Completions and compactifications of quasi-uniform spaces

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Abstract

A *-compactification of a $T_1$ quasi-uniform space $(X, \mathcal{U})$ is a compact $T_1$ quasi-uniform space $(Y, \mathcal{V})$ that has a $T(\mathcal{V}^*)$-dense subspace quasi-isomorphic to $(X, \mathcal{U})$, where $\mathcal{V}^*$ denotes the coarsest uniformity finer than $\mathcal{V}$.

With the help of the notion of $T_1$ *-half completion of a quasi-uniform space, which is introduced and studied here, we show that if a $T_1$ quasi-uniform space $(X, \mathcal{U})$ has a *-compactification, then it is unique up to quasi-isomorphism. We identify the *-compactification of $(X, \mathcal{U})$ with the subspace of its bicompletion $(\tilde{X}, \tilde{T}(\tilde{\mathcal{U}}))$ consisting of all points which are closed in $(\tilde{X}, T(\tilde{\mathcal{U}}))$ and prove that $(X, \mathcal{U})$ is *-compactifiable if and only if it is point symmetric and $(\tilde{X}, \tilde{\mathcal{U}})$ is compact. Finally, we discuss some properties of locally fitting $T_0$ quasi-uniform spaces, a large class of quasi-uniform spaces whose bicompletion is $T_1$, and, hence, they are $T_1$ *-half completable. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction and preliminaries

Our basic reference for quasi-uniform spaces is [7].

Let us recall that if $\mathcal{U}$ is a quasi-uniformity on a set $X$, then $\mathcal{U}^{-1} = \{U^{-1}; U \in \mathcal{U}\}$ is also a quasi-uniformity on $X$ called the conjugate of $\mathcal{U}$. The uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$ will be denoted by $\mathcal{U}^*$. If $U \in \mathcal{U}$, the entourage $U \cap U^{-1}$ of $\mathcal{U}^*$ will be denoted by $U^*$.

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Each quasi-uniformity \( U \) on \( X \) induces a topology \( T(U) \) on \( X \), defined as follows:

\[
T(U) = \{ A \subseteq X : \text{ for each } x \in A \text{ there is } U \in U \text{ such that } U(x) \subseteq A \}.
\]

If \( (X, T) \) is a topological space and \( U \) is a quasi-uniformity on \( X \) such that \( T = T(U) \) we say that \( U \) is compatible with \( T \).

A quasi-uniform space \( (X, U) \) is totally bounded provided that the uniform space \( (X, U^*) \) is totally bounded. In this case we say that \( U \) is a totally bounded quasi-uniformity on \( X \).

A quasi-uniformity \( U \) on \( X \) is called point symmetric if \( T(U) \subseteq T(U^{-1}) \), and \( U \) is locally symmetric provided that for each \( x \in X \), \( \{U^{-1}(U(x)) : U \in U \} \) is a base for the \( T(U) \)-neighborhood filter of \( x \).

Clearly, every locally symmetric quasi-uniformity is point symmetric.

Let \( (X, U) \) be a quasi-uniform space. An entourage \( U \) of \( U \) is called transitive if \( U \circ U = U \).

A quasi-uniformity \( U \) on \( X \) is called transitive if it has a base of transitive entourages.

The Pervin quasi-uniformity of any \( T_1 \) topological space provides an example of a point symmetric totally bounded transitive quasi-uniformity (see [7]).

Let us recall that the Pervin quasi-uniformity of a topological space \( X \) is the quasi-uniformity \( \mathcal{P} \) on \( X \) which is generated by all sets of the form \( (G \times G) \cup ((X \setminus G) \times X) \), where \( G \) is open in \( X \).

Let \( (X, U) \) and \( (Y, V) \) be two quasi-uniform spaces. A mapping \( f : (X, U) \to (Y, V) \) is said to be quasi-uniformly continuous if for each \( V \in V \) there is \( U \in U \) such that \( (f(x), f(y)) \in V \) whenever \( (x, y) \in U \). A bijection \( f : (X, U) \to (Y, V) \) is called a quasi-isomorphism if \( f \) and \( f^{-1} \) are quasi-uniformly continuous. In this case we say that \( (X, U) \) and \( (Y, V) \) are quasi-isomorphic.

Following [8] (see also [7]), a compactification of a \( T_1 \) quasi-uniform space \( (X, U) \) is a compact \( T_1 \) quasi-uniform space \( (Y, V) \) that has a \( T(V) \)-dense subspace quasi-isomorphic to \( (X, U) \). It is proved in [8] that a totally bounded \( T_1 \) quasi-uniform space has a compactification if and only if it is point symmetric.

In this paper we introduce and study the notion of a \(*\)-compactification of a \( T_1 \) quasi-uniform space. In fact, while a point symmetric totally bounded \( T_1 \) quasi-uniform space may have many totally bounded compactifications (see [8, p. 34]), a \(*\)-compactifiable \( T_1 \) quasi-uniform space has an (up to quasi-isomorphism) unique \(*\)-compactification. This provides a motivation for our study and justifies, in part, the interest in constructing \(*\)-compactification(s) rather than compactifications of (totally bounded) \( T_1 \) quasi-uniform spaces.

Thus, in Section 2 we introduce the auxiliary notion of a \( T_1 \) \(*\)-half completion of a quasi-uniform space and show that if a quasi-uniform space \( (X, U) \) admits a \( T_1 \) \(*\)-half completion, then it is unique up to quasi-isomorphism. We also identify the \( T_1 \) \(*\)-half completion of a \( T_1 \) quasi-uniform space \( (X, U) \) with the subspace of the bicompletion \( (\check{X}, \check{U}) \) of \( (X, U) \) consisting of all points which are closed in \( (\check{X}, T(\check{U})) \).

Since every \(*\)-compactification is a \( T_1 \) \(*\)-half completion, it follows that if \( (X, U) \) has a
\[ \text{-compactification, then it is unique up to quasi-isomorphism. Some results on extension of quasi-uniformly continuous mappings are also established.} \]

In Section 3 we characterize those quasi-uniform spaces which are \text{-compactifiable}. This is done with the help of the notions of a \( T_1 \) \text{-right} \( K \)-completion and of a \( T_1 \) \text{-left} \( K \)-completion of a quasi-uniform space. In particular we prove that a \( T_1 \) quasi-uniform space is \text{-compactifiable} if and only if it is point symmetric and its bicompletion is compact. An internal characterization of such spaces is also obtained. Furthermore, and solving a question posed by Fletcher and Lindgren in [8], we characterize those point symmetric totally bounded \( T_1 \) quasi-uniform spaces whose \text{-compactification is Hausdorff.}

Finally, in Section 4, we discuss some properties of locally fitting \( T_0 \) quasi-uniform spaces, a large class of quasi-uniform spaces whose bicompletion is \( T_1 \), and hence they are \( T_1 \) \text{-half completable.}

2. \( T_1 \) \text{-half completion and \text{-compactification of quasi-uniform spaces}

A quasi-uniform space \((X, \mathcal{U})\) is called \textit{bicomplete} if each \( \mathcal{U}^* \)-Cauchy filter converges with respect to the topology \( T(\mathcal{U}^*) \), i.e., if the uniform space \((X, \mathcal{U}^*)\) is complete ([7,22]).

A \textit{bicompletion} of a quasi-uniform space \((X, \mathcal{U})\) is a bicomplete quasi-uniform space \((Y, \mathcal{V})\) that has a \( T(\mathcal{V}^*) \)-dense subspace quasi-isomorphic to \((X, \mathcal{U})\).

Recall that a \( \mathcal{U}^* \)-Cauchy filter on a quasi-uniform space \((X, \mathcal{U})\) is \textit{minimal} provided that it contains no \( \mathcal{U}^* \)-Cauchy filter other than itself.

\[ \text{Lemma 1} \ [7]. \text{Let } \mathcal{F} \text{ be a } \mathcal{U}^* \text{-Cauchy filter on a quasi-uniform space } (X, \mathcal{U}). \text{ Then there is exactly one minimal } \mathcal{U}^* \text{-Cauchy filter coarser than } \mathcal{F}. \text{ Furthermore, if } \mathcal{H} \text{ is any base for } \mathcal{F}, \text{ then } \{U(H): H \in \mathcal{H} \text{ and } U \text{ is a symmetric member of } \mathcal{U}^*\} \text{ is a base for the minimal } \mathcal{U}^* \text{-Cauchy filter coarser than } \mathcal{F}. \]

\[ \text{Lemma 2} \ [7]. \text{Let } (X, \mathcal{U}) \text{ be a } T_0 \text{ quasi-uniform space and let } \tilde{X} = \{\mathcal{F}: \mathcal{F} \text{ is a minimal } \mathcal{U}^* \text{-Cauchy filter on } X\}. \text{ For each } U \in \mathcal{U} \text{ let } \tilde{U} = \{(\mathcal{F}, G) \in \tilde{X} \times \tilde{X}: \text{ there exist } F \in \mathcal{F} \text{ and } G \in G \text{ such that } F \times G \subseteq U\}, \text{ and let } \tilde{\mathcal{U}} = \{\tilde{U}: U \in \mathcal{U}\}. \text{ Then the following statements hold:} \]

\begin{enumerate}
\item \((\tilde{X}, \tilde{\mathcal{U}})\) is a bicomplete \( T_0 \) quasi-uniform space and \((X, \mathcal{U})\) is quasi-uniformly embedded as a \( T(\tilde{\mathcal{U}}^*) \)-dense subspace of \((\tilde{X}, \tilde{\mathcal{U}})\) by the mapping \( i: X \to \tilde{X} \) such that, for each \( x \in X, i(x) \) is the \( T(\mathcal{U}^*) \)-neighborhood filter at \( x \). Furthermore \( \tilde{i}^{-1} = \tilde{\mathcal{U}}^{-1} \) and \( \tilde{i}^* = \tilde{\mathcal{U}}^* \) on \( \tilde{X} \).
\item Any \( T_0 \) bicompletion of \((X, \mathcal{U})\) is quasi-isomorphic to \((\tilde{X}, \tilde{\mathcal{U}})\).
\end{enumerate}

The bicomplete \( T_0 \) quasi-uniform space \((\tilde{X}, \tilde{\mathcal{U}})\) of the preceding lemma is called the \textit{bicompletion} of \((X, \mathcal{U})\) and we identify \( X \) with \( i(X) \) if no confusion arises. If \( \mathcal{U} \) is a uniformity on \( X, (\tilde{X}, \tilde{\mathcal{U}}) \) is the uniform completion of \((X, \mathcal{U})\).

A quasi-uniform space \((X, \mathcal{U})\) is said to be \textit{half complete} if every \( \mathcal{U}^* \)-Cauchy filter is \( T(\mathcal{U}) \)-convergent [1].
It is obvious that each bicomplete quasi-uniform space is half complete and it is well known that the converse is not true in general. Furthermore, both bicompleteness and half completeness coincide with the usual notion of completeness when \((X, \mathcal{U})\) is a uniform space.

Let \((X, \mathcal{U})\) be a quasi-uniform space. A \(T_1\) quasi-uniform space \((Y, \mathcal{V})\) is called a \(T_1\) half completion of \((X, \mathcal{U})\) if \((Y, \mathcal{V})\) is half complete and \((X, \mathcal{U})\) is quasi-isomorphic to a \(T(\mathcal{V})\)-dense subspace of \(Y\) [19].

A quasi-uniform space \((X, \mathcal{U})\) is said to be \(T_1\) half completable if it admits a \(T_1\) half completion.

Note that if \((X, \mathcal{U})\) is a \(T_1\) half completable quasi-uniform space, then the topological space \((X, T(\mathcal{U}))\) is \(T_1\).

The following strong form of \(T_1\) half completion will be the key of our main results.

**Definition 1.** Let \((X, \mathcal{U})\) be a quasi-uniform space. A \(T_1\) quasi-uniform space \((Y, \mathcal{V})\) is called a \(T_1\)*-half completion of \((X, \mathcal{U})\) if \((Y, \mathcal{V})\) is half complete and \((X, \mathcal{U})\) is quasi-isomorphic to a \(T(\mathcal{V}^*)\)-dense subspace of \((Y, \mathcal{V})\).

**Definition 2.** We say that a quasi-uniform space is \(T_1\)*-half completable if it has a \(T_1\)*-half completion.

Next we analyze some properties of \(T_1\)*-half completable quasi-uniform spaces. In particular, we shall prove that if a quasi-uniform space \((X, \mathcal{U})\) admits a \(T_1\)*-half completion \((Y, \mathcal{V})\), then it is unique (up to quasi-isomorphism). Furthermore, the space \((Y, \mathcal{V})\) is described as a subspace of the bicompletion of \((X, \mathcal{U})\) (see Theorem 1 below).

**Proposition 1.** If a quasi-uniform space \((X, \mathcal{U})\) has a \(T_1\)*-half completion \((Y, \mathcal{V})\), then the bicompletion of \((Y, \mathcal{V})\) is quasi-isomorphic to the bicompletion of \((X, \mathcal{U})\).

**Proof.** Let \((\tilde{Y}, \tilde{\mathcal{V}})\) be the bicompletion of \((Y, \mathcal{V})\). Clearly \((X, \mathcal{U})\) is quasi-isomorphic to a \(T(\tilde{\mathcal{V}}^*)\)-dense subspace of \((\tilde{Y}, \tilde{\mathcal{V}})\). Therefore \((\tilde{Y}, \tilde{\mathcal{V}})\) is a \(T_0\) bicompletion of \((X, \mathcal{U})\). So \((\tilde{Y}, \tilde{\mathcal{V}})\) is quasi-isomorphic to the bicompletion of \((X, \mathcal{U})\). \(\square\)

Let \((X, \mathcal{U})\) be a \(T_0\) quasi-uniform space and \((\tilde{X}, \tilde{\mathcal{U}})\) its bicompletion. We will denote by \(G(X)\) the set of closed points of \((\tilde{X}, T(\mathcal{U}))\).

Clearly \(G(X) = \tilde{X}\) whenever \((\tilde{X}, \tilde{\mathcal{U}})\) is a \(T_1\) quasi-uniform space.

The following result is proved in [19].

**Lemma 3.** A \(T_1\) quasi-uniform space \((X, \mathcal{U})\) is \(T_1\) half completable if and only if whenever \(\mathcal{F}\) is a \(U^*\)-Cauchy filter on \(X\) that \(T(U^{-1})\)-converges to a point \(x \in X\), then \(\mathcal{F}\) is \(T(U)\)-convergent to \(x\).
It immediately follows from Lemma 3 the well-known fact that every point symmetric $T_1$ quasi-uniform space is $T_1$ half completable.

**Proposition 2.** Let $(X, \mathcal{U})$ be a $T_1$ half completable quasi-uniform space. Then $X \subseteq G(X)$.

**Proof.** Let $x \in X$ and suppose that $x \notin G(X)$. Then there exists $y \in \tilde{X}\setminus\{x\}$ such that $x \notin U(y)$ for every $\tilde{U} \in \mathcal{U}$.

Since $X$ is $T(\tilde{U}^*)$-dense in $\tilde{X}$, there exists a filter base $\mathcal{F}$ on $X$ which is $T(\tilde{U}^*)$-convergent to $y$. Hence $\mathcal{F}$ is $T(\tilde{U}^{-1})$-convergent to $x$ and, by Lemma 3, $\mathcal{F}$ is $T(\tilde{U}^*)$-convergent to $x$. Since $(X, T(\mathcal{U}^*))$ is Hausdorff, it follows that $x = y$, a contradiction. We conclude that $X \subseteq G(X)$.

**Proposition 3.** Let $(X, \mathcal{U})$ be a half complete $T_1$ quasi-uniform space. Then $X = G(X)$.

**Proof.** It is clear that $(X, \mathcal{U})$ is $T_1$ half completable, and hence $X \subseteq G(X)$ by Proposition 2. Now let $y \in G(X)$, and let us prove that $y \in X$. Since $X$ is $T(\tilde{U}^*)$-dense in $\tilde{X}$, then there exists a filter base $\mathcal{F}$ on $X$ which is $T(\tilde{U}^*)$-convergent to $y$. Since $(X, \mathcal{U})$ is half complete, there exists $x \in X$ such that $\mathcal{F}$ is $T(\mathcal{U})$-convergent to $x$. Let $U \in \mathcal{U}$, then there exists $F \in \mathcal{F}$ such that $F \subseteq \tilde{U}^*(y) \cap U(x)$, and hence $\tilde{U}^*(y) \cap U(x) \neq \emptyset$ for each $U \in \mathcal{U}$.

It easily follows that $y \in \tilde{U}(x)$ for each $U \in \mathcal{U}$, and since $(G(X), T(\tilde{\mathcal{U}})| G(X))$ is a $T_1$ space, we deduce that $y = x$. Therefore $X = G(X)$.

**Example 1.** It is proved in [17] that the well-monotone quasi-uniformity of a topological space is left $K$-complete and thus half complete (see [14] or [17] for the notion of the well-monotone quasi-uniformity and Section 3 for the notion of a left $K$-complete quasi-uniformity). Since the well-monotone quasi-uniformity is transitive, it follows that both the fine quasi-uniformity and the fine transitive quasi-uniformity (see [7]) of a topological space are left $K$-complete. Hence, if $X$ is a $T_1$ topological space and $\mathcal{U}$ denotes the well-monotone (respectively, fine, fine transitive) quasi-uniformity of $X$, then $X = G(X)$ by Proposition 3.

In the sequel the quasi-uniform space $(G(X), \tilde{\mathcal{U}}| G(X) \times G(X))$, will be simply denoted by $(G(X), \tilde{\mathcal{U}}| G(X))$.

**Definition 3.** A *-compactification of a $T_1$ quasi-uniform space $(X, \mathcal{U})$ is a compact $T_1$ quasi-uniform space $(Y, \mathcal{V})$ that has a $T(\mathcal{V}^*)$-dense subspace quasi-isomorphic to $(X, \mathcal{U})$.

**Definition 4.** We say that a $T_1$ quasi-uniform space is *-compactifiable if it has a *-compactification.

Obviously, if $(X, \mathcal{U})$ has a *-compactification $(Y, \mathcal{V})$, then $(Y, \mathcal{V})$ is a $T_1$ *-half completion of $(X, \mathcal{U})$.

**Theorem 1.** Let $(X, \mathcal{U})$ be a $T_1$ *-half completable quasi-uniform space. Then any $T_1*$-half completion of $(X, \mathcal{U})$ is quasi-isomorphic to $(G(X), \tilde{\mathcal{U}}| G(X))$. Hence, it is unique.
up to quasi-isomorphism. Moreover if $U$ is a uniformity, $(G(X), \tilde{U} \mid G(X))$ is the uniform completion of $(X, U)$.

**Proof.** Let $(Y, V)$ be a $T_1$ *-half completion of $(X, U)$. By Proposition 1 it follows that $(\tilde{X}, \tilde{U})$ is quasi-isomorphic to $(\tilde{Y}, \tilde{V})$, and hence $(G(X), \tilde{U} \mid G(X))$ is quasi-isomorphic to $(G(Y), \tilde{V} \mid G(Y))$. Since $(Y, V)$ is half complete and $T_1$, it follows from Proposition 3 that $Y = G(Y)$, whence $(G(X), \tilde{U} \mid G(X))$ is quasi-isomorphic to $(Y, V)$.

Finally, if $U$ is a uniformity on $X$, $G(X) = \tilde{X}$, so $(G(X), \tilde{U} \mid G(X))$ is the uniform completion of $(X, U)$. ♦

**Corollary.** If a $T_1$ quasi-uniform space $(X, U)$ has a *-compactification, then any *-compactification of $(X, U)$ is quasi-isomorphic to $(G(X), \tilde{U} \mid G(X))$. Hence, it is unique up to quasi-isomorphism.

**Proof.** Let $(Y, V)$ be a *-compactification of $(X, U)$. Then $(Y, V)$ is a $T_1$ *-half completion of $(X, U)$. By Theorem 1 $(Y, V)$ is quasi-isomorphic to $(G(X), \tilde{U} \mid G(X))$, and thus it is unique up to quasi-isomorphism. ♦

**Remark 1.** It is interesting to recall that a Tychonoff point symmetric totally bounded quasi-uniform space may have many (totally bounded) compactifications (see [8, p. 34]), although the *-compactification is unique up to quasi-isomorphism, as we just proved.

In the following, if $(X, U)$ is $T_1$ *-complete, $(G(X), \tilde{U} \mid G(X))$ will be called the $T_1$ *-half completion of $(X, U)$, and if $(X, U)$ is *-compactifiable, $(G(X), \tilde{U} \mid G(X))$ will be called the *-compactification of $(X, U)$.

**Proposition 4.** If a totally bounded $T_1$ quasi-uniform space $(X, U)$ is $T_1$ *-half completable, then it is *-compactifiable.

**Proof.** Since by assumption $(X, U)$ is totally bounded, it follows that $(\tilde{X}, \tilde{U})$ is totally bounded and hence $(G(X), \tilde{U} \mid G(X))$ is also totally bounded. From the fact that every totally bounded half complete quasi-uniform space is compact we deduce that $(G(X), \tilde{U} \mid G(X))$ is compact and thus it is the *-compactification of $(X, U)$. Consequently $(X, U)$ is *-compactifiable. ♦

At the end of this section we deal with the following general question: If $f$ is a quasi-uniformly continuous mapping from a quasi-uniform space $(X, U)$ to a quasi-uniform space $(Y, V)$, when can it be extended to a quasi-uniformly continuous mapping from $(G(X), \tilde{U} \mid G(X))$ to $(G(Y), \tilde{V} \mid G(Y))$?

Although a complete answer to this question might be difficult in the light of the theory of Wallman compactification (see [9,10]), we have some partial results which will be stated below.

It is well known (see [7, Theorem 3.29]) that if $(X, U)$ is a $T_0$ quasi-uniform space and $f$ is a quasi-uniformly continuous mapping from $(X, U)$ to a bicomplete $T_0$ quasi-uniform
space \((Y, V)\), then \(f\) has a unique continuous extension \(F\) from \((\widetilde{X}, \mathcal{T}(\widetilde{U}^*))\) to \((Y, \mathcal{T}(V^*))\) and \(F\) is quasi-uniformly continuous.

From this result and the fact, proved in Proposition 2, that for a \(T_1\) half completable quasi-uniform space \((X, \mathcal{U})\), we have \(X \subseteq G(X)\) and thus \((\widetilde{X}, \widetilde{U})\) is the bicompletion of \((G(X), \widetilde{U} | G(X))\), we deduce the following

**Proposition 5.** Let \((X, \mathcal{U})\) be a \(T_1\) half completable quasi-uniform space, let \((Y, V)\) be a bicomplete \(T_0\) quasi-uniform space and let \(f : (X, \mathcal{U}) \to (Y, V)\) be a quasi-uniformly continuous mapping. Then \(f\) has a unique continuous extension \(F\) from \((G(X), \mathcal{T}(\widetilde{U}^*)) | G(X)\) to \((Y, \mathcal{T}(V^*))\) and \(F : (G(X), \widetilde{U} | G(X)) \to (Y, V)\) is quasi-uniformly continuous.

The following corollary is not a trivial consequence of Proposition 5 because there exist locally symmetric compact (and, hence, half complete) Hausdorff quasi-uniform spaces which are not bicomplete (see Example 2 below).

**Corollary.** Let \((X, \mathcal{U})\) be a \(T_1\) half completable quasi-uniform space, let \((Y, V)\) be a locally symmetric half complete \(T_0\) quasi-uniform space and let \(f : (X, \mathcal{U}) \to (Y, V)\) be a quasi-uniformly continuous mapping. Then \(f\) has a continuous extension \(F : (G(X), \mathcal{T}(\widetilde{U}) | G(X)) \to (Y, \mathcal{T}(V))\).

**Proof.** We may assume that \(f\) is a quasi-uniformly continuous mapping from \((X, \mathcal{U})\) to the bicompletion \((\widetilde{Y}, \widetilde{V})\) of \((Y, V)\). So, by Proposition 5, \(f\) has a quasi-uniformly continuous extension \(h : (G(X), \widetilde{U} | G(X)) \to (\widetilde{Y}, \widetilde{V})\). Now define \(r : \widetilde{Y} \to Y\) as the mapping which carries each minimal \(V^*\)-Cauchy filter \(F\) to its limit point in \((Y, \mathcal{T}(V))\). It is easily checked that \(r(\widetilde{V}(F)) \subseteq V^{-1}(V(r(F)))\) for all \(V \in \mathcal{V}\) and \(F \in \widetilde{F}\), so \(r\) is a continuous mapping from \((\widetilde{Y}, \mathcal{T}(\widetilde{V}))\) to \((Y, \mathcal{T}(V))\) by local symmetry of \((Y, V)\). Therefore \(r\) is a retraction. Finally, the composition mapping \(F = r \circ h\) is clearly a continuous extension of \(f\) from \((G(X), \mathcal{T}(\widetilde{U}) | G(X))\) to \((Y, \mathcal{T}(V))\).

**Corollary.** Let \((X, \mathcal{U})\) be a \(T_1\) half completable quasi-uniform space, let \((Y, V)\) be a compact Hausdorff quasi-uniform space and let \(f : (X, \mathcal{U}) \to (Y, V)\) be a quasi-uniformly continuous mapping. Then \(f\) has a continuous extension \(F : (G(X), \mathcal{T}(\widetilde{U}) | G(X)) \to (Y, \mathcal{T}(V))\).

**Proof.** By [7, Theorem 2.27], every quasi-uniformity compatible with a compact Hausdorff topological space is locally symmetric. The result follows from the preceding corollary.

**Example 2.** Let \(X = \{1/n; \ n \in \mathbb{N}\}\) and \(Y = \{0\} \cup X\). Let \(\mathcal{U}\) be the uniformity induced on \(X\) by the Euclidean metric and let \(d\) be the quasi-metric on \(Y\) given by \(d(0, 1/n) = 1/n\) and \(d(1/n, 0) = 1\) for all \(n \in \mathbb{N}\), \(d(1/n, 1/m) = |1/n - 1/m|\) for all \(n, m \in \mathbb{N}\) and \(d(0, 0) = 0\).

Denote by \(V\) the quasi-uniformity on \(Y\) induced by \(d\). Clearly, \((Y, V)\) is a Hausdorff compact nonbicomplete quasi-uniform space. Let \(f\) be the identity function on \(X\). Obviously \(f : (X, \mathcal{U}) \to (Y, V)\) is quasi-uniformly continuous. Suppose that \(f\) has a
quasi-uniformly continuous extension $F : (G(X), \tilde{U}) \to (Y, \mathcal{V})$. Since $\tilde{U}$ is the uniformity induced on $G(X)$ by the Euclidean metric and $G(X) = \{0\} \cup X$, i.e., $G(X) = \tilde{X}$, it follows that $F(0) = 0$ by continuity of $F$. Hence $F$ is the identity function on the set $Y$. Since for each $\tilde{U} \in \tilde{U}$ there is $n \in \mathbb{N}$ such that $(1/n, 0) \in \tilde{U}$ but $d(1/n, 0) = 1$, we deduce that $F$ cannot be quasi-uniformly continuous.

Consequently “continuous” cannot be replaced by “quasi-uniformly continuous” in the statement of the two corollaries of Proposition 5. Furthermore, if $\tilde{f}$ denotes the (unique) quasi-uniformly continuous extension of $f$ from $(\tilde{X}, \tilde{U})$ to $(\tilde{Y}, \tilde{V})$, this example shows that the restriction of $\tilde{f}$ to $G(X)$ does not necessarily coincide with the continuous extension $F$ obtained in the two corollaries of Proposition 5.

**Example 3.** Let $X$ be a $T_1$ topological space, $Y$ a compact Hausdorff topological space and $f : X \to Y$ a continuous mapping. Let $\mathcal{U}$ denote the Pervin (respectively, semicontinuous, point finite, locally finite) quasi-uniformity of $X$ and $\mathcal{V}$ the corresponding quasi-uniformity of $Y$ (see [7]). Then $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is quasi-uniformly continuous [7, Proposition 2.17]. Since $T(\mathcal{U}^{-1})$ is the discrete topology on $X$, $(X, \mathcal{U})$ is $T_1$ half completable, so by the second corollary of Proposition 5, $f$ has a continuous extension $F$ from $(G(X), T(\tilde{U}) | G(X))$ to $Y$.

3. **Characterizations of $T_1$ *-half completable and *-compactifiable quasi-uniform spaces**

In this section we shall characterize both $T_1$ *-half complettable and *-compactifiable quasi-uniform spaces. The following auxiliary lemmas will be useful.

**Lemma 4.** Let $(Y, \mathcal{V})$ be a quasi-uniform space and let $X$ be a $T(\mathcal{V}^*)$-dense subset of $Y$. If every Cauchy filter on $(X, \mathcal{V}^* | X \times X)$ is convergent in $(Y, T(\mathcal{V}))$, then $(Y, \mathcal{V})$ is half complete.

**Proof.** Let $\mathcal{F}$ be a $\mathcal{V}^*$-Cauchy filter on $Y$. Then the family $\{V^*(F) \cap X : F \in \mathcal{F}, V \in \mathcal{V}\}$ is clearly a base for a filter $\mathcal{G}$ on $X$. We show that $\mathcal{G}$ is a $\mathcal{V}^* | X \times X$-Cauchy filter. Indeed, given $V \in \mathcal{V}$ choose $W \in \mathcal{V}$ such that $W^3 \subseteq V$. Then there exists $F \in \mathcal{F}$ such that $F \times F \subseteq W$. Thus $(W^*(F) \cap X) \times (W^*(F) \cap X) \subseteq V \cap (X \times X)$, so $\mathcal{G}$ is a $\mathcal{V}^* | X \times X$-Cauchy filter.

Therefore $\mathcal{G}$ is $T(\mathcal{V})$-convergent to some point $x \in X$. From this fact and our assumption that $X$ is $T(\mathcal{V}^*)$-dense in $Y$ it easily follows that $\mathcal{F}$ is $T(\mathcal{V})$-convergent to $x$. Hence $(Y, \mathcal{V})$ is half complete. $\square$

Let $(G, \mathcal{F})$ be an ordered pair of filters on a quasi-uniform space $(X, \mathcal{U})$. Then $(G, \mathcal{F})$ is called a Cauchy filter pair if for each $U \in \mathcal{U}$ there exist $G \in \mathcal{G}$ and $F \in \mathcal{F}$ such that $G \times F \subseteq U$ [5].
In the following we shall write \((G, F) \rightarrow 0\) whenever \((G, F)\) is a Cauchy filter pair.

**Lemma 5.** Let \((X, \mathcal{U})\) be a \(T_0\) quasi-uniform space, let \(\mathcal{F}\) be a filter on \(X\) and let \(\mathcal{F}_0\) be a minimal \(\mathcal{U}^*\)-Cauchy filter on \(X\). Then \((\mathcal{F}_0, \mathcal{F}) \rightarrow 0\) if and only if the filter base \(i(\mathcal{F})\) is \(T(\mathcal{U})\)-convergent to \(\mathcal{F}_0\).

**Proof.** Suppose that \((\mathcal{F}_0, \mathcal{F}) \rightarrow 0\). Let \(U \in \mathcal{U}\). Choose \(V \in \mathcal{U}\) such that \(V^2 \subseteq U\). Then there exist \(F_0 \in \mathcal{F}_0\) and \(F \in \mathcal{F}\) such that \(F_0 \times F \subseteq V\) and \(F_0 \times F \subseteq V\). Let \(x \in F\). Then \(F_0 \times V^*(x) \subseteq U\), so \(i(F) \subseteq \hat{U}(\mathcal{F}_0)\). We conclude that \(i(\mathcal{F})\) is \(T(\mathcal{U})\)-convergent to \(\mathcal{F}_0\).

Conversely, let \(U \in \mathcal{U}\). Choose \(V \in \mathcal{U}\) such that \(V^2 \subseteq U\). Then there is \(F_0 \in \mathcal{F}_0\) such that \(F_0 \times F_0 \subseteq V\). Since \(i(\mathcal{F})\) is \(T(\mathcal{U})\)-convergent to \(\mathcal{F}_0\), there is \(F \in \mathcal{F}\) such that \(i(F) \subseteq \hat{V}(\mathcal{F}_0)\), so \(F \subseteq V(F_0)\) and thus \(F_0 \times F \subseteq U\). Therefore \((\mathcal{F}_0, \mathcal{F}) \rightarrow 0\). \(\square\)

**Definition 5.** Let \((X, \mathcal{U})\) be a quasi-uniform space. A \(\mathcal{U}^*\)-Cauchy filter \(\mathcal{F}\) on \(X\) is said to be symmetrizable if whenever \(G\) is a \(\mathcal{U}^*\)-Cauchy filter on \(X\) such that \((G, \mathcal{F}) \rightarrow 0\), then \((\mathcal{F}, G) \rightarrow 0\).

Note that if \(G\) is a symmetrizable \(\mathcal{U}^*\)-Cauchy filter on the quasi-uniform space \((X, \mathcal{U})\), then the (unique) minimal \(\mathcal{U}^*\)-Cauchy filter \(G\) contains is also symmetrizable.

On the other hand, if \((X, \mathcal{U})\) is a quasi-uniform space, then the relation \(\mathcal{F} \leq \mathcal{G}\) between filters on \(X\) defined by the condition that \((\mathcal{F}, \mathcal{G})\) is a Cauchy filter pair, for minimal \(\mathcal{U}^*\)-Cauchy filters agrees with the specialization order on the bicompletion and the closed singletons are just the minimal (= symmetrizable) elements in this order, as is explained in our next lemma.

**Lemma 6.** Let \(\mathcal{G}\) be a minimal \(\mathcal{U}^*\)-Cauchy filter on a \(T_0\) quasi-uniform space \((X, \mathcal{U})\). Then \(\mathcal{G}\) is symmetrizable if and only if \(\mathcal{G} \in G(X)\).

**Proof.** Suppose that \(\mathcal{G}\) is symmetrizable. Let \(\mathcal{H} \in \tilde{X}\) such that \(\mathcal{G} \in \bigcap_{\mathcal{U} \in \mathcal{U}} \hat{U}(\mathcal{H})\). Then \((\mathcal{H}, \hat{\mathcal{G}}) \rightarrow 0\). So, by assumption, \((\mathcal{G}, \mathcal{H}) \rightarrow 0\). Hence \(\mathcal{G} \in \bigcap_{\mathcal{U} \in \mathcal{U}} \hat{U}^{-1}(\mathcal{H})\). Since \((\tilde{X}, T(\mathcal{U}^*))\) is a Hausdorff space, \(\mathcal{G} = \mathcal{H}\). Consequently \(\mathcal{G} \in G(X)\).

Conversely, let \(\mathcal{H}\) be a \(\mathcal{U}^*\)-Cauchy filter on \(X\) such that \((\mathcal{H}, \hat{\mathcal{G}}) \rightarrow 0\). Let \(\mathcal{H}_0\) be the (unique) minimal \(\mathcal{U}^*\)-Cauchy filter on \(X\) such that \(\mathcal{H}_0 \subseteq \mathcal{H}\). From Lemma 1 it follows that \((\mathcal{H}_0, \mathcal{H}) \rightarrow 0\), so \((\mathcal{H}_0, \hat{\mathcal{G}}) \rightarrow 0\). Thus \(\mathcal{G} \in \hat{U}(\mathcal{H}_0)\) for all \(U \in \mathcal{U}\). Since \(\mathcal{G} \in G(X)\), we have that \(\mathcal{G} = \mathcal{H}_0\); hence \((\mathcal{G}, \mathcal{H}) \rightarrow 0\), and thus \(\mathcal{G}\) is symmetrizable. \(\square\)

**Theorem 2.** A quasi-uniform space \((X, \mathcal{U})\) is \(T_1\) \(*\)-half completable if and only if it is \(T_1\) half completable and for each \(\mathcal{U}^*\)-Cauchy filter \(\mathcal{F}\) there is a symmetrizable \(\mathcal{U}^*\)-Cauchy filter \(\mathcal{G}\) such that \((\mathcal{G}, F) \rightarrow 0\).

**Proof.** If \((X, \mathcal{U})\) is \(T_1\) \(*\)-half completable, then it is \(T_1\) half completable. On the other hand, it follows from Theorem 1 that \((G(X), \hat{U} | G(X))\) is the unique (up to quasi-isomorphism) \(T_1\) \(*\)-half completion of \((X, \mathcal{U})\). Let \(\mathcal{F}\) be a \(\mathcal{U}^*\)-Cauchy filter on \(X\). Since by Proposition 2
Let us recall that a filter $F$ on a quasi-uniform space $(X, \mathcal{U})$ is said to be right $K$-Cauchy if there is a so-called co-filter $\mathcal{G}$ of $F$ such that $(\mathcal{G}, \mathcal{F}) \to 0$.

A quasi-uniform space $(X, \mathcal{U})$ is called right $K$-complete (respectively left $K$-complete, $D$-complete) if each right $K$-Cauchy (respectively left $K$-Cauchy, $D$-Cauchy) filter is $T (\mathcal{U})$-convergent.

Obviously, every right $K$-complete (respectively left $K$-complete, $D$-complete) quasi-uniform space is half complete, but the converse implication does not hold, in general.

Furthermore, it is well known that the notions of right $K$-completeness, left $K$-completeness and $D$-completeness are independent for quasi-uniform spaces, but they coincide with the classical notion of completeness for uniform spaces.

Since every $T_1 *$-right $K$-completable (respectively $T_1 *$-left $K$-completable, $T_1 *$-$D$-completable) quasi-uniform space is $T_1 *$-half completable, the proof of Theorem 1 permits us to state the following variant of it.

**Theorem 3.** Let $(X, \mathcal{U})$ be a $T_1 *$-right $K$-completable (respectively $T_1 *$-left $K$-completable, $T_1 *$-$D$-completable) quasi-uniform space. Then any $T_1 *$-right $K$-completion (respectively $T_1 *$-left $K$-completion, $T_1 *$-$D$-completion) of $(X, \mathcal{U})$ is quasi-isomorphic to $(G(X), \mathcal{U} \upharpoonright G(X))$. Hence, it is unique (up to quasi-isomorphism). Moreover if $\mathcal{U}$ is a uniformity, $(G(X), \mathcal{U} \upharpoonright G(X))$ is the uniform completion of $(X, \mathcal{U})$.

On the other hand, a slight modification of the proof of Lemma 4 shows the following result.
Lemma 7. Let \((Y, V)\) be a quasi-uniform space and let \(X\) be a \(T(\mathcal{V}^*)\)-dense subset of \(Y\). If every right \(K\)-Cauchy (respectively left \(K\)-Cauchy, \(D\)-Cauchy) filter on \((X, V | X \times X)\) is convergent in \((Y, T(\mathcal{V}))\), then \((Y, V)\) is right \(K\)-complete (respectively left \(K\)-complete, \(D\)-complete).

Therefore, we can obtain the following characterization of \(T_1\) *-right \(K\)-completable (respectively \(T_1\) *-left \(K\)-completable, \(T_1\) *-*completable) quasi-uniform spaces.

Theorem 4. A \(T_1\) quasi-uniform space \((X, \mathcal{U})\) is \(T_1\) *-right \(K\)-completable (respectively \(T_1\) *-left \(K\)-completable, \(T_1\) *-*completable) if and only if it is \(T_1\) half completable and for each right \(K\)-Cauchy (respectively left \(K\)-Cauchy, \(D\)-Cauchy) filter \(\mathcal{F}\) there is a symmetrizable \(\mathcal{U}^*\)-Cauchy filter \(\mathcal{G}\) such that \((\mathcal{G}, \mathcal{F}) \to 0\).

Next we characterize \(T_1\) *-right \(K\)-completable quasi-uniform spaces in terms of right \(K\)-completeness of their bicompletions. This result and its method of proof will be crucial in characterizing \(*\)-compactifiable quasi-uniform spaces.

Let us recall that a net \((x_\lambda)_{\lambda \in \Lambda}\) in a quasi-uniform space \((X, \mathcal{U})\) is said to be right \(K\)-Cauchy if for each \(U \in \mathcal{U}\) there is \(\lambda_U \in \Lambda\) such that \((x_\mu, x_\lambda) \in U\) whenever \(\mu \geq \lambda \geq \lambda_U\) [18]. It was proved in [18] that a quasi-uniform space \((X, \mathcal{U})\) is right \(K\)-complete if and only if every right \(K\)-Cauchy net is \(T(\mathcal{U})\)-convergent.

Theorem 5. A quasi-uniform space \((X, \mathcal{U})\) is \(T_1\) *-right \(K\)-completable if and only if it is \(T_1\) half completable and its bicompletion is right \(K\)-complete.

Proof. Suppose that \((X, \mathcal{U})\) is \(T_1\) *-right \(K\)-completable. Then it is clear that it is \(T_1\) half completable, and its bicompletion \((\bar{X}, \bar{\mathcal{U}})\) is right \(K\)-complete, since it contains the \(T(\mathcal{U}^*)\)-dense subspace \(G(X)\) which is right \(K\)-complete.

Conversely, suppose that \((X, \mathcal{U})\) is \(T_1\) half completable and its bicompletion \((\bar{X}, \bar{\mathcal{U}})\) is right \(K\)-complete. Let \(\mathcal{F}\) be a right \(K\)-Cauchy filter on \((X, \mathcal{U})\). We shall find a symmetrizable (i.e., minimal with respect to \(\subseteq\)) \(\mathcal{U}^*\)-Cauchy filter \(\mathcal{F}_0\) such that \((\mathcal{F}_0, \mathcal{F}) \to 0\).

Indeed, since \(\mathcal{F}\) is a right \(K\)-Cauchy filter, \(i(\mathcal{F})\) is a right \(K\)-Cauchy filter base on \((\bar{X}, \bar{\mathcal{U}})\) and since \((\bar{X}, \bar{\mathcal{U}})\) is right \(K\)-complete, there exists a minimal \(\mathcal{U}^*\)-Cauchy filter \(\mathcal{F}_0\) on \(X\) such that \(i(\mathcal{F})\) converges to \(\mathcal{F}_0\) in \((\bar{X}, T(\bar{\mathcal{U}}))\) and hence \((\mathcal{F}_0, \mathcal{F}) \to 0\) by Lemma 5.

If \(\mathcal{F}_0\) is symmetrizable, the characterization of \(T_1\) *-right \(K\)-completable quasi-uniform spaces stated in Theorem 4 concludes the proof. If \(\mathcal{F}_0\) is not symmetrizable, then there exists a transfinite sequence \((\mathcal{G}_\lambda)_{\lambda \in \Lambda}\) (with \(\Lambda \neq \emptyset\) an ordinal) such that \(\mathcal{G}_\lambda\) is minimal \(\mathcal{U}^*\)-Cauchy, \((\mathcal{G}_\lambda, \mathcal{F}_0) \to 0\) for any \(\lambda \in \Lambda\), \((\mathcal{G}_\lambda, \mathcal{G}_\mu) \to 0\) for any \(\lambda \geq \mu\) and if there exists a \(\mathcal{U}^*\)-Cauchy filter \(\mathcal{G}\) with \((\mathcal{G}, \mathcal{G}_\lambda) \to 0\) for every \(\lambda \in \Lambda\) then there exists \(\lambda \in \Lambda\) with \((\mathcal{G}_\lambda, \mathcal{G}) \to 0\). (Indeed, since \(\mathcal{F}_0\) is not symmetrizable, there exists a \(\mathcal{U}^*\)-Cauchy filter \(\mathcal{G}_0\) such that \((\mathcal{G}_0, \mathcal{F}_0) \to 0\) but \((\mathcal{F}_0, \mathcal{G}_0) \to 0\) and \(\mathcal{G}_0 \neq \emptyset\), and suppose that we have defined \(\mathcal{G}_\lambda\). If \(\mathcal{G}_\lambda\) is symmetrizable, then the family \(\{\mathcal{G}_\mu : \mu \leq \lambda\}\) fulfills the requirements; we call it Case 1. Otherwise, there exists a \(\mathcal{U}^*\)-Cauchy filter \(\mathcal{G}_{\lambda+1}\) with \((\mathcal{G}_{\lambda+1}, \mathcal{G}_\lambda) \to 0\) but
Corollary. Let \((\mathcal{G}_\lambda, \mathcal{G}_{\lambda} + 1) \to 0\) and hence \((\mathcal{G}_{\lambda} + 1, \mathcal{G}_{\mu}) \to 0\) for each \(\mu \leq \lambda\) and \((\mathcal{G}_{\lambda} + 1, \mathcal{F}_\emptyset) \to 0\). Let \(\lambda \in \Lambda\) be a limit ordinal, and suppose that we have defined \(\mathcal{G}_\mu\) for each \(\mu < \lambda\). Suppose that there exists a \(\mathcal{U}^*\)-Cauchy filter \(\mathcal{G}_\lambda\) with \((\mathcal{G}_\lambda, \mathcal{G}_\mu) \to 0\) for each \(\lambda < \mu\). Then \((\mathcal{G}_\lambda, \mathcal{F}_\emptyset) \to 0\).

Otherwise, the family \([\mathcal{G}_\mu: \mu < \lambda]\) fulfills the requirements; we call it Case 2. In fact, in this case there is no \(\mathcal{U}^*\)-Cauchy filter \(\mathcal{G}\) satisfying \((\mathcal{G}, \mathcal{G}_\mu) \to 0\) for every \(\mu < \lambda\). Finally, it is clear that there exists an ordinal \(\lambda\) which verifies Cases 1 or 2.

It is easily seen that \((\mathcal{G}_\lambda)_{\lambda \in \Lambda}\) is a right \(K\)-Cauchy net in \((\tilde{X}, \tilde{\mathcal{U}})\), and since it is right \(K\)-complete we deduce that \((\mathcal{G}_\lambda)_{\lambda \in \Lambda}\) is \(T(\tilde{\mathcal{U}})\)-convergent to a point \(\mathcal{F}_{\emptyset 0} \in \tilde{X}\). Hence \(\mathcal{F}_{\emptyset 0}\) is a minimal \(\mathcal{U}^*\)-Cauchy filter on \(X\) such that \((\mathcal{F}_{\emptyset 0}, \mathcal{G}_\lambda) \to 0\) for all \(\lambda \in \Lambda\). By the properties of \((\mathcal{G}_\lambda)_{\lambda \in \Lambda}\) it follows that \(\mathcal{F}_{\emptyset 0}\) is symmetrizable. Since \((\mathcal{G}_\lambda, \mathcal{F}_{\emptyset}) \to 0\) for all \(\lambda \in \Lambda\) and \((\mathcal{F}_{0 0}, \mathcal{F}) \to 0\), we deduce that \((\mathcal{F}_{\emptyset 0}, \mathcal{F}) \to 0\), and hence \((X, \mathcal{U})\) is \(T_1^*\)-right \(K\)-completable by Theorem 4.

Let us recall that if \((X, \mathcal{U})\) is a \(T_0\) quasi-uniform space, then the quasi-uniformity \(\mathcal{U}\) is Smyth completable if and only if every left \(K\)-Cauchy filter on \((X, \mathcal{U})\) is \(\mathcal{U}^*\)-Cauchy and \(\mathcal{U}\) is Smyth complete if and only if every left \(K\)-Cauchy filter is \(T(\mathcal{U}^*)\)-convergent (to a unique point of \(X\)). Furthermore, if \(\mathcal{U}\) is Smyth completable, \(\tilde{\mathcal{U}}\) is a Smyth complete quasi-uniformity on \(\tilde{X}\) [13].

**Corollary.** Let \((X, \mathcal{U})\) be a \(T_1\) half completable quasi-uniform space such that \(\mathcal{U}^{-1}\) is Smyth completable. Then \((X, \mathcal{U})\) is \(T_1^*\)-right \(K\)-completable.

**Proof.** Let \(\mathcal{F}\) be a right \(K\)-Cauchy filter on \((\tilde{X}, \tilde{\mathcal{U}})\). Then \(\mathcal{F}\) is left \(K\)-Cauchy on \((\tilde{X}, \tilde{\mathcal{U}})\), so it is \(T(\tilde{\mathcal{U}}^*)\)-convergent. We deduce that \((\tilde{X}, \tilde{\mathcal{U}})\) is right \(K\)-complete. By Theorem 5 \((X, \mathcal{U})\) is \(T_1^*\)-right \(K\)-completable.

Since the conjugate quasi-uniformity of a hereditarily precompact quasi-uniformity is Smyth completable [13] we deduce the following consequence of the preceding result.

**Corollary.** Every hereditarily precompact \(T_1\) half completable quasi-uniform space is \(T_1^*\)-right \(K\)-completable.

**Lemma 8** [13]. A quasi-uniform space is compact if and only if it is precompact and left \(K\)-complete.

**Lemma 9.** Let \((X, \mathcal{U})\) be a \(T_1\) half completable quasi-uniform space such that its bicompletion is left \(K\)-complete and right \(K\)-complete. Then \((X, \mathcal{U})\) is \(T_1^*\)-left \(K\)-completable and \(T_1^*\)-right \(K\)-completable.

**Proof.** By Theorem 5 \((X, \mathcal{U})\) is \(T_1^*\)-right \(K\)-completable. Now let \(\mathcal{F}\) be a left \(K\)-Cauchy filter on \((X, \mathcal{U})\). Then \(i(\mathcal{F})\) is a left \(K\)-Cauchy filter base on \((\tilde{X}, \tilde{\mathcal{U}})\) and since \((\tilde{X}, \tilde{\mathcal{U}})\) is left \(K\)-complete, there exists a minimal \(\mathcal{U}^*\)-Cauchy filter \(\mathcal{F}_0\) on \(X\) such that \(i(\mathcal{F})\) converges to \(\mathcal{F}_0\) in \((\tilde{X}, T(\tilde{\mathcal{U}}))\) and hence \((\mathcal{F}_0, \mathcal{F}) \to 0\) by Lemma 5. If \(\mathcal{F}_0\) is symmetrizable, the characterization of \(T_1^*\)-left \(K\)-completable quasi-uniform spaces stated in Theorem 4
concludes the proof. If \( \mathcal{F}_0 \) is not symmetrizable, as in the proof of Theorem 5, there exists a transfinite sequence \((\mathcal{G}_\lambda)_{\lambda \in \Lambda}\) (with \( \Lambda \neq \emptyset \) an ordinal) such that \( \mathcal{G}_\lambda \) is minimal \( \mathcal{U}^*-\)Cauchy, \((\mathcal{G}_\lambda, \mathcal{F}_0) \rightarrow 0\) for any \( \lambda \in \Lambda \), \((\mathcal{G}_\lambda, \mathcal{G}_\mu) \rightarrow 0\) for any \( \lambda \geq \mu \) and if there exists a \( \mathcal{U}^*-\)Cauchy filter \( \mathcal{G} \) with \((\mathcal{G}, \mathcal{G}_\lambda) \rightarrow 0\) for every \( \lambda \in \Lambda \) then there exists \( \lambda \in \Lambda \) with \((\mathcal{G}_\lambda, \mathcal{G}) \rightarrow 0\). Clearly \((\mathcal{G}_\lambda)_{\lambda \in \Lambda}\) is a right \( K \)-Cauchy net in \((\tilde{X}, \tilde{T})\), and since it is right \( K \)-complete we deduce that \((\mathcal{G}_\lambda)_{\lambda \in \Lambda}\) is \( T(\tilde{T})\)-convergent to a point \( \mathcal{F}_{00} \in \tilde{X} \). Hence \( \mathcal{F}_{00} \) is a minimal \( \mathcal{U}^*-\)Cauchy filter on \( X \) such that \((\mathcal{F}_{00}, \mathcal{G}_\lambda) \rightarrow 0\) for all \( \lambda \in \Lambda \). By the properties of \((\mathcal{G}_\lambda)_{\lambda \in \Lambda}\) it follows that \( \mathcal{F}_{00} \) is symmetrizable. Since \((\mathcal{G}_\lambda, \mathcal{F}_0) \rightarrow 0\) for all \( \lambda \in \Lambda \) and \((\mathcal{F}_0, \mathcal{F}) \rightarrow 0\), we deduce that \((\mathcal{F}_{00}, \mathcal{F}) \rightarrow 0\), and hence \((X, \mathcal{U})\) is \( T_1 \) \(*\)-left \( K \)-completable by Theorem 4. \( \square \)

**Theorem 6.** A \( T_1 \) quasi-uniform space is \(*\)-compactifiable if and only if it is point symmetric and its bicompletion is compact.

**Proof.** Let \((X, \mathcal{U})\) be a \(*\)-compactifiable \( T_1 \) quasi-uniform space. Then \((G(X), \tilde{\mathcal{U}} \upharpoonright G(X))\) is the \(*\)-compactification of \((X, \mathcal{U})\). So \((G(X), \tilde{\mathcal{U}} \upharpoonright G(X))\) is a compact \( T_1 \) quasi-uniform space, and thus it is point symmetric. Therefore \((X, \mathcal{U})\) is point symmetric. Let \((\tilde{X}, \tilde{T})\) be the bicompletion of \((X, \mathcal{U})\). Since \((G(X), \tilde{\mathcal{U}} \upharpoonright G(X))\) is \( T(\tilde{T})\)-dense in \( \tilde{X} \), it follows that \((\tilde{X}, \tilde{T})\) is compact [15, Proposition 5].

Conversely suppose that \((X, \mathcal{U})\) is a point symmetric \( T_1 \) quasi-uniform space such that its bicompletion \((\tilde{X}, \tilde{T})\) is compact. Then \((X, \mathcal{U})\) is \( T_1 \) half completable, so \( X \subseteq G(X) \). Moreover \((\tilde{X}, \tilde{T})\) is precompact, right \( K \)-complete and left \( K \)-complete. By [15, Proposition 4(c)], \((G(X), \tilde{\mathcal{U}} \upharpoonright G(X))\) is precompact and by Lemma 9 \((X, \mathcal{U})\) is \( T_1 \) \(*\)-left \( K \)-completable, i.e., \((G(X), \tilde{\mathcal{U}} \upharpoonright G(X))\) is left \( K \)-complete. It follows from Lemma 8 that \((G(X), \tilde{\mathcal{U}} \upharpoonright G(X))\) is compact. Consequently \((X, \mathcal{U})\) is \(*\)-compactifiable. \( \square \)

**Corollary.** Every point symmetric precompact Smyth completable \( T_1 \) quasi-uniform space is \(*\)-compactifiable.

**Proof.** Let \((X, \mathcal{U})\) be a point symmetric precompact Smyth completable \( T_1 \) quasi-uniform space. By [15, Proposition 4(c)], \((\tilde{X}, \tilde{T})\) is precompact. Now let \( \mathcal{F} \) be a left \( K \)-Cauchy filter on \((X, \mathcal{U})\). By assumption \( \mathcal{F} \) is \( \mathcal{U}^*-\)Cauchy, so it is convergent in \((\tilde{X}, T(\tilde{T}))\). It follows from Lemma 7 that \((\tilde{X}, \tilde{T})\) is left \( K \)-complete. Therefore \((X, \mathcal{U})\) is \(*\)-compactifiable by Lemma 8 and Theorem 6. \( \square \)

Since every totally bounded quasi-uniform space is precompact and Smyth completable, we obtain the following consequence of the preceding result.

**Corollary.** Every point symmetric totally bounded \( T_1 \) quasi-uniform space is \(*\)-compactifiable.

In [21] we characterize Wallman \( T_1 \) compactifications of \( T_1 \) topological spaces in terms of the \(*\)-compactification of certain point symmetric totally bounded transitive compatible quasi-uniformities.
It was proved in [8, Theorem 3.5], that each point symmetric totally bounded \( T_1 \) quasi-uniform space \((X,\mathcal{U})\) has a compactification \((Y,\mathcal{V})\) that is a subspace of the bicompletion of \((X,\mathcal{U})\). Thus \((Y,\mathcal{V})\) is quasi-isomorphic to the \( \ast \)-compactification of \((X,\mathcal{U})\) (see also the preceding corollary). In particular \( Y \) consists of all maximal regular filters on \((X,\mathcal{U})\). In Proposition 6 below we show that actually \((Y,\mathcal{V})\) coincides with \((G(X),\overline{\mathcal{U}}|G(X))\).

Let us recall that a filter \( \mathcal{F} \) on a quasi-uniform space \((X,\mathcal{U})\) is regular (or co-regular) provided that \( \mathcal{F} \) coincides with its co-envelope \( \text{co}(\mathcal{F}) \), where \( \text{co}(\mathcal{F}) \) is the filter on \( X \) which has as a base the collection \( \{U^{-1}(F) : U \in \mathcal{U}, F \in \mathcal{F}\} \).

Note that for each filter \( \mathcal{F} \) on \((X,\mathcal{U})\), \( \text{co}(\mathcal{F}) = \text{co}(\text{co}(\mathcal{F})) \), so \( \text{co}(\mathcal{F}) \) is always regular.

**Lemma 10.** Let \((X,\mathcal{U})\) be a quasi-uniform space and let \( \mathcal{F} \) and \( \mathcal{G} \) be filters on \( X \) such that \( \mathcal{G} \) is \( \mathcal{U}^\ast \)-Cauchy. Then \((\mathcal{F},\mathcal{G}) \to 0 \) if and only if \( \text{co}(\mathcal{G}) \subseteq \mathcal{F} \).

**Proof.** Suppose that \((\mathcal{F},\mathcal{G}) \to 0 \), and let \( U \in \mathcal{U} \) and \( G \in \mathcal{G} \). Let \( F \in \mathcal{F} \) and \( G' \in \mathcal{G} \) be such that \( G' \subseteq G \) and \( F \times G' \subseteq U \). Then \( F \subseteq U^{-1}(G') \subseteq U^{-1}(G) \), and hence \( \text{co}(\mathcal{G}) \subseteq \mathcal{F} \).

Conversely, suppose that \( \text{co}(\mathcal{G}) \subseteq \mathcal{F} \), and let \( U \in \mathcal{U} \). Let \( V \in \mathcal{U} \) be such that \( V^2 \subseteq U \).

Since \( \mathcal{G} \) is \( \mathcal{U}^\ast \)-Cauchy, there exists \( G' \in \mathcal{G} \) such that \( G \times G' \subseteq V \). Let \( F \in \mathcal{F} \) be such that \( F \subseteq V^{-1}(G') \). Then it is easy to check that \( F \times G \subseteq V^2 \subseteq U \), and hence \((\mathcal{F},\mathcal{G}) \to 0 \). \( \square \)

**Proposition 6.** Let \((X,\mathcal{U})\) be a totally bounded \( T_0 \) quasi-uniform space. A filter on \( X \) is a maximal regular filter if and only if it is a symmetrizable minimal \( \mathcal{U}^\ast \)-Cauchy filter.

**Proof.** Let \( \mathcal{F} \) be a maximal regular filter. By [8, Proposition 3.3], \( \mathcal{F} \) is a minimal \( \mathcal{U}^\ast \)-Cauchy filter on \( X \). Suppose that there exists a \( \mathcal{U}^\ast \)-Cauchy filter \( \mathcal{G} \) with \( \text{co}(\mathcal{G}) \to 0 \).

By Lemma 10 \( \text{co}(\mathcal{F}) \subseteq \mathcal{G} \), but since \( \mathcal{F} \) is regular, then \( \mathcal{F} = \text{co}(\mathcal{F}) \), so \( \mathcal{F} \subseteq \mathcal{G} \) and hence \( \mathcal{F} = \text{co}(\mathcal{F}) \subseteq \text{co}(\mathcal{G}) \), but since \( \text{co}(\mathcal{G}) \) is regular and \( \mathcal{F} \) is a maximal regular filter, \( \mathcal{F} = \text{co}(\mathcal{G}) \), thus by Lemma 10 \((\mathcal{F},\mathcal{G}) \to 0 \). Consequently \( \mathcal{F} \) is symmetrizable.

Conversely, let \( \mathcal{F} \) be a symmetrizable minimal \( \mathcal{U}^\ast \)-Cauchy filter. Since \( \text{co}(\mathcal{F}) \subseteq \text{co}(\mathcal{F}) \), it follows from Lemma 10 that \( \text{co}(\mathcal{F},\mathcal{F}) \to 0 \). Then there exists a maximal regular filter \( \mathcal{G} \) with \( \text{co}(\mathcal{G}) \subseteq \mathcal{G} \) [8, Proposition 2.2] and hence \( \mathcal{G},\mathcal{F}) \to 0 \). By [8, Lemma 3.1], \( \mathcal{G} \) is a \( \mathcal{U}^\ast \)-Cauchy filter. Since \( \mathcal{F} \) is symmetrizable, \((\mathcal{F},\mathcal{G}) \to 0 \), so by Lemma 10 \( \mathcal{G} = \text{co}(\mathcal{G}) \subseteq \mathcal{F} \), and hence \( \mathcal{G} = \text{co}(\mathcal{G}) \subseteq \text{co}(\mathcal{F}), \mathcal{G} = \text{co}(\mathcal{F}). \) Then \( \text{co}(\mathcal{F}) \) is \( \mathcal{U}^\ast \)-Cauchy, but \( \text{co}(\mathcal{F}) \subseteq \mathcal{F} \) and since \( \mathcal{F} \) is a minimal \( \mathcal{U}^\ast \)-Cauchy filter, it follows that \( \mathcal{F} = \text{co}(\mathcal{F}) \). We conclude that \( \mathcal{F} \) is a maximal regular filter. \( \square \)

Consider the following condition for a quasi-uniform space \((X,\mathcal{U})\):

\((\ast)\) if \( A \) and \( B \) are subsets of \( X \) such that there is \( U \in \mathcal{U} \) with \( U^{-1}(A) \cap U^{-1}(B) = \emptyset \), then there exists \( V \in \mathcal{U} \) such that \( V(A) \cap V(B) = \emptyset \).

In [8] it is proved that if \((X,\mathcal{U})\) is a point symmetric totally bounded \( T_1 \) quasi-uniform space satisfying condition \((\ast)\), then the \( \ast \)-compactification of \((X,\mathcal{U})\) is a Hausdorff compactification of \((X,\mathcal{T}(\mathcal{U}))\), and it is asked whether condition \((\ast)\) is equivalent to the condition that the \( \ast \)-compactification of \((X,\mathcal{U})\) is a Hausdorff space. Next we prove that this is the case.

The following three lemmas are easy to prove.
Lemma 11 [7, Proposition 1.7]. Let \((X, \mathcal{U})\) be a quasi-uniform space, \(A \subseteq X\) and \(U \in \mathcal{U}\). Then \(\overline{A} \subseteq U^{-1}(A)\).

Lemma 12. Let \((X, \mathcal{U})\) be a quasi-uniform space. \(D\) be a \(T(\mathcal{U}^*)\)-dense subspace of \(X\), and \(A, B \subseteq D\), \(U \in \mathcal{U}\) with \(U^{-1}(A) \cap U^{-1}(B) \cap D = \emptyset\). Then there exists \(V \in \mathcal{U}\) such that \(V^{-1}(A) \cap V^{-1}(B) = \emptyset\).

Recall that a quasi-uniform space \((X, \mathcal{U})\) is said to be equinormal if for each pair of disjoint nonempty \(T(\mathcal{U})\)-closed subsets \(A\) and \(B\) of \(X\) there is \(U \in \mathcal{U}\) such that \(U(A) \cap B = \emptyset\) [7].

Lemma 13. Let \((X, \mathcal{U})\) be an equinormal Hausdorff quasi-uniform space. Then \((X, T(\mathcal{U}))\) is normal if and only if for each disjoint closed subsets \(A, B\) of \(X\), there exists \(U \in \mathcal{U}\) such that \(U(A) \cap U(B) = \emptyset\).

Proposition 7. Let \((X, \mathcal{U})\) be a point symmetric totally bounded \(T_1\) quasi-uniform space. Then the *-compactification of \((X, \mathcal{U})\) is a Hausdorff space if and only if \((X, \mathcal{U})\) satisfies condition \((\ast)\).

Proof. If \((X, \mathcal{U})\) satisfies condition \((\ast)\) then \((G(X), \widetilde{\mathcal{U}} | G(X))\) is a Hausdorff space by Proposition 6 and [8, Proposition 3.9].

Conversely, suppose that \((G(X), \tilde{\mathcal{U}} | G(X))\) is Hausdorff, and let \(A, B \subseteq X\) and \(U \in \mathcal{U}\) with \(U^{-1}(A) \cap U^{-1}(B) = \emptyset\). By Lemma 12 there is \(W \in \mathcal{U}\) such that \(G(X) \cap \tilde{W}^{-1}(A) \cap \tilde{W}^{-1}(B) = \emptyset\), and by Lemma 11 we have that \(\overline{\text{Cl}_{G(X)}(A)} \cap \text{Cl}_{G(X)}(B) = \emptyset\), where \(\text{Cl}_{G(X)}(A)\) (respectively \(\text{Cl}_{G(X)}(B)\)) denotes the closure of \(A\) (respectively of \(B\)) with respect to \(T(\tilde{\mathcal{U}} | G(X))\). Since \((G(X), \tilde{\mathcal{U}} | G(X))\) is a compact Hausdorff space it is equinormal and normal, so by Lemma 13, there exists \(V \in \mathcal{U}\) such that \(G(X) \cap \tilde{V}(\text{Cl}_{G(X)}(A)) \cap \tilde{V}(\text{Cl}_{G(X)}(B)) = \emptyset\), and hence \(V(A) \cap V(B) = \emptyset\). Therefore \((X, \mathcal{U})\) satisfies condition \((\ast)\). \(\square\)

We conclude this section obtaining an internal characterization of *-compactifiable quasi-uniform spaces.

Recall that a quasi-uniform space \((X, \mathcal{U})\) is said to be Cauchy bounded [11] if for each ultrafilter \(\mathcal{F}\) on \(X\) there is a filter \(\mathcal{G}\) on \(X\) such that \((\mathcal{G}, \mathcal{F}) \to 0\).

Definition 6. A quasi-uniform space \((X, \mathcal{U})\) is called *-Cauchy bounded if for each ultrafilter \(\mathcal{F}\) on \(X\) there is a \(\mathcal{U}^*\)-Cauchy filter \(\mathcal{G}\) on \(X\) such that \((\mathcal{G}, \mathcal{F}) \to 0\).

Remark 2. It is clear that both a compact quasi-uniform space and a totally bounded quasi-uniform space is *-Cauchy bounded. Furthermore, each *-Cauchy bounded quasi-uniform space is Cauchy bounded.

Proposition 8. A quasi-uniform space is compact if and only if it is *-Cauchy bounded and half complete.
**Proof.** Let \((X, \mathcal{U})\) be a \(*\)-Cauchy bounded half complete quasi-uniform space and let \(\mathcal{F}\) be an ultrafilter on \(X\). Then there is a \(\mathcal{U}\)-Cauchy filter \(\mathcal{G}\) on \(X\) such that \((\mathcal{G}, \mathcal{F}) \rightarrow 0\). By assumption, the filter \(\mathcal{G}\) converges with respect to \(\mathcal{T}(\mathcal{U})\). So \(\mathcal{F}\) converges with respect to \(\mathcal{T}(\mathcal{U})\) and thus \((X, \mathcal{U})\) is compact. The converse is obvious. ☐

**Proposition 9.** The bicompletion of a \(T_0\) quasi-uniform space \((X, \mathcal{U})\) is compact if and only if \((X, \mathcal{U})\) is \(*\)-Cauchy bounded.

**Proof.** Suppose that the bicompletion \((\tilde{X}, \tilde{\mathcal{U}})\) of \((X, \mathcal{U})\) is compact. Let \(\mathcal{F}\) be an ultrafilter on \(X\). Then \(i(\mathcal{F})\) is a base for an ultrafilter on \(\tilde{X}\). Then there is a minimal \(\mathcal{U}\)-Cauchy filter \(\mathcal{G}\) on \(X\) such that \(i(\mathcal{F})\) converges to \(\mathcal{G}\) with respect to \(\mathcal{T}(\tilde{\mathcal{U}})\). By Lemma 5 \((\mathcal{G}, \mathcal{F}) \rightarrow 0\). We have shown that \((X, \tilde{\mathcal{U}})\) is \(*\)-Cauchy bounded.

Conversely, let \(\mathcal{F}\) be an ultrafilter on \(X\). Then \(\{\tilde{U}^*(F) \cap X : U \in \mathcal{U}, F \in \mathcal{F}\}\) is a base for a filter \(\mathcal{F}'\) on \(X\). Let \(\mathcal{G}'\) be an ultrafilter on \(X\) containing \(\mathcal{F}'\) and let \(\mathcal{G}\) be a \(\mathcal{U}\)-Cauchy filter on \(X\) such that \((\mathcal{G}, \mathcal{F}') \rightarrow 0\). Let \(\mathcal{G}_0\) be the minimal \(\mathcal{U}\)-Cauchy filter coarser than \(\mathcal{G}\). Then \((\mathcal{G}_0, \mathcal{F}') \rightarrow 0\). By Lemma 5 \(i(\mathcal{F}')\) converges to \(\mathcal{G}_0\) with respect to \(\mathcal{T}(\tilde{\mathcal{U}})\). It easily follows that \(\mathcal{F}\) clusters to \(\mathcal{G}_0\) with respect to \(\mathcal{T}(\tilde{\mathcal{U}})\). We conclude that \((\tilde{X}, \tilde{\mathcal{U}})\) is compact. ☐

Combining Proposition 9 and Theorem 6 we obtain the following result.

**Theorem 7.** A \(T_1\) quasi-uniform space is \(*\)-compactifiable if and only if it is point symmetric and \(*\)-Cauchy bounded.

### 4. Locally fitting quasi-uniform spaces

In order to obtain a consistent theory of \(D\)-completion, Doitchinov introduced the notion of a quiet quasi-uniform space \([3,4]\).

A quasi-uniform space \((X, \mathcal{U})\) is called quiet if for each \(U \in \mathcal{U}\) there is \(V \in \mathcal{U}\) such that whenever \((\mathcal{G}, \mathcal{F})\) is a Cauchy filter pair and \(x\) and \(y\) are points of \(X\) satisfying \(V(x) \in \mathcal{F}\) and \(V^{-1}(y) \in \mathcal{G}\), then \((x, y) \in U\).

In this case, we say that \(V\) is quiet for \(U\) and that \(U\) is a quiet quasi-uniformity.

Clearly, each uniform space is quiet. It is also well known that each quiet quasi-uniform space is regular and that a quasi-uniform space \((X, \mathcal{U})\) is quiet if and only if \((X, \mathcal{U}^{-1})\) is quiet. Thus quietness is a conjugate invariant property.

Next we observe that quietness provides a useful condition to obtain a large class of \(*\)-compactifiable quasi-uniform spaces with Hausdorff \(*\)-compactification.

**Example 4.** Let \((X, \mathcal{U})\) be a Cauchy bounded quiet \(T_1\) quasi-uniform space. By \([15,\text{Propositions 4(b) and 12}]\), \((\tilde{X}, \tilde{\mathcal{U}})\) is a Cauchy bounded quiet \(T_0\) quasi-uniform space. Since every quiet quasi-uniform space is regular, it follows from \([16, \text{Proposition 13}]\), that \((\tilde{X}, \tilde{\mathcal{U}})\) is a Hausdorff compact quasi-uniform space. Hence \((\tilde{X}, \tilde{\mathcal{U}})\) is the \(*\)-compactification of \((X, \mathcal{U})\) and it is a Hausdorff compactification of \((X, \mathcal{T}(\mathcal{U}))\).
Motivated by the existence of many interesting examples of nonregular quasi-uniform spaces having useful conjugate invariant properties the first author introduced in [20] the following generalization of the notion of a quiet quasi-uniform space: A quasi-uniform space \((X, \mathcal{U})\) is called fitting if for each \(U \in \mathcal{U}\) there is \(V \in \mathcal{U}\) such that whenever \((\mathcal{G}, \mathcal{F})\) is a Cauchy filter pair and \(x\) and \(y\) are points of \(X\) satisfying \(V^*(x) \in \mathcal{F}\) and \(V^*(y) \in \mathcal{G}\), then \((x, y) \in U\).

In this case, we say that \(V\) is fitting for \(U\) and \(U\) is called a fitting quasi-uniformity.

Every fitting quasi-uniform space is \(R_0\), and, hence, every fitting \(T_0\) quasi-uniform space is \(T_1\). Furthermore, a quasi-uniform space \((X, \mathcal{U})\) is fitting if and only if \((X, \mathcal{U}^{-1})\) is fitting [20].

It is known that both quietness and fittingness are hereditary and productive properties.

A local generalization of quietness, namely local quietness, was introduced in [2], where it was shown that a co-stable locally quiet quasi-uniform space is bicomplete if and only if it is \(D\)-complete (see [2] for the notion of a co-stable quasi-uniform space).

Let us recall that a quasi-uniform space \((X, \mathcal{U})\) is said to be locally quiet if for each Cauchy filter pair \((\mathcal{G}, \mathcal{F})\) and each \(U \in \mathcal{U}\) there is a \(V \in \mathcal{U}\) (depending on \((\mathcal{G}, \mathcal{F})\) and \(U\)) such that whenever \(x\) and \(y\) are points of \(X\) satisfying \(V(x) \in \mathcal{F}\) and \(V^{-1}(y) \in \mathcal{G}\), then \((x, y) \in U\).

The above notions suggest, in a natural way, the following simultaneous generalizations of (local) quietness and fittingness.

**Definition 7.** A quasi-uniform space \((X, \mathcal{U})\) is said to be locally fitting if for each Cauchy filter pair \((\mathcal{G}, \mathcal{F})\) and each \(U \in \mathcal{U}\) there is a \(V \in \mathcal{U}\) (depending on \((\mathcal{G}, \mathcal{F})\) and \(U\)) such that whenever \(x\) and \(y\) are points of \(X\) satisfying \(V^*(x) \in \mathcal{F}\) and \(V^*(y) \in \mathcal{G}\), then \((x, y) \in U\).

**Remark 3.** The following facts are immediate consequences of the definitions:

(a) Every locally fitting (respectively locally quiet) quasi-uniform space is \(R_0\) (respectively regular).

(b) A quasi-uniform space \((X, \mathcal{U})\) is local fitting (respectively locally quiet) if and only if \((X, \mathcal{U}^{-1})\) is locally fitting (respectively locally quiet).

(c) Both local fittingness and local quietness are hereditary and productive properties.

It was proved in [15] that any bicompletion of a quiet quasi-uniform space is quiet. Fittingness is also preserved by bicompleteness as it was shown in [19]. Here, we show that a similar situation occurs for locally quiet and locally fitting quasi-uniform spaces.

**Proposition 10.** Let \((Y, \mathcal{U})\) be a quasi-uniform space and let \(X\) be a \(T(\mathcal{U})\) -dense subset of \(Y\). Then \((Y, \mathcal{U})\) is locally fitting (respectively locally quiet) if and only if \((X, \mathcal{U} \mid X \times X)\) is locally fitting (respectively locally quiet).

**Proof.** Suppose that \((Y, \mathcal{U})\) is a locally fitting quasi-uniform space. Then \((X, \mathcal{U} \mid X \times X)\) is locally fitting.
Conversely, let \((G, F)\) be a Cauchy filter pair on \((Y, U)\) and let \(U_0 \in U\). Choose \(U_1 \in U\) such that \(U_1^3 \subseteq U_0\). Since \(X\) is a \(T(U^*)\)-dense subset of \(Y\), \(F_1\) and \(G_1\) are filter bases on \(X\), where

\[
F_1 = \{ U^*(F) \cap X : F \in F, U \in U \} \quad \text{and} \quad G_1 = \{ U^*(G) \cap X : G \in G, U \in U \}.
\]

Furthermore \((G_1, F_1) \to 0\) because \((G, F) \to 0\). So, by assumption, there is \(V \in U\), with \(V \subseteq U_1\), such that \((a, b) \in U_1\) whenever \(a\) and \(b\) are points of \(X\) satisfying \(V^*(a) \cap X \in F_1\) and \(V^*(b) \cap X \in G_1\).

Finally, let \(W \in U\) such that \(W^3 \subseteq V\), and let \(x\) and \(y\) be points of \(Y\) such that \(W^*(x) \in F\) and \(W^*(y) \in G\). Since \(X\) is \(T(U^*)\)-dense in \(Y\), there exist points \(a\) and \(b\) of \(X\) such that \(a \in W^*(x)\) and \(b \in W^*(y)\). Therefore \(W^*(W^*(x)) \cap X \subseteq V^*(a) \cap X\). Since \(W^*(x) \in F\) we deduce that \(V^*(a) \cap X \in F_1\). Similarly, we obtain that \(V^*(b) \cap X \in G_1\). So \((a, b) \in U_1\). Thus \((x, y) \in U_1^3 \subseteq U_0\). We conclude that \((Y, U)\) is locally fitting.

We omit the proof of the parenthetical result because it follows by a slight modification of the locally fitting case.

From Proposition 10 we immediately deduce the following result.

**Theorem 8.** Let \((Y, V)\) be a bicompletion of a locally fitting (respectively locally quiet) quasi-uniform space \((X, U)\). Then \((Y, V)\) is a locally fitting (respectively locally quiet) quasi-uniform space.

**Corollary.** The bicompletion of any locally fitting (respectively locally quiet) \(T_0\) quasi-uniform space is a locally fitting (respectively locally quiet) \(T_1\) quasi-uniform space.

**Proof.** Apply Theorem 8 and the well-known fact that every \(T_0\) and \(R_0\) topological space is \(T_1\).

Fletcher and Hunsaker proved in [6] that every quiet totally bounded quasi-uniform space is a uniform space (see [12] for an alternative proof). This result was generalized to locally quiet quasi-uniform spaces and fitting quasi-uniform spaces in [13] and [20], respectively.

We observe that since any bicompletion of a locally fitting quasi-uniform space is locally fitting (Theorem 8) and every locally fitting quasi-uniform space is \(R_0\), the technique used in the proof of Theorem 2 in [20], actually shows the following more general result.

**Theorem 9.** Let \((X, U)\) be a locally fitting totally bounded quasi-uniform space. Then \((X, U)\) is a uniform space.
We shall show that $T_0$ quasi-uniform spaces whose bicompletion is a $T_1$ quasi-uniform space, have some interesting properties. To this end, the following consequence of Lemma 6 will be useful.

**Proposition 11.** Let $(X, \mathcal{U})$ be a $T_0$ quasi-uniform space. Then its bicompletion is $T_1$ if and only if every $\mathcal{U}^*$-Cauchy filter on $X$ is symmetrizable.

**Proposition 12.** Every half complete $T_0$ quasi-uniform space whose bicompletion is $T_1$ is bicomplete.

**Proof.** Let $(X, \mathcal{U})$ be a half complete $T_0$ quasi-uniform space whose bicompletion is $T_1$. Then $(X, \mathcal{U})$ is half complete and $T_1$ and obviously its bicompletion is quasi-isomorphic to its $T_1^*$-half completion, i.e., to $(X, \mathcal{U})$. We conclude that $(X, \mathcal{U})$ is bicomplete. 

**Corollary.** Every locally fitting half complete $T_0$ quasi-uniform space is bicomplete.

**Proof.** Let $(X, \mathcal{U})$ be a locally fitting half complete $T_0$ quasi-uniform space. By the corollary of Theorem 8, the bicompletion of $(X, \mathcal{U})$ is $T_1$. Since $(X, \mathcal{U})$ is half complete, it follows from Proposition 12 that $(X, \mathcal{U})$ is actually bicomplete.

**Corollary.** Let $(X, \mathcal{U})$ be a quiet (respectively locally quiet, fitting, locally fitting) $T_0$ quasi-uniform space. Then $(X, \mathcal{U})$ is $T_1^*$-half completable and its $T_1^*$-half completion is quiet (respectively locally quiet, fitting, locally fitting) and coincides with the bicompletion of $(X, \mathcal{U})$.

**Proof.** From the corollary of Theorem 8 it follows that the bicompletion of $(X, \mathcal{U})$ is a $T_1$ quasi-uniform space. Now the conclusion follows from the well-known fact that the bicompletion of a $T_0$ quiet (respectively locally quiet, fitting, locally fitting) is quiet (respectively locally quiet, fitting, locally fitting).

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**References**


