\alpha > -1$, $\lambda > 0$ and the leading coefficient of the S_n is equal to the leading coefficient of the Laguerre polynomial $L_n^{(\alpha)}$. Then, if $x \in \mathbb{C} \setminus [0, +\infty)$,

$$\lim_{n\to\infty} \frac{S_n(x)}{L_n^{(\alpha-1)}(x)}$$

is a constant depending on λ.

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1. Introduction

Consider the Sobolev inner product

$$(f,g)_S = \int_0^{+\infty} f(x)g(x)x^{\alpha}e^{-x}\,dx + \lambda \int_0^{+\infty} f'(x)g'(x)x^{\alpha}e^{-x}\,dx, \tag{1.1}$$

with $\alpha > -1$ and $\lambda > 0$. Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to (1.1), normalized by the condition that S_n and the Laguerre polynomial $L_n^{(\alpha)}$ have the same leading coefficient ($n = 0, 1, 2, \ldots$).

The special case $\alpha = 0$ has already been studied by Brenner in [1]. In [5], Schäfke and Wolf introduced *einfache verallgemeinerte klassische Orthogonalpolynome* and the above defined sequence $\{S_n\}$ is a special case of them. The inner product (1.1) can also be studied as a special case of inner products defined by a *coherent pair of measures* as introduced by Iserles et al. [2].

The most complete treatment of the sequence $\{S_n\}$ orthogonal with respect to (1.1) is a recent paper of Marcellán et al. [3]. The paper gives among others several algebraic and differential relations with $\{L_n^{(\alpha)}\}$, a four-term recurrence relation, a Rodrigues-type formula and some properties concerning the zeros. The paper states one asymptotic result for $S_n(x)$ with $x \in \mathbb{C}\backslash[0, +\infty)$ and $n \to \infty$, but it is only for the special case $\alpha = 0$ and no proof is given (only a reference to a private communication by V.N. Sorokin).

The aim of the present paper is to derive an asymptotic result for $S_n(x)$ with $x \in \mathbb{C}\backslash[0, +\infty)$ and $n \to \infty$. Our main result is stated in the following theorem.

Theorem 1.1. *If x is in complex plane cut along the positive real axis, then*

$$S_n(x) = \frac{2}{\sqrt{\lambda^2 + 4\lambda} - \lambda} L_n^{(\alpha-1)}(x)\{1 + O(n^{-1/2})\};$$

the bound for the remainder holds uniformly on compact subsets of $\mathbb{C}\backslash[0, +\infty)$.

The asymptotic behaviour of $L_n^{(\alpha-1)}(x)$ with $x \in \mathbb{C}\backslash[0, +\infty)$ and $n \to \infty$ has been found by Perron; we mention his result in Section 2. We point out that Perron's result holds for arbitrary real α, so $-1 < \alpha \leqslant 0$ is allowed in the theorem; in this case Laguerre polynomials are defined by means of their explicit representation as given in [6, pp. 100–102].

Finally we remark that a similar result for polynomials orthogonal with respect to a Sobolev inner product defined by a coherent pair of measures of compact support has been derived by Martínez-Finkelshtein et al. [4].

2. Classical Laguerre polynomials

Laguerre polynomials, for arbitrary real α, are defined by (see [6, pp. 100–102])

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}, \qquad n = 0, 1, 2, \ldots .$$

This definition gives $L_n^{(\alpha)}(x)$ for arbitrary real α as a polynomial of degree n with leading coefficient $(-1)^n/n!$.

If $\alpha > -1$, then $\{L_n^{(\alpha)}\}$ is orthogonal with respect to the inner product

$$(f,g) = \int_0^{+\infty} f(x)g(x)x^\alpha e^{-x}\,dx.$$

Moreover, if $\alpha > -1$, then

$$\int_0^{+\infty} (L_n^{(\alpha)}(x))^2 x^\alpha\, e^{-x}\,dx = \frac{\Gamma(n+\alpha+1)}{n!}, \quad n = 0,1,2,\dots . \tag{2.1}$$

For arbitrary real α the following relations are satisfied

$$L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x) = L_n^{(\alpha-1)}(x), \tag{2.2}$$

$$\frac{d}{dx}(L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x)) = -L_{n-1}^{(\alpha)}(x). \tag{2.3}$$

The following asymptotic result is due to Perron (see [6, p. 199]).

Lemma 2.1. *Let α be an arbitrary real number. Then*

$$L_n^{(\alpha)}(x) = \frac{1}{2\sqrt{\pi}} e^{\frac{x}{2}}(-x)^{-\frac{\alpha}{2}-\frac{1}{4}} n^{\frac{\alpha}{2}-\frac{1}{4}} e^{2\sqrt{-nx}} \left\{1 + O(n^{-\frac{1}{2}})\right\},$$

the relation holds if x is in the complex plane cut along the positive part of the real axis; $(-x)^{-\frac{\alpha}{2}-\frac{1}{4}}$ and $\sqrt{-x}$ must be taken real and positive if $x<0$. The bound for the remainder holds uniformly in every closed domain with no points in common with $x \geqslant 0$.

As a direct consequence of Lemma 2.1 we have

Lemma 2.2. *Let α be an arbitrary real number. Then*

$$\lim_{n\to\infty} \frac{L_n^{(\alpha)}(x)}{L_{n-1}^{(\alpha)}(x)} = 1, \tag{2.4}$$

$$\lim_{n\to\infty} \frac{n^{\frac{1}{2}} L_n^{(\alpha-1)}(x)}{L_n^{(\alpha)}(x)} = \sqrt{-x}, \tag{2.5}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,+\infty)$.

3. Laguerre–Sobolev orthogonal polynomials

Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to the Sobolev inner product

$$(f,g)_S = \int_0^{+\infty} f(x)g(x)x^\alpha e^{-x}\,dx + \lambda \int_0^{+\infty} f'(x)g'(x)x^\alpha e^{-x}\,dx \tag{3.1}$$

with $\alpha > -1$ and $\lambda > 0$. The S_n are normalized by the condition that the leading coefficient of S_n equals the leading coefficient of $L_n^{(\alpha)}$.

Observe that $S_0 = L_0^{(\alpha)}$ and $S_1 = L_1^{(\alpha)}$.

Several authors obtained the following result.

Lemma 3.1. *There exist positive constants a_n depending on α and λ, such that*

$$L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x) = S_n(x) - a_{n-1}S_{n-1}(x), \quad n \geq 1. \tag{3.2}$$

Proof. Put

$$L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x) = S_n(x) + \sum_{i=0}^{n-1} c_i^{(n)} S_i(x).$$

Then

$$c_i^{(n)}(S_i, S_i)_S = (L_n^{(\alpha)} - L_{n-1}^{(\alpha)}, S_i)_S.$$

Applying (3.1) and (2.3) to the right-hand side we obtain

$$c_i^{(n)} = 0, \quad 0 \leq i \leq n-2,$$

$$c_{n-1}^{(n)}(S_{n-1}, S_{n-1})_S = -\int_0^{+\infty} L_{n-1}^{(\alpha)}(x) S_{n-1}(x) x^\alpha e^{-x}\, dx = -\int_0^{+\infty} (L_{n-1}^{(\alpha)}(x))^2 x^\alpha e^{-x}\, dx. \quad \square$$

Marcellán et al. [3] found the following recurrence relation.

Lemma 3.2. *The sequence $\{a_n\}$ in (3.2) satisfies*

$$a_n = \frac{n+\alpha}{n(2+\lambda) + \alpha - n a_{n-1}}, \quad n \geq 1 \tag{3.3}$$

with $a_0 = 1$.

Proof. Write

$$R_0 = S_0, \qquad R_n = S_n - a_{n-1}S_{n-1}, \quad n \geq 1,$$

then for $n \geq 1$,

$$(R_{n+1}, R_n)_S + a_n(R_n, R_n)_S + a_n a_{n-1}(R_n, R_{n-1})_S = 0.$$

After computing the Sobolev inner products with (3.1), (3.2), (2.1) and (2.3) we obtain (3.3) for $n \geq 1$.

Since $S_0 = L_0^{(\alpha)}$ and $S_1 = L_1^{(\alpha)}$, relation (3.2) implies $a_0 = 1$. $\square$

In order to derive the asymptotic behaviour of S_n we need more information on the sequence $\{a_n\}$.

Lemma 3.3. *The sequence* $\{a_n\}$ *is convergent, and*

$$\ell = \lim_{n\to\infty} a_n = \tfrac{1}{2}(\lambda + 2 - \sqrt{\lambda^2 + 4\lambda}) < 1.$$

Moreover, for all $p<1$, *we have*

$$\lim_{n\to\infty} n^p(a_n - \ell) = 0. \tag{3.4}$$

Proof. First we observe that a simple induction argument applied on Lemma 3.2 gives $a_n \leqslant 1$ for all $n \geqslant 0$.

Suppose that $\ell = \lim_{n\to\infty} a_n$ exists, then (3.3) implies

$$\ell^2 - \ell(2+\lambda) + 1 = 0.$$

Since $a_n \leqslant 1$ for all $n \geqslant 0$, we have $\ell \leqslant 1$. Hence

$$\ell = \tfrac{1}{2}(\lambda + 2 - \sqrt{\lambda^2 + 4\lambda}) < 1.$$

Now, we prove that (3.4) is satisfied; in particular this implies that $\{a_n\}$ is indeed convergent. With (3.3) and $\ell(2+\lambda) = \ell^2 + 1$ we have

$$a_n - \ell = \frac{\alpha - \alpha\ell + n\ell(a_{n-1} - \ell)}{n(2+\lambda) + \alpha - na_{n-1}}.$$

Then, using $a_{n-1} \leqslant 1$,

$$|a_n - \ell| \leqslant \frac{|\alpha - \alpha\ell|}{n(1+\lambda) + \alpha} + \frac{n\ell|a_{n-1} - \ell|}{n(1+\lambda) + \alpha}.$$

Put $t_n = n^p|a_n - \ell|$, with $p<1$, then

$$t_n \leqslant \frac{n^p|\alpha - \alpha\ell|}{n(1+\lambda) + \alpha} + \frac{n^{p+1}}{(n-1)^p[n(1+\lambda) + \alpha]}\ell t_{n-1}.$$

Let $\varepsilon > 0$ and $\ell < r < 1$. Then there exists an integer N such that

$$t_n < \varepsilon + r t_{n-1}, \qquad n \geqslant N + 1.$$

By repeated application, for $k \geqslant 1$, we deduce

$$t_{N+k} < \varepsilon(1 + r + r^2 + \cdots + r^{k-1}) + r^k t_N < \frac{\varepsilon}{1-r} + r^k t_N.$$

This implies

$$\lim_{n\to\infty} t_n = 0. \qquad \square$$

Now, we are able to prove our main result.

Theorem 3.4. *Put*

$$S_n = \frac{1}{1-\ell}L_n^{(\alpha-1)} - \frac{\ell}{1-\ell}F_n. \tag{3.5}$$

If $x \in \mathbb{C}\backslash[0,+\infty)$, *then*

$$\lim_{n\to\infty} \frac{F_n(x)}{L_n^{(\alpha-2)}(x)} = \frac{1}{1-\ell},$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,+\infty)$.

Proof. With (2.2) we rewrite (3.2) to

$$L_n^{(\alpha-1)} = S_n - a_{n-1}S_{n-1}.$$

Substituting (3.5) for S_n and S_{n-1} we obtain

$$L_n^{(\alpha-1)} - L_{n-1}^{(\alpha-1)} = F_n - a_{n-1}F_{n-1} + \frac{1}{\ell}(a_{n-1}-\ell)L_{n-1}^{(\alpha-1)}.$$

Again we use (2.2)

$$L_n^{(\alpha-2)} = F_n - a_{n-1}F_{n-1} + \frac{1}{\ell}(a_{n-1}-\ell)L_{n-1}^{(\alpha-1)}.$$

We abbreviate the last relation to

$$A_n = 1 + b_{n-1}A_{n-1} + \rho_{n-1}, \tag{3.6}$$

with

$$A_n = A_n(x) = \frac{F_n}{L_n^{(\alpha-2)}}, \qquad b_{n-1} = b_{n-1}(x) = a_{n-1}\frac{L_{n-1}^{(\alpha-2)}}{L_n^{(\alpha-2)}},$$

$$\rho_{n-1} = \rho_{n-1}(x) = -\sqrt{n}(a_{n-1}-\ell)\frac{L_{n-1}^{(\alpha-1)}}{\ell\sqrt{n}L_n^{(\alpha-2)}}.$$

Let K denote a compact subset of $\mathbb{C}\backslash[0,+\infty)$. From (2.4) and Lemma 3.3 we obtain

$$\lim_{n\to\infty} b_{n-1}(x) = \ell < 1,$$

and with (3.4), (2.4) and (2.5)

$$\lim_{n\to\infty} \rho_n(x) = 0;$$

in both limits the convergence is uniform on K.

Put

$$A_n^* = A_n - \frac{1}{1-\ell}.$$

Then (3.6) implies

$$A_n^* = \frac{b_{n-1} - \ell}{1 - \ell} + \rho_{n-1} + b_{n-1}A_{n-1}^*.$$

Let $\varepsilon > 0$ and $\ell < r < 1$. Then there exists an N, such that, if $n \geq N + 1$ and $x \in K$, we have

$$|A_n^*| < \varepsilon + r|A_{n-1}^*|.$$

By repeated application, for $k \geq 1$ and $x \in K$, we deduce

$$|A_{N+k}^*| < \varepsilon(1 + r + \cdots + r^{k-1}) + r^k|A_N^*| < \frac{\varepsilon}{1 - r} + r^k|A_N^*|.$$

This implies

$$\lim_{n \to \infty} A_n^* = 0,$$

uniformly on K. This proves the theorem. □

Remark. In the special case $\alpha = 0$ we have with (3.3)

$$|a_n - \ell| = \frac{n\ell|a_{n-1} - \ell|}{n(2 + \lambda) - na_{n-1}},$$

thus

$$|a_n - \ell| < \ell^n|a_0 - \ell|, \quad n \geq 1.$$

Then (3.4) can be improved to

$$\lim_{n \to \infty} n^p(a_n - \ell) = 0, \text{ for every } p.$$

Proceeding as in the proof of Theorem 3.4, we obtain a complete asymptotic expansion

$$S_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \ell^k}{(1 - \ell)^{k+1}} L_n^{(-k-1)}(x),$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, +\infty)$.

Finally, observe that Theorem 1.1 is a direct consequence of Theorem 3.4 and (2.5).

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