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doi:10.1006/jabr.2001.9059, available online at http://www.idealibrary.com on IDEAL®

Lie Algebras of CL Type[1]

Guang-Yu Shen

Department of Mathematics, East China Normal University, Shanghai 200062, People's Republic of China
E-mail: shengy@online.sh.cn

Communicated by Alexander Premet

Received January 22, 2001

The notion of a strongly nilpotent element of a Lie algebra is introduced. According to the existence or nonexistence of nontrivial strongly nilpotent elements, the simple modular Lie algebras are divided into two categories, CA type and CL type, which coincide with Lie algebras of generalized Cartan type and classical type, respectively, when the characteristic is greater than 7. Examples of nonclassical simple Lie algebras of CL type are given which all have affinities to the classical Lie algebras. © 2002 Elsevier Science (USA)

1. INTRODUCTION

According to the Block–Strade–Wilson classification theorem [St-Wi], simple Lie algebras over an algebraically closed field of characteristic $p > 7$ are divided into two categories: Lie algebras of (generalized) Cartan type and Lie algebras of classical type. A characterizing property that distinguishes the former from the latter is the existence of sandwich elements (a nonzero element x is a sandwich element if $(\operatorname{ad} x)^2 = 0$). Premet proved in [P] that a simple Lie algebra of characteristic $p > 3$ is classical if and only if it is not strongly degenerate; i.e., it contains no sandwich elements. However, this distinction becomes ambiguous in lower characteristics. For instance, when $p = 2$ every classical Lie algebra contains sandwich elements (e.g., the highest root vectors) and the class of Cartan-type Lie algebras and the class of classical Lie algebras overlap; e.g., $G_2 \cong H(4, \mathbf{1}) \cong S(3, \mathbf{1})$ (cf. e.g., [Sh]). Let L be a Lie algebra,

[1] Supported by National Natural Science Foundation of China.

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$\mathrm{nil}(L) := \{y \in L | \mathrm{ad}\, y \text{ is nilpotent}\}$, $\overline{\mathfrak{N}}(L) := \{x \in L | \mathrm{ad}\, x \cdot L \subset \mathrm{nil}(L)\}$, and $\mathfrak{N}(L) = \overline{\mathfrak{N}}(L) \cap \mathrm{nil}(L)$. A simple Lie algebra L is said to be of CL type if $\mathfrak{N}(L) = 0$; otherwise, L is of CA type. It will be shown that classical Lie algebras are of CL type for any characteristic, and (generalized) Cartan-type Lie algebras are of CA type with a few exceptions in characteristics 2 and 3. In particular, CL type and CA type coincide with classical type and (generalized) Cartan type, respectively, if the characteristic is greater than 7. Some classes of simple Lie algebras of characteristics 2 and 3 which have affinities to the classical Lie algebras are listed in Section 3 and are shown to be of CL type.

In the present article, unless otherwise stated, all spaces and algebras considered are finite dimensional over an algebraically closed field $\mathfrak{F}$ of characteristic p.

Notation. If V is a linear space and S, S' are subsets of V, then $\langle S \rangle$ denotes the subspace spanned by S and $S + S' := \{s + s' | s \in S, s' \in S'\}$.

If L is a Lie algebra and S, S' are subsets of L, then $\langle\langle S \rangle\rangle$ denotes the Lie subalgebra generated by S and $[S, S'] := \{[s, s'] | s \in S, s' \in S'\}$. If, in addition, $L = L_{[-r]} \oplus \cdots \oplus L_{[s]}$ (resp. $L \supset L_{-r} \supset \cdots \supset L_s \supset 0$) is a graded (resp. filtered) Lie algebra, then $S_{[i]} := S \cap L_{[i]}$ (resp. $S_i := S \cap L_i$). If L is a filtered Lie algebra, then $\mathrm{gr}\, L$ denotes the graded Lie algebra canonically associated to L.

2. STRONGLY NILPOTENT ELEMENT

(2.1) DEFINITION. Let L be a Lie algebra, $\mathrm{nil}(L) := \{y \in L | \mathrm{ad}\, y$ is nilpotent$\}$, $\overline{\mathfrak{N}}(L) := \{x \in L | \mathrm{ad}\, x \cdot L \subset \mathrm{nil}(L)\}$, and $\mathfrak{N}(L) = \overline{\mathfrak{N}}(L) \cap \mathrm{nil}(\mathrm{L})$. If $x \in \mathfrak{N}(L)$, then x will be called a strongly nilpotent element of L.

(2.2) PROPOSITION. *Let p be arbitrary. A Lie algebra L is semisimple if* (a) $\mathfrak{N}(L) = 0$ *or* (b) $\overline{\mathfrak{N}}(L) = 0$.

Proof. (a) If L is not semisimple, then L contains a nonzero abelian ideal I. Let $0 \neq x \in I$. Then $(\mathrm{ad}_L x)^2 = 0$. Let y be an arbitrary element of L and $z = \mathrm{ad}\, x \cdot y \in I$. Then $z \in \mathrm{nil}(L)$ and $x \in \overline{\mathfrak{N}}(L) \cap \mathrm{nil}(L) = \mathfrak{N}(L)$, a contradiction. (b) follows from (a) immediately. ∎

(2.3) THEOREM. *If $p = 0$, the following assertions are equivalent*:

(a) *L is semisimple*;

(b) $\overline{\mathfrak{N}}(L) = 0$;

(c) $\mathfrak{N}(L) = 0$.

Proof. By Proposition 2.2, it suffices to show (a) $\Rightarrow$ (b). Let Φ^+ and Φ^- be, respectively, the sets of positive and negative roots with respect to a Cartan subalgebra H. Let x be a nonzero element of L and let $x = \sum_{\alpha\in\Phi^-} c_\alpha x_\alpha + h + \sum_{\beta\in\Phi^+} c_\beta x_\beta$ be the standard root decomposition of x. If $x \notin H$, without loss of generality, we may suppose $c_\beta \neq 0$ for some $\beta \succ 0$. Let γ be maximal in the set $\{\beta \succ 0 | c_\beta \neq 0\}$. Then $z := \operatorname{ad} x \cdot x_{-\gamma} = h_\gamma + x^- \in B^-$, the negative Borel subalgebra, with $0 \neq h_\gamma \in H$ and x^- a linear combination of negative root vectors. Hence z is not nilpotent and $x \notin \overline{\mathfrak{N}}(L)$. If $0 \neq x = h \in H$, let x_α be a root vector belonging to the root α such that $\alpha(h) \neq 0$ and A denotes the automorphism $\exp(\operatorname{ad} x_\alpha)$. Let $y = Ax = h + \alpha(h)e_\alpha \notin H$. Then $y \notin \overline{\mathfrak{N}}(L)$ and $x \notin \overline{\mathfrak{N}}(L)$. ∎

(2.4) THEOREM. (1) *$\overline{\mathfrak{N}}(L) = L$ if and only if $[L, L]$ is nilpotent. In particular, if $p = 0$, $\overline{\mathfrak{N}}(L) = L$ if and only if L is solvable.* (2) *$\mathfrak{N}(L) = L$ if and only if L is nilpotent.*

Proof. (1) $\overline{\mathfrak{N}}(L) = L \Leftrightarrow \operatorname{ad}_L[x, y]$ is nilpotent $\forall x, y \in L \Leftrightarrow \operatorname{ad}_{[L,L]} \times [x, y]$ is nilpotent $\forall x, y \in L \Leftrightarrow [L, L]$ is nilpotent. When $p = 0$, by Lie's theorem, L is solvable if and only if $[L, L]$ is nilpotent. (2) is obvious. ∎

When $p > 0$, we say a simple Lie algebra is classical (or of classical type) if it is obtained from a Chevally basis of a simple complex Lie algebra by a modulo p process (and possibly a modulo center process). For detailed discussions of the classical Lie algebras, readers may refer to [Hi, Ho]. By an argument similar to the proof of Theorem 2.3, we have

(2.5) THEOREM. *If $p > 0$, L is a simple Lie algebra of classical type, then $\overline{\mathfrak{N}}(L) = 0$ and $\mathfrak{N}(L) = 0$.*

(2.6) LEMMA. *Let $L = L_{-r} \supset \cdots \supset L_s \supset L_{s+1} = 0$ be a filtered Lie algebra, $r \geq 0$. If $s > r$, then $\mathfrak{N}(L) \neq 0$.*

Proof. Let $0 \neq x \in L_s$. Then $x \in \operatorname{nil}(L)$ and $\operatorname{ad}_L x \cdot L \subset L_{s-r} \subset L_1 \subset \operatorname{nil}(L)$, i.e., $x \in \mathfrak{N}(L)$. ∎

Let $L \supset L_{-r} \supset \cdots \supset L_s \supset L_{s+1} \supset 0$ be a filtered Lie algebra and $\overline{L} = \operatorname{gr} L = \bigoplus_{i=-r}^{s} \overline{L}_{[i]}$. If $0 \neq x \in L$, denote $\deg x := \max\{i | x \in L_i\}$. If $\deg x = t$, let $\bar{x} = \operatorname{gr} x := x + L_{t+1} \in \overline{L}_{[t]}$. When $[\bar{x}, \bar{y}] \neq 0$, we have $[\bar{x}, \bar{y}] = \overline{[x, y]}$.

(2.7) LEMMA. *Let L and $\overline{L}$ be as above.* (a) *If $x_1, x_2 \in L_0$ and* $\operatorname{gr} x_1 = \operatorname{gr} x_2$, *then $x_1 \in \operatorname{nil}(L)$ if and only if $x_2 \in \operatorname{nil}(L)$*; (b) *If $x \in L$ and $\bar{x} := \operatorname{gr} x \notin \operatorname{nil}(\overline{L})$, then $x \notin \operatorname{nil}(L)$*; (c) *If $x \in L$ and $\bar{x} \notin \mathfrak{N}(\overline{L})$ (resp. $\overline{\mathfrak{N}}(\overline{L})$), then $x \notin \mathfrak{N}(L)$ (resp. $\overline{\mathfrak{N}}(L)$).*

Proof. (a) We need only consider the case $\operatorname{gr} x_1 = \operatorname{gr} x_2 = 0$. If $x_1 \in \operatorname{nil}(L)$, then $\{\operatorname{ad} x_1\} \cup \{\operatorname{ad} L_1\}$ is a nil weakly closed set. Hence $\langle \operatorname{ad} x_1 \rangle + \operatorname{ad} L_1$, which contains $\operatorname{ad} x_2$, consists of nilpotent elements.

(b) There exists $y \in L$ such that $(\operatorname{ad} \bar{x})^n \cdot \bar{y} = \overline{(\operatorname{ad}_L x)^n \cdot y} \neq 0$ for every $n > 0$. Hence $(\operatorname{ad}_L x)^n \cdot y \neq 0$ and $x \notin \operatorname{nil}(L)$.

(c) If $\bar{x} \notin \operatorname{nil}(\overline{L})$, then $x \notin \operatorname{nil}(L) \supset \mathfrak{N}(L)$. If $\bar{x} \in \operatorname{nil}(\overline{L})$, then $\bar{x} \notin \overline{\mathfrak{N}}(\overline{L})$ and there exists y such that $\operatorname{ad}_{\overline{L}} \bar{x} \cdot \bar{y} = \overline{\operatorname{ad}_L x \cdot y} \notin \operatorname{nil}(\overline{L})$. By (b), $\operatorname{ad}_L x \cdot y \notin \operatorname{nil}(L)$ and $x \notin \overline{\mathfrak{N}}(L)$. ∎

(2.8) COROLLARY. *Let L and $\overline{L}$ be as above. If $\mathfrak{N}(\overline{L}) = 0$ (resp. $\overline{\mathfrak{N}}(\overline{L}) = 0$), then $\mathfrak{N}(L) = 0$ (resp. $\overline{\mathfrak{N}}(L) = 0$).*

Proof. This is a direct consequence of Lemma 2.7(c). ∎

For the definitions and notation of the simple graded Lie algebras of Cartan type, we essentially follow [St-Fa] (for the contact algebras $K(m, \mathbf{n})$ of characteristic 2, which are excluded in [St-Fa], we refer to [Fe-Sh, Lin 1]). Let $\overline{X}(m, \mathbf{n}) := X'(m, \mathbf{n})$, $X = W, S, K$, and $\overline{H}(m, \mathbf{n}) = H''(m, \mathbf{n})$ (cf. [St-Fa]).

(2.9) *Note.* When $p = 2$, $W(1, n)$ is not simple. For notational consistency, in this case, we shall write $\overline{W}(1, n)$ instead of $W(1, n)$ and let $W(1, n) := [\overline{W}(1, n), \overline{W}(1, n)]$, which is simple of dimension $2^n - 1$.

If L is a filtered Lie algebra, $\operatorname{gr} L = \overline{L}$, we shall call L a filtered deformation of $\overline{L}$. A simple Lie algebra L is called of (generalized) Cartan type $X(m, \mathbf{n})$ (or simply a (generalized) Cartan-type Lie algebra), $X = W, S, H$, or K, if L is a filtered deformation of a homogeneous subalgebra $\overline{L}$ of $\overline{X}(m, \mathbf{n})$ such that $\overline{L} \supset X(m, \mathbf{n})$.

(2.10) *Remark.* In the author's personal opinion, the terminology "generalized Cartan-type Lie algebras" is somewhat cumbersome. And what exactly are the "Cartan-type Lie algebras" that the "generalized Cartan-type Lie algebras" generalize? Might it be better to just call them the Cartan-type Lie algebras and specify with modifiers, e.g., "restricted," "graded," etc., in special cases?

(2.11) THEOREM. (1) *Let L be a (generalized) Cartan-type Lie algebra. Then $\mathfrak{N}(L) \neq 0$ if L is not of type* (a) $W(1, 1)$, $p = 3$; (b) $W(2, \mathbf{1})$, $p = 2$; (c) $S(3, \mathbf{1})$, $p = 2$; (d) $H(4, \mathbf{1})$, $p = 2$; (e) $K(3, \mathbf{1})$, $p = 2$; (f) $W(1, 2)$, $p = 2$.

(2) *If L is a filtered deformation of* (a)–(e) *in* (1), *then $\overline{\mathfrak{N}}(L) = \mathfrak{N}(L) = 0$.*

(3) *If $p = 2$, $L = W(1, 2)$, then $\mathfrak{N}(L) = 0$ and $\overline{\mathfrak{N}}(L) = \langle x_1 D_1 \rangle \neq 0$.*

Proof. (1) L satisfies the condition of Lemma 2.6.

(2) When $p = 3$, $W(1, 1) \cong A_1$. When $p = 2$, $W(2, \mathbf{1}) \cong K(3, \mathbf{1}) \cong A_2$ and $S(3, \mathbf{1}) \cong H(4, \mathbf{1}) \cong G_2$. Our assertion follows from Theorem 2.5 and Corollary 2.8.

(3) Let $e = x_1^{(2)} D_1$, $h = x_1 D_1$, and $f = D_1$. By an argument similar to Theorem 2.3, we can show that if $y \notin \langle h \rangle$, $y \notin \overline{\mathfrak{N}}(L)$. We have $\operatorname{ad} h \cdot L = \langle e, f \rangle \subset \operatorname{nil}(L)$ since $(\operatorname{ad}(\alpha e + \beta f))^4 = 0$, $\forall\, \alpha, \beta \in \mathfrak{F}$. ∎

(2.12) *Note.* When $p = 2$, $W(1, 2)$ is the three-dimensional simple Lie algebra which is unique up to isomorphism. Hence $W(1, 2)$ does not possess any nontrivial filtered deformation.

(2.13) DEFINITION. A (generalized) Cartan-type Lie algebra is called exceptional if it is of type (a)–(f).

In view of the Block–Strade–Wilson classification theorem, Theorems 2.5 and 2.11 suggest the following:

(2.14) DEFINITION. A simple Lie algebra L of characteristic $p > 0$ is of CL type (resp. CA type) if $\mathfrak{N}(L) = 0$ (resp. $\mathfrak{N}(L) \neq 0$).

(2.15) EXAMPLES. The following Lie algebras are of CL type:

(i) classical Lie algebras;

(ii) contragredient Lie algebras in [Sk2, We-Ka];

(iii) the class $L(\varepsilon, \delta, \rho)$ of simple Lie algebras of characteristic 3 in [Ko];

(iv) variations of G_2 ($V_i G$, $i = 3, 4, 5, 6$) of characteristic 2 in [Sh];

(v) filtered deformations of the exceptional graded Cartan-type Lie algebras.

Proof. For (ii)–(iv), see Section 3 below. ∎

(2.16) *Remark.* Let L be an exceptional (generalized) Cartan-type Lie algebra which is a filtered deformation of a homogeneous subalgebra $\overline{L}$ of $\overline{X} := \overline{X}(m, \mathbf{n})$ and $\overline{L} \supset X := X(m, \mathbf{n})$. If $\operatorname{length}(\overline{X}) > \operatorname{length}(X)$, then L is of CA type by Lemma 2.6. However, if $\operatorname{length}(\overline{X}) = \operatorname{length}(X)$ and $\overline{X} \neq X$, we cannot decide the CL–CA type of L without more detailed knowledge of L. (To the author's knowledge, there is not much information about the structures of the nongraded (generalized) Cartan-type Lie algebras in characteristics 2 and 3.)

(2.17) EXAMPLES. The following Lie algebras are of CA type:

(i) the nonexceptional (generalized) Cartan-type algebras, $p > 0$;

(ii) the Melikyan algebras of characteristic 5 [M];

(iii) $T(n)$ of characteristic 3 discussed in [B1, B2, Ch, Ku, Sk1];

(iv) $R(2 : (n_1, n_2))^{(1)}$ of characteristic 3 discussed in [B2, Ch, E, Ku, Sk1];

(v) $Z'(F)$, $Y(F)$, and $X'''(F, \omega)$ of characteristic 3 in [Sk1];

(vi) the nonalternating Hamiltonian Lie algebra $P(n, \mathbf{m})$ ($\mathbf{m} \neq \mathbf{1}$ if $n < 4$) of characteristic 2 in [Lin1];

(vii) $D_4(3 : m_1, m_2, m_3)$, $(m_1, m_2, m_3) \neq (1, 1, 1)$, $G_2(2 : m_1, m_2)$, $(m_1, m_2) \neq (1, 1)$, and $C_{3,\alpha}(2 : n, 1)$, $n \neq 1$, of characteristic 2 in [B3].

Proof. They have filtered Lie algebra stuctures satisfying the condition of Lemma 2.6. ∎

(2.18) *Remark.* The $\mathfrak{N}(L)$ criterion and the strong degeneracy criterion are not compatible in the cases $p = 3$, $L = W(1, 1)$ and $p = 2$, $L = W(2, \mathbf{1})$, $S(3, \mathbf{1})$, $H(4, \mathbf{1})$, or $K(3, \mathbf{1})$. By the former, they should be viewed as classical, while, by the latter, they should be counted as of Cartan type.

(2.19) *Remark.* The above consideration applies to infinite-dimensional Lie algebras as well. Following [J], we call an element x of an infinite-dimensional Lie algebra L quasi-nilpotent if $\cap_{i=1}^{\infty}(\operatorname{ad} x)^i \cdot L = 0$. Let q-nil(L) be the set of quasi-nilpotent elements of L and $q\mathfrak{N}(L) = \{x \in L | \operatorname{ad} x \cdot L \subset q\text{-nil}(L)\} \cap q\text{-nil}(L)$. A simple Lie algebra L is of qCL type (resp. qCA type) if $q\mathfrak{N}(L) = 0$ (resp. $q\mathfrak{N}(L) \neq 0$). For instance, let $L = L(A)$ be a simple Kac–Moody Lie algebra over $\mathbf{C}$ associated to a symmetrizable generalized Cartan matrix A (which is nonsingular; cf. Proposition 1.10B of [Wa]). Then L possesses a nontrivial invariant bilinear form and, by an argument similar to Theorem 2.5, L is of qCL type. On the other hand, if $L = X(m)$, $X = W, S, H$, or K, is an infinite-dimensional Cartan-type Lie algebra over $\mathbf{C}$, it is easily shown that L is of qCA type. (However, for ordinary ad-nilpotency, we have $\mathfrak{N}(L(A)) = 0$ and $\mathfrak{N}(X(m)) = 0$). It is also easy to see that the Witt or centerless Virasoro algebra is of qCL type.

(2.20) *Problem.* Is it possible for every Lie algebra of CA type to be endowed with a "nice" filtered Lie algebra structure that satisfies the condition of Lemma 2.6?

(2.21) *Problem.* If $p > 0$, what can we say about the Lie algebras of CL type? Every Lie algebra in (2.15) is, more or less, affiliated with classical Lie algebras, by structure and/or by origin. Does this phenomenon reflect something essential to the Lie algebras of CL type?

(2.22) *Problem.* Classify over the complex field the simple $\mathbf{Z}$-graded Lie algebras with finite-dimensional grading spaces which are of qCL type.

3. SOME CLASSES OF LIE ALGEBRAS OF CL TYPE

In this section we shall give a detailed description of V_iG, $i = 3, 4, 5, 6$, in terms of the root system of G_2 and show in light of the proof of Theorem 2.3 that the Lie algebras (ii)–(iv) in (2.15) are of CL type.

(3.1) DEFINITION. Let L be a Lie algebra. A direct sum decomposition $L = L^- \oplus H \oplus L^+$ is called a quasi-triangular decomposition if (1) L^+ is a nilpotent Lie subalgebra of L; (2) L^- is a subspace of L; (3) H is a Lie subalgebra of L; (4) $[H, L^+] \subset L^+$ and $[H, L^-] \subset L^-$; and (5) if $0 \neq h \in H$, then $(\mathrm{ad}_L h)|_{L^+}$ is not nilpotent. If, in addition, L^- is also a nilpotent Lie subalgebra of L and $(\mathrm{ad}_L h)|_{L^-}$ is not nilpotent for $0 \neq h \in H$, then $L = L^- \oplus H \oplus L^+$ will be called a triangular decomposition of L.

(3.2) PROPOSITION. *Suppose* $p > 0$. (a) *If* $L = L^- \oplus H \oplus L^+$ *is a quasi-triangular decomposition of the Lie algebra* L *and for every nonzero* $y \in L^-$ (*resp.* L^+) *there exists* $x \in L^+$ (*resp.* L^-) *such that* $[x, y] = h + x^+$ *where* $0 \neq h \in H$ *and* $x^+ \in L^+$, *then* $\mathfrak{R}(L) = 0$. (b) *If* $L = L^- \oplus H \oplus L^+$ *is a triangular decomposition of* L *and for every nonzero* $y \in L^-$ (*resp.* L^+) *there exists* $x \in L^+$ (*resp.* L^-) *such that* $[x, y] = h + x^+$ (*resp.* $h + x^-$) *where* $0 \neq h \in H$ *and* $x^+ \in L^+$ (*resp.* $x^- \in L^-$), *then* $\mathfrak{R}(L) = 0$.

Proof. (a) Let w be a nonzero element of L. (1) Suppose $w \in H \oplus L^+$ and $w \notin L^+$, i.e., $w = h + x^+$, $0 \neq h \in H$, $x^+ \in L^+$. Denote $\xi(a) := (\mathrm{ad}_L a)|_{L^+}$ for $a \in H \oplus L^+$. Then $\xi(w)^{p^r} = \xi(h)^{p^r} + z$, $z \in \xi(L^+)_p$, where $\xi(L^+)_p$ is the p-envelop of $\xi(L^+)$ in $gl(L^+)$ which consists of nilpotent transformations. Hence $\xi(w)^{p^r} \neq 0$ for every r. This implies $w \notin \mathrm{nil}(L)$ and, in particular, $w \notin \mathfrak{R}(L)$. (2) $w \in L^+$. There exists $y \in L^-$ such that $[w, y] = h + x^+ \notin \mathrm{nil}(L)$ and hence $w \notin \mathfrak{R}(L)$. (3) $w \notin H \oplus L^+$. Then $w = x^- + h + x^+$, $0 \neq x^- \in L^-$, $h \in H$, $x^+ \in L^+$. By assumption, there exists $y \in L^+$ such that $[x^-, y] = h' + y^+$, $0 \neq h' \in H$, $y^+ \in L^+$. Then $[w, y] = h' + w^+$ where $w^+ \in L^+$. Hence $w \notin \mathfrak{R}(L)$. (b) can be proved similarly. ∎

(3.3) PROPOSITION. *The contragredient algebras in* [Sk2, We-Ka] *are of CL type*.

Proof. Let L be a contragredient algebra. Then L has a quasi-triangular (in fact, triangular) decomposition $L = L^- \oplus H \oplus L^+$ where H is a torus, L^+ is the subalgebra of "positive root vectors," i.e., L^+ is spanned by a basis $\{e_\alpha | \alpha \in \Phi^+\}$ where Φ^+ is the set of positive (formal) roots, and L^- is the subalgebra of "negative root vectors" spanned by a basis $\{e_{-\alpha} | \alpha \in \Phi^+\}$. The set $\Phi := \Phi^+ \cup (-\Phi^+)$ will be called the (formal) root system of L. As in the classical cases, Φ is partially ordered. We have the multiplications

$$[e_\alpha, e_\beta] = c_{\alpha,\beta} e_{\alpha+\beta}, \quad \alpha, \beta \text{ unproportional}, \tag{3.3.1}$$

where

$$c_{\alpha,\beta} = 0 \quad \text{if } \alpha + \beta \notin \Phi, \tag{3.3.2}$$

$$[h, e_\alpha] = \bar{\alpha}(h), \qquad h \in H, \alpha \in \Phi, \tag{3.3.3}$$

where $\bar{\alpha}$ (a "true" root) is a linear function on H corresponding to α; and

$$0 \neq [e_\alpha, e_{-\alpha}] = h_\alpha \in H, \qquad \alpha \in \Phi^+. \tag{3.3.4}$$

(Note: Here we let h_α denote the $(e_\alpha, e_{-\alpha})h_\alpha$ in [We-Ka]. For details, cf. [Ka1, We-Ka].) Let $y = \sum a_{\pm\alpha} e_{\pm\alpha} \in L^{\pm}$ and let $\pm\beta$ be minimal in $\{\pm\alpha | a_{\pm\alpha} \neq 0\}$. Then $[e_{\mp\beta}, y] = \mp a_{\pm\beta} h_\beta + x^+$, $x^+ \in L^+$. We have $\mathfrak{N}(L) = 0$ by Proposition 3.2(a). ∎

(3.4) Proposition. *The Lie algebras $L(\varepsilon, \delta, \rho)$ of characteristic* 3 *in* [Ko] *are of CL type.*

Proof. Let Φ be the root system of C_2. In [Ko] a class $L(\varepsilon)$ of simple Lie algebra was constructed which is spanned by a basis $\{E_\alpha, \alpha \in \Phi; H_\beta, Z\}$, $L(\varepsilon)$ was shown to be contragredient in [We-Ka]. Here, in particular, $L(\varepsilon)$ possesses a multiplication table of the form (3.3.1)–(3.3.4) (where $H = \langle H_\beta, Z \rangle$). The algebras $L(\varepsilon, \delta, \rho)$ are deformations of $L(\varepsilon)$ ($L(0, 0, \varepsilon) = L(\varepsilon)$) spanned by the same basis elements with the same multiplication table except in the cases

$$\begin{gathered} [E_{-(2\alpha+\beta)}, E_{-(\alpha+\beta)}] = \delta E_\beta, \quad [E_{-(2\alpha+\beta)}, E_{-\beta}] = \delta E_{\alpha+\beta}, \\ [E_{-(2\alpha+\beta)}, E_\beta] = \rho E_\alpha, \quad [E_{-(2\alpha+\beta)}, E_{-\alpha}] = \rho E_{-\beta}, \\ [E_{-(\alpha+\beta)}, E_{-\beta}] = -\varepsilon^{-1} \delta E_{2\alpha+\beta}, \quad [E_{-\alpha}, E_\beta] = -\varepsilon^{-1} \rho E_{2\alpha+\beta}. \end{gathered} \tag{3.4.1}$$

Let $L^+ := \langle E_\omega | \omega > 0 \rangle$, $L^- := \langle E_\omega | \omega < 0 \rangle$, and $H = \langle H_\beta, Z \rangle$. Then $L(\varepsilon, \delta, \rho) = L^- \oplus H \oplus L^+$ is a quasi-triangular decomposition. Note that products of the form $[E_\gamma, E_{-\gamma'}]$ in (3.4.1) are in L^+. Hence the proof of Proposition 3.3 applies here as well and we have $\mathfrak{N}(L(\varepsilon, \delta, \rho)) = 0$. ∎

(3.5) Proposition. *The simple Lie algebras $V_4G(a_2, a_3)$, $a_2, a_3 \in \mathfrak{F}$, $a_2 a_3 \neq 0$, of characteristic* 2 *in* [Sh] *are of CL type.*

Proof. Let $L := V_4G(a_2, a_3)$ which is spanned by the basis $\{1, x_i,\ i = 1, 2, 3, 4;\ f_i,\ i = 1, 2, 3, 4, 5;\ h_i, i = 1, 2;\ e_i,\ i = 1, 2\}$ (for details, cf. [Sh]). Let Φ be the root system of G_2 and let α_1 and α_2 be the short and long simple roots, respectively. To see more clearly its affinity to (and deviation from) G_2, denote

$$\begin{gathered} E_{-(3\alpha_1+2\alpha_2)} := 1, \quad E_{-\alpha_2} := x_1, \quad E_{-(\alpha_1+\alpha_2)} := x_2, \\ E_{-(3\alpha_1+\alpha_2)} := x_3, \quad E_{-(2\alpha_1+\alpha_2)} := x_4; \end{gathered} \tag{3.5.1}$$

$$\begin{gathered} E_{(3\alpha_1+\alpha_2)} := f_1, \quad E_{2\alpha_1+\alpha_2} := f_2, \quad E_{\alpha_2} := f_3, \\ E_{\alpha_1+\alpha_2} := f_4, \quad E_{3\alpha_1+2\alpha_2} := f_5; \end{gathered} \tag{3.5.2}$$

$$E_{\alpha_1} := e_1, \quad E_{-\alpha_1} := e_2. \tag{3.5.3}$$

Let $L^- := \langle E_\gamma | \gamma > 0 \rangle$, $L^+ := \langle E_{-\gamma} | \gamma < 0 \rangle$, and $H = \langle h_1, h_2 \rangle$. Define linear functions $\bar{\alpha}_1$ and $\bar{\alpha}_2$ on H by letting

$$\bar{\alpha}_i(h_j) = \delta_{ij}, \qquad i = 1, 2. \tag{3.5.4}$$

Extending linearly, we let

$$\overline{c_1\alpha_1 + c_2\alpha_2} = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2, \qquad c_1, c_2 \in \mathfrak{F}. \tag{3.5.5}$$

In particular, for every $\alpha \in \Phi$ there corresponds a linear function $\bar{\alpha}$. Dually, we let

$$h_\gamma := kh_1 + lh_2, \qquad \gamma = k\alpha_1 + l\alpha_2 \in \Phi. \tag{3.5.6}$$

We have

$$\begin{aligned} &[E_\beta, E_\gamma] = c(\beta, \gamma)E_{\beta+\gamma} \neq 0, \\ &\qquad\qquad \beta, \gamma \text{ unproportional and } \beta + \gamma \in \Phi, \end{aligned} \tag{3.5.7}$$

except

$$[E_{-\alpha_1}, E_{-(\alpha_1+\alpha_2)}] = [E_{-\alpha_1}, E_{2\alpha_1+\alpha_2}] = 0, \tag{3.5.7$'$}$$

where

$$\begin{aligned} c(-(2\alpha_1 + \alpha_2), \alpha_1 + \alpha_2) = c(\alpha_1, -(2\alpha_1 + \alpha_2)) = a_2, \\ \text{and all other } c(\beta, \gamma) = 1; \end{aligned} \tag{3.5.7.1}$$

$$[E_\beta, E_\gamma] = 0, \quad \beta, \gamma \text{ unproportional and } \beta + \gamma \notin \Phi, \tag{3.5.8}$$

except

$$\begin{aligned} [E_{-\alpha_2}, E_{3\alpha_1+\alpha_2}] = a_3E_{-\alpha_1}, \quad [E_{-\alpha_2}, E_{\alpha_1}] = a_3E_{-(3\alpha_1+\alpha_2)}, \\ [E_{\alpha_1}, E_{3\alpha_1+\alpha_2}] = a_3E_{\alpha_2}; \end{aligned} \tag{3.5.8$'$}$$

$$[h, E_\gamma] = \bar{\gamma}(h)E_\gamma, \qquad h \in H, \gamma \in \Phi; \tag{3.5.9}$$

$$[E_{-\gamma}, E_\gamma] = h_\gamma, \qquad \gamma \in \Phi. \tag{3.5.10}$$

It follows that $L = L^- \oplus H \oplus L^+$ is a quasi-triangular decomposition. Similar to Proposition 3.4 we can show that $\mathfrak{N}(L) = 0$. ∎

(3.6) PROPOSITION. *The simple Lie algebras* $V_3G(a)$, $a \in \mathfrak{F}$, $a \neq 0$, *of characteristic* 2 *in* [Sh] *are of CL type.*

Proof. Let $L = V_3G(a)$ which has a basis $\{1, x_i, i = 1, 2, 3, 4; f_i, i = 1, 2, 3, 4, 5; h_i, i = 1, 2, ; e_i, i = 1, 2\}$. Adopting the notation (3.5.1)–(3.5.3), we can obtain the multiplication table of $V_3G(a)$ from that of $V_4G(a_2, a_3)$ by setting $a_2 = a$, $a_3 = 0$. Then the exceptional cases in (3.5.8′) do not occur. Our conclusion can be obtained by an argument similar to (in fact, simpler than) that of Proposition 3.5. ∎

(3.7) PROPOSITION. *The Lie algebras $V_5G(a, a_1, b_1)$, $aa_1b_1 \neq 0$, $d := a + a_1b_1 \neq 0$, of characteristic 2 in* [Sh] *are of CL type.*

Proof. Let $L := V_5G(a, a_1, b_1)$ which is spanned by the basis $\{1; x_i, i = 1, 2, 3, 4;\ f_i, i = 1, 2, 3, 4, 5;\ h_i, i = 1, 2; e_i, i = 1, 2, 3\}$ (cf. [Sh]). We adopt the notation (3.5.1)–(3.5.5) and let

$$E_{\pm 2\alpha_1} := e_3. \tag{3.7.1}$$

Then the multiplication table is

$$[E_\beta, E_\gamma] = c(\beta, \gamma)E_{\beta+\gamma} \neq 0, \qquad \beta, \gamma \text{ unproportional and } \beta + \gamma \in \Phi, \tag{3.7.2}$$

where

$$\begin{aligned}
c(-\alpha_1, -(\alpha_1+\alpha_2)) &= c(-(\alpha_1+\alpha_2), 2\alpha_1+\alpha_2)\\
&= c(-\alpha_1, 2\alpha_1+\alpha_2) = a,\\
c(\alpha_1, -(\alpha_1+\alpha_2)) &= c(\alpha_1, -(3\alpha_1+\alpha_2)) = c(\alpha_1, \alpha_2)\\
&= c(\alpha_1, 2\alpha_1+\alpha_2) = c(-(\alpha_1+\alpha_2), \alpha_2)\\
&= c(-(3\alpha_1+\alpha_2), 2\alpha_1+\alpha_2) = a_1,\\
c(-\alpha_1, \alpha_1+\alpha_2) &= c(-\alpha_1, 3\alpha_1+\alpha_2) = c(-\alpha_1, -\alpha_2)\\
&= c(-\alpha_1, -(2\alpha_1+\alpha_2)) = c(\alpha_1+\alpha_2, -\alpha_2)\\
&= c(3\alpha_1+\alpha_2, -(2\alpha_1+\alpha_2)) = b_1,\\
c(-(3\alpha_1+2\alpha_2), 3\alpha_1+\alpha_2) &= c(-(3\alpha_1+2\alpha_2), 2\alpha_1+\alpha_2)\\
&= c(-(3\alpha_1+2\alpha_2), \alpha_2)\\
&= c(-(3\alpha_1+2\alpha_2), \alpha_1+\alpha_2)\\
&= d \text{ and all other } c(\beta, \gamma) = 1;
\end{aligned} \tag{3.7.2.1}$$

$$[E_{\pm 2\alpha_1}, E_\beta] = \begin{cases} E_{\beta+2\alpha_1}, & \beta, \alpha_1 \text{ unproportional, } \beta+2\alpha_1 \in \Phi,\\ aE_{\beta-2\alpha_1}, & \beta, \alpha_1 \text{ unproportional, } \beta-2\alpha_1 \in \Phi,\\ 0, & \beta, \alpha_1 \text{ unproportional, } \beta \pm 2\alpha_1 \notin \Phi; \end{cases} \tag{3.7.2.2}$$

(3.7.2.3) $$[E_{\pm 2\alpha_1}, E_{\alpha_1}] = [E_{\pm 2\alpha_1}, E_{-\alpha_1}] = 0;$$

(3.7.2.4) $$[E_\beta, E_\gamma] = aE_{\pm 2\alpha_1}, \quad \beta, \gamma \text{ unproportional, } \beta + \gamma = 2\alpha_1,$$

(3.7.2.4′) $$[E_\beta, E_\gamma] = E_{\pm 2\alpha_1}, \quad \beta, \gamma \text{ unproportional, } \beta + \gamma = -2\alpha_1;$$

(3.7.2.5) $[E_\beta, E_\gamma] = 0, \quad \beta, \gamma$ unproportional and $\beta, \gamma \notin \Phi \cup \{2\alpha_1, -2\alpha_1\}$, except

(3.7.2.6) $$\begin{aligned} [E_{-\alpha_2}, E_{\alpha_1}] &= aE_{-(3\alpha_1+\alpha_2)}, & [E_{\alpha_2}, E_{-\alpha_1}] &= E_{3\alpha_1+\alpha_2}, \\ [E_{-\alpha_2}, E_{3\alpha_1+\alpha_2}] &= aE_{-\alpha_1}, & [E_{\alpha_2}, E_{-(3\alpha_1+\alpha_2)}] &= E_{\alpha_1}; \end{aligned}$$

(3.7.2.7) $$[h, E_\gamma] = \bar{\gamma}(h) E_\gamma, \qquad \gamma \in \Phi, \qquad h \in H' := \langle h_1, h_2 \rangle;$$

(3.7.2.8) $$[h, E_{\pm 2\alpha_1}] = 0, \qquad h \in H';$$

(3.7.2.9) $$[E_\gamma, E_{-\gamma}] = h_\gamma,$$

where (different from (3.5.6))

(3.7.2.10) $$h_\gamma = idh_1 + j(ah_1 + dh_2), \qquad \gamma = i\alpha_1 + j\alpha_2.$$

Let $L^+ = \langle E_\gamma | \gamma > 0 \rangle$, $L^- = \langle E_\gamma | \gamma < 0 \rangle$, and $H := \langle h_1, h_2, E_{\pm 2\alpha_1} \rangle$. We have, restricted on L^+ or L^-, $(\mathrm{ad} E_{\pm 2\alpha_1})^4 = a(\mathrm{ad} E_{\pm 2\alpha_1})^2 \neq 0$. In view of (3.7.2.7) and (3.7.2.8), condition (5) of Definition 3.1 is satisfied and $L = L^- \oplus H \oplus L^+$ is a triangular decomposition of L. Suppose $0 \neq x := \sum_{\gamma > 0} c_{-\gamma} E_{-\gamma} \in L^-$. Let $\mathscr{S} := \{\gamma > 0 | c_{-\gamma} \neq 0\}$. Let β be a maximal element in $\mathscr{S}$. From the multiplication table, in particular (3.7.2.6), we see that if $\beta \neq 3\alpha_1 + \alpha_2$ or α_1, then $[E_\beta, E_{-\gamma}] \in \langle E_{\pm 2\alpha_1} \rangle + L^+$ for all γ not greater than β and $[E_\beta, x] \in c_{-\beta} h_\beta + \langle E_{\pm 2\alpha_1} \rangle + L^+$. If $\beta = \alpha_1$ but $c_{-\alpha_2} = 0$, the above argument applies also. If $\beta = \alpha_1$, $c_{-\alpha_2} \neq 0$, we can take $\beta = \alpha_2$ instead of α_1. If $\beta = 3\alpha_1 + \alpha_2$, let $y = sE_{3\alpha_1+\alpha_2} + tE_{\alpha_2}$. By (3.7.2.4), (3.7.2.6), and (3.7.2.1) we have $[y, x] \equiv sc_{-(3\alpha_1+\alpha_2)} h_{3\alpha_1+\alpha_2} + tc_{-\alpha_2} h_{-\alpha_2} + (sac_{-\alpha_2} + ta_1 c_{-(\alpha_1+\alpha_2)}) E_{-\alpha_1} (\mathrm{mod} \langle E_{\pm 2\alpha_1}, L^+ \rangle)$. We can take s and t such that $(sc_{-(3\alpha_1+\alpha_2)}, tc_{-\alpha_2}) \neq (0,0)$ and $sac_{-\alpha_2} + ta_1 c_{-(\alpha_1+\alpha_2)} = 0$. Then $[y, x] \equiv h$ $(\mathrm{mod} \langle E_{\pm 2\alpha_1}, L^+ \rangle)$ where $h \in H'$ and $h \neq 0$ since $h_{3\alpha_1+\alpha_2}$ and h_{α_2} are linearly independent. By the symmetry of L^+ and L^- the conditions of Proposition 3.2(b) are satisfied and we have $\mathfrak{R}(L) = 0$. ∎

(3.7.3) *Remark.* If $a = 1$, $a_1 = b_1$, then the positive and negative parts of V_5G are completely symmetrical: The linear map φ: $E_\beta \mapsto E_{-\beta}$, $\beta \neq \pm(3\alpha_1 + 2\alpha_2)$, $E_{-(3\alpha_1+2\alpha_2)} \mapsto dE_{3\alpha_1+2\alpha_2}$, $E_{3\alpha_1+2\alpha_2} \mapsto \frac{1}{d} E_{-(3\alpha_1+2\alpha_2)}$, $E_{\pm 2\alpha_1} \mapsto E_{\pm 2\alpha_1}$, $h_i \mapsto h_i, i = 1, 2$, is an involution. Another involution has been obtained in [Liu] by an inspection of the graded Lie algebra structure of V_5G considered in [Sh].

(3.8) PROPOSITION. *The Lie algebras* $V_6G(a,s)$, $a,s\in\mathfrak{F}$, $as\neq 0$, *of characteristic* 2 *in* [Sh] *are of CL type*.

Proof. Let $L:=V_6G(a,s)$ which is spanned by the basis $\{1;x_i,\ i=1,2,3,4;\ f_i,\ i=1,2,3,4,5;\ h_1',h_2;\ e_1,e_2''\}$ (cf. [Sh]). We adopt the notation (3.5.1) and (3.5.2) and let

$$E_{-\alpha_1}:=e_1,\qquad E_{\alpha_1}:=e_2'', \tag{3.8.1}$$

$$E_{0,\pm 2\alpha_1}:=h_1'. \tag{3.8.2}$$

Then the multiplication table is as follows.

(I) $\beta,\gamma\in\Phi$ unproportional, $\beta>0,\gamma>0$ or $\beta<0,\gamma<0$.

$$[E_\beta,E_\gamma]=c(\beta,\gamma)E_{\beta+\gamma}\neq 0,\qquad \beta+\gamma\in\Phi, \tag{3.8.I.1}$$

except

$$\begin{gathered}[E_{-\alpha_1},E_{-(\alpha_1+\alpha_2)}]=0,\qquad [E_{-\alpha_1},E_{-\alpha_2}]=aE_{-(3\alpha_1+\alpha_2)},\\ [E_{\alpha_1},E_{\alpha_2}]=E_{3\alpha_1+\alpha_2},\qquad [E_{-\alpha_1},E_{-(2\alpha_1+\alpha_2)}]=E_{-(\alpha_1+\alpha_2)},\\ [E_{\alpha_1},E_{2\alpha_1+\alpha_2}]=aE_{\alpha_1+\alpha_2},\end{gathered} \tag{3.8.I.2}$$

where

$$\text{all } c(\beta,\gamma)=1\quad \text{except } c(\alpha_1,\alpha_1+\alpha_2)=a^{-1}s; \tag{3.8.I.3}$$

$$[E_\beta,E_\gamma]=0,\qquad \beta+\gamma\notin\Phi, \tag{3.8.I.1$'$}$$

except

$$[E_{\alpha_1},E_{(3\alpha_1+\alpha_2)}]=sE_{\alpha_2}. \tag{3.8.I.2$'$}$$

(II) $\beta,\gamma\in\Phi,\beta,\gamma$ unproportional, $\beta<0$, $\gamma>0$.

$$[E_\beta,E_\gamma]=d(\beta,\gamma)E_{\beta+\gamma}\neq 0,\qquad \beta+\gamma\in\Phi, \tag{3.8.II.1}$$

except

$$\begin{gathered}[E_{-(\alpha_1+\alpha_2)},E_{2\alpha_1+\alpha_2}]=0,\qquad [E_{-\alpha_1},E_{2\alpha_1+\alpha_2}]=0,\\ [E_{-(2\alpha_1+\alpha_2)},E_{\alpha_1}]=E_{-(3\alpha_1+\alpha_2)},\quad [E_{-(2\alpha_1+\alpha_2)},E_{3\alpha_1+2\alpha_2}]=E_{3\alpha_1+\alpha_2},\\ [E_{-(3\alpha_1+2\alpha_2)},E_{2\alpha_1+\alpha_2}]=aE_{-(3\alpha_1+\alpha_2)},\quad [E_{-\alpha_2},E_{3\alpha_1+2\alpha_2}]=E_{\alpha_1+\alpha_2},\\ [E_{-(3\alpha_1+2\alpha_2)},E_{\alpha_2}]=E_{-(\alpha_1+\alpha_2)},\end{gathered} \tag{3.8.II.2}$$

where, for $\beta=-(i\alpha_1+j\alpha_2)$, $\gamma=i'\alpha_1+j'\alpha_2$,

$$d(\beta,\gamma)=\begin{cases}1, & \text{if } j+j'\equiv 0 \pmod 2,\\ & \qquad (\beta,\gamma)\neq(-(2\alpha_1+\alpha_2),\alpha_1+\alpha_2),\\ a^{-1}, & \beta=-(2\alpha_1+\alpha_2),\gamma=\alpha_1+\alpha_2,\\ s, & \text{if } j+j'\equiv 1 \pmod 2);\end{cases} \tag{3.8.II.3}$$

(3.8.II.1′) $$[E_\beta, E_\gamma] = 0, \beta + \gamma \notin \Phi,$$

except

(3.8.II.2′) $$\begin{gathered}[E_{-\alpha_2}, E_{\alpha_1}] = E_{-(\alpha_1+\alpha_2)}, \quad [E_{-\alpha_2}, E_{3\alpha_1+2\alpha_2}] = E_{\alpha_1}, \\ [E_{-\alpha_2}, E_{2\alpha_1+\alpha_2}] = ah_2, \quad [E_{-(2\alpha_1+\alpha_2)}, E_{\alpha_2}] = h_2.\end{gathered}$$

(III) $\beta \in \Phi$, $\beta = i\alpha_1 + j\alpha_2$,

(3.8.III.1) $$[E_{-\beta}, E_\beta] = \begin{cases} sh_2, & \text{if } i+j \equiv 0 \pmod 2, \\ sE_{0,\pm 2\alpha_1}, & \text{if } i+j \equiv 1 \pmod 2 \\ & \quad \text{and } j \equiv 0 \pmod 2, \\ E_{0,\pm 2\alpha_1}, & \text{if } i+j \equiv 1 \pmod 2 \\ & \quad \text{and } j \equiv 1 \pmod 2). \end{cases}$$

(IV) For $\gamma = i\alpha_1 + j\alpha_2 \in \Phi$, let $\mathrm{ht}(\gamma) = i + j$.

(3.8.IV.1) $$[h_2, E_\gamma] = \mathrm{ht}(\gamma) E_\gamma.$$

(V) $\gamma \in \Phi$.

(3.8.V.1) $$[E_{0,\pm 2\alpha_1}, E_\gamma] = \begin{cases} aE_{\gamma - 2\alpha_1}, & \text{if } \gamma - 2\alpha_1 \in \Phi, \gamma \neq \alpha_1, \\ E_{\gamma + 2\alpha_1}, & \text{if } \gamma + 2\alpha_1 \in \Phi, \gamma \neq -\alpha_1, \\ 0, & \text{otherwise.} \end{cases}$$

(VI) Finally,

(3.8.VI.1) $$[E_{0,\pm 2\alpha_1}, h_2] = 0.$$

Let $H := \langle E_{0,\pm 2\alpha_1}, h_2 \rangle, L^- = \langle E_\gamma | \gamma > 0 \rangle$, and $L^+ = \langle E_\gamma | \gamma < 0 \rangle$. We have

(3.8.3) $$(\mathrm{ad} E_{0,\pm 2\alpha_1}|_{L^\pm})^4 = a(\mathrm{ad} E_{0,\pm 2\alpha_1}|_{L^\pm})^2 \neq 0;$$

(3.8.4) $$(\mathrm{ad} h_2|_{L^\pm})^2 = \mathrm{ad} h_2|_{L^\pm}.$$

We see that $L = L^- \oplus H \oplus L^+$ is a quasi-triangular decomposition of L. Let $0 \neq x := \sum_{\gamma<0} c_\gamma E_\gamma \in L^+$, $\mathcal{S} := \{\gamma | c_\gamma \neq 0\}$, and let $-\omega$ be a maximal element in $\mathcal{S}$. If $\omega \neq \alpha_2$, then from the multiplication table $[E_\omega, E_{-\gamma}] \in L^+$ for any γ not less than ω. Hence $[x, E_\omega] = c_\omega h_\omega + y^+$ where $0 \neq h_\omega = [E_{-\omega}, E_\omega] \in H$ and $y^+ \in L^+$. If $\omega = \alpha_2$, then there may appear a multiple of h_2 (cf. (3.8.II.2′)) which, however, is linearly independent of $[E_{-\alpha_2}, E_{\alpha_2}] = E_{0,\pm 2\alpha_1}$. Thus, in any case, $[x, E_\omega] = h + y^+$ where $0 \neq h \in H$ and $y^+ \in L^+$. For $x \in L^-$ the argument is similar. Then the conditions of Proposition 3.2(a) are satisfied and $\mathfrak{N}(L) = 0$. ∎

(3.8.5) *Remark.* V_6G varies from G_2 quite far. In some cases, the roles of $\pm\alpha_2$ and $\pm(2\alpha_1 + \alpha_2)$ and α_1 and $-\alpha_1$ are mixed, respectively. At any rate, the affinity of V_6G to G_2 is still clearly recognizable.

Summing up, we have

(3.9) THEOREM. *The Lie algebras in* (2.15) *are of CL type*.

(3.10) *Erratum to* [Sh]. As pointed out by D. Liu (cf. [Liu]), the Lie algebras V_4', V_5', and V_6' are restricted and all isomorphic to $sl(3)$. Hence the statements about the "newness" of V_i', $i=4,5,6$, in [Sh, Theorem 3.1] are incorrect and should be omitted. The author regrets these unfortunate mistakes.

ACKNOWLEDGMENTS

The author thanks Liu Dong for providing the multiplication tables of V_iG, $i=3,4,5,6$, in the original notation, and Jiang Zhi-Hong for his help in the preparation of the manuscript. The author thanks the referee for his/her valuable comments and suggestions.

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