NORTH-HOLLAND

Upper Triangular Similarity of Upper Triangular Matrices

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ABSTRACT

We consider the following equivalence relation in the set of all complex upper triangular $n \times n$ matrices: A and B are called $\mathscr{U}$-similar if there exists an invertible upper triangular matrix S such that $A = S^{-1}BS$. If A, B are $\mathscr{U}$-similar, then they must have the same diagonal and the same Jordan form. It is known that for $n \geqslant 6$ there are infinitely many mutually non-$\mathscr{U}$-similar nilpotent upper triangular matrices with the same Jordan form. We introduce an appropriate generalization of the Jordan block (called an *irreducible* matrix), and we prove that each upper triangular matrix is $\mathscr{U}$-similar to a "generalized" direct sum of irreducible blocks, where the location and the order of the blocks is fixed and each block is determined uniquely up to $\mathscr{U}$-similarity. © Elsevier Science Inc., 1997

INTRODUCTION

In this paper we consider the following problem: Given an upper triangular matrix A, what upper triangular matrices $U^{-1}AU$ are similar to A if U is assumed to be an invertible upper triangular matrix (we shall call the matrices A and $U^{-1}AU$ *$\mathscr{U}$-similar*; notation $A \sim_u U^{-1}AU$). Of course, one is interested in simple forms for $U^{-1}AU$, in analogy to the Jordan form.

Though not without interest in its own right, an external motivation for considering this question came from problems involving triangular forms of

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matrices (see, e.g., 1, 2, 4, 10): If $A \in \mathbb{C}^{n\times n}$ and $\{0\} \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = \mathbb{C}^n$ is a chain of A-invariant subspaces (i.e., $AM_i \subseteq M_i$, $\dim M_i = i$ for $i = 1,2,\dots,n$), then the matrix representation A_W of A in an ordered basis $\{\mathbf{w}_1,\dots,\mathbf{w}_n\}$ such that $M_k = \operatorname{span}(\{\mathbf{w}_1,\dots,\mathbf{w}_k\})$, $k = 1,2,\dots,n$, is an upper triangular matrix, and if $\{\mathbf{v}_1,\dots,\mathbf{v}_n\}$ is another basis such that $M_k = \operatorname{span}(\{\mathbf{v}_1,\dots,\mathbf{v}_k\})$, then the matrix representation A_V of A in that basis is $\mathscr{U}$-similar to A_W; conversely, if the matrix B is $\mathscr{U}$-similar to A_W, then it is the matrix representation of A in some ordered basis $\{\mathbf{m}_1,\dots,\mathbf{m}_n\}$ such that $M_k = \operatorname{span}(\{\mathbf{m}_1,\dots,\mathbf{m}_k\})$ for each k.

Given the matrix A and the chain (M_i) of invariant subspaces, one might want to choose a basis $\{\mathbf{w}_1,\dots,\mathbf{w}_n\}$ as above with an additional property, e.g., $\{\mathbf{w}_1,\dots,\mathbf{w}_n\}$ should be a reordering of a Jordan basis for A. Since the pair $(A,(M_i))$ determines a $\mathscr{U}$-similarity equivalence class, this is possible only if this class contains a "generalized" Jordan matrix, which is usually not the case, as we shall see below.

Another relation between $\mathscr{U}$-similarity and triangular forms arose in [3]; it is of a more specialist nature, and a short description can be found in the introduction of [9].

In the setting of upper triangular matrices over finite fields the problem was considered by G. Higman [6, 7] before 1960 and, more recently, by A. Vera-López and J. M. Arregi [11, 12]; both deal with the question of (estimates for) the total number of equivalence classes (called *conjugacy classes* by these authors), and Vera-López and Arregi provide a full listing of all conjugacy classes up to order 5 for the case where the number of elements in the field is a power of a prime number. The number of conjugacy classes of nilpotent upper triangular $n \times n$ matrices over infinite fields was shown to be infinite for large enough n by M. Roitman [8], who described an infinite family of mutually nonconjugated upper triangular nilpotent 12×12 matrices; later D. Z. Djoković and J. Malzan [5] provided a similar example consisting of 6×6 matrices, and this result cannot be improved.

Let us settle some terminology and notation. The set of all upper triangular (complex) matrices of order n will be denoted by $\mathscr{U}_n$, and $\mathscr{G}\mathscr{U}_n$ will denote the group of invertible elements in $\mathscr{U}_n$. If $A, B \in \mathscr{U}_n$ and $A = U^{-1}BU$ for some $U \in \mathscr{G}\mathscr{U}_n$, then we shall call the matrices A and B *$\mathscr{U}$-similar*: notation $A \sim_u B$.

Next, we generalize the notion of a direct sum of matrices: Let $I_1 \dot\cup \cdots \dot\cup I_s = \{1,\dots,n\}$ be a *decomposition* of $\{1,\dots,n\}$ (that is, $I_1,\dots,I_s$ are subsets with $I_i \cap I_j = \varnothing$ if $i \neq j$ and $\{1,\dots,n\} = I_1 \cup \cdots \cup I_s$). If $A_j = (a^j_{xy})^{\#I_j}_{x,y=1}$ are matrices of order $\#I_j$, $j = 1,2,\dots,s$, then the *generalized direct sum* is defined by

$$A = (a_{ij})^n_{i,j=1} = (A_1)_{I_1} \oplus (A_2)_{I_2} \oplus \cdots \oplus (A_s)_{I_s}, \tag{0.1}$$

where $a_{ij} = 0$ if $i \in I_k$, $j \in I_l$, $k \neq l$, and $a_{ij} = a^k_{xy}$ if $i = i_x$, $j = i_y \in I_k = \{i_1, \dots, i_m\}$, $i_p < i_{p+1}$, $m = \# I_k$. If all $I_j \neq \varnothing$ and $1 + \max I_j = \min I_{j+1}$, $j = 1, \dots, s-1$, then one has the usual direct sum, and we shall write $A = A_1 \oplus \cdots \oplus A_s$ in that case. A generalized direct sum of Jordan blocks will be called a *generalized Jordan matrix*. If $I_1 \dot{\cup} I_2$ is a nontrivial decomposition of $\{1, \dots, n\}$, i.e., if $I_1 \neq \varnothing$, $I_2 \neq \varnothing$, and $A \in \mathscr{U}_n$ is $\mathscr{U}$-similar to $(A_1)_{I_1} \oplus (A_2)_{I_2}$, $A_i \in \mathscr{U}_{\# I_i}$, $i = 1, 2$, then we shall call A *$\mathscr{U}$-reducible*. If a matrix $A \in \mathscr{U}_n$ is not $\mathscr{U}$-reducible, it will be called *$\mathscr{U}$-irreducible*. The $\mathscr{U}$-irreducible matrices will be the building blocks in the theory of $\mathscr{U}$-similarity: Each $A \in \mathscr{U}_n$ is $\mathscr{U}$-similar to a generalized direct sum of $\mathscr{U}$-irreducible matrices.

In order to deal with generalized direct sums we use a generalized block-matrix notation: Writing $\mathbb{C}^n = \operatorname{span}(\{\mathbf{e}_1, \dots, \mathbf{e}_k\}) \oplus \operatorname{span}(\{\mathbf{e}_{k+1}, \dots, \mathbf{e}_n\})$ (here $\mathbf{e}_i$ denotes the ith unit vector in $\mathbb{C}^n$), the matrix $A \in \mathscr{U}_n$ has the usual partitioning

$$A = \begin{pmatrix} A_1 & A_r \\ 0 & A_2 \end{pmatrix}.$$

If $I_1 = \{i_1, \dots, i_p\}$, $I_2 = \{j_1, \dots, j_q\}$, $i_x < i_{x+1}$, $j_y < j_{y+1}$, $p + q = n$, and $I_1 \dot{\cup} I_2 = \{1, \dots, n\}$, then, with respect to the decomposition $\mathbb{C}^n = \operatorname{span}(\{e_k \mid k \in I_1\}) \oplus \operatorname{span}(\{e_k \mid k \in I_2\})$, the matrix $A \in \mathscr{U}_n$ has the *generalized partitioning*

$$A = \begin{pmatrix} (A_1) & (A'_r) \\ (A''_r) & (A_2) \end{pmatrix}, \tag{0.2}$$

where $A'_r (a'_{xy})^{p}_{x=1,}{}^{q}_{y=1}$, $A''_r = (a''_{xy})^{q}_{x=1,}{}^{p}_{y=1}$. Since $A \in \mathscr{U}_n$, one has that $a'_{xy} = 0$ if $i_x > j_y$, $a''_{xy} = 0$ if $j_x > i_y$. If for example, $n = 4$ and $I_1 = \{1, 3\}$, $I_2 = \{2, 4\}$, then

$$A = \begin{pmatrix} 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 6 \\ 0 & 3 \end{pmatrix} \\ \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

whereas with respect to $I'_1 = \{1, 4\}$, $I'_2 = \{2, 3\}$ one would have

$$A = \begin{pmatrix} 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 5 \\ 0 & 3 \end{pmatrix} & \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

For $A, B \in \mathscr{U}_n$ one has, with respect to the same decomposition $\{1,\dots,n\} = I_1 \dot{\cup} I_2$,

$$AB = \begin{pmatrix} (A_1) & (A'_r) \\ (A''_r) & (A_2) \end{pmatrix} \begin{pmatrix} (B_1) & (B'_r) \\ (B''_r) & (B_2) \end{pmatrix}$$

$$= \begin{pmatrix} (A_1B_1 + A'_rB''_r) & (A_1B'_r + A'_rB_2) \\ (A''_rB_1 + A_2B''_r) & (A''_rB'_r + A_2B_2) \end{pmatrix}. \tag{0.3}$$

If $C = (C_1)_{I_1} \oplus (C_2)_{I_2} \in \mathscr{U}_n$, i.e., if $C'_r = 0$, $C''_r = 0$, then we shall call C *reducible along* the decomposition $I_1 \dot{\cup} I_2$. For example, if

$$A = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

then A is reducible along $I_1 \dot{\cup} I_2 = \{1,3\} \dot{\cup} \{2,4\}$, but B isn't:

$$A = \begin{pmatrix} \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix} \end{pmatrix}, \qquad B = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

whereas B is reducible along $I'_1 \dot{\cup} I'_2 = \{1,4\} \dot{\cup} \{2,3\}$, but A isn't.

We end this introduction with a survey of the main results. In Section 1 we shall prove that $A \in \mathscr{U}_n$ is $\mathscr{U}$-similar to a generalized direct sum $(\alpha_1 I + N_1)_{I_1} \oplus \cdots \oplus (\alpha_s I + N_s)_{I_s}$, $\alpha_i \neq \alpha_j$, $i \neq j$, $N_j \in \mathscr{U}$ nilpotent, and that the corresponding index decomposition $I_1 \dot{\cup} \cdots \dot{\cup} I_s$ is unique. This allows us to concentrate on *nilpotent* upper triangular matrices. Further, it is shown that $A \in \mathscr{U}_n$ is $\mathscr{U}$-similar to a (unique) generalized Jordan matrix if A is nonderogatory or if no partial multiplicity of A exceeds 2 [that is, if $\operatorname{Ker}(A - \lambda I)^2 = \operatorname{Ker}(A - \lambda I)^3$ for each λ]. In Section 2 we obtain for given $A \in \mathscr{U}_n$ the existence of a unique index decomposition $I_1 \dot{\cup} \cdots \dot{\cup} I_s = \{1,\dots,n\}$, all $I_j \neq \varnothing$, such that $A \sim_u (A_1)_{I_1} \oplus \cdots \oplus (A_s)_{I_s}$, $A_j \in \mathscr{U}_{\#I_j}$ irreducible, where all A_j are uniquely determined up to $\mathscr{U}$-similarity. In Section 3 we study conditions in order that $A \in \mathscr{U}_{n+1}$, A nilpotent, is irreducible if A is considered as an extension of the nilpotent matrix $A' \in \mathscr{U}_n$ by adding the

$(n + 1)$st column to A'. Using these conditions one obtains infinitely many mutually non-$\mathscr{U}$-similar nilpotent irreducible matrices in $\mathscr{U}_n$ for $n \geqslant 6$ (this is, in fact, the example of D. Z. Djoković and J. Malzan [5]). Special types of extensions will be considered in Section 4, and the paper ends with a complete list of all $\mathscr{U}$-similarity equivalence classes of nilpotent irreducible upper triangular matrices of orders up to 6.

Many proofs are abridged or omitted, especially in Sections 3 and 4; for the full proofs the reader is referred to [9].

If no ambiguity is to be feared, we drop the explicit reference to $\mathscr{U}$ in our terminology, so "similar" will mean "$\mathscr{U}$-similar," "$\sim$" mean "$\sim_u$" and "(ir)reducible" mean "$\mathscr{U}$-(ir)reducible." In writing generalized direct sums $A = (A_1)_{I_1} \oplus (A_2)_{I_2} \oplus \cdots \oplus (A_k)_{I_k}$, we omit the index sets $I_1, \dots, I_k$ where no confusion can arise.

1. TECHNICAL RESULTS

In this section we prepare the ground for the main results which will be proved in the next two sections. Our first result is the observation that similar matrices $A, B \in \mathscr{U}_n$ have the same diagonal:

LEMMA 1.1. *If $A, B \in \mathscr{U}_n$, $A \sim B$, then $a_{ii} = b_{ii}$, $i = 1, 2, \dots, n$.*

The diagonal entries of $A \in \mathscr{U}_n$ are the eigenvalues of A. The position of the eigenvalues on the diagonal gives rise to the *spectral reduction* of A (cf. M. Roitman [8]):

PROPOSITION 1.2. *Let $A \in \mathscr{U}_n$ have the different eigenvalues $\alpha_1, \dots, \alpha_k$, and set $E_j = \{i \mid a_{ii} = \alpha_j\}$. Then there exist $A_1, \dots, A_k$, $A_j \in \mathscr{U}_{\#E_j}$, $\sigma(A_j) = \{\alpha_j\}$ such that $A \sim (A_1)_{E_1} \oplus \cdots \oplus (A_k)_{E_k}$; moreover, if $A \sim (\hat{A}_1)_{E_1} \oplus \cdots \oplus (\hat{A}_k)_{E_k}$, where $\hat{A}_j \in \mathscr{U}_{\#E_j}$, $\sigma(\hat{A}_j) = \{\alpha_j\}$, then $A_j \sim \hat{A}_j$, $j = 1, 2, \dots, k$.*

Proof. If $\mathbf{v} = (v_j)_{j=1}^n$, $v_i \neq 0$, $v_j = 0$, $j > i$, then we call $\mathbf{v}$ an *i-vector*. If $\mathbf{v}$ is an i-vector and $A\mathbf{v} = \alpha_j \mathbf{v}$, then $i \in E_j$; in fact, if $E_j = \{i_{j1}, \dots, i_{jm_j}\}$, $i_{jx} < i_{j\,x+1}$, $m_j = \#E_j$, then the generalized eigenspace $\operatorname{Ker}(A - \alpha_j I)^n$ has a basis $\{\mathbf{v}_{j1}, \dots, \mathbf{v}_{jm_j}\}$ such that $\mathbf{v}_{jx}$ is an i_{jx}-vector: This can be seen using the Gauss reduction algorithm on a given basis of $\operatorname{Ker}(A - \alpha_j I)^n$ and keeping in mind that an i-vector cannot be a generalized eigenvector of A associated with α_j if $i \notin E_j$. Observe that $A\mathbf{v}_{jx} \in \operatorname{span}(\{\mathbf{v}_{j1}, \dots, \mathbf{v}_{jx}\})$, since $A \in \mathscr{U}_n$

and $i_{j1},\dots,i_{jm_j}$ are in increasing order. Define $V\in\mathscr{GU}_n$ by

$$V=\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{pmatrix},\qquad \mathbf{v}_{i_{jx}}=\mathbf{v}_{jx}.$$

Then $V^{-1}AV\sim A$ has the desired properties. If $\hat V=(\hat{\mathbf{v}}_1\ \cdots\ \hat{\mathbf{v}}_n)\in\mathscr{GU}_n$ is such that $\hat V^{-1}A\hat V=(\hat A_1)_{E_1}\oplus\cdots\oplus(\hat A_k)_{E_k}$, then $\{\hat{\mathbf{v}}_{i_{j1}},\dots,\hat{\mathbf{v}}_{i_{jm_j}}\}$ is a basis for $\operatorname{Ker}(A-\alpha_jI)^n$ consisting of i_{jx}-vectors, whereas $A\hat{\mathbf{v}}_{i_{jx}}\in\operatorname{span}(\{\hat{\mathbf{v}}_{i_{j1}},\dots,\hat{\mathbf{v}}_{i_{jx}}\})$. This proves that $A_j\sim\hat A_j$. ∎

A nice application of the spectral reduction is the following

PROPOSITION 1.3. *Let* $A,B\in\mathscr{U}_n$, $b\in\mathbb{C}$. *Define* $I_1=\{i\mid b_{ii}=b\}$, $I_2=\{i\mid b_{ii}\neq b\}$. *Assume that* $AB=BA$. *Then* $A\sim(A_1)_{I_1}\oplus(A_2)_{I_2}$. *In particular, if A is irreducible, then either* $I_1=\varnothing$ (*i.e., b is not an eigenvalue of B*) *or* $I_2=\varnothing$ (*i.e.,* $B-bI$ *is nilpotent*).

Proof. Without loss of generality we can assume that $I_1\neq\varnothing$ and $b=0$. There exists $V\in\mathscr{GU}_n$ such that $V^{-1}BV=(B_1)_{I_1}\oplus(B_2)_{I_2}$ with $0\notin\sigma(B_2)$, $\sigma(B_1)=\{0\}$, so B_2 is invertible and B_1 is nilpotent. Consider

$$\hat A=V^{-1}AV=\begin{pmatrix}(A_1) & (A'_r)\\ (A''_r) & (A_2)\end{pmatrix}$$

with respect to $\{1,\dots,n\}=I_1\,\dot\cup\,I_2$. From $AB=BA$ one has $\hat A((B_1)\oplus(B_2))=((B_1)\oplus(B_2))\hat A$, so $A'_rB_2=B_1A'_r$, $A''_rB_1=B_2A''_r$ and $A'_rB_2^k=B_1^kA'_r=0$, $B_2^kA''_r=A''_rB_1^k=0$ for $k\geqslant\#I_1$. Since B_2^k is invertible, this proves that $A'_r=A''^T_r=0$ and $A\sim\hat A=(A_1)_{I_1}\oplus(A_2)_{I_2}$. ∎

A consequence of the spectral reduction in Proposition 1.2 is that for the description of the equivalence class of A, it is sufficient to describe the equivalence class of each "spectral component" A_j, $j=1,2,\dots,k$. Since $A\sim B$ if and only if $A-\beta I\sim B-\beta I$, we can restrict ourselves to the description of the equivalence classes of upper triangular nilpotent matrices; we shall use the notation $\mathscr{U}_n^0=\{\mathscr{A}\in\mathscr{U}_n\mid\sigma(A)=\{0\}\}$. Obvious candidates for representatives of the equivalence classes seem to be the nilpotent generalized Jordan matrices. Indeed, if $J,\hat J\in\mathscr{U}_n^0$ are generalized Jordan matrices, then $J\sim\hat J$ if and only if $J=\hat J$: Assume that $J\neq\hat J$. Setting $J=(s_{ij})_{i,j=1}^n$, $\hat J=(s'_{ij})_{i,j=1}^n$, there exist $k<l$ such that $s_{kl}\neq s'_{kl}$; for defi-

niteness, $s_{kl} = 1$, $s'_{kl} = 0$. Further, interchanging $J, \hat{J}$, if necessary, we can achieve that $s'_{kj} = 0$ for all $j \leqslant l$, since no row in $J, \hat{J}$ contains more than one nonzero entry. Assume that $V = (v_{ij})_{i,j=1}^{n} \in \mathscr{U}_n$ and that $VJ = \hat{J}V$. Then

$$(VJ)_{kl} = v_{kk} = (\hat{J}V)_{kl} = \sum_{i=k+1}^{n} s'_{ki}v_{il} = \sum_{i=k+1}^{l} s'_{ki}v_{il} = 0,$$

and V is not invertible.

Unfortunately, for $n \geqslant 4$ there exist $\mathscr{U}$-similarity classes in $\mathscr{U}_n^0$ not containing generalized Jordan matrices. Consider

$$J_{3,1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ & 0 & 0 & 1 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in \mathscr{U}_4^0, \tag{1.1}$$

with the partial multiplicities 3 and 1. It is not difficult to show that $J_{3,1}$ is irreducible [cf. also Example 3.7(iii) below]. In particular, $J_{3,1}$ is not similar to any of the generalized Jordan matrices

$$J_1 \oplus J_3, \qquad (J_1)_{\{2\}} \oplus (J_3)_{\{1,3,4\}}, \qquad (J_1)_{\{3\}} \oplus (J_3)_{\{1,2,4\}}, \qquad J_3 \oplus J_1,$$

where $J_k = (\delta_{i+1,j})_{i,j=1}^{k}$ denotes the nilpotent Jordan block of order k.

A simple test whether a given matrix $A \in \mathscr{U}_n^0$ is similar to a generalized Jordan matrix is the following: *A is similar to a generalized Jordan matrix if and only if there exists a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of $\mathbb{C}^n$ consisting of Jordan chains of A such that each $\mathbf{v}_i$ is an i-vector.*

EXAMPLE 1.4. Let $A \in \mathscr{U}_k^0$ be unicellular. Then $A \sim J_k$. Indeed, $A^{k-1} \neq 0 = A^k$; let $\tilde{A} = (a_{ij})_{i,j=1}^{k-1} \in \mathscr{U}_{k-1}^0$ be the matrix consisting of the initial $k-1$ rows and columns of A. Then $\tilde{A}^{k-1} = 0$, and hence $A^{k-1}\mathbf{e}_k \neq 0$. This proves that $A^{k-1}\mathbf{e}_k, A^{k-2}\mathbf{e}_k, \dots, A\mathbf{e}_k, \mathbf{e}_k$ is a Jordan chain for A, and $\{A^{k-1}\mathbf{e}_k, A^{k-2}\mathbf{e}_k, \dots, A\mathbf{e}_k, \mathbf{e}_k\}$ is a basis for $\mathbb{C}^k$. If $A^{k-j}\mathbf{e}_k$ is a i_j-vector, then $i_j < i_{j+1}$, as $A \in \mathscr{U}_n^0$. But $A^{k-1}\mathbf{e}_k \neq \mathbf{0}$, $i_k = k$, so $i_j = j$, $j = 1, 2, \dots, k$.

It follows from this example that a nonderogatory matrix $A \in \mathscr{U}_n$ is similar to a generalized Jordan matrix. The same turns out to be true if no partial multiplicity of $A \in \mathscr{U}_n$ exceeds two:

THEOREM 1.5. *Let $A \in \mathscr{U}_n$, and assume that one of the following two conditions is met:*

(i) *A is nonderogatory;*
(ii) $\dim \operatorname{Ker}(A - \lambda I)^2 = \dim \operatorname{Ker}(A - \lambda I)^3$ *for each* $\lambda \in \mathbb{C}$.

Then there exists a generalized Jordan matrix J which is $\mathscr{U}$-similar to A.

Proof. It suffices to consider the case where $A \in \mathscr{U}_n^0$ and $A^2 = 0$. We proceed by induction on n. For $n = 1$ we have $A = (0)$ and the result is trivially true. Assume that the desired result has been proved for $1, 2, \ldots, n-1$. Let $A = (a_{ij})_{i,j=1}^n$, and set $\tilde{A} = (a_{ij})_{i,j=1}^{n-1} \in \mathscr{U}_{n-1}^0$. Then $\tilde{A}^2 = 0$, so there exists a generalized Jordan matrix $\tilde{J} = (J_{\#I_1})_{I_1} \oplus (J_{\#I_2})_{I_2} \oplus \cdots \oplus (J_{\#I_k})_{I_k} \in \mathscr{U}_{n-1}^0$ such that $\tilde{V}^{-1}\tilde{A}\tilde{V} = \tilde{J}$ for some $\tilde{V} \in \mathscr{G}\mathscr{U}_{n-1}$. Since $\tilde{J}^2 = 0$, one has $\#I_j \leqslant 2$, i.e.,

$$J_{\#I_j} = (0) \quad \text{or} \quad J_{\#I_j} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Replacing A by $[\tilde{V} \oplus (1)]^{-1}A[\tilde{V} \oplus (1)]$ we may assume A to have the partitioning

$$A = \begin{pmatrix} \tilde{J} & \mathbf{a} \\ \mathbf{0} & 0 \end{pmatrix}, \qquad \mathbf{a} = (\mathbf{a}_1)_{I_1} \oplus \cdots \oplus (\mathbf{a}_k)_{I_k} \in \mathbb{C}^{n-1},$$

and replacing A by WAW^{-1}, where

$$W = \begin{pmatrix} I & \mathbf{w} \\ \mathbf{0} & 1 \end{pmatrix} \in \mathscr{U}_n, \qquad \mathbf{w} \in \operatorname{Im} \tilde{J} \text{ appropriate},$$

we may assume that

$$\mathbf{a}_j = \begin{pmatrix} 0 \\ a_{i_{j2}} \end{pmatrix} \quad \text{if} \quad \#I_j = 2, \quad I_j = \{i_{j1}, i_{j2}\}, \quad i_{j1} < i_{j2}.$$

Using the fact that $A\mathbf{e}_n = \mathbf{a} \oplus (0)$, $\mathbf{0} = A^2\mathbf{e}_n = \tilde{J}\mathbf{a} \oplus (0)$, it is clear that $a_{i_{j2}} = 0$, i.e., $a_j = 0$ if $\#I_j = 2$. If $\mathbf{a} = \mathbf{0}$, then $A = \tilde{J} \oplus (0)$ is a generalized Jordan matrix; so assume that $\mathbf{a} \neq \mathbf{0}$, and let $i_0 = \max\{i \mid a_i \neq 0\}$. By dividing the nth column and multiplying the nth row in A by a_{i_0} we replace A by a similar matrix such that $a_{i_0} = 1$. If $a_i \neq 0$, then the ith column of A is $\mathbf{0}$, the

ith row is $a_i \mathbf{e}_n^T = (0, \ldots, 0, a_i)$ (since $\{i\} = I_j$ for some j). Now we subtract in A, for each $i \neq i_0$ such that $a_i \neq 0$, a_i times the i_0th row from the ith row, and we add a_i times the ith column (which is $\mathbf{0}$) to the i_0th column. The ensuing matrix $\hat{A}$ is similar to A, and, taking $I_k = \{i_0\}$ for simplicity, one has that

$$\hat{A} = (J_{\# I_1})_{I_1} \oplus \cdots \oplus (J_{\# I_{k-1}})_{I_{k-1}} \oplus \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)_{\{i_0, n\}},$$

that is, $\hat{A}$ is a generalized Jordan matrix. ■

In the proof we have used the *elementary transformations of $\mathscr{U}$-similarity*: The matrix A is transformed to a matrix which is $\mathscr{U}$-similar with A if (a) the kth row is multiplied by $c \neq 0$ and simultaneously the kth column is divided by c, or (b) a multiple of the kth row is subtracted from the lth row, $l < k$, and then the same multiple of the lth column is added to the kth column.

The other relevant element in the proof of Theorem 1.5 is the concept of *restriction to matrices of lower order*: Assume that $A \in \mathscr{U}_n$, and let $M_k = \operatorname{span}(\{\mathbf{e}_1, \ldots, \mathbf{e}_k\})$ denote the subspace of $\mathbb{C}^n$ spanned by the initial k unit vectors; since M_k is A-invariant, one can consider $A|_{M_k}$ as an element of $\mathscr{U}_k$; if $A \sim B$, then $A|_{M_k} \sim B|_{M_k}$, and if A has a reduction $A \sim (C_1)_{I_1} \oplus \cdots \oplus (C_s)_{I_s}$, then $A|_M$ is similar to

$$A|_M \sim (\hat{C}_1)_{I_1 \cap \{1, \ldots, k\}} \oplus \cdots \oplus (\hat{C}_s)_{I_s \cap \{1, \ldots, k\}},$$

where $\hat{C}_j = C_j|_{\operatorname{span}(\{\mathbf{e}_i | i \leqslant \# I_j \cap \{1, \ldots, k\}\})}$. The validity of this claim is easily seen, writing the standard partitioning of the matrices A, $B = V^{-1}AV$, $V \in \mathscr{G}\mathscr{U}_n$:

$$A = \begin{pmatrix} A_1 & A_r \\ 0 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & B_r \\ 0 & B_2 \end{pmatrix}, \qquad V = \begin{pmatrix} V_1 & V_r \\ 0 & V_2 \end{pmatrix},$$

which implies $V_1 B_1 = A_1 V_1$; observe that the compressions A_2, B_2 of A, B to the subspace $\operatorname{span}(\{\mathbf{e}_{k+1}, \ldots, \mathbf{e}_n\})$ are also similar. In general, this conclusion is not correct for the compressions to subspaces of the type $M = \operatorname{span}(\{\mathbf{e}_j \,|\, j \in J\})$ where the index set J is neither $\{1, \ldots, k\}$ nor $\{k+1, \ldots, n\}$. With respect to the decomposition $\{1, \ldots, n\} = J \,\dot{\cup}\, (\{1, \ldots, n\} \setminus J)$ one has for A, $B = V^{-1}AV$, $V \in \mathscr{G}\mathscr{U}_n$ the generalized partitioning

$$A = \begin{pmatrix} (A_1) & (A'_r) \\ (A''_r) & (A_2) \end{pmatrix}, \qquad B = \begin{pmatrix} (B_1) & (B'_r) \\ (B''_r) & (B_2) \end{pmatrix}, \qquad V = \begin{pmatrix} (V_1) & (V'_r) \\ (V''_r) & (V_2) \end{pmatrix},$$

which yields, e.g., $A_1V_1 + A'_rV''_r = V_1B_1 + V'_rB''_r$; and $A'_rV''_r \neq V'_rB''_r$ can easily occur: Consider the example

$$A = J_{3,1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ & 0 & 0 & 1 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \sim B = \begin{pmatrix} 0 & 1 & 1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}, \qquad J = \{2, 4\},$$

where $A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not similar to $B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

However, if $V = (V_1) \oplus (V_2)$ or both $A = (A_1) \oplus (A_2)$ and $B = (B_1) \oplus (B_2)$ (see Lemma 2.2 below) then one has $A_iV_i = V_iB_i$, $i = 1, 2$, and the compressions of A, B to span($\{\mathbf{e}_j \mid j \in J\}$), span($\{\mathbf{e}_j \mid j \notin J\}$), respectively, are similar.

In Section 2 we shall use restrictions and compressions of various orders. In Section 3 we have the following setting: Given $\tilde{A} \in \mathscr{U}^0_{n-1}$, describe the so-called *direct extensions* $A \in \mathscr{U}^0_n$ of $\tilde{A}$, i.e., those $A \in \mathscr{U}^0_n$ such that $A|_{M_{n-1}} = \tilde{A}$.

2. THE UNIQUENESS OF THE REDUCTION TO SUMS OF IRREDUCIBLE MATRICES

In Section 1 we have seen that the spectral reduction of $A \in \mathscr{U}_n$ is essentially unique. Here we shall prove that a reduction of A to a generalized sum of irreducible matrices is unique in the same sense: All such reductions are based on the same decomposition of the index set $\{1, \dots, n\}$, and the irreducible matrices belonging to the same component are $\mathscr{U}$-similar. This shows that the irreducible matrices are the elementary building blocks for the $\mathscr{U}$-equivalence classes.

THEOREM 2.1. *Let* $A = (A_1)_{I_1} \oplus \cdots \oplus (A_s)I_s$, $B = (B_1)_{J_1} \oplus \cdots \oplus (B_t)_{J_t} \in \mathscr{U}_n$, *where* $A_k \in \mathscr{U}_{\#I_k}$ *and* $B_l \in \mathscr{U}_{\#J_l}$ *are nonempty and irreducible. Assume that* $A \sim_u B$. *Then* $t = s$ *and* $I_k \cap J_l \neq \varnothing$ *implies* $I_k = J_l$, $A_k \sim_u B_l$. *In particular, if* $I_1, \dots, I_s, J_1, \dots, J_t$ *are ordered so that* $\min I_k < \min I_{k+1}$, $\min J_l < \min J_{l+1}$, *then* $s = t$, $J_k = I_k$, $A_k \sim B_k$, $k = 1, 2, \dots, s$.

In the proof of this result we need two lemmas:

LEMMA 2.2. *Let* $A = (A_1)_{I_1} \oplus (A_2)_{I_2}$, $B = (B_1)_{I_1} \oplus (B_2)_{I_2} \in \mathscr{U}_n$, $I_1 \dot\cup I_2 = \{1, \dots, n\}$. *Then* $A \sim B$ *if and only if* $A_1 \sim B_1$ *and* $A_2 \sim B_2$.

Proof. The sufficiency of the condition is clear; assume that $VA = BV$ for $V \in \mathscr{G}\mathscr{U}_n$. The decomposition $I_1 \dot{\cup} I_2$ yields the generalized partitioning

$$A = \begin{pmatrix} (A_1) & (0) \\ (0) & (A_2) \end{pmatrix}, \qquad B = \begin{pmatrix} (B_1) & (0) \\ (0) & (B_2) \end{pmatrix}, \qquad V = \begin{pmatrix} (V_1) & (V_r') \\ (V_r'') & (V_2) \end{pmatrix},$$

and $VA = BV$ implies $V_1A_1 + V_r'0 = B_1V_1 + 0V_r''$, $V_r''0 + V_2A_2 = 0V_r' + B_2V_2$. Clearly, V_1, V_2 are invertible, so $A_i \sim B_i$, $i = 1, 2$. ■

LEMMA 2.3. *Let $I_1 \dot{\cup} I_2 = \{1, \dots, n-1\}$ and $A_j \in \mathscr{U}_{\#I_j}$, $j = 1, 2$. Consider*

$$A = \begin{pmatrix} (A_1)_{I_1} \oplus (A_2)_{I_2} & (\mathbf{a})_{I_1} \oplus (\mathbf{0})_{I_2} \\ \mathbf{0} & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} (A_1)_{I_1} \oplus (A_2)_{I_2} & (\mathbf{0})_{I_1} \oplus (\mathbf{b})_{I_2} \\ \mathbf{0} & 0 \end{pmatrix} \in \mathscr{U}_n.$$

Then $A \sim B$ implies $A \sim (A_1)_{I_1} \oplus (A_2)_{I_2} \oplus ((0))_{\{n\}}$.

Proof. Let $V \in \mathscr{G}\mathscr{U}_n$, $VA = BV$. Decomposing V according to $\{1, \dots, n\} = I_1 \dot{\cup} I_2 \dot{\cup} \{n\}$, one has

$$V = \begin{pmatrix} (V_1) & (V_r') & (\mathbf{v}_1) \\ (V_r'') & (V_2) & (\mathbf{v}_2) \\ & \mathbf{0} & v \end{pmatrix},$$

where $v \neq 0$, $V_j \in \mathscr{G}\mathscr{U}_{\#I_j}$. From $VA = BV$ one has $V_1A_1 = A_1V_1$ and $V_1\mathbf{a} = A_1\mathbf{v}_1$. Define $U \in \mathscr{G}\mathscr{U}_n$ by

$$U = \begin{pmatrix} (I) \oplus (I) & (-V_1^{-1}\mathbf{v}_1) \oplus (\mathbf{0}) \\ \mathbf{0} & 1 \end{pmatrix},$$

$$\text{i.e.} \quad U^{-1} = \begin{pmatrix} (I) \oplus (I) & (V_1^{-1}\mathbf{v}_1) \oplus (\mathbf{0}) \\ \mathbf{0} & 1 \end{pmatrix}.$$

Then $U^{-1}AU = (A_1) \oplus (A_2) \oplus ((0))$, as $a - A_1V_1^{-1}\mathbf{v}_1 = 0$. ■

Lemma 2.3 is a special case of more general results on the direct extensions of reduced matrices $(A_1) \oplus (A_2)$ which we shall meet in the next section.

Proof of Theorem 2.1. According to Proposition 1.2 it is sufficient to prove the theorem under the extra assumption that $A, B \in \mathscr{U}_n^0$. We apply induction on the order n of the matrices A, B. For $n = 1$ we have $A = B = (0)$ and the result is trivially true; next, we assume that $n \geqslant 2$ and that the desired result has been obtained for the orders $1, 2, \dots, n-1$. Let $\min I_k < \min I_{k+1}$, $\min J_l < \min J_{l+1}$, and let $n \in I_x$, $n \in J_y$. Set $M = \operatorname{span}(\{\mathbf{e}_1, \dots, \mathbf{e}_{n-1}\})$, $A' = A|_M$, $B' = B|_M \in \mathscr{U}_{n-1}^0$. Writing $I'_x = I_x \backslash \{n\}$, $J'_y = J_y \backslash \{n\}$, one has

$$A' = \bigoplus_{l \neq x} (A_l)_{I_l} \oplus (A'_x)_{I'_x}, \qquad B' = \bigoplus_{l \neq y} (B_l)_{J_l} \oplus (B'_y)_{J'_y},$$

where A'_x and B'_y are respectively the restrictions of A_x and B_y to the initial $\#I_x - 1$ and $\#J_y - 1$ coordinates. Using appropriate similarities, based on matrices of the type $(I) \oplus (U_A)_{I'_x}$ and $(I) \oplus (U_B)_{J'_y}$, respectively, one can replace A', B' by

$$A'' = \left[(I) \oplus \left(U_A^{-1}\right)\right] A' [(I) \oplus (U_A)],$$

$$B'' = \left[(I) \oplus \left(U_B^{-1}\right)\right] B' [(I) \oplus (U_B)],$$

where

$$A'' = \bigoplus_{l \neq x} (A_l)_{I_l} \oplus \bigoplus_{j=1}^{s'} (A_{xj})_{I_{xj}} \sim B'' = \bigoplus_{l \neq y} (B_l)_{J_l} \oplus \bigoplus_{j=1}^{t'} (B_{yj})_{I_{yj}},$$

with $s' = 0$ $(t' = 0)$ if A'_x (B'_y) is empty. If $I_x \cup J_y \neq \{1, \dots, n\}$, then our induction hypothesis implies that $I_u = J_w$, $A_u \sim B_w$ for some $u \neq x$, $w \neq y$, and according to Lemma 2.2 the desired result then follows through application of the induction hypothesis to $\oplus_{l \neq u} (A_l)_{I_l} \sim \oplus_{l \neq w} (B_l)_{J_l}$. So we can assume that $J_y \cup I_x = \{1, \dots, n\}$. Let $x \leqslant y$ for definiteness. Then $x = 1$, i.e. $n \in I_1$, and there exist $1 \leqslant j_1 < \cdots < j_{s'-1} \leqslant t'$, $1 \leqslant i_1 < \cdots < i_{t'-1} \leqslant s'$ such that $I_{k+1} = J_{yj_k}$, $J_k = I_{xi_k}$, $k < y$, and $J_{k+1} = I_{xi_k}$, $k \geqslant y$, whereas the sets $\{I_{xl} \mid l \neq i_1, \dots, i_{t'-1}\}$ and $\{J_{yl} \mid l \neq j_1, \dots, j_{s'-1}\}$ coincide, with A_{k+1}

$\sim B_{yj_k}$, $k = 1, 2, \dots, s-1$, $B_k \sim A_{xi_k}$, $k < y$, $B_{k+1} \sim A_{xi_k}$, $k \geqslant y$, and $B_{yl'} \sim A_{xl}$ if $I_{xl'} = J_{yl}$. This follows from the induction hypothesis, applied to $A'' \sim B''$. Using a similarity based on a matrix $V' \in \mathscr{G}\mathscr{U}_{n-1}$ of the form

$$V' = \bigoplus_{i=2}^{s} (V_i)_{I_i} \oplus \bigoplus_{j=1}^{t-1} \left(V_{xi_j}\right)_{I_{xi_j}} \oplus \bigoplus_{l \neq y} (V_{xl})_{I_{xl}},$$

one has $A'' = V'^{-1} B'' V'$. Define $V = V' \oplus (1) \in \mathscr{G}\mathscr{U}_n$. Replacing A, B by

$$\tilde{A} = \left[(I) \oplus \left(U_A^{-1}\right) \oplus (1)\right] A \left[(I) \oplus (U_A) \oplus (1)\right],$$

$$\tilde{B} = \left[(I) \oplus \left(U_B^{-1}\right) \oplus (1)\right] B \left[(I) \oplus (U_B) \oplus (1)\right]$$

(which means replacing A_x by $\hat{A}_x \sim A_x$ and B_y by $\hat{B}_y \sim B_y$ in A and B, respectively), and defining

$$K_1 = \bigcup_{l=1}^{t-1} I_{xi_l}, \qquad K_2 = \bigcup_{l=2}^{s} I_l, \qquad K_3 = I_x \backslash (\{n\} \cup K_1),$$

one has

$$\tilde{A} = \begin{pmatrix} \left(\tilde{A}_1\right)_{K_1} \oplus \left(\tilde{A}_2\right)_{K_2} \oplus \left(\tilde{A}_3\right)_{K_3} & (\mathbf{a}_1)_{K_1} \oplus (\mathbf{0})_{K_2} \oplus (\mathbf{a}_3)_{K_3} \\ \mathbf{0} & 0 \end{pmatrix}$$

$$\sim V^{-1} \tilde{B} V = \begin{pmatrix} \left(\tilde{A}_1\right)_{K_1} \oplus \left(\tilde{A}_2\right)_{K_2} \oplus \left(\tilde{A}_3\right)_{K_3} & (\mathbf{0})_{K_1} \oplus (\mathbf{b}_2)_{K_2} \oplus (\mathbf{b}_3)_{K_3} \\ \mathbf{0} & 0 \end{pmatrix}.$$

Since $V = (\tilde{V}_1)_{K_1} \oplus (\tilde{V}_2)_{K_2} \oplus (\tilde{V}_3)_{K_3} \oplus ((1))_{\{n\}}$, the structure of $\tilde{A}$, $\tilde{B}$, and V allows us to consider the compressions to spaces $\operatorname{span}(\{\mathbf{e}_i \mid i \in K_p \cup K_q \cup \{n\}\})$, $p \neq q \in \{1, 2, 3\}$. In particular

$$\hat{A}_x = \begin{pmatrix} \left(\tilde{A}_1\right)_{K_1} \oplus \left(\tilde{A}_3\right)_{K_3} & (\mathbf{a}_1)_{K_1} \oplus (\mathbf{a}_3)_{K_3} \\ \mathbf{0} & 0 \end{pmatrix},$$

$$\hat{B}_y \sim \begin{pmatrix} \left(\tilde{A}_2\right)_{K_2} \oplus \left(\tilde{A}_3\right)_{K_3} & (\mathbf{0})_{K_2} \oplus (\mathbf{b}_3)_{K_3} \\ \mathbf{0} & 0 \end{pmatrix},$$

and the compressions of $\tilde{A}$ and $V^{-1}\tilde{B}V$ to the subspace span($\{\mathbf{e}_i \mid i \notin J_y \cap I_x\} \cup \{\mathbf{e}_n\}$) = span($\{\mathbf{e}_i \mid i \in K_1 \cup K_2 \cup \{n\}\}$) yields

$$\hat{A} = \begin{pmatrix} (\tilde{A}_1)_{K_1} \oplus (\tilde{A}_2)_{K_2} & (\mathbf{a}_1)_{K_1} \oplus (\mathbf{0})_{K_2} \\ \mathbf{0} & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} (\tilde{A}_1)_{K_1} \oplus (\tilde{A}_2)_{K_2} & (\mathbf{0})_{K_1} \oplus (\mathbf{b})_{K_2} \\ \mathbf{0} & 0 \end{pmatrix}.$$

According to Lemma 2.3 there exist

$$\hat{U} = \begin{pmatrix} (I)_{K_1} \oplus (I)_{K_2} & (\mathbf{u})_{K_1} \oplus (\mathbf{0})_{K_2} \\ \mathbf{0} & 1 \end{pmatrix} \in \mathcal{GU}_{\#K_1 + \#K_2 + 1}$$

such that $\hat{U}^{-1}\hat{A}\hat{U} = (\tilde{A}_1)_{K_1} \oplus (\tilde{A}_2)_{K_2} \oplus ((0))_{\{n\}}$; defining $U = (\hat{U})_{K_1 \cup K_2 \cup \{n\}} \oplus (I)_{K_3}$ and setting $\tilde{U} = (I) \oplus (I)$, $\mathbf{u}_0 = (\mathbf{u}) \oplus (\mathbf{0})$, one has

$$U = \begin{pmatrix} (\tilde{U}) & (0) & (\mathbf{u}_0) \\ (0) & (I) & (\mathbf{0}) \\ \multicolumn{2}{c}{\mathbf{0}} & 1 \end{pmatrix}$$

and

$U^{-1}\tilde{A}U$

$$= (\hat{U}^{-1}) \oplus (I) \begin{pmatrix} ([(\tilde{A}_1) \oplus (\tilde{A}_2)]\tilde{U}) \oplus (\tilde{A}_3) & ([(\tilde{A}_1) \oplus (\tilde{A}_2)]\mathbf{u}_0 + (\mathbf{a}_1) \oplus (\mathbf{0})) \oplus (\mathbf{a}_3) \\ \mathbf{0} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (\tilde{U}^{-1}[(\tilde{A}_1) \oplus (\tilde{A}_2)]\tilde{U}) \oplus (\tilde{A}_3) & (\tilde{U}^{-1}([(\tilde{A}_1) \oplus (\tilde{A}_2)]\mathbf{u}_0 + (\mathbf{a}_1) \oplus (\mathbf{0}))) \oplus (\mathbf{a}_3) \\ \mathbf{0} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (\tilde{A}_1) \oplus (\tilde{A}_2) \oplus (\tilde{A}_3) & (\mathbf{0}) \oplus (\mathbf{0}) \oplus (\mathbf{a}_3) \\ & 0 \end{pmatrix},$$

as $[(A_1) \oplus (A_2)]\tilde{U} = \tilde{U}[(A_1) \oplus (A_2)]$, $\tilde{U}^{-1}\{[(A_1) \oplus (A_2)]\mathbf{u}_0 + (\mathbf{a}_1) \oplus (\mathbf{0})\} = (\mathbf{0}) \oplus (\mathbf{0})$. Using the special structure of U and U^{-1}, one can consider the compression of $\tilde{A}$ and $U^{-1}\tilde{A}U$ to span($\{\mathbf{e}_i \mid i \in K_1 \cup K_3 \cup \{n\}\}$) and thus obtain

$$A_x \sim \begin{pmatrix} (\tilde{A}_1) \oplus (\tilde{A}_3) & (\mathbf{a}_1) \oplus (\mathbf{a}_3) \\ \mathbf{0} & 0 \end{pmatrix} \sim \begin{pmatrix} (\tilde{A}_1) \oplus (\tilde{A}_3) & (\mathbf{0}) \oplus (\mathbf{a}_3) \\ \mathbf{0} & 0 \end{pmatrix}.$$

Since A_x is irreducible, this implies $\tilde{A}_1 = \varnothing$. The same type of argument shows

$$B_y \sim \begin{pmatrix} (\tilde{A}_1) \oplus (\tilde{A}_3) & (\mathbf{b}_1) \oplus (\mathbf{b}_3) \\ \mathbf{0} & 0 \end{pmatrix} \sim \begin{pmatrix} (\tilde{A}_1) \oplus (\tilde{A}_3) & (\mathbf{0}) \oplus (\mathbf{b}_3) \\ \mathbf{0} & 0 \end{pmatrix},$$

implying $\tilde{A}_2 = \varnothing$. Hence $s = t = 1 = x = y$, and $A \sim B$ is irreducible. ■

3. EXTENSION THEORY

Let $A \in \mathscr{U}_n^0$ be a nilpotent upper triangular matrix. The matrix $\tilde{A} \in \mathscr{U}_{n+1}^0$ will be called a *direct extension* of A if $\tilde{A}|_{\mathrm{span}(\{\mathbf{e}_1,\ldots,\mathbf{e}_n\})} = A$. If

$$\begin{pmatrix} \mathbf{a} \\ 0 \end{pmatrix}, \qquad \mathbf{a} \in \mathbb{C}^n,$$

is the final column of $\tilde{A}$, then $\mathbf{a}$ determines $\tilde{A}$ as a direct extension of A, and we shall call $\tilde{A}$ the **a**-*extension* of A. If A is similar to $V^{-1}AV = B$, $V \in \mathscr{G}\mathscr{U}_n$, then the **a**-extension of A is similar to the $V^{-1}\mathbf{a}$-extension of B. Conversely, if a direct extension $\tilde{A}$ of A is similar to $C \in \mathscr{U}_{n+1}^0$, then A is similar to the restriction $C|_{\mathrm{span}(\{\mathbf{e}_1,\ldots,\mathbf{e}_n\})}$.

In this section we shall outline a construction process for the (equivalence classes of) irreducible elements in $\mathscr{U}_{n+1}^0$ as extensions of representatives of the equivalence classes in $\mathscr{U}_n^0$. If the **a**-extension of $A \in \mathscr{U}_n^0$ is reducible, then there can be two reasons for this:

(1) $\tilde{A} \sim A \oplus (0)$, i.e., there is a reduction along the decomposition $\{1,\ldots,n,n+1\} = \{1,\ldots,n\} \cup \{n+1\}$,

(2) $\tilde{A}$ and $A \oplus (0)$ are not similar, but A has a reduction along $\{1,\ldots,n\} = I_1 \dot{\cup} I_2$, then $\tilde{A}$ has a reduction along $(I_1 \cup \{n+1\}) \dot{\cup} I_2$.

If A is irreducible, then only the first possibility exists, and we shall deal with that situation first.

PROPOSITION 3.1. *Let* $A \in \mathscr{U}_n^0$, $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$.

(i) *The* **a**-*extension and the* **b**-*extension of* A *are similar if and only if there exist* $V \in \mathscr{G}\mathscr{U}_n$, $v \neq 0$, *such that* $VA = AV$ *and* $V\mathbf{a} - v\mathbf{b} \in \mathrm{Im}\, A$.

(ii) *The* **a**-*extension of* A *is similar to* $A \oplus (0)$ *if and only if* $\mathbf{a} \in \operatorname{Im} A$.

Proof. (i):

$$\begin{pmatrix} V & \mathbf{v} \\ \mathbf{0} & v \end{pmatrix}\begin{pmatrix} A & \mathbf{a} \\ \mathbf{0} & 0 \end{pmatrix} = \begin{pmatrix} A & \mathbf{b} \\ \mathbf{0} & 0 \end{pmatrix}\begin{pmatrix} V & \mathbf{v} \\ \mathbf{0} & v \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} V & \mathbf{v} \\ \mathbf{0} & v \end{pmatrix} \in \mathscr{G}\mathscr{U}_{n+1}$$

if and only if $V \in \mathscr{G}\mathscr{U}_n$, $v \neq 0$, and $VA = AV$, $V\mathbf{a} = A\mathbf{v} + v\mathbf{b}$.

(ii): Apply (i) with $\mathbf{b} = \mathbf{0}$, and use $V^{-1}A = AV^{-1}$. ∎

COROLLARY 3.2. *Let* $A \in \mathscr{U}_n^0$ *be irreducible and* $\mathbf{a} \in \mathbb{C}^n$. *Then the* **a**-*extension of* A *is irreducible if and only if* $\mathbf{a} \notin \operatorname{Im} A$.

Observe that the **a**-extension and the **b**-extension of A are always similar if $\mathbf{b} - \mathbf{a} \in \operatorname{Im} A$. For irreducible $A \in \mathscr{U}_n^0$ it follows from Proposition 1.3 that the requirement $VA = AV$ with $V \in \mathscr{G}\mathscr{U}_n$ implies

$$V = uI + N, \quad N \in \mathscr{U}_n^0, \qquad NA = AN, \tag{3.1a}$$

and the other requirement in Proposition 3.1(i) reads

$$u\mathbf{a} - v\mathbf{b} + N\mathbf{a} \in \operatorname{Im} A. \tag{3.1b}$$

EXAMPLE 3.3.

(i) Let $J_n \in \mathscr{U}_n^0$ denote the nilpotent Jordan block. If $\mathbf{b} = (b_i)_{i=1}^n \in \mathbb{C}^n$, then the **b**-extension of J_n is irreducible if and only if $b_n \neq 0$. If $b_n \neq 0$, then $I\mathbf{e}_n - b_n^{-1}\mathbf{b} \in \operatorname{Im} J_n$, and up to similarity J_{n+1}, the $\mathbf{e}_n$-extension of J_n is the only irreducible extension of J_n.

(ii) Consider the irreducible matrix $J_{3,1} \in \mathscr{U}_4^0$ introduced in Section 1. One has $\operatorname{Im} J_{3,1} = \operatorname{span}(\{\mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3\})$, and modulo $\operatorname{Im} J_{3,1}$ a vector $\mathbf{a} \notin \operatorname{Im} J_{3,1}$ is of the form $\mathbf{a} = x\mathbf{e}_3 + y\mathbf{e}_4$, $(x, y) \neq (0, 0)$. In order to apply (3.1) we consider $N = (n_{ij})_{i,j=1}^4 \in \mathscr{U}_4^0$ such that $NJ_{3,1} = J_{3,1}N$. This implies $n_{23} = 0$, $n_{24} = n_{12} + n_{13}$, but n_{34} is arbitrary. Observe that $\operatorname{Im} N|_{\operatorname{span}(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})} \subseteq \operatorname{span}(\{\mathbf{e}_1\}) \subseteq \operatorname{Im} J_{3,1}$. Hence

$$\left(\frac{1}{y}I - \frac{x}{y^2}\mathbf{e}_3\mathbf{e}_4^T\right)\mathbf{a} - \mathbf{e}_4 = \mathbf{0} \qquad \text{if} \quad y \neq 0,$$

and the **a**-extension and the $\mathbf{e}_4$-extension of $J_{3,1}$ are similar. We shall denote the $\mathbf{e}_4$-extension of $J_{3,1}$ by B_1. If $y = 0$, then $x \neq 0$ and $(1/x)I\mathbf{a} - \mathbf{e}_3 = \mathbf{0}$.

Thus, for $y = 0$, the $\mathbf{a}$-extension of $J_{3,1}$ is similar to the $\mathbf{e}_3$-extension, which we shall denote by B_2:

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 0 \\ & & 0 & 1 & 0 \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 0 \\ & & 0 & 1 & 1 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}.$$

Since B_1 has the partial multiplicities $4, 1$ and B_2 has the partial multiplicities $3, 2$, the matrices B_1 and B_2 are clearly nonsimilar.

(iii) Next we consider the extensions of B_1 and B_2.

(a) For B_1 the stuation is analogous to that for $J_{3,1}$: One has $\operatorname{Im} B_1 = \operatorname{span}(\{\mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_4\})$, so $\mathbf{a} \notin \operatorname{Im} B_1$ is $\mathbf{a} = x\mathbf{e}_3 + y\mathbf{e}_5$ modulo $\operatorname{Im} B_1$; the same type of argument as in (ii) leads to the conclusion that up to similarity the $\mathbf{e}_5$-extension $B_{1,1}$ and the $\mathbf{e}_3$-extension $B_{1,2}$ are the only possible irreducible extensions of B_1; these matrices are

$$B_{1,1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 0 & 0 \\ & & 0 & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}, \qquad B_{1,2} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 0 & 0 \\ & & 0 & 1 & 0 & 1 \\ & & & 0 & 1 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}.$$

Since $B_{1,1}$ has the partial multiplicities $5, 1$ and $B_{1,2}$ has the partial multiplicites $4, 2$, the matrices $B_{1,1}$ and $B_{1,2}$ are nonsimilar.

(b) It turns out that B_2 has infinitely many nonsimilar irreducible direct extensions: $\operatorname{Im} B_2 = \operatorname{span}(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$ has codimension 2, and the equation $NB_2 = B_2 N$, $N \in \mathscr{U}_5^0$ implies $\operatorname{Im} N \subseteq \operatorname{Im} B_2$. Thus, the $\mathbf{a}$-extension and the $\mathbf{b}$-extension of B_2 are not similar if $\mathbf{a}, \mathbf{b}$ are linearly independent modulo $\operatorname{Im} B_2$. Writing $\mathbf{a} = x\mathbf{e}_4 + y\mathbf{e}_4 \notin \operatorname{Im} B_2$, and choosing $x = 1$ if $x \neq 0$, $y = 1$ if $x = 0$, one obtains the irreducible extensions

$$B_{2,z} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 0 & 0 \\ & & 0 & 1 & 1 & 0 \\ & & & 0 & 0 & 1 \\ & & & & 0 & z \\ & & & & & 0 \end{pmatrix}, \qquad z \in \mathbb{C},$$

$$B_{2,\infty} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 0 & 0 \\ & & 0 & 1 & 1 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix},$$

where $B_{2,z}$ has the partial multiplicities 4, 2 for each $z \in \mathbb{C}$, and $B_{2,\infty}$ has the partial multiplicities 3, 3. It is not difficult to see that these matrices are $\mathscr{U}$-equivalent to the transposes X_α^T of the matrices X_α found by Djoković and Malzan [5]; here $B_{2,z}$ corresponds to X_α with $\alpha = -z/(1+z)$, $B_{2,\infty}$ to X_{-1}. ■

COROLLARY 3.4. *For $n \geqslant 6$ there are infinitely many mutually non-$\mathscr{U}$-similar $\mathscr{U}$-irreducible matrices in $\mathscr{U}_n^0$.*

Indeed, since codim Im $A \geqslant 1$ for A nilpotent, each irreducible $A \in \mathscr{U}_n^0$ has at least one irreducible extension, and if A and B are not $\mathscr{U}$-similar, then no direct extension of A can be $\mathscr{U}$-similar to a direct extension of B.

For $n \leqslant 5$ there are only finitely many irreducible equivalence classes in $\mathscr{U}_n^0$: for $n = 4$ these are represented by J_4 and $J_{3,1}$, whereas besides J_5, B_1, and B_2 there are two further irreducible classes in $\mathscr{U}_5^0$ which have reducible restrictions to span$(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\})$; these will be constructed below.

Next, we consider the extensions of a matrix $A \in \mathscr{U}_n^0$ which as a reduction $A \sim (A_1) \oplus (A_2)$. Decomposing the vectors $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ in the same way as $\mathbf{a} = (\mathbf{a}_1) \oplus (\mathbf{a}_2)$, $\mathbf{b} = (\mathbf{b}_1) \oplus (\mathbf{b}_2)$, we describe the relations which must exist if the $\mathbf{a}$-extension and the $\mathbf{b}$-extension of A are similar. A first result in this direction was Lemma 2.3.

PROPOSITION 3.5. *Let $I_1 \dot{\cup} I_2 = \{1, \dots, n\}$ and $A_j \in \mathscr{U}_{\#I_j}^0$, $\mathbf{a}_j, \mathbf{b}_j \in \mathbb{C}^{\#I_j}$, $j = 1, 2$. Then the $(\mathbf{a}_1)_{I_1} \oplus (\mathbf{a}_2)_{I_2}$-extension of and the $(\mathbf{b}_1)_{I_1} \oplus (\mathbf{b}_2)_{I_2}$-extension $A = (A_1)_{I_1} \oplus (A_2)_{I_2}$ are $\mathscr{U}$-similar if and only if there exist $V_j \in \mathscr{G}\mathscr{U}_{\#I_j}$, $\mathbf{v}_j \in \mathbb{C}^{\#I_j}$, $j = 1, 2$, $v \neq 0$, $V_r'(v_{xy}')_{x=1,y=1}^{\#I_1\ \ \#I_2}$, $V_r'' = (v_{pq}'')_{p=1,q=1}^{\#I_2\ \ \#I_1}$, $v_{xy}' = 0$ for $i_x > j_y$, and $v_{pq}'' = 0$ for $j_p > i_q$ (where $I_1 = \{i_1, \dots, i_{\#I_1}\}$, $I_2 = \{j_1, \dots, j_{\#I_2}\}$, $i_x < i_{x+1}, j_y < j_{y+1}$) such that*

(i) $V_1 A_1 = A_1 V_1$, $V_2 A_2 = A_2 V_2$, $V_r' A_2 = A_1 V_r'$, $V_r'' A_1 = A_2 V_r''$,
(ii) $V_1 \mathbf{a}_1 + V_r' \mathbf{a}_2 = A_1 \mathbf{v}_1 + v\mathbf{b}_1$, $V_r'' \mathbf{a}_1 + V_2 \mathbf{a}_2 = A_2 \mathbf{v}_2 + v\mathbf{b}_2$.

Proof. It suffices to write out the equations implied by the fact that $V \in \mathscr{G}\mathscr{U}_{n+1}$ intertwines the $(\mathbf{a}_1) \oplus (\mathbf{a}_2)$-extension and the $(\mathbf{b}_1) \oplus (\mathbf{b}_2)$-

extension of $(A_1) \oplus (A_2)$:

$$\begin{pmatrix} (V_1) & (V_r') & (\mathbf{v}_1) \\ (V_r'') & (V_2) & (\mathbf{v}_2) \\ \multicolumn{2}{c}{\mathbf{0}} & v \end{pmatrix} \begin{pmatrix} (A_1) & (0) & (\mathbf{a}_1) \\ (0) & (A_2) & (\mathbf{a}_2) \\ \multicolumn{2}{c}{\mathbf{0}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (A_1) & (0) & (\mathbf{b}_1) \\ (0) & (A_2) & (\mathbf{b}_2) \\ \multicolumn{2}{c}{\mathbf{0}} & 0 \end{pmatrix} \begin{pmatrix} (V_1) & (V_r') & (\mathbf{v}_1) \\ (V_r'') & (V_2) & (\mathbf{v}_2) \\ \multicolumn{2}{c}{\mathbf{0}} & v \end{pmatrix}. \qquad \blacksquare$$

The equations presented in Proposition 3.5 are quite often solvable if concrete additional information allows to simplify them, e.g. if

(1) for given A_1, A_2, I_1, I_2 the relations (i) reduce the possible choices of V_1, V_2, V_r', V_r'', making the description of the solutions of (ii) possible, or

(2) a given irreducible extension is known and one can find conditions for another irreducible extension to be similar to it.

In order to decide on the reducibility of the $(\mathbf{a}_1) \oplus (\mathbf{a}_2)$-extension of $(A_1) \oplus (A_2)$ along the decomposition $\{1, \dots, n+1\} = (I_1 \cup \{n+1\}) \dot{\cup} I_2$ $[I_1 \dot{\cup} (I_2 \cup \{n+1\})]$, one sets $\mathbf{b}_2 = \mathbf{0}$ $[\mathbf{b}_1 = \mathbf{0}]$, $V_j = I$, $v = 1$, and treats $\mathbf{b}_1$ $[\mathbf{b}_2]$ as an additional unknown:

PROPOSITION 3.6. *Let $I_1, I_2, A_1, A_2, \mathbf{a}_1, \mathbf{a}_2$ be as in Proposition* 3.5.

(i) *The $(\mathbf{a}_1)_{I_1} \oplus (\mathbf{a}_2)_{I_2}$-extension of $(A_1)_{I_1} \oplus (A_2)_{I_2}$ is reducible along the decomposition $(I_1 \cup \{n+1\}) \dot{\cup} I_2$ if and only if there exist V_r'', $\mathbf{v}_2 \in \mathbb{C}^{\# I_2}$ such that*

(a) $v''_{pq} = 0$ *for* $j_p > i_q$, $V_r'' A_1 = A_2 V_r''$,
(b) $\mathbf{a}_2 = A_2 \mathbf{v}_2 - V_r'' \mathbf{a}_1$.

(ii) *The $(\mathbf{a}_1)_{I_1} \oplus (\mathbf{a}_2)_{I_2}$-extension of $(A_1)_{I_1} \oplus (A_2)_{I_2}$ is reducible along the decomposition $I_1 \dot{\cup} (I_2 \cup \{n+1\})$ if and only if there exist V_r', $v_1 \in \mathbb{C}^{\# I_1}$ such that*

(a′) $v'_{xy} = 0$ *for* $i_x > j_y$, $V_r' A_2 = A_1 V_r'$,
(b′) $\mathbf{a}_1 = A_1 \mathbf{v}_1 - V_r' \mathbf{a}_2$.

The proof can be found in [9, Proposition 3.6].

EXAMPLE 3.7.

(i) *Irreducible extensions of* $(J_2) \oplus (J_2)$. Here

$$A_1 = A_2 = J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $I_1 = \{i_1, i_2\}$, $I_2 = \{j_1, j_2\}$ are to be specified. For the $(\mathbf{a}_1) \oplus (\mathbf{a}_2)$-extension of $(J_2) \oplus (J_2)$ to be irreducible, the requirement is $(\mathbf{a}_1) \oplus (\mathbf{a}_2) \notin \operatorname{Im}(J_2) \oplus (J_2)$, and up to $\mathscr{U}$-similarity one may assume

$$\mathbf{a}_1 = \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If $I_1 = \{1,2\}$, $I_2 = \{3,4\}$, then $V_r'' = 0$, but V_r' can be any matrix commuting with J_2; taking $V_r' = -I$, $\mathbf{v}_1 = \mathbf{0}$ solves (b′). So the $\mathbf{e}_2 \oplus \mathbf{e}_2$-extension of $J_2 \oplus J_2$ is reducible along $\{1,2\} \cup \{3,4,5\}$; in fact, it is similar to $J_2 \oplus J_3$. If $I_1 = \{1,3\}$, $I_2 = \{2,4\}$, then

$$V_r' = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$$

commutes with J_2; taking $x = -1$, $y = 0$ solves (b′), and the $\mathbf{e}_3 + \mathbf{e}_4$-extension of $(J_2)_{\{1,3\}} \oplus (J_2)_{\{2,4\}}$ is $\mathscr{U}$-similar to $(J_2)_{\{1,3\}} \oplus (J_3)_{\{2,4,5\}}$. If, however, $I_1 = \{1,4\}$, $I_2 = \{2,3\}$, then

$$V_r'' = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}, \qquad V_r' = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix};$$

requiring $J_2 V_r^{\prime(\prime)} = V_r^{\prime(\prime)} J_2$ implies that both V_r'', V_r' are multiples of J_2, and the equations (b), (b′) are unsolvable. Hence

$$C_{3,2} = \begin{pmatrix} 0 & & & 1 & 0 \\ & 0 & 1 & & 0 \\ & & 0 & & 1 \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \qquad \text{with partial multiplicities } 3, 2,$$

is a representative of the unique class of irreducible direct extensions of a matrix with partial multiplicities 2, 2.

(ii) *Irreducible extensions of* $(J_3)_{I_1} \oplus (J_2)_{I_2}$, $I_1 \dot\cup I_2 = \{1,\ldots,5\}$. As in (i), one only has to consider the $(\mathbf{e}_3) \oplus (\mathbf{e}_2)$-extension of $(J_3) \oplus (J_2)$. The same type of argument as in (i) leads to the conclusion that out of the ten possible choices for I_2 only five lead to irreducible extensions: $I_2 = \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}$. Thus we have five mutually nonsimilar 6×6 matrices with the partial multiplicities 4, 2 which are extensions of a reducible matrix $(J_3) \oplus (J_2)$; they are listed in [9, Example 3.7(ii)]. Since these matrices are direct extensions of reducible matrices, none of them is $\mathscr{U}$-similar to the matrices $B_{1,2}$ or $B_{2,z}$ from Example 3.3(iii), as those are extensions of the irreducible matrices B_1, B_2.

(iii) *Irreducible extensions of* $(J_2)_{I_1} \oplus ((0))_{\{k\}}$, $1 \leqslant k \leqslant 3$, $I_1 = \{1,2,3\} \setminus \{k\}$. Take $\mathbf{a} \notin \operatorname{Im} A$; one may assume $\mathbf{a} = (\mathbf{e}_2) \oplus ((1))$. The relevant equation for the reducibility of the $\mathbf{a}$-extension is $(1) = (0)v_2 - V_r''\mathbf{e}_2$, $V_r'' = (v_1 \ v_2)$ such that $0V_r'' = (0,0) = V_r''J_2 = (0 \ v_1)$, i.e., $v_1 = 0$. For $k = 1,2$ one can choose $v_2 \neq 0$, and the given extension is reducible, unless $k = 3$. For $k = 3$ one has $V_r'' = (0 \ 0)$, and the $\mathbf{e}_2 \oplus (1)$-extension of $J_2 \oplus (0)$, that is,

$$J_{3,1} = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & & 1 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix},$$

is irreducible.

The above examples suggest several special cases where the analysis of the extension problem is relatively simple. These will be considered in the next section. We conclude this section with a few remarks concerning the extensions of matrices $A = (A_1)_{I_1} \oplus \cdots \oplus (A_k)_{I_k}$ where $k \geqslant 3$; the proofs of the first two results can be found in [9].

PROPOSITION 3.8. *Let* $A \in \mathscr{U}_n^0$ *be given by* $A = (A_1)_{I_1} \oplus (A_2)_{I_2} \oplus \cdots \oplus (A_k)_{I_k}$, *each* $I_j \neq \varnothing$, *and let* $\mathbf{a} = (\mathbf{a}_1)_{I_1} \oplus (\mathbf{a}_2)_{I_2} \oplus \cdots \oplus (\mathbf{a}_k)_{I_k}$, $\mathbf{a}_j \in \mathbb{C}^{\# I_j}$. *In order for the* $\mathbf{a}$*-extension of* A *to be irreducible it is necessary that for each nonempty subset* $S \subseteq \{1,\ldots,k\}$ *the* $\bigoplus_{j \in S}(\mathbf{a}_j)_{I_j}$*-extension of* $\bigoplus_{j \in S}(A_j)_{I_j}$ *be irreducible.*

The validity of this proposition is immediately clear, taking $\mathbf{b} = (\mathbf{0})_{I_1'} \oplus (\mathbf{b}')_{I_2'}$, $I_1' \dot\cup I_2' = \bigcup_{j \in S} I_j$ in the next lemma.

LEMMA 3.9. *Let A, $\mathbf{a}$, and S be as in Proposition 3.8. If the $\bigoplus_{j \in S}(\mathbf{a}_j)_{I_j}$-extension and the $\mathbf{b}$-extension of $\bigoplus_{j \in S}(A_j)_{I_j}$ are $\mathscr{U}$-similar, then the $\mathbf{a}$-extension of A is $\mathscr{U}$-similar to the $(\bigoplus_{j \notin S}(\mathbf{a}_j)_{I_j}) \oplus (\mathbf{b})$-extension of A.*

The above proposition is useful in the search for irreducible extensions in several ways: Evdiently, one must require $\mathbf{a}_i \notin \operatorname{Im} A_i$ for the matrix A and the vector $\mathbf{a}$ as in the proposition; further, only certain orderings of the index sets are possible to allow irreducible extension: $(J_2)_{I_1} \oplus (J_2)_{I_2}$ could only occur for $i_{11} < i_{21} < i_{22} < i_{12}$, where $I_j = \{i_{j1}, i_{j2}\}$. Finally, the presence of certain combinations of components A_i sometimes simply contradicts the existence of an irreducible extension. Having formulated all necessary conditions implicated by Proposition 3.8 the research can then be finalized by means of Proposition 3.6, using the following lemma:

LEMMA 3.10. *Let A $\bigoplus_{j=1}^k (A_j)_{I_j}$, $k \geqslant 2$, A_j irreducible, and $\mathbf{a} \notin \operatorname{Im} A$. If the $\mathbf{a}$-extension of A is reducible, then it has a reduction along the decomposition $I_{j_0} \dot\cup ((\bigcup_{j \neq j_0} I_j) \cup \{n+1\})$ for some j_0.*

Proof. Let $(\hat{A}_1)_{K_1} \oplus \cdots \oplus (\hat{A}_l)_{K_l}$ be a complete reduction of the a-extension of A. For definiteness, let $n+1 \in K_l$. Then $m = \# K_l \geqslant 2$. Evidently, A is similar to

$$\left(\hat{A}_1\right)_{K_1} \oplus \cdots \oplus \left(\hat{A}_{l-1}\right)_{K_{l-1}} \oplus \left(\hat{A}_l|_{\operatorname{lin}(\{\mathbf{e}_1, \ldots, \mathbf{e}_{m-1}\})}\right)_{K_l \setminus \{n+1\}}.$$

Since $\hat{A}_1, \ldots, \hat{A}_{l-1}$ are irreducible, there must exist $1 \leqslant j_1, \ldots, j_{l-1} \leqslant k$ such that $K_x = I_{j_x}$, $\hat{A}_x = A_{j_x}$, $x = 1, 2, \ldots, l-1$. As $l - 1 \geqslant 1$, this completes the proof. ∎

EXAMPLE 3.11.

(i) *Extensions of matrices containing* $((0)) \oplus ((0))$. If $A = \bigoplus_{j=1}^k (A_j)_{I_j}$ and $A_j = (0)$ for at least two different indices, then A has no irreducible direct extensions: For $n \geqslant 2$ the zero matrix $O \in \mathscr{U}_n^0$ has no irreducible extensions, as can be verified by a direct calculation (see [9]).

(ii) *Extensions of* $(J_2) \oplus (J_2) \oplus ((0))$. Because of Example 3.7(i) we must consider $A = (J_2)_{\{i_1, i_4\}} \oplus (J_2)_{\{i_2, i_3\}} \oplus ((0))_{\{k\}}$ with $i_1 < i_2 < i_3 < i_4$, and $\mathbf{a} = (\mathbf{e}_2)_{\{i_1, i_4\}} \oplus (\mathbf{e}_2)_{\{i_2, i_3\}} \oplus ((1))_{\{k\}}$. Let $I_1 = \{i_1, i_2, i_3, i_4\}$, $I_2 = \{k\}$. An argument as in Example 3.7 shows that one must have $k = 5$, so $I_1 = \{1, 4\}$,

$I_2 = \{2, 3\}$, and

$$J_{3,2,1} = \begin{pmatrix} 0 & & & 1 & & 0 \\ & 0 & 1 & & & 0 \\ & & 0 & & & 1 \\ & & & 0 & & 1 \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix},$$

with partial multiplicities 3, 2, 1 is up to similarity the only possible irreducible extension of a matrix $(J_2) \oplus (J_2) \oplus ((0))$. That $J_{3,2,1}$ is indeed irreducible is shown in [9, Example 3.11(ii)].

(iii) *Extensions of* $(J_2) \oplus (J_2) \oplus (J_2)$. Here, evidently, the only option is the matrix

$$C_{3,2,2} = \begin{pmatrix} 0 & & & & & 1 & 0 \\ & 0 & & & 1 & & 0 \\ & & 0 & 1 & & & 0 \\ & & & 0 & & & 1 \\ & & & & 0 & & 1 \\ & & & & & 0 & 1 \\ & & & & & & 0 \end{pmatrix},$$

with the partial multiplicities 3, 2, 2 which is the $\mathbf{e}_6 + \mathbf{e}_5 + \mathbf{e}_4$-extension of $(J_2)_{\{1,6\}} \oplus (J_2)_{\{2,5\}} \oplus (J_2)_{\{3,4\}}$. That $C_{3,2,2}$ is indeed irreducible is shown in [9, Example 3.11(iii)].

In the next section we shall describe extensions of matrices of the type $(A_1) \oplus ((0))$.

4. EXTENSIONS OF SPECIAL TYPES

In this section we consider situations where conditions with respect to the solvability of the equations on Proposition 3.6 are simple. For example, if $A = A_1 \oplus A_2$ is a (nongeneralized) direct sum, and $\mathbf{a}_2 \notin \operatorname{Im} A_2$, then the $\mathbf{a}_1 \oplus \mathbf{a}_2$-extension of A has no reduction along the decomposition $(I_1 \cup \{n+1\}) \dot{\cup} I_2$, as $V_r'' = 0$ if $\max I_1 = \min I_2 - 1$; V_r', on the other hand, is potentially a "full" matrix, and the intertwining relation $V_r' A_2 = A_1 V_r'$ should be used for analyzing the solvability of the equation $\mathbf{a}_1 = A_1 \mathbf{v}_1 - V_r' \mathbf{a}_2$.

Another interesting situation occurs where $A = (A_1) \oplus (0)$. Choosing $\mathbf{a}_2 = (1)$, one has to consider the solvability of $1 = 0 - V_r''\mathbf{a}_1$, V_r'' a row vector with $V_r''A_1 = \mathbf{0}^T$, and $\mathbf{a}_1 = A_1\mathbf{v}_1 - V_r'$, V_r' a column vector with $A_1V_r' = \mathbf{0}$. This leads to the next proposition, for which a proof can be found in [9, Proposition 4.1].

PROPOSITION 4.1. *Let* $A = (A_1)_{I_1} \oplus ((0))_{\{k\}}$, *where* $1 \leqslant k \leqslant n$, $I_1 = \{1, \dots, n\} \backslash \{k\}$, *and* $A_1 \in \mathscr{U}^0_{n-1}$, *and let* $\mathbf{a}, \mathbf{b} \in \mathbb{C}^{n-1}$.

(i) *The* $(\mathbf{a})_{I_1} \oplus ((1))_{\{k\}}$*-extension of* A *is not reducible along the decomposition* $(I_1 \cup \{n+1\}) \dot{\cup} \{k\}$ *if and only if* $\mathbf{a} \in \operatorname{span}(\{\mathbf{e}_1, \dots, \mathbf{e}_{k-1}\}) + \operatorname{Im} A_1$.

(ii) *The* $(\mathbf{a})_{I_1} \oplus ((1))_{\{k\}}$*-extension of* A *is not reducible along the decomposition* $I_1 \dot{\cup} \{k, n+1\}$ *if and only if* $\mathbf{a} \notin \operatorname{Im} A_1 + (\operatorname{Ker} A_1 \cap \operatorname{span}(\{\mathbf{e}_1, \dots, \mathbf{e}_{k-1}\}))$.

(iii) *The* $(\mathbf{a})_{I_1} \oplus ((1))_{\{k\}}$*- and* $(\mathbf{b})_{I_1} \oplus ((1))_{\{k\}}$*-extensions of* A *are* $\mathscr{U}$*-similar if and only if the vectors* $V\mathbf{a}, \mathbf{b}$ *are linearly dependent modulo* $\operatorname{Im} A_1 + (\operatorname{Ker} A_1 \cap \operatorname{span}(\{\mathbf{e}_1, \dots, \mathbf{e}_{k-1}\}))$ *for some* $V \in \mathscr{G}\mathscr{U}_{n-1}$ *commuting with* A_1.

We shall call the type of extension considered in Proposition 4.1 an $\mathbf{a}$-$(0)_k$-extension of A_1. If $k = n$, then the condition under (i) is automatically satisfied; we call the extension the $\mathbf{a}$-0-extension of A_1 in that case. If A_1 is irreducible, then Proposition 4.1 contains necessary and sufficient conditions for the irreducibility of the $\mathbf{a}$-$(0)_k$-extensions of A_1.

COROLLARY 4.2. *Let* $A_1 \in \mathscr{U}^0_{n-1}$ *be a irreducible and* $\mathbf{a}, \mathbf{b} \in \mathbb{C}^{n-1}$.

(i) *The* $\mathbf{a}$-$(0)_k$*-extension of* A_1 *is irreducible if and only if*

$$\mathbf{a} \in \left[\operatorname{Im} A_1 + \operatorname{span}(\{\mathbf{e}_1, \dots, \mathbf{e}_{k-1}\})\right] \backslash$$

$$\left[\operatorname{Im} A_1 + \left(\operatorname{Ker} A_1 \cap \operatorname{span}(\{\mathbf{e}_1, \dots, \mathbf{e}_{k-1}\})\right)\right];$$

the $\mathbf{a}$-$(0)_k$*-extension and the* $\mathbf{b}$-$(0)_k$*-extension of* A_1 *are* $\mathscr{U}$*-similar if and only if there exists some* $V \in \mathscr{G}\mathscr{U}_{n-1}$ *which commutes with* A_1 *such that* $V\mathbf{a}$ *and* $\mathbf{b}$ *are linearly dependent modulo* $\operatorname{Im} A_1 + \operatorname{Ker} A_1 \cap \operatorname{lin}(\{\mathbf{e}_1, \dots, \mathbf{e}_{k-1}\})$.

(ii) *The* $\mathbf{a}$-0*-extension of* A_1 *is irreducible if and only if* $\mathbf{a} \notin \operatorname{Im} A_1 + \operatorname{Ker} A_1$, *and it is* $\mathscr{U}$*-similar to the* $\mathbf{b}$-0*-extension of* A_1 *if and only if there exists some* $V \in \mathscr{G}\mathscr{U}_{n-1}$ *which commutes with* A_1 *such that* $V\mathbf{a}, \mathbf{b}$ *are linearly dependent modulo* $\operatorname{Im} A_1 + \operatorname{Ker} A_1$.

EXAMPLE 4.3.

(i) Let $A_1 = J_{n-1}$, $n \geqslant 3$; then $\operatorname{span}(\{\mathbf{e}_1, \ldots, \mathbf{e}_{k-1}\}) \subseteq \operatorname{Im} A_1 = \operatorname{span}(\{\mathbf{e}_1, \ldots, \mathbf{e}_{n-2}\})$ for $k \leqslant n-1$; thus for $k \leqslant n-1$ an $\mathbf{a}$-$(0)_k$-extension is always reducible. For $k = n$ it is not difficult to see that the $\mathbf{e}_{n-1}$-0-extension of J_{n-1}, which is the $\mathbf{e}_{n-1} + \mathbf{e}_n$-extension of $J_{n-1} \oplus (0)$, is irreducible and that up to $\mathscr{U}$-similarity it is the only $\mathbf{a}$-$(0)_k$-extension of J_{n-1} which is irreducible; we shall denote it by $J_{n,1} \in \mathscr{U}^0_{n+1}$, as it has the partial multiplicities $n, 1$. The matrix $J_{3,1}$ is the simplest example (with $n - 1 = 2$), as $n = 2$ would lead to extensions of $(0) \oplus (0)$, which has no irreducible extensions.

(ii) Next, consider the possible $\mathbf{a}$-$(0)_k$-extensions for $J_{n,1}$. Observing that

$$\operatorname{Im} J_{n,1} = \operatorname{span}(\{\mathbf{e}_1, \ldots, \mathbf{e}_{n-2}, \mathbf{e}_{n-1} + \mathbf{e}_n\}) \subseteq \mathbb{C}^{n+1},$$

$$\operatorname{Ker} J_{n,1} = \operatorname{span}(\{\mathbf{e}_1, \mathbf{e}_n\}) \subseteq \mathbb{C}^{n+1},$$

and using that

$$\operatorname{Im} J_{n,1} + \operatorname{span}(\{\mathbf{e}_1, \ldots, \mathbf{e}_{k-1}\} = \operatorname{Im} J_{n,1} + (\operatorname{Ker} J_{n,1} \cap \operatorname{span}(\{\mathbf{e}_1, \ldots, \mathbf{e}_{k-1}\}))$$

for $k \leqslant n-1$ and $k = n+1$; hence only $k = n$ and $k = n+2$ can lead to irreducible (0)-extensions; it turns out that the $\mathbf{e}_n$-(0)-extension of $J_{n,1}$ [which is the $\mathbf{e}_{n+1} + \mathbf{e}_n$-extension of $(J_{n,1})_{\{1,\ldots,n-1,n+1,n+2\}} \oplus ((0))_{\{n\}}]$ is irreducible, and up to $\mathscr{U}$-similarity it is the unique irreducible $\mathbf{a}$-$(0)_n$-extension of $J_{n,1}$; it has the partial multiplicities $n, 2, 1$. For $k = n+2$ the $\mathbf{e}_{n+1}$-0-extension of $J_{n,1}$ is up to $\mathscr{U}$-similarity the unique irreducible $\mathbf{a}$-0-extensions of $J_{n,1}$; it has the partial multiplicities $n+1, 1, 1$. For $n = 3$ the relevant extensions are

$$J_{((3),2),1} = \begin{pmatrix} 0 & 1 & & 0 & 0 & 0 \\ & 0 & & 0 & 1 & 0 \\ & & 0 & & & 1 \\ & & & 0 & 1 & 1 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \quad J_{4,1,1} = \begin{pmatrix} 0 & 1 & 0 & 0 & & 0 \\ & 0 & 0 & 1 & & 0 \\ & & 0 & 1 & & 0 \\ & & & 0 & & 1 \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}.$$

Evidently, these matrices are nonsimilar.

(iii) In Example 3.11(ii) we have analysed the possible $\mathbf{a}$-(0)-extensions for the reducible matrix $A_1 = (J_2) \oplus (J_2)$. We have seen that the $\mathbf{e}_3 + \mathbf{e}_4$-0-

extension of $(J_2)_{\{1,4\}} \oplus (J_2)_{\{2,3\}}$, denoted by $J_{3,2,1}$ in Example 3.11(ii), is the only possible option.

It follows from Corollary 4.2 that for irreducible A_1 there is (up to $\mathscr{U}$-similarity) just one irreducible $\mathbf{a}$-$(0)_k$-extension if the quotient space

$$T_k := \big[\operatorname{Im} A_1 + \operatorname{span}(\{\mathbf{e}_1, \dots, \mathbf{e}_{k-1}\})\big] \Big/ \big[\operatorname{Im} A_1 + \big(\operatorname{Ker} A_1 \cap \operatorname{span}(\{\mathbf{e}_1, \dots, \mathbf{e}_{k-1}\})\big)\big]$$

is one-dimensional (see [9, Proposition 4.4]). If $\dim T_k \geqslant 2$, then one may have infinitely many mutually nonsimilar $\mathbf{a}$-$(0)_k$-extensions of A_1, but also only finitely many. The situation is completely analogous to that of direct extensions [cf. Example 3.3(iii)]: The structure of those matrices $N \in \mathscr{U}_{n-1}^0$ which commute with A_1 determines the situation.

EXAMPLE 4.4.

(i) The matrix $B_1 \in \mathscr{U}_5^0$ which was constructed in Example 3.3(ii) has (up to $\mathscr{U}$-similarity) two irreducible -(0)-extensions; these are represented by the $\mathbf{e}_3 + \mathbf{e}_4$-$(0)_3$-extension (which has the partial multiplicities 4, 2, 1) and the $\mathbf{e}_5 + \mathbf{e}_6$-0-extension (i.e., the extension with $k = 6$), which has the partial multiplicities 5, 1, 1 (cf. [9, Example 4.5(i)]).

(ii) The matrix $B_2 \in \mathscr{U}_5^0$ which was also constructed in Example 3.3(ii) has (up to $\mathscr{U}$-similarity) one irreducible -$(0)_5$-extension, represented by the $\mathbf{e}_4 + \mathbf{e}_5$-$(0)_5$-extension with the partial multiplicities 4, 2, 1; but infinitely many mutually non-$\mathscr{U}$-similar irreducible $\mathbf{a}$-0-extensions, represented by the $\mathbf{e}_4 + z\mathbf{e}_5 + \mathbf{e}_6$-0-extensions, $z \in \mathbb{C}$, which all have the partial multiplicities 4, 2, 1, and the $\mathbf{e}_5 + \mathbf{e}_6$-0-extension, which has the partial multiplicities 3, 3, 1 (see [9, Example 4.5(ii)]).

(iii) The matrix $C_{3,2}$ introduced in Example 3.7(i) has irreducible $\mathbf{a}$-$(0)_k$-extensions for $k = 4$, 5, and 6; these are represented, respectively, by the $\mathbf{e}_4 + \mathbf{e}_5$-$(0)_4$-extension, the $\mathbf{e}_4 + \mathbf{e}_5$-$(0)_5$-extension, both with the partial multiplicities 3, 3, 1, and the two extensions for $k = 6$, namely the $\mathbf{e}_4 + \mathbf{e}_6$-0-extension with the partial multiplicities 3, 3, 1, and the $\mathbf{e}_5 + \mathbf{e}_6$-0-extension with the partial multiplicities 4, 2, 1. The complete verification of these results can be found in [9, Example 4.5]; there it is also stated that the $\mathbf{e}_4$-extension (with partial multiplicities 3, 3) and the $\mathbf{e}_5$-extension (with partial multiplicities 4, 2) are the possible direct extensions of $C_{3,2}$.

It is not difficult to generalize the results concerning the **a**-0-extensions of a matrix A_1 to extensions of the direct sum $A_1 \oplus J_k$; the proof of this result can be found in [9, Proposition 4.6].

PROPOSITION 4.5. *Let* $A \in \mathscr{U}_n^0$ *be irreducible,* $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$.

(i) *The* $\mathbf{a} \oplus \mathbf{e}_k$*-extension of* $A \oplus J_k$ *is irreducible if and only if* $a \notin \operatorname{Im} A + \operatorname{Ker} A^k$.

(ii) *The* $\mathbf{a} \oplus \mathbf{e}_k$*- and* $\mathbf{b} \oplus \mathbf{e}_k$*-extensions of* $A \oplus J_k$ *are* $\mathscr{U}$*-similar if and only if* $V\mathbf{a}, \mathbf{b}$ *are linearly dependent modulo* $\operatorname{Im} A + \operatorname{Ker} A^k$ *for some* $V \in \mathscr{G}\mathscr{U}_n$ *commuting with* A.

EXAMPLE 4.6. The matrices $B_1 \oplus J_2, B_2 \oplus J_2, C_{3,2} \oplus J_2$, where $B_1, B_2, C_{3,2}$ are the matrices occurring in Example 4.5, have the irreducible extensions

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & & & 0 \\ & 0 & 0 & 1 & 0 & & & 0 \\ & & 0 & 1 & 0 & & & 0 \\ & & & 0 & 1 & & & 0 \\ & & & & 0 & & & 1 \\ & & & & & 0 & 1 & 0 \\ & & & & & & 0 & 1 \\ & & & & & & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & & & 0 \\ & 0 & 0 & 1 & 0 & & & 0 \\ & & 0 & 1 & 1 & & & 0 \\ & & & 0 & 0 & & & 1 \\ & & & & 0 & & & 0 \\ & & & & & 0 & 1 & 0 \\ & & & & & & 0 & 1 \\ & & & & & & & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & & & 0 \\ & 0 & 1 & 0 & 0 & & & 0 \\ & & 0 & 0 & 1 & & & 0 \\ & & & 0 & 1 & & & 0 \\ & & & & 0 & & & 1 \\ & & & & & 0 & 1 & 0 \\ & & & & & & 0 & 1 \\ & & & & & & & 0 \end{pmatrix},$$

respectively.

Describing the irreducible extensions (if any exist) of $(A)_{I_1} \oplus (J_k)_{I_2}$, $I_1 \dot{\cup} I_2 = \{1, \ldots, n+k\}$, is more complicated. For the case J_2 necessary and sufficient conditions, based on Proposition 3.5, are derived in [9]; we present the result without proof (Proposition 4.8 in [9]) and a few examples.

PROPOSITION 4.7. *Let $A \in \mathscr{U}_n^0$ be irreducible and $\{k,l\} \dot{\cup} I_1 = \{1,\dots,n+2\}$ with $k<l$. Denote by P_k the orthogonal projection on* $\operatorname{lin}(\{e_k,\dots,e_n\})$. *Then the $(\mathbf{a})_{I_1} \oplus (\mathbf{e}_2)_{\{k,l\}}$-extension of $(A)_{I_1} \oplus (J_2)_{\{k,l\}}$ is irreducible if and only if*

$$\mathbf{a} \in \operatorname{Im} A + \operatorname{span}(\{\mathbf{e}_1,\dots,\mathbf{e}_{l-2}\}) + \operatorname{Ker} P_k A$$

and

$$\mathbf{a} \notin \operatorname{Im} A + \big(\operatorname{Ker} A^2 \cap \operatorname{Ker} P_k A \cap \operatorname{span}(\{\mathbf{e}_1,\dots,\mathbf{e}_{l-2}\})\big);$$

further, the $(\mathbf{a})_{I_1} \oplus (\mathbf{e}_2)_{\{k,l\}}$-extension and the $(\mathbf{b})_{I_1} \oplus (\mathbf{e}_2)_{\{k,l\}}$-extension of $(A)_{I_1} \oplus (J_2)_{\{k,l\}}$ are $\mathscr{U}$-similar if and only if $V\mathbf{a},\mathbf{b}$ are linearly dependent modulo $\operatorname{Im} A + (\operatorname{Ker} A^2 \cap \operatorname{Ker} P_k A \cap \operatorname{span}(\{\mathbf{e}_1,\dots,\mathbf{e}_{l-2}\}))$ *for some $V \in \mathscr{G}\mathscr{U}_n$ which commutes with A.*

EXAMPLE 4.8.

(i) Let $A = J_3$; then $\operatorname{Im} A = \operatorname{span}(\{e_1, e_2\}) = \operatorname{Ker} A^2$. Thus the $(e_3)_{I_1} \oplus (e_2)_{\{k,l\}}$-extension of $(J_3)_{I_1} \oplus (J_2)_{\{k,l\}}$ is irreducible if and only if $\mathbb{C}^3 = \operatorname{Im} A + \operatorname{span}(\{e_1,\dots,e_{l-2}\}) + \operatorname{Ker} P_k J_3$. Observe that $P_3 J_3 = P_4 J_3 = 0$, so $\operatorname{Ker} P_k J_3 = \mathbb{C}^3$, $k = 3,4$, but $\operatorname{Ker} P_k J_3 \subseteq \operatorname{span}(\{\mathbf{e}_1,\mathbf{e}_2\}) \subseteq \operatorname{Im} A$ for $k = 1,2$. Further, $\operatorname{span}(\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_{l-2}\}) = \mathbb{C}^3$ for $l = 5$. Thus the described extensions are irreducible for the following decompositions: $\{1,2,5\} \dot{\cup} \{3,4\}$, $\{1,2,3\} \dot{\cup} \{4,5\}$, $\{1,2,4\} \dot{\cup} \{3,5\}$, $\{1,3,4\} \dot{\cup} \{2,5\}$, $\{2,3,4\} \dot{\cup} \{1,5\}$. The corresponding mutually nonsimilar matrices were described in Example 3.7(iii).

(ii) Let $A = J_{3,1}$. Then $\operatorname{Im} A = \operatorname{span}(\{\mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3\})$, $\operatorname{Ker} A = \operatorname{span}(\{\mathbf{e}_1,\mathbf{e}_3\}) = \operatorname{Ker} P_1 A$, $\operatorname{Ker} A^2 = \operatorname{span}(\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}) = \operatorname{Ker} P_k A$, $k = 2,3$, $\operatorname{Ker} P_k A = \mathbb{C}^4$, $k = 4,5$. Observe that $\operatorname{Im} A + \operatorname{Ker} P_k A = \operatorname{span}(\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\})$ for $k \leqslant 3$, $\operatorname{Im} A + \operatorname{Ker} P_k A = \mathbb{C}^4$ for $k = 4,5$. Using the same type of argument as before, one can conclude that the $(\mathbf{e}_3)_I \oplus (\mathbf{e}_2)_{\{k,l\}}$-extension of $(J_{3,1})_I \oplus (J_2)_{\{k,l\}}$ is irreducible for $k = 1$, $l \leqslant 4$ and for $k = 2$, $l = 3$, whereas the $(\mathbf{e}_4)_I \oplus (\mathbf{e}_2)_{\{k,l\}}$-extension of $(J_{3,1})_I \oplus (J_2)_{\{k,l\}}$ is irreducible for $k = 4$, $l = 5$ and for k arbitrary, $l = 2$. For the remaining five combinations $\{k,l\}$ such that $I \dot{\cup} \{k,l\} = \{1,\dots,6\}$ the matrix $(J_{3,1})_I \oplus (J_2)_{\{k,l\}}$ has no irreducible extensions. The details can be found in [9, Example 4.9(ii)].

Observing that $\operatorname{span}(\{\mathbf{e}_1,\dots,\mathbf{e}_{l-2}\}) = \operatorname{Ker} P_{l-1} A^0$, it is natural to guess the following test for the irreducibility of extensions of $(A) \oplus (J_3)$, where A is irreducible: The $(\mathbf{a})_I \oplus (\mathbf{e}_3)_{\{k,l,m\}}$-extension of $(A)_I \oplus (J_3)_{\{k,l,m\}}$ (with $k<l$

$< m$) is irreducible if and only if

$$\mathbf{a} \in \operatorname{Im} A + \operatorname{Ker} P_k A^2 + \operatorname{Ker} P_{l-1} A + \operatorname{lin}(\{\mathbf{e}_1, \dots, \mathbf{e}_{m-3}\}),$$

$$\mathbf{a} \notin \operatorname{Im} A + \left(\operatorname{Ker} A^3 \cap \operatorname{Ker} P_k A^2 \cap \operatorname{Ker} P_{l-1} A \cap \operatorname{lin}(\{\mathbf{e}_1, \dots, \mathbf{e}_{m-3}\}\right).$$

In [9] the validity of this criterion is tested for the case where $A = J_3$ itself, considering the $(\mathbf{e}_3) \oplus (\mathbf{e}_3)$-extension of $(J_3) \oplus (J_3)$: if the corresponding decomposition is $\{1, \dots, 6\} = \{a, b, c\} \dot\cup \{k, l, m\}$ (with $a < b < c$, $k < l < m$), then one can assume $a = 1$ because of the symmetry. It turns out that the $(\mathbf{e}_3) \oplus (\mathbf{e}_3)$-extension can only be irreducible if $l > b$ or $c > m$; using $k \geqslant 2$, one derives from the above test that one must have $m - 2 \leqslant 3$ or $\operatorname{Ker} P_{l-1} J_3 \subseteq \operatorname{span}(\{\mathbf{e}_1, \mathbf{e}_2\})$, implying $l \leqslant 3$; as $l \geqslant 3$, $m \geqslant 4$, this means that $l = 3$ or $m \leqslant 5$; clearly, $m \leqslant 5$ means $c = 6 > m$, and $l = 3$ means $b \geqslant 4 > l$. The test thus leads to the same result. Applying the test with $k = 1$ leads to the condition $m = 6 > c$ or $l \geqslant 4 > b$, which is the same result with the two index sets interchanged. The conclusion is that $(J_3) \oplus (J_3)$ has an irreducible extension in five configurations: Fixing $a = 1$, one can have $k = 2$ and either $b = 4$, $c = 5$ or 6 or $b = 5$, $c = 6$, whereas $k = 2$, $b = 3$, $c = 6$ and $k = 3$, $b = 2$, $c = 6$ are also possible.

5. CONCLUSIONS

We have outlined a way of constructing irreducible matrices in $\mathscr{U}_{n+1}^0$ as extensions of matrices $A \in \mathscr{U}_n^0$. Already for irreducible A there may be several, even infinitely many, nonsimilar irreducible extensions. If A is reducible, then the number of possibilities also seems to increase with the sizes of the irreducible blocks: if $\cdots((0)) \oplus ((0))\cdots$ appears in the reduction of A, then no irreducible extension is possible, whereas $\cdots(J_2) \oplus (J_2)\cdots$ can appear only in one way. This implies, e.g., that

$$A = (J_2) \oplus (J_2) \oplus \cdots \oplus (J_2),$$

a generalized sum of n copies of J_2, has an irreducible extension only if the associated decomposition is $\{1, 2n\} \dot\cup \{2, 2n - 1\} \dot\cup \cdots \dot\cup \{n, n + 1\}$. However, both $\cdots(J_2) \oplus (J_3)\cdots$ and $\cdots(J_3) \oplus (J_3)\cdots$ can appear in five different ways, and $\cdots(J_2) \oplus (J_{3,1})\cdots$ allows for no less than ten variants.

We conclude this report with a listing of all irreducible matrices in $\mathscr{U}_n^0$ for $n \leqslant 6$. In the previous sections many of these matrices have been described.

$n \leqslant 3$: J_n.

$n = 4$: $J_4, J_{3,1}$.

$n = 5$: (a) Direct extensions of irreducible matrices in $\mathscr{U}_4^0$:
J_5 (of J_4);
B_1, B_2 [of $J_{3,1}$; cf. Example 3.3(ii)].
(b) Direct extensions of reducible matrices in $\mathscr{U}_4^0$:
$J_{4,1}$ [of $J_3 \oplus (0)$; cf. Example 4.3(i)];
$C_{3,2}$ [of $(J_2) \oplus (J_2)$; cf. Example 3.7(i)].

$n = 6$: (a) Direct extensions of irreducible matrices in $\mathscr{U}_5^0$:
J_6 (of J_5);
$B_{1,1}$ and $B_{1,2}$ [of B_1; cf. Example 3.3(iii)];
$B_{2,\infty}$ and $B_{2,z}$, $z \in \mathbb{C}$ [of B_2; cf. Example 3.3(iii)];
the $\mathbf{e}_5$-extension of $J_{4,1}$ (partial multiplicities 5, 1);
the $\mathbf{e}_4$-extension of $J_{4,1}$ (partial multiplicities 4, 2);
the $\mathbf{e}_4$-extension and the $\mathbf{e}_5$-extension of $C_{3,2}$ [cf. Example 4.4(iii)].
(b) Direct extensions of reducible matrices in $\mathscr{U}_5^0$:
(i) $\mathbf{a}$-(0)-extensions of matrices in $\mathscr{U}_4^0$:
$J_{5,1}$, the $\mathbf{e}_5$-0-extension of J_4;
$J_{((3,2,1))}$, the $\mathbf{e}_3$-$(0)_3$-extension of $J_{3,1}$ [cf. Example 4.3(ii)];
$J_{4,1,1}$, the $\mathbf{e}_4$-0-extensions of $J_{3,1}$ [cf. Example 4.3(ii)];
$J_{3,2,1}$ [cf. Example 3.11(ii) and Example 4.3(iii)];
(ii) Direct extensions of $(J_2) \oplus (J_3)$: The five matrices described in Example 3.7(ii) are the representatives of this class.

Of course, many examples from $\mathscr{U}_7^0$ have been presented above. At the same time it is clear that a complete analysis of the case $n = 7$ is much more complicated than the case $n = 6$.

In a paper by A. Vera-López and J. M. Arriga [11] one can find a complete listing of all conjugacy (i.e., equivalence) classes up to order 5 in the case of matrices over a field with p^k elements, p a prime number.

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