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On the number of minima of a random polynomial☆

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Abstract

We give the upper bound $\sqrt{2}(d-1)^{(n+1)/2}$ for the expected number of critical points of a normal random polynomial with degree at most d and n variables. Using the large deviation principle for the spectral value of large random matrices we obtain the bound

$$K \exp\left(-n^2 \frac{\ln 3}{4} + \frac{n+1}{2} \ln(d-1)\right)$$

for the expected number of minima of such a polynomial (here K is a positive constant). This proves that most normal random polynomials of fixed degree have only saddle points. Finally, we give a closed form expression for the expected number of maxima (resp. minima) of a random univariate polynomial, in terms of hypergeometric functions.

Keywords: Random polynomials; Number of minima; Number of maxima; Critical points

1. Introduction

We consider a random polynomial f over the reals with $n \geqslant 1$ variables and degree $d \geqslant 2$. The problem is to compute, on the average, the number of its critical points (the number of real roots of the system $Df(x) = 0$), and the number of its local minima. Since a generic polynomial has

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only nondegenerate stationary points, this last number is also given by the real roots of the system $Df(x) = 0$ such that $D^2 f(x)$ is positive definite. This reduces our problem to the computation of the number of real roots of a polynomial system under certain constraints.

Generally speaking, let $F = (F_1, \ldots, F_n)$ be a random system of real polynomial equations with n variables and degree $F_i \leqslant d_i$. Let $N^F(U)$ denote the number of zeros of the system $F(x) = 0$ lying in the subset $U \subset \mathbb{R}^n$ and $N^F(\mathbb{R}^n) = N^F$. Little is known on the distribution of the random variable $N^F(U)$. A classical result in the case of one polynomial of one variable is given by Kac [10,11], who gives the asymptotic value

$$E(N^F) \approx \frac{2}{\pi} \ln d$$

as d tends to infinity when the coefficients of F are Gaussian centered independent random variables with variances equal to 1. But, when the variance of the ith coefficient is equal to $\binom{d}{i}$ (Weyl's distribution), we have (see [5,7])

$$E(N^F) = \sqrt{d}.$$

In 1992, Shub and Smale extended this result to a real polynomial system F where

$$F_i(x_1, \ldots, x_n) = \sum_{\alpha_1 + \cdots + \alpha_n \leqslant d_i} a_{i,\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

when the coefficients $a_{i,\alpha}$ are Gaussian centered independent random variables with variances equal to

$$\binom{d_i}{\alpha} = \frac{d_i!}{\alpha_1! \ldots \alpha_n! (d_i - |\alpha|)!},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-integer and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ (see [12] on this distribution and its properties). Their result is

$$E(N^F) = \sqrt{d_1 \cdots d_n}$$

that is the square root of the Bézout number of the system.

A general formula for the expected value of $N^F(U)$ when the random functions F_i, $1 \leqslant i \leqslant n$, are stochastly independent and their law is centered and invariant under the isometries of $\mathbb{R}^n$ can be found in Azaïs–Wschebor [3]. This includes the Shub–Smale formula as a special case.

This result has also been extended by Rojas [15] to multi-homogeneous polynomial systems, and then partially by Malajovich and Rojas [13] to sparse polynomial systems.

Wschebor in [18] studies the moments of N^F and Armentano–Wschebor [2] consider random systems of equations of the type $P_i(x) + X_i(x)$, $1 \leqslant i \leqslant n$, $x \in \mathbb{R}^n$, where the P_is are non-random polynomials (the signal) and the X_is are independent Gaussian random variables (the noise).

Notice a major difference between these studies and the case considered here: the n equations of the system $Df(x) = 0$ are not independent!

Through this paper we denote by $\mathcal{P} = \mathcal{P}_{d,n}$ the space of degree at most d, n-variate polynomials with real coefficients. This space is endowed with the inner product:

$$\langle f, g \rangle_{\mathcal{P}} = \sum_{|\alpha| \leqslant d} \binom{d}{\alpha}^{-1} f_\alpha g_\alpha$$

with

$$f(x) = \sum_{|\alpha| \leqslant d} f_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \sum_{|\alpha| \leqslant d} f_\alpha x^\alpha.$$

We make $\mathcal{P}$ a probability space in considering the probability measure

$$\frac{1}{\sqrt{2\pi}^{\dim \mathcal{P}}} e^{-\|f\|_{\mathcal{P}}^2/2}\, d\mathcal{P} = \frac{1}{\sqrt{2\pi}^{\dim \mathcal{P}}} e^{-\|f\|_{\mathcal{P}}^2/2} \bigwedge_{|\alpha| \leqslant d} \binom{d}{\alpha}^{-1/2} df_\alpha,$$

i.e. a random polynomial has here Gaussian centered independent random coefficients with variances equal to $\binom{d}{\alpha}$.

Let $\mathcal{S}_n$ be the space of $n \times n$ real symmetric matrices, endowed with the Frobenius inner product $\langle R, S\rangle = \mathrm{Trace}(R^T S)$ and its induced norm

$$\|S\|^2 = \sum_{1 \leqslant i,j \leqslant n} S_{ij}^2.$$

For a matrix $S \in \mathcal{S}_n$ we write "$S > 0$" when S is positive definite and we denote the cone of such matrices by $\mathcal{S}_n^{++}$.

The *Gaussian Orthogonal Ensemble* is the space $\mathcal{S}_n$ together with the probability measure

$$\frac{e^{-\|S\|^2/2}}{(2\pi)^{n(n+1)/4}}\, dS = \frac{e^{-\|S\|^2/2}}{2^{n/2}\pi^{n(n+1)/4}} \bigwedge_{1 \leqslant i \leqslant j \leqslant n} dS_{ij}.$$

Thus, the entries of a matrix in $\mathcal{S}_n$ are independent Gaussian random variables with mean 0 and variance 1 for a diagonal entry, and mean 0 and variance $\frac{1}{2}$ for a non-diagonal entry.

Our first main result is the following:

Theorem 1. *Let $C_{d,n}$ denote the expected number of critical points of a random polynomial of degree at most d in n variables, and $E_{d,n}$ the expected number of minima. Let P_n be the probability that a matrix in the Gaussian Orthogonal Ensemble is positive definite. Then, for every $n \geqslant 2$,*

$$C_{2,n} = 1 \quad \textit{and} \quad E_{2,n} = P_n,$$

and for $d \geqslant 3$

$$C_{d,n} \leqslant \sqrt{2}(d-1)^{(n+1)/2} \quad \textit{and} \quad E_{d,n} \leqslant \sqrt{2}(d-1)^{(n+1)/2} P_n.$$

When $n = 1$ one has

$$C_{d,1} = 2E_{d,1} = \frac{2\sqrt{d-1}}{\pi} \int_0^\infty \frac{\sqrt{d(d-1)r^4 + 2dr^2 + 2}}{(dr^2+1)(r^2+1)} dr \leqslant 1 + \sqrt{d-2}.$$

Moreover, when $d \to \infty$,

$$\frac{C_{d,1}}{1 + \sqrt{d-2}} \to 1.$$

Let P_n be the probability that a matrix in the Gaussian Orthogonal Ensemble $GOE(n)$ is positive definite:

$$P_n = \int_{\mathcal{S}_n{}^{++}} \frac{e^{-\|S\|^2/2}}{2^{n/2}\pi^{n(n+1)/4}} \bigwedge_{1 \leqslant i \leqslant j \leqslant n} dS_{ij}.$$

Via the change of variable $S = Q\Lambda Q^T$ with $Q \in \mathbb{O}_n$ and $\Lambda = \mathrm{diag}(\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0)$ one has

$$P_n = \frac{\mathrm{Vol}\mathbb{O}_\mathrm{n}}{2^n} \int_{\mathbb{R}^n_>} \prod_{i<j} (\lambda_i - \lambda_j) \frac{e^{-\|\lambda\|^2/2}}{(2\pi)^{n(n+1)/4}} \, d\lambda,$$

where $\lambda \in \mathbb{R}^n_>$ if and only if $\lambda_1 > \cdots > \lambda_n > 0$ and

$$\mathrm{Vol}\mathbb{O}_n = \frac{2^{n(n+3)/4}\Gamma(1/2)^{n(n+1)/2}}{\prod_{j=1}^n \Gamma((n-j+1)/2)}$$

(see [14] for the description of P_n as an integral over $\mathbb{R}^n$ and [8] for the volume of the orthogonal group). The following values are easy to obtain:

$$P_1 = \frac{1}{2}, \quad P_2 = \frac{2-\sqrt{2}}{4}, \quad P_3 = \frac{\pi - 2\sqrt{2}}{4\pi}.$$

P_3 was computed by Carlos Beltrán.

Using the large deviation principle for the spectral value of large random matrices Dean and Majumdar give in [6] the asymptotic value of P_n for large values of n (see also [9] on that subject)

$$P_n \sim \exp\left(-n^2 \frac{\ln 3}{4}\right).$$

Thus, there exists a positive constant C such that, for every $n \geqslant 1$,

$$P_n \leqslant C \exp\left(-n^2 \frac{\ln 3}{4}\right).$$

This gives our second main theorem:

Theorem 2. *There exists a positive constant K such that for every n and d the number of minima of a random polynomial satisfies*

$$E_{d,n} \leqslant K \exp\left(-n^2 \frac{\ln 3}{4} + \frac{n+1}{2} \ln(d-1)\right).$$

Remark 1. This is a quite surprising result: it shows that most of random polynomials of reasonable degree have only saddle points. Thus, in general, the solution of a polynomial programming problem will be found on the boundary of the feasible set and not in its interior.

2. The space of n-variate polynomials

The inner product space $\mathcal{P}$, $\langle \cdot, \cdot \rangle_{\mathcal{P}}$ has several interesting properties resumed in the following:

Lemma 1. 1. *It admits the reproducing kernel* $K(z, x) = (1 + \langle z, x\rangle)^d$:

$$f(x) = \langle K(., x), f\rangle_{\mathcal{P}} \tag{1}$$

for any $x \in \mathbb{R}^n$ *and* $f \in \mathcal{P}$.

2. *It has a representation formula for the derivatives: for any integer* $k \geqslant 1$ *and* $x, u_1, \dots, u_k \in \mathbb{R}^n$ *we have*

$$D^k f(x)(u_1, \dots, u_k) = \langle K_k(., x, u_1, \dots, u_k), f\rangle_{\mathcal{P}} \tag{2}$$

with

$$\begin{aligned} K_k(z, x, u_1, \dots, u_k) &= D_x^k K(z, x)(u_1, \dots, u_k) \\ &= d \cdots (d - k + 1)\langle z, u_1\rangle \cdots \langle z, u_k\rangle \, (1 + \langle z, x\rangle)^{d-k}. \end{aligned} \tag{3}$$

3. *This scalar product is orthogonally invariant*:

$$\langle f \circ U, g \circ U\rangle_{\mathcal{P}} = \langle f, g\rangle_{\mathcal{P}} \tag{4}$$

for any $f, g \in \mathcal{P}$ *and the orthogonal transformation* $U \in \mathbb{O}_n$.

Proof. The first two formulas are well known and easily obtained via a direct computation. For the orthogonal invariance see [4, Section 12.1], or [12]. □

A second interest of Weyl's distribution for polynomials is due to the following identity: let $f(x) = x^T Sx$ (here S is a symmetric $n \times n$ matrix) be a homogeneous degree 2 polynomial, then $\|f\|_{\mathcal{P}} = \|S\|$. This is the reason why

Proposition 1. $C_{2,n} = 1$ *and* $E_{2,n} = P_n$.

Proof. Since a generic degree 2 polynomial has only one critical point we have $C_{2,n} = 1$. Given $f \in \mathcal{P}_{2,n}$ we can write it

$$f(x) = \alpha + \sum_{1\leqslant i\leqslant n} b_i x_i + \sum_{1\leqslant i\leqslant n} a_{ii} x_i^2 + \sum_{1\leqslant i<j\leqslant n} a_{ij} x_i x_j.$$

One has

$$\|f\|_{\mathcal{P}}^2 = \alpha^2 + \frac{1}{2}\sum_{1\leqslant i\leqslant n} b_i^2 + \sum_{1\leqslant i\leqslant n} a_{ii}^2 + \frac{1}{2}\sum_{1\leqslant i<j\leqslant n} a_{ij}^2$$

so that

$$\begin{aligned} E_{2,n} &= \int_{D^2 f(0)>0} \frac{e^{-\|f\|_{\mathcal{P}}^2/2}}{2^{n(n+1)/4}(2\pi)^{(n+1)(n+2)/4}}\, d\alpha\, db\, da \\ &= \int_{D^2 f(0)>0} \frac{e^{-\left(\sum_i a_{ii}^2 + \frac{1}{2}\sum_{i<j} a_{ij}^2\right)/2}}{2^{n(n-1)/4}(2\pi)^{n(n+1)/4}}\, da. \end{aligned}$$

To compute this last integral we let $S = \frac{1}{2}D^2 f(0)$; this gives

$$E_{2,n} = \int_{S>0} \frac{e^{-\|S\|^2/2}}{(2\pi)^{n(n+1)/4}}\, dS = P_n. \qquad \square$$

3. An integral formulation

Let us define

$$\mathrm{eval}_1 : \mathcal{P} \times \mathbb{R}^n \to \mathbb{R}^n, \quad \mathrm{eval}_1(f, x) = Df(x).$$

The incidence variety for real critical points of a polynomial is defined by

$$V = \left\{(f, x) \in \mathcal{P} \times \mathbb{R}^n \;:\; \mathrm{eval}_1(f, x) = 0\right\}.$$

The derivative of eval_1 is given by

$$D\mathrm{eval}_1(f, x)(\dot{f}, \dot{x}) = D\dot{f}(x) + D^2 f(x)\dot{x}$$

for any $f, \dot{f} \in \mathcal{P}$ and $x, \dot{x} \in \mathbb{R}^n$. Since this derivative is onto, V is a submanifold and its dimension is

$$\dim V = \dim \mathcal{P} = \binom{n+d}{d}.$$

The tangent space at $(f, x) \in V$ is given by

$$T_{(f,x)}V = \ker D\mathrm{eval}_1(f, x) = \left\{(\dot{f}, \dot{x}) \in \mathcal{P} \times \mathbb{R}^n : D\dot{f}(x) + D^2 f(x)\dot{x} = 0\right\}.$$

The restriction $\pi_2 : V \to \mathbb{R}^n$ of the projection $\mathcal{P} \times \mathbb{R}^n \to \mathbb{R}^n$ is surjective and is also a regular map because for any $(f, x) \in V$ the derivative $D\pi_2(f, x) : T_{(f,x)}V \to \mathbb{R}^n$ is surjective. The fiber of π_2 above $x \in \mathbb{R}^n$

$$V_x = \left\{(f, x) \in \mathcal{P} \times \mathbb{R}^n : \mathrm{eval}_1(f, x) = 0\right\}$$

is isomorphic to a $\dim \mathcal{P} - n$ linear space. V_x is equipped with the volume form inherited from the induced metric.

The restriction $\pi_1 : V \to \mathcal{P}$ of the projection $\mathcal{P} \times \mathbb{R}^n \to \mathcal{P}$ is a smooth map. A given $f \in \mathcal{P}$ is a regular value of π_1 when either f has no critical point or when, for any x such that $(f, x) \in V$, $D\pi_1(f, x) : T_{(f,x)}V \to \mathcal{P}$ is surjective. This last condition is satisfied when the second derivative $D^2 f(x)$ is an isomorphism which is the generic situation:

$$\Sigma' = \left\{(f, x) \in V \;:\; \det D^2 f(x) = 0\right\}$$

is an algebraic variety in V and $\dim \Sigma' < \dim V$. Thus Σ' and its image $\Sigma = \pi_1(\Sigma')$ have zero measure and we may ignore them. For any $(f, x) \in V \setminus \Sigma'$ and any $\dot{f} \in \mathcal{P}$ we have $D\pi_1(f, x)(\dot{f}, \dot{x}) = \dot{f}$ for $\dot{x} = -D^2 f(x)^{-1} D\dot{f}(x)$ and the fiber above f

$$V_f = \left\{(f, x) \in \mathcal{P} \times \mathbb{R}^n \;:\; \mathrm{eval}_1(f, x) = 0\right\}$$

consists of a finite number of points.

Given $(f, x) \in V \setminus \Sigma'$ we are in the context of the implicit function theorem that is V is locally around (f, x) the graph of the function

$$G = \pi_2 \circ \pi_1^{-1},$$

where π_1^{-1} is the local inverse of π_1 such that $\pi_1^{-1}(f) = (f, x)$. Since the graph of $DG(f)$ is the tangent space $T_{(f,x)}V$ we get

$$DG(f)\dot{f} = -D^2 f(x)^{-1} D\dot{f}(x) \tag{5}$$

for any $\dot{f} \in \mathcal{P}$.

Like in [4, Section 13.2, Theorem 3], we have the following:

Proposition 2. *Let U be a measurable subset of V. Let us denote by* $\#(f, U)$ *the number of pairs* $(f, x) \in U$ *and by* E_U *the expectation of* $\#(f, U)$ *when f is taken at random*:

$$E_U = \int_{\mathcal{P}} \#(f, U) \frac{e^{-\|f\|^2_{\mathcal{P}}/2}}{(2\pi)^{\dim \mathcal{P}/2}} \, \mathrm{d}\mathcal{P}. \tag{6}$$

With these notations, one has

$$E_U = \int_{\mathbb{R}^n} \mathrm{d}x \int_{V_x \cap U} \det(DG(f)DG(f)^*)^{-1/2} \frac{e^{-\|f\|^2_{\mathcal{P}}/2}}{(2\pi)^{\dim \mathcal{P}/2}} \, \mathrm{d}V_x. \tag{7}$$

Remark 2. In our context two sets are of particular interest: $U = V$ to compute the average number of critical points of a polynomial $C_{d,n}$, and $U = V_+$ with

$$V_+ = \left\{(f, x) \in \mathcal{P} \times \mathbb{R}^n : Df(x) = 0 \text{ and } D^2 f(x) > 0\right\}$$

for the average number of local minima $E_{d,n}$.

We have now to compute the determinant appearing in Eq. (7). This is done in the following:

Proposition 3. *Under the notations above*

$$\det(DG(f)DG(f)^*) = d^n (1 + \|x\|^2)^{n(d-1)-1} (1 + d\,\|x\|^2) \left|\det D^2 f(x)\right|^{-2}. \tag{8}$$

Proof. Let us denote $D\dot{f}(x) = D_x \dot{f}$. Since $DG(f)\dot{f} = -D^2 f(x)^{-1} D_x \dot{f}$ and since $D^2 f(x)$ is symmetric, we get

$$DG(f)DG(f)^* = D^2 f(x)^{-1} D_x D_x^* D^2 f(x)^{-1}$$

so that

$$\det(DG(f)DG(f)^*) = \det(D_x D_x^*) \left|\det D^2 f(x)\right|^{-2}. \tag{9}$$

To compute $\det(D_x D_x^*)$ we use the representation formula for the derivative (Eq. (2)) with $k = 1$. Let us denote by e_i, $1 \leqslant i \leqslant n$, the canonical basis in $\mathbb{R}^n$. Then, for any $\dot{f} \in \mathcal{P}$,

$$D_x \dot{f} = \sum_i e_i \left\langle K_1(., x, e_i), \dot{f}\right\rangle_{\mathcal{P}}$$

so that, with $\dot{x} \in \mathbb{R}^n$, $\dot{x} = \sum_i \dot{x}_i e_i$,

$$\langle D_x^* \dot{x}, \dot{f} \rangle_{\mathcal{P}} = \langle \dot{x}, D_x \dot{f} \rangle = \left\langle \dot{x}, \sum_i e_i \langle K_1(., x, e_i), \dot{f} \rangle_{\mathcal{P}} \right\rangle = \sum_i \dot{x}_i \langle K_1(., x, e_i), \dot{f} \rangle_{\mathcal{P}}.$$

Thus, we get

$$D_x^* \dot{x} = \sum_i \dot{x}_i K_1(., x, e_i)$$

and consequently

$$\begin{aligned} D_x D_x^* \dot{x} &= \sum_i e_i \left\langle K_1(., x, e_i), \sum_j \dot{x}_j K_1(., x, e_j) \right\rangle_{\mathcal{P}} \\ &= \sum_i e_i \frac{\partial}{\partial z_i} \left(\sum_j \dot{x}_j d \langle z, e_j \rangle (1 + \langle z, x \rangle)^{d-1} \right) \Bigg|_{z=x} \\ &= \sum_{i,j} e_i \dot{x}_j \times \begin{cases} d(d-1) x_i x_j (1 + \|x\|^2)^{d-2} & \text{if } i \neq j, \\ d(d-1) x_i^2 (1 + \|x\|^2)^{d-2} + d(1 + \|x\|^2)^{d-1} & \text{if } i = j, \end{cases} \end{aligned}$$

which correspond to the matrix

$$d(d-1)(1 + \|x\|^2)^{d-2} x x^T + d(1 + \|x\|^2)^{d-1} I_n.$$

Its eigenvectors are x and any nonzero vector in the orthogonal subspace $x^\perp$. The corresponding eigenvalues are

$$d(d-1)(1 + \|x\|^2)^{d-2} \|x\|^2 + d(1 + \|x\|^2)^{d-1} = d(1 + \|x\|^2)^{d-2}(1 + d\|x\|^2)$$

with multiplicity 1, and

$$d(1 + \|x\|^2)^{d-1}$$

with multiplicity $n - 1$ so that

$$\det D_x D_x^* = d^n (1 + \|x\|^2)^{n(d-1)-1} (1 + d\|x\|^2).$$

Our proposition combines this value and Eq. (9). □

If we combine Propositions 2 and 3 we obtain the following integral formulation:

Proposition 4. *Let U be a measurable subset of V. One has*

$$E_U = \int_{\mathbb{R}^n} dx \int_{V_x \cap U} \frac{\left|\det D^2 f(x)\right|}{d^{n/2}(1 + \|x\|^2)^{(n(d-1)-1)/2}(1 + d\|x\|^2)^{1/2}} \frac{e^{-\|f\|_{\mathcal{P}}^2/2}}{(2\pi)^{\dim \mathcal{P}/2}} \mathrm{d}V_x. \quad (10)$$

An action of the orthogonal group $\mathbb{O}_n$ on $\mathcal{P} \times \mathbb{R}^n$ is defined by

$$(Q, f, x) \in \mathbb{O}_n \times \mathcal{P} \times \mathbb{R}^n \to (f \circ Q, Q^T x) \in \mathcal{P} \times \mathbb{R}^n.$$

This action leaves the incidence variety V invariant and also the scalar product $\langle ., . \rangle_{\mathcal{P}}$ (Lemma 1). For this reason, when the measurable set U is itself invariant, the integral on $V_x \cap U$ in Proposition 4 only depends on $r = \|x\|$. Thus, taking spherical coordinates in $\mathbb{R}^n$, we get:

Proposition 5. *Let U be a measurable subset of V invariant under the action of $\mathbb{O}_n$ (for any $(Q, f, x) \in \mathbb{O}_n \times U$ we have $(f \circ Q, Q^T x) \in U$). Under this condition*

$$E_U = \frac{\alpha_n}{d^{n/2}} \int_0^\infty \frac{r^{n-1}\,\mathrm{d}r}{R^{(d-1)n-1}(dr^2+1)^{1/2}} \int_{V_{re_1}\cap U} |\det D^2 f(re_1)| \frac{e^{-\|f\|^2_{\mathcal{P}}/2}}{(2\pi)^{\dim V_{re_1}/2}}\,\mathrm{d}V_{re_1},$$

where $\alpha_n = \frac{\mathrm{Vol}\, S^{n-1}}{(2\pi)^{n/2}} = \frac{2}{2^{n/2}\Gamma(n/2)}$, $R = \sqrt{r^2+1}$ and $re_1^T = (r, 0, \ldots, 0)$.

Remark 3. The measurable sets considered here: $U = V$ and $U = V_+ = \{(f, x) \in V \ : \ D^2 f(x) > 0\}$, are clearly invariant under the action of $\mathbb{O}_n$.

4. The inner integral

Our objective is now to compute the integral over $V_{re_1} \cap U$ appearing in Proposition 5.

Let $D^2 : V_{re_1} \to \mathcal{S}_n$ denote the operator $f \mapsto D^2 f(re_1)$. We would like to compute its pseudo-inverse $\Psi : \mathcal{S}_n \to (\ker D^2)^\perp$. This means that Ψ is the minimum norm right inverse of D^2 ($D^2 \circ \Psi = \mathrm{id}_{\mathcal{S}_n}$).

This will allow us to "integrate out" $\ker D^2$:

$$\begin{aligned}\int_{V_{re_1}\cap U} &|\det D^2 f| \frac{e^{-\|f\|^2_{\mathcal{P}}/2}}{(2\pi)^{\dim V_{re_1}/2}}\,\mathrm{d}V_{re_1} \\ &= \int_{D^2(V_{re_1}\cap U)} |\det S| \left|\det \Psi^*\Psi\right|^{1/2} \frac{e^{-\|\Psi(S)\|^2_{\mathcal{P}}/2}}{(2\pi)^{\dim \mathcal{S}_n/2}}\,\mathrm{d}S. \qquad (11)\end{aligned}$$

To compute $\Psi(S)$ and $\left|\det \Psi^*\Psi\right|$ we need the following lemma:

Lemma 2. *Let us denote*

- e_i, $1 \leqslant i \leqslant n$, *the canonical basis in* $\mathbb{R}^n$,
- $\partial_{e_i} = K_1(z, re_1, e_i)$,
- $\partial_{e_i e_j} = K_2(z, re_1, e_i, e_j)$,
- $R = \sqrt{1+r^2}$.

Then,

1. $\langle \partial_{e_1}, \partial_{e_1} \rangle_{\mathcal{P}} = d(1+dr^2)R^{2d-4}$.
2. *If* $i \neq 1$, *then* $\langle \partial_{e_i}, \partial_{e_i} \rangle_{\mathcal{P}} = dR^{2d-2}$.
3. *If* $i \neq j$, *then* $\langle \partial_{e_i}, \partial_{e_j} \rangle_{\mathcal{P}} = 0$.
4. $\langle \partial_{e_1}, \partial_{e_1 e_1} \rangle_{\mathcal{P}} = d(d-1)(dr^2+2)rR^{2d-6}$.
5. *If* $(i, j, k) \neq (1, 1, 1)$, *then* $\langle \partial_{e_j}, \partial_{e_i e_k} \rangle_{\mathcal{P}} = 0$.
6. $\langle \partial_{e_1 e_1}, \partial_{e_1 e_1} \rangle_{\mathcal{P}} = d(d-1)\left(d(d-1)r^4 + 4(d-1)r^2 + 2\right) R^{2d-8}$.
7. *If* $k \neq 1$, *then* $\langle \partial_{e_1 e_k}, \partial_{e_1 e_k} \rangle_{\mathcal{P}} = d(d-1)((d-1)r^2+1)R^{2d-6}$.

8. *If* $i \neq 1$ *and* $k \neq 1$, *then* $\langle \partial_{e_i e_k}, \partial_{e_i e_k} \rangle_{\mathcal{P}} = (1 + \delta_{ik})\, d(d-1)R^{2d-4}$ (δ_{ik} *is the Kronecker symbol*).
9. *If* $\{i, k\} \neq \{j, l\}$, *then* $\langle \partial_{e_i e_k}, \partial_{e_j e_l} \rangle_{\mathcal{P}} = 0$.

Proof. It is a consequence of the representation formulas given in Lemma 1:

- $\langle \partial_{e_1}, \partial_{e_1} \rangle_{\mathcal{P}} = \langle K_1(., re_1, e_1), K_1(., re_1, e_1) \rangle_{\mathcal{P}} = \frac{\partial}{\partial z_1} K_1(z, re_1, e_1) \big|_{z=re_1} = \frac{\partial}{\partial z_1} dz_1(1 + rz_1)^{d-1} \big|_{z=re_1} = d(1+r^2)^{d-2}(1+dr^2)$,

and similarly

- $\langle \partial_{e_i}, \partial_{e_i} \rangle_{\mathcal{P}} = \frac{\partial}{\partial z_i} K_1(z, re_1, e_i) \big|_{z=re_1} = \frac{\partial}{\partial z_i} dz_i(1+rz_1)^{d-1} \big|_{z=re_1} = d(1+r^2)^{d-1}$,
- $\langle \partial_{e_i}, \partial_{e_j} \rangle_{\mathcal{P}} = \frac{\partial}{\partial z_i} K_1(z, re_1, e_j) \big|_{z=re_1} = \frac{\partial}{\partial z_i} dz_j(1+rz_1)^{d-1} \big|_{z=re_1} = 0$ when $i \neq j$,
- $\langle \partial_{e_1}, \partial_{e_1 e_1} \rangle_{\mathcal{P}} = \frac{\partial}{\partial z_1} K_2(z, re_1, e_1, e_1) \big|_{z=re_1} = \frac{\partial}{\partial z_1} d(d-1) z_1^2 (1+rz_1)^{d-2} \big|_{z=re_1} = d(d-1)r(2+dr^2)(1+r^2)^{d-3}$,
- $\langle \partial_{e_j}, \partial_{e_i e_k} \rangle_{\mathcal{P}} = \frac{\partial}{\partial z_j} d(d-1) z_i z_k (1+rz_1)^{d-2} \big|_{z=re_1} = 0$ when $(i, j, k) \neq (1, 1, 1)$,
- $\langle \partial_{e_1 e_1}, \partial_{e_1 e_1} \rangle_{\mathcal{P}} = \frac{\partial^2}{\partial z_1^2} d(d-1) z_1^2 (1+rz_1)^{d-2} \big|_{z=re_1} = d(d-1)(1+r^2)^{d-4}(2+4(d-1)r^2 + d(d-1)r^4)$,
- $\langle \partial_{e_1 e_k}, \partial_{e_1 e_k} \rangle_{\mathcal{P}} = \frac{\partial^2}{\partial z_1 z_k} d(d-1) z_1 z_k (1+rz_1)^{d-2} \big|_{z=re_1} = d(d-1)(1+r^2)^{d-3}(1+(d-1)r^2)$,
- $\langle \partial_{e_i e_k}, \partial_{e_i e_k} \rangle_{\mathcal{P}} = \frac{\partial^2}{\partial z_i z_k} d(d-1) z_i z_k (1+rz_1)^{d-2} \big|_{z=re_1} = (1+\delta_{ik}) d(d-1)(1+r^2)^{d-2}$,
- $\langle \partial_{e_i e_k}, \partial_{e_j e_l} \rangle_{\mathcal{P}} = \frac{\partial^2}{\partial z_i z_k} d(d-1) z_j z_l (1+rz_1)^{d-2} \big|_{z=re_1} = 0$ when $\{i, k\} \neq \{j, l\}$. $\square$

Let us now evaluate Ψ. Recall that

$$V_{re_1} = \{ f \in \mathcal{P} \ : \ Df(re_1) = 0 \}$$

or, in other words, $f \in V_{re_1}$ if and only if

$$\langle f, \partial_{e_i} \rangle_{\mathcal{P}} = 0, \ 1 \leqslant i \leqslant n.$$

Thus, by Lemmas 2 and 3, ∂_{e_i}, $1 \leqslant i \leqslant n$, constitute an orthogonal basis of $V_{re_1}^{\perp}$. We also have

$$\ker D^2 = \operatorname{Span} \left\{ \partial_{e_i e_j}, \ 1 \leqslant i \leqslant j \leqslant n \right\}^{\perp} \cap V_{re_1},$$

hence

$$(\ker D^2)^{\perp} = \operatorname{Span} \left\{ P \partial_{e_i e_j}, \ 1 \leqslant i \leqslant j \leqslant n \right\},$$

where P stands for the orthogonal projection onto V_{re_1}. We have seen that for $(i, j, k) \neq (1, 1, 1)$, $\partial e_i e_j \perp \partial e_k$ (Lemmas 2–5). Hence,

$$P \partial_{e_1 e_1} = \partial_{e_1 e_1} - \partial_{e_1} \frac{\langle \partial_{e_1 e_1}, \partial_{e_1} \rangle_{\mathcal{P}}}{\| \partial e_1 \|_{\mathcal{P}}^2}$$

and for $(i, j) \neq (1, 1)$,

$$P\partial_{e_i e_j} = \partial_{e_i e_j}.$$

Let us now show that

$$\Psi(S) = \sum_{1 \leqslant i \leqslant j \leqslant n} S_{ij} \frac{P\partial_{e_i e_j}}{\|P\partial_{e_i e_j}\|_{\mathcal{P}}^2}.$$

Since this expression is clear in $(\ker D^2)^\perp$ it suffices to prove that $D^2 \circ \Psi(S) = S$ for any $S \in \mathcal{S}_n$, i.e.

$$D^2\Psi(S)(re_1)(e_k, e_l) = S_{kl}$$

or, using Lemma 1, that

$$\left\langle \partial_{e_k e_l}, \sum_{1 \leqslant i \leqslant j \leqslant n} S_{ij} \frac{P\partial_{e_i e_j}}{\|P\partial_{e_i e_j}\|_{\mathcal{P}}^2} \right\rangle_{\mathcal{P}} = S_{kl}.$$

This last equality holds because $P\partial_{e_i e_j}$, $1 \leqslant i \leqslant j \leqslant n$, is an orthogonal basis of $(\ker D^2)^\perp$.

It is important to have in mind that Ψ is not an isometry, we have

$$\|\Psi(S)\|_{\mathcal{P}}^2 = \sum_{1 \leqslant i \leqslant j \leqslant n} \frac{S_{ij}^2}{\|P\partial_{e_i e_j}\|_{\mathcal{P}}^2}.$$

We introduce now the functions

$$A(d, r) = \sqrt{\frac{d(d-1)r^4 + 2dr^2 + 2}{(dr^2 + 1)R^4}}$$

and

$$B(d, r) = \sqrt{\frac{(d-1)r^2 + 1}{R^2}},$$

where again $R = \sqrt{1 + r^2}$.

Lemma 3. *Let $i \leqslant j$. Then,*

$$\|P\partial_{e_i e_j}\|_{\mathcal{P}}^2 = d(d-1)R^{2d-4} \times \begin{cases} A(d, r)^2 & \text{if } i = 1 \text{ and } j = 1 \\ B(d, r)^2 & \text{if } i = 1 \text{ and } j \neq 1 \\ (1 + \delta_{ij}) & \text{if } i \neq 1 \text{ and } j \neq 1 \end{cases}$$

with $\delta_{ij} = 1$ when $i = j$ and 0 otherwise.

Let us now compute $\det \Psi^*\Psi$. For any $f = \sum_{1 \leqslant i \leqslant j \leqslant n} f_{ij} P\partial_{e_i e_j} \in (\ker D^2)^\perp$ and for any $S \in \mathcal{S}_n$ we have

$$\langle \Psi^*(f), S \rangle = \langle f, \Psi(S) \rangle_{\mathcal{P}} = \sum_{1 \leqslant i \leqslant j \leqslant n} f_{ij} S_{ij}.$$

Therefore, we have always for any $T \in \mathcal{S}_n$:

$$\langle \Psi^*\Psi(T), S\rangle = \sum_{1 \leqslant i \leqslant j \leqslant n} \frac{T_{ij} S_{ij}}{\| P\partial_{e_i e_j} \|_{\mathcal{P}}^2}.$$

We write the matrix of the operator $\Psi^*\Psi$ with respect to the orthonormal basis of $\mathcal{S}$ given by $\mathrm{e}_1\mathrm{e}_1^T, \ldots, \mathrm{e}_n\mathrm{e}_n^T$ and then, for $i < j$, $\frac{1}{\sqrt{2}}\left(\mathrm{e}_i\mathrm{e}_j^T + \mathrm{e}_j\mathrm{e}_i^T\right)$:

$$\begin{bmatrix} \frac{1}{\|P\partial_{e_1e_1}\|^2} & & & & & \\ & \ddots & & & & \\ & & \frac{1}{\|P\partial_{e_ne_n}\|^2} & & & \\ & & & \frac{1}{2\|P\partial_{e_1e_2}\|^2} & & \\ & & & & \ddots & \\ & & & & & \frac{1}{2\|P\partial_{e_{n-1}e_n}\|^2} \end{bmatrix}.$$

Using Lemma 3 we obtain:

Lemma 4.

$$(\det \Psi^*\Psi)^{1/2} = 2^{-\frac{(n+2)(n-1)}{4}} \left(d(d-1)R^{2d-4}\right)^{-\frac{n(n+1)}{4}} A(d,r)^{-1} B(d,r)^{-(n-1)}.$$

At this point

Proposition 6. *Under the conditions above,*

$$E_U = \frac{\alpha_n}{d^{n/2}} \int_0^\infty \frac{\left(\det \Psi^*\Psi\right)^{\frac{1}{2}} r^{n-1}\, dr}{(dr^2+1)^{1/2} R^{(d-1)n-1}} \int_{D^2(U \cap V_{re_1})} \frac{|\det S|}{(2\pi)^{\dim \mathcal{S}_n/2}} e^{-\|\Psi(S)\|_{\mathcal{P}}^2/2}\, \mathrm{d}\mathcal{S}_n.$$

In particular,

$$C_{d,n} = \frac{\alpha_n}{d^{n/2}} \int_0^\infty \frac{\left(\det \Psi^*\Psi\right)^{\frac{1}{2}} r^{n-1}\, dr}{(dr^2+1)^{1/2} R^{(d-1)n-1}} \int_{\mathcal{S}_n} \frac{|\det S|}{(2\pi)^{\dim \mathcal{S}_n/2}} e^{-\|\Psi(S)\|_{\mathcal{P}}^2/2}\, \mathrm{d}\mathcal{S}_n$$

and

$$E_{d,n} = \frac{\alpha_n}{d^{n/2}} \int_0^\infty \frac{\left(\det \Psi^*\Psi\right)^{\frac{1}{2}} r^{n-1}\, dr}{(dr^2+1)^{1/2} R^{(d-1)n-1}} \int_{\mathcal{S}_n{}^{++}} \frac{\det S}{(2\pi)^{\dim \mathcal{S}_n/2}} e^{-\|\Psi(S)\|_{\mathcal{P}}^2/2}\, \mathrm{d}\mathcal{S}_n,$$

where $\mathcal{S}_n{}^{++}$ denotes the set of positive definite matrices. When $n = 1$,

$$C_{d,1} = 2E_{d,1} = \frac{2\sqrt{d-1}}{\pi} \int_0^\infty \frac{\sqrt{d(d-1)r^4 + 2dr^2 + 2}}{(dr^2+1)(r^2+1)}\, \mathrm{d}r.$$

Proof. The three first formulas are obtained by combining Proposition 5, Eq. (11) and Lemma 4. For the case $n = 1$ we obtain

$$E_{d,1} = \frac{2}{d\sqrt{d-1}\sqrt{2\pi}} \int_0^\infty \frac{dr}{A(dr^2+1)^{1/2} R^{2d-4}} \int_0^\infty \frac{s}{\sqrt{2\pi}} e^{-\frac{s^2}{2d(d-1)R^{2d-4}A^2}}\, \mathrm{d}s$$
$$= \frac{\sqrt{d-1}}{\pi} \int_0^\infty \frac{\sqrt{d(d-1)r^4 + 2dr^2 + 2}}{(dr^2+1)(r^2+1)}\, \mathrm{d}r.$$

The identity $C_{d,1} = 2E_{d,1}$ is easy. $\square$

5. Some integral lemmas

The term $e^{-\|\Psi(S)\|^2_{\mathcal{P}}/2}$ in the inner integrals of Proposition 6 can be simplified through additional changes of coordinates. We reparametrize the spaces $\mathcal{S}_n$ and $\mathcal{S}_n^{++}$ through a stretching $S \mapsto T = \Delta^{-1} S \Delta^{-1}$.

The stretching coefficients are $\Delta_i = \left(2d(d-1)R^{2d-4}\right)^{1/4}$ for $i \geqslant 2$, $\Delta_1 = B(d,r)\Delta_2$ and $\Delta = \mathrm{Diag}(\Delta_1, \Delta_2, \ldots, \Delta_n)$. We obtain

$$\|\Psi(S)\|^2_{\mathcal{P}} = \frac{1}{d(d-1)R^{2d-4}} \left(\frac{S_{11}^2}{A^2} + \sum_{j=2}^n \frac{S_{1j}^2}{B^2} + \sum_{1<i\leqslant j\leqslant n} \frac{1}{1+\delta_{ij}} S_{ij}^2 \right)$$

and

$$\|\Delta^{-1} S \Delta^{-1}\|^2 = \frac{1}{d(d-1)R^{2d-4}} \left(\frac{S_{11}^2}{2B^4} + \sum_{j=2}^n \frac{S_{1j}^2}{B^2} + \sum_{1<i\leqslant j\leqslant n} \frac{1}{1+\delta_{ij}} S_{ij}^2 \right)$$

so that

$$\|\Psi(S)\|^2_{\mathcal{P}} = \|\Delta^{-1} S \Delta^{-1}\|^2 + \left(\frac{1}{A^2} - \frac{1}{2B^4} \right) \frac{S_{11}^2}{d(d-1)R^{2d-4}}.$$

Let us define $T = \Delta^{-1} S \Delta^{-1}$. We get

$$\|\Psi(S)\|^2_{\mathcal{P}} = \|T\|^2 + \left(\frac{2B^4}{A^2} - 1 \right) T_{11}^2$$

so that, via this change of variable,

$$\int_{D^2(U\cap V_{re_1})} \frac{|\det S|}{\sqrt{2\pi}^{\dim \mathcal{S}_n}} e^{-\|\Psi(S)\|^2/2}\, \mathrm{d}S$$
$$= \left(\prod_{i=1}^n \Delta_i \right)^{n+3} \int_{\Delta^{-1} D^2(U\cap V_{re_1})\Delta^{-1}} \frac{|\det T|}{\sqrt{2\pi}^{\dim \mathcal{S}_n}} e^{-\frac{1}{2}\left(\|T\|^2 + \left(\frac{2B(d,r)^4}{A(d,r)^2} - 1 \right) T_{11}^2 \right)}\, \mathrm{d}T.$$

If $U \subset V$, we define the auxiliary quantity

$$C_U(d,r,n) = \int_{\Delta^{-1} D^2(U\cap V_{re_1})\Delta^{-1}} \frac{|\det T|}{\sqrt{2\pi}^{\dim \mathcal{S}_n}} e^{-\frac{1}{2}\left(\|T\|^2 + \left(\frac{2B(d,r)^4}{A(d,r)^2} - 1 \right) T_{11}^2 \right)}\, \mathrm{d}T.$$

There are two cases of interest corresponding to $U = V$ for the average of critical points and $U = V_+$ for the average number of local minima. The corresponding functions are denoted as $C_V(d, r, n)$ and $C_{V_+}(d, r, n)$. Using Proposition 6 we get (the proof is easy and left to the reader)

Proposition 7.

$$E_U = \frac{2\sqrt{2}(d-1)^{n/2}}{\Gamma(n/2)} \int_0^\infty \frac{((d-1)r^2+1)^2}{R^2\sqrt{d(d-1)r^4+2dr^2+2}} \frac{r^{n-1}}{R^{n-1}} C_U(d, r, n)\, \mathrm{d}r.$$

Moreover

$$C_V(d, r, n) = \int_{\mathcal{S}_n} \frac{|\det T|}{\sqrt{2\pi}^{\dim \mathcal{S}_n}} e^{-\frac{1}{2}\left(\|T\|^2 + \left(\frac{2B(d,r)^4}{A(d,r)^2} - 1\right) T_{11}^2\right)}\, \mathrm{d}T$$

and

$$C_{V_+}(d, r, n) = \int_{\mathcal{S}_n{}^{++}} \frac{\det T}{\sqrt{2\pi}^{\dim \mathcal{S}_n}} e^{-\frac{1}{2}\left(\|T\|^2 + \left(\frac{2B(d,r)^4}{A(d,r)^2} - 1\right) T_{11}^2\right)}\, \mathrm{d}T.$$

6. Proof of Theorem 1

To prove our main theorem we use both Proposition 7 and the case $d = 2$ already investigated in Proposition 1. We have

$$1 = C_{2,n} = \frac{2\sqrt{2}}{\Gamma(n/2)} \int_0^\infty \frac{r^{n-1}}{R^{n-1}} \frac{C_V(2, r, n)}{\sqrt{2}}\, \mathrm{d}r$$

and

$$C_V(2, r, n) = \int_{\mathcal{S}_n} \frac{|\det T|}{(2\pi)^{n(n+1)/4}} e^{-\frac{1}{2}\left(\|T\|^2 + 2r^2 T_{11}^2\right)}\, \mathrm{d}T.$$

Lemma 5. *The quantity* $\Lambda(d, r) = \frac{2B(d,r)^4}{A(d,r)^2} - 1$ *satisfies, for all* $r > 0$ *and* $d \geqslant 2$, *the scaling law*:

$$\Lambda(2, r\sqrt{d-1}) \leqslant \Lambda(d, r) \leqslant \Lambda\left(2, \frac{\sqrt{5}}{2} r\sqrt{d-1}\right).$$

Proof. We write

$$\Lambda(d, r) = 2(d-1)r^2 + \frac{d-2}{d} \frac{(d-1)r^4}{(d-1)r^4 + 2r^2 + \frac{2}{d}}.$$

The lower bound is now obvious. The upper bound is obtained as follows:

$$\begin{aligned}\Lambda(d, r) &= 2(d-1)r^2 + \frac{d-2}{d} \frac{(d-1)r^4}{(d-1)r^4 + 2r^2 + \frac{2}{d}} \\ &\leqslant 2(d-1)r^2 + \frac{d-2}{2d}(d-1)r^2\end{aligned}$$

$$\leqslant \frac{5}{4}\Lambda(2, r^2\sqrt{d-1})$$
$$= \Lambda\left(2, \frac{\sqrt{5}}{2}r^2\sqrt{d-1}\right). \qquad \square$$

It follows from Lemma 5 that

$$\begin{aligned} C_V(d,r,n) &= \int_{S_n} \frac{|\det T|}{\sqrt{2\pi}^{\dim S_n}} e^{-\frac{1}{2}\left(\|T\|^2+\Lambda(d,r)T_{11}^2\right)} \\ &\leqslant \int_{S_n} \frac{|\det T|}{\sqrt{2\pi}^{\dim S_n}} e^{-\frac{1}{2}\left(\|T\|^2+\Lambda(2,r\sqrt{d-1})T_{11}^2\right)} \\ &= C_V(2, r\sqrt{d-1}, n) \end{aligned}$$

and similarly $C_{V+}(d,r,n) \leqslant C_{V+}(2, r\sqrt{d-1}, n)$. Now we have:

$$\begin{aligned} C_{d,n} &= \frac{2\sqrt{2}(d-1)^{n/2}}{\Gamma(n/2)} \int_0^\infty \frac{((d-1)r^2+1)^2}{R^2\sqrt{d(d-1)r^4+2dr^2+2}} \frac{r^{n-1}}{R^{n-1}} C_V(d,r,n)\,dr \\ &\leqslant \frac{2\sqrt{2}(d-1)^{n/2}}{\Gamma(n/2)} \int_0^\infty \frac{((d-1)r^2+1)^2}{R^2\sqrt{d(d-1)r^4+2dr^2+2}} \frac{r^{n-1}}{R^{n-1}} C_V(2, r\sqrt{d-1}, n)\,dr. \end{aligned}$$

We set $s = r\sqrt{d-1}$ and $S = \sqrt{d-1+s^2}$ to obtain

$$\begin{aligned} C_{d,n} &\leqslant \frac{2\sqrt{2}(d-1)^{n/2}}{\Gamma(n/2)} \int_0^\infty \frac{(d-1)(s^2+1)^2}{S^2\sqrt{\frac{d}{d-1}s^4+2\frac{d}{d-1}s^2+2}} \frac{s^{n-1}}{S^{n-1}}\, C_V(2,s,n)\, \frac{ds}{\sqrt{d-1}} \\ &= \frac{2\sqrt{2}}{\Gamma(n/2)} \int_0^\infty \frac{s^{n-1}}{(1+s^2)^{(n-1)/2}} \frac{C_V(2,s,n)}{\sqrt{2}} A\,ds \end{aligned}$$

with

$$A = (d-1)^{n/2} \frac{(d-1)(s^2+1)^2}{S^2\sqrt{\frac{d}{d-1}s^4+2\frac{d}{d-1}s^2+2}} \frac{(1+s^2)^{(n-1)/2}}{S^{n-1}} \frac{\sqrt{2}}{\sqrt{d-1}}.$$

Since

$$\frac{s^2+1}{S^2} \leqslant 1 \quad \text{and} \quad \frac{(s^2+1)^2}{S^2\sqrt{\frac{d}{d-1}s^4+2\frac{d}{d-1}s^2+2}} \leqslant 1,$$

we obtain $A \leqslant \sqrt{2}(d-1)^{(n+1)/2}$ so that

$$C_{d,n} \leqslant \sqrt{2}(d-1)^{(n+1)/2} C_{2,n} = \sqrt{2}(d-1)^{(n+1)/2}.$$

The same argument holds for $E_{d,n}$ and we are done.

7. The Riemann surface

We rewrite the case $n = 1$ (Proposition 6) for convenience as

$$E_{d,1} = \frac{(d-1)\sqrt{d}}{2\pi} \int_{\mathbb{R}} g(z)\,\mathrm{d}z \tag{12}$$

with

$$g(z) = \frac{\sqrt{z^4 + \frac{2}{d-1}z^2 + \frac{2}{d(d-1)}}}{(1+z^2)(1+dz^2)}.$$

At this point we encounter a classical situation: we want to compute a line integral of a function $g(z)$, which is a two-branched meromorphic function of $\mathbb{C}$. In order to apply the residue theorem, we need first to replace g by a regular meromorphic function, defined in the relevant Riemann surface R. The branching points of the Riemann surface are the roots of the polynomial inside the square root. If we set

$$\zeta = \sqrt{\frac{-1 + i\sqrt{1 - \frac{2}{d}}}{d-1}}$$

with the branch of the external square root in such a way that ζ belongs to the positive quadrant, we can now factorize

$$z^4 + \frac{2}{d-1}z^2 + \frac{2}{d(d-1)} = (z-\zeta)(z-\bar{\zeta})(z+\zeta)(z+\bar{\zeta}).$$

It follows that the Riemann surface R is a twofold cover of $\mathbb{C}$ with branch points $\zeta, -\bar{\zeta}, -\zeta, \bar{\zeta}$.

Let γ be the arc of circle (centered in the origin) joining $-\bar{\zeta}$ to ζ crossing the positive imaginary axis. Notice that it crosses the segment $[i/\sqrt{d}, i]$. Let $\mathcal{D}$ denote the upper half plane with γ removed.

Then, the positive branch of $\sqrt{z^4 + \frac{2}{d-1}z^2 + \frac{2}{d(d-1)}}$ on $\mathbb{R}$ extends to a unique branch on $\mathcal{D}$. The square root is real and positive on $[0, i|\zeta|]$ and real and negative on $[i|\zeta|, i\infty)$.

The residue theorem is now:

$$\int_{\mathbb{R}} g(z)\,\mathrm{d}z - 2\int_{\gamma} g(z)\,\mathrm{d}z 2\pi i \mathrm{Res}_{[z=i/\sqrt{d}]} g(z) + 2\pi i \mathrm{Res}_{[z=i]} g(z).$$

Residues are respectively $\frac{-i}{2(d-1)\sqrt{d}}$ and $\frac{-i\sqrt{d-2}}{2(d-1)\sqrt{d}}$. Therefore,

$$E_{d,1} = \frac{1}{2} + \frac{\sqrt{d-2}}{2} + \frac{(d-1)\sqrt{d}}{\pi}\int_{\gamma} g(z)\,\mathrm{d}z. \tag{13}$$

(we mean the integral of the branch that is positive on $i|\zeta|$).

Now, in order to integrate $g(z)$, we introduce a linear fractional transformation mapping the real line onto the circle containing γ (Fig. 1). Namely,

$$\Psi(w) = \frac{Aw + B}{Cw + D}$$

with $A = |\zeta|$, $B = i|\zeta|$, $C = i$, $D = 1$. For the record, $AD - BC = 2|\zeta|$.

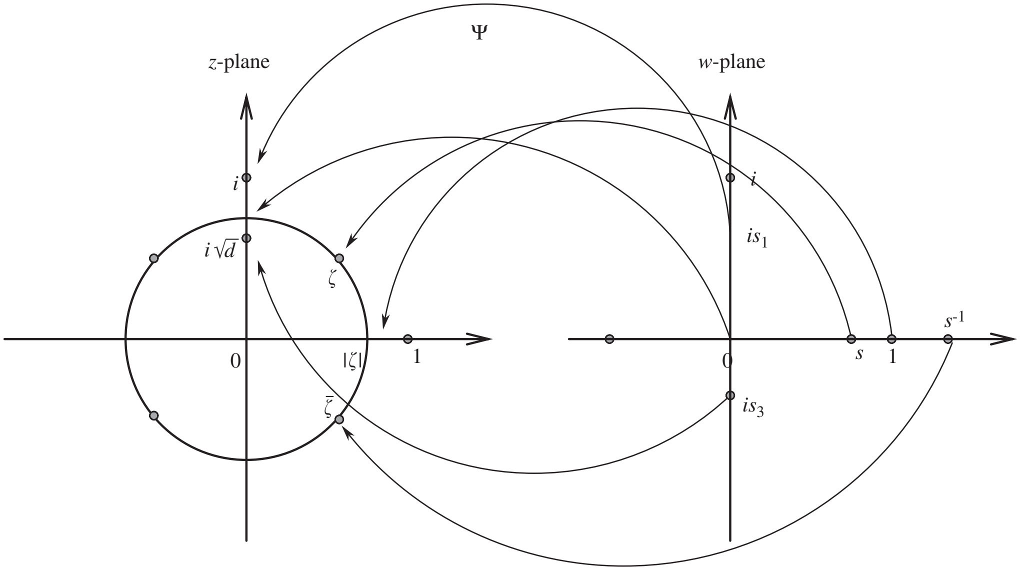

Fig. 1. The linear fractional map $w \mapsto z = \Psi(w)$.

Let $s = \frac{\mathrm{Re}(\zeta)}{|\zeta|+\mathrm{Im}(\zeta)}$. Define also $s_1 = \frac{1-|\zeta|}{1+|\zeta|}$, $s_2 = s_1^{-1}$, $s_3 = \frac{1-|\zeta|\sqrt{d}}{1+|\zeta|\sqrt{d}}$ and $s_4 = s_3^{-1}$. We have the following mapping table for Ψ:

w	$\Psi(w)$	w	$\Psi(w)$
-1	$-\lvert\zeta\rvert$	is_1	i
0	$i\lvert\zeta\rvert$	is_2	$-i$
1	$\lvert\zeta\rvert$	is_3	$i/\sqrt{d}$
$-s^{-1}$	$-\zeta$	is_4	$-i/\sqrt{d}$
$-s$	$-\bar{\zeta}$		
s	ζ		
s^{-1}	$\bar{\zeta}$		

Changing coordinates,

$$\int_\gamma g(z)\,\mathrm{d}z = 2c(d)\mathrm{Re}\int_{[0,s]} \frac{\sqrt{(w^2-s^2)(w^2-s^{-2})}}{\prod_{k=1}^4 (w-is_k)}\,\mathrm{d}w$$

with

$$c(d) = \frac{(AD-BC)\sqrt{A^4 + \frac{2}{d-1}A^2C^2 + \frac{2}{d(d-1)}C^4}}{(A^2+C^2)(dA^2+C^2)} \in O(d^{-3/2})$$

(more precisely: $\lim d^{3/2}C(d) = -2^{7/4}\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}-1} \simeq -6.2151$).

Table 1
Residues and arguments

Pole	Residue R_k	Argument n_k
is_1	$i\dfrac{(s_1-s^2)(s_1-s^{-2})}{(s_1-s_2)(s_1-s_3)(s_1-s_4)}$	$-\dfrac{s^2}{s_1^2}$
is_2	$i\dfrac{(s_2-s^2)(s_2-s^{-2})}{(s_2-s_1)(s_2-s_3)(s_2-s_4)}$	$-\dfrac{s^2}{s_2^2}$
is_3	$i\dfrac{(s_3-s^2)(s_3-s^{-2})}{(s_3-s_1)(s_3-s_2)(s_3-s_4)}$	$-\dfrac{s^2}{s_3^2}$
is_4	$i\dfrac{(s_4-s^2)(s_4-s^{-2})}{(s_4-s_1)(s_4-s_2)(s_4-s_3)}$	$-\dfrac{s^2}{s_4^2}$

At this point, good practice seems to be:

1. Multiply numerator and denominator by the conjugate of the denominator, in order to obtain a real polynomial in the denominator.
2. Multiply numerator and denominator by the square root.
3. Expand in partial fractions.
4. Put into Legendre normal form.
5. Write down the integral in terms of elliptic functions K and Π.

We expand the integrand in partial fractions:

$$\begin{aligned}
\int_\gamma g(z)\,\mathrm{d}z &= 2c(d)\int_{[0,s]} \frac{1}{\sqrt{(w^2-s^2)(w^2-s^{-2})}}\left(1+\sum_{k=1}^{4}\mathrm{Re}\frac{R_k}{w-is_k}\right)\mathrm{d}w \\
&= 2c(d)\int_{[0,s]} \frac{1}{\sqrt{(w^2-s^2)(w^2-s^{-2})}}\left(1+\sum_{k=1}^{4}\frac{s_k^{-2}\mathrm{Re}\,(R_k(w+is_k))}{1+w^2s_k^{-2}}\right)\mathrm{d}w \\
&= 2c(d)\int_{[0,s]} \frac{1}{\sqrt{(w^2-s^2)(w^2-s^{-2})}}\left(1+\sum_{k=1}^{4}\frac{s_k^{-1}R_k i}{1+w^2s_k^{-2}}\right)\mathrm{d}w
\end{aligned}$$

(the last step uses the fact that all residues R_k are pure imaginary). Residues are given in Table 1. We use formula [1, 17.4.45] to compute the *parameter* $m=s^4$. Then we set $\sin\alpha=s^2$ above, and also $w=s\sin\theta$ to obtain the Legendre normal form:

$$\int_\gamma g(z)\,\mathrm{d}z = 2c(d)s\int_0^{\pi/2}\frac{1}{\sqrt{1-\sin^2\alpha\,\sin^2\theta}}\left(1+\sum_{k=1}^{4}\frac{R_ks_ki}{1-n_k\sin^2\theta}\right)\mathrm{d}\theta.$$

This is a combination of one complete elliptic integral of the first kind and four complete elliptic integrals of the third kind. The *arguments* $n_k=-s^2s_k^{-2}$ of the integrals of the third kind are given in Table 1.

Therefore,

$$\int_\gamma g(z)\,\mathrm{d}z = 2c(d)\left(K(m) + \sum_{k=1}^{4} R_k s_k i\,\Pi(n_k; m)\right),$$

where K and Π denote the complete elliptic integrals of the first and third kinds, respectively.

ASYMPTOTICS: $s \to \sqrt{2}-1$, so $m \to 0.029437251$, $\alpha \to 0.172425997$ rad $\simeq 9°52'45.42''$. Also, $s_1, s_2 \to 1$ and $s_3 = s_4^{-1} \Longrightarrow (1-\sqrt{2})/(1+\sqrt{2})$.

EXPERIMENTAL DATA: The hypergeometric functions were evaluated using Romberg iteration. Coefficients and residues were obtained symbolically and then numerically. Digits are not guaranteed to be all significative.

d	$E_U - \frac{1}{2} - \frac{\sqrt{d-2}}{2}$
3	−0.280134
4	−0.319279
5	−0.337448
6	−0.348064
7	−0.355053
8	−0.360010
9	−0.363712
10	−0.366583
10^2	−0.387335
10^3	−0.389199
10^4	−0.389384
10^5	−0.389402
10^6	−0.389404
10^7	−0.389405
10^8	−0.389405
10^9	−0.389405

Remark 4. Rybowicz [16] provided the following alternative formula for $C_{d,1} = 2E_{d,1}$:

$$\begin{aligned} C_{d,1} = &- \frac{4d(u-2)}{\sqrt{u}(u-1)(u-d)\pi} K(v) \\ &+ \frac{u+1}{\sqrt{u}(u-1)\pi}\Pi\left(-\frac{(u-1)^2}{4u}, v\right) \\ &+ \frac{(2-d)(u+d)}{\sqrt{u}(d-u)\pi}\Pi\left(-\frac{(d-u)^2}{4du}, v\right), \end{aligned}$$

where

$$u = \sqrt{\frac{2d}{d-1}} \quad \text{and} \quad v = \frac{\sqrt{2-u}}{2}.$$

His formula agrees with ours up to six decimal places.

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