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Representations of J-central J-Potapov functions in both nondegenerate and degenerate cases

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ABSTRACT

Let J be an $m \times m$ signature matrix (i.e. $J^* = J$ and $J^2 = I_m$) and let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Denote $\mathcal{P}_J(\mathbb{D})$ the class of all J-Potapov functions in $\mathbb{D}$, i.e. the set of all meromorphic $m \times m$ matrix-valued functions f in $\mathbb{D}$ with J-contractive values at all points of $\mathbb{D}$ at which f is holomorphic. Further, denote $\mathcal{P}_{J,0}(\mathbb{D})$ the subclass of all $f \in \mathcal{P}_J(\mathbb{D})$ which are holomorphic at the origin. Let $f \in \mathcal{P}_{J,0}(\mathbb{D})$, and let $f(w) = \sum_{j=0}^{\infty} A_j w^j$ be the Taylor series representation of f in some neighborhood of 0. Then it was proved in [B. Fritzsche, B. Kirstein, U. Raabe, On the structure of J-Potapov sequences, Linear Algebra Appl., in press] that for each $n \in \mathbb{N}$ the matrix A_n can be described by its position in a matrix ball depending on the sequence $(A_j)_{j=0}^{n-1}$. The J-Potapov function f is called J-central if there exists some $k \in \mathbb{N}$ such that for each integer $j \geqslant k$ the matrix A_j coincides with the center of the corresponding matrix ball.

In this paper, we derive left and right quotient representations of matrix polynomials for J-central J-Potapov functions in $\mathbb{D}$. Moreover, we obtain recurrent formulas for the matrix polynomials involved in these quotient representations.

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0. Introduction

This paper continues the systematic study of J-Potapov functions which was started in the papers [8,9]. The class of J-Potapov functions originates in the fundamental paper [10] by V.P. Potapov, who developed a far-reaching factorization theory for these functions. The main object of the present paper is a distinguished subclass of J-Potapov functions which was introduced in [9, Section 5], namely the subclass of J-central J-Potapov functions. This subclass is a natural generalization of the class of all central $p \times q$ Schur functions which were introduced in [5] (see also [2]). The investigation of the matricial Schur problem clarified the distinguished role of central $p \times q$ Schur functions. In the paper [7], there was worked out an approach to the matricial Schur problem in both nondegenerate and degenerate cases which is based on an intensive use of central $p \times q$ Schur functions. Here special representations of central $p \times q$ Schur functions as left and right quotients of suitable matrix polynomials, which were obtained in [6], played the key role in the computation of the solution set of the matricial Schur problem in the general case. In the nondegenerate case, the description of the solution set coincides with that parametrization which was given by Arov/Krein [1] (see also [2, Section 3.10]). The particular matrix polynomials which realize the left and right quotient representations of central $p \times q$ Schur functions coincide in the nondegenerate case up to a normalizing (left or right) factor with the particular matrix polynomials which were used by Arov/Krein [1].

The main goal of this paper is to construct appropriate matrix polynomials with the aid of which left and right quotient representations of J-central J-Potapov functions can be realized. On this way, we are guided by the strategy which was used in [6]. In the particular case that the signature matrix J coincides with the $m \times m$ unit matrix I_m we reobtain representations of central $m \times m$ Schur functions as left and right quotients of matrix polynomials, which were constructed in [6].

The main results of this paper form the basis of our approach to the treatment of the interpolation problem for J-Potapov functions in both nondegenerate and degenerate cases. This will be done in subsequent work.

This paper is organized as follows. In Section 1, we summarize some basic facts on J-Potapov functions in the open unit disk, J-Potapov sequences and their interrelations. These results originate from [8,9].

In Section 2, we prove the announced representations of J-central J-Potapov functions as left and right quotients of matrix polynomials (see Theorems 2.7 and 2.8). It will be shown (see Propositions 2.18 and 2.19) that the left and right quotient representations can be chosen in such way that the zeros of the determinant of the "denominator matrix polynomials" are exactly the poles of the J-central J-Potapov function under consideration.

Section 3 is devoted to create recursive constructions of the matrix polynomials which were found in Section 2 to generate left and right quotient representations of J-central J-Potapov functions.

In Section 4, we will consider those matrix polynomials which will turn out to play later the role of a resolvent matrix for the general (possibly degenerate) interpolation problem for J-Potapov functions. In particular, we will obtain representations of these matrix polynomials as a product of elementary factors (see Proposition 4.3). The results of Section 4 generalize some facts which were obtained in [7, Section 4] for the case of a given finite $p \times q$ Schur sequence.

In the final Section 5, we consider the case of a given finite strict J-Potapov sequence which allows to simplify some of the foregoing results obtained for a general J-Potapov sequence.

1. Some preliminaries on J-Potapov functions in the open unit disk and on J-Potapov sequences

Throughout this paper, let m be a positive integer. We will use the notations $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{C}$ for the set of all positive integers, the set of all nonnegative integers, and the set of all complex numbers, respectively. If $s \in \mathbb{N}_0$ and $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, then $\mathbb{N}_{s,\kappa}$ denotes the set of all integers n satisfying $s \leqslant n \leqslant \kappa$. Further, let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

Let $p, q \in \mathbb{N}$. Then $\mathbb{C}^{p\times q}$ designates the set af all complex $p \times q$ matrices. The notation $0_{p\times q}$ stands for the null matrix which belongs to $\mathbb{C}^{p\times q}$, and the identity matrix which belongs to $\mathbb{C}^{q\times q}$ will be designated by I_q. In cases where the size of a null matrix or the size of an identity matrix is obvious, we will omit the indeces. If $A \in \mathbb{C}^{p\times q}$, then A^+ stands for the Moore–Penrose inverse of A. Furthermore, for each $A \in \mathbb{C}^{p\times q}$, let $\mathcal{R}(A)$ be the range of A and let $\mathcal{N}(A)$ be the nullspace of A.

We will write $\mathbb{C}^{q\times q}_{\geqslant}$ (respectivly, $\mathbb{C}^{q\times q}_{>}$) to denote the set of all nonnegative (respectively, positive) Hermitian matrices belonging to $\mathbb{C}^{q\times q}$. In the set $\mathbb{C}^{q\times q}_{H}$ of all Hermitian $q \times q$ matrices we will use the Löwner semi-ordering, i.e., we will write $A \leqslant B$ or $B \geqslant A$ to indicate that A and B are Hermitian matrices of the same size such that $B - A$ is nonnegative Hermitian.

Let n and $p_1, \dots, p_n$ be positive integers, and let $A_j \in \mathbb{C}^{p_j\times p_j}$ for each $j \in \mathbb{N}_{1,n}$. Then $\operatorname{diag}(A_1, \dots, A_n)$ denotes the block diagonal matrix with diagonal blocks $A_1, \dots, A_n$.

If f is an $m \times m$ matrix-valued function which is meromorphic in the open unit disk $\mathbb{D}$, then let $\mathbb{H}_f$ be the set of all points at which f is holomorphic. Let J be an $m \times m$ signature matrix and let f be a $\mathbb{C}^{m\times m}$-valued function which is meromorphic in $\mathbb{D}$. Then f is called a *J-Potapov function in* $\mathbb{D}$ (respectively, a *strong J-Potapov function in* $\mathbb{D}$), if for each $w \in \mathbb{H}_f$ the matrix $f(w)$ is J-contractive (respectively, strictly J-contractive). Here a matrix $A \in \mathbb{C}^{m\times m}$ is called *J-contractive* (respectively, *strictly J-contractive*), if the matrix $J - A^*JA$ is nonnegative Hermitian (respectively, positive Hermitian). For each $m \times m$ signature matrix J, we will use the notation $\mathcal{P}_J(\mathbb{D})$ (respectively, $\mathcal{P}'_J(\mathbb{D})$) to denote the set of all J-Potapov functions in $\mathbb{D}$ (respectively, strong J-Potapov functions in $\mathbb{D}$). We will turn particular attention to a distinguished subclass of $\mathcal{P}_J(\mathbb{D})$, namely the class

$$\mathcal{P}_{J,0}(\mathbb{D}) := \{f \in \mathcal{P}_J(\mathbb{D}) : 0 \in \mathbb{H}_f\}.$$

In the case $J = I_m$ the classes $\mathcal{P}_J(\mathbb{D})$ and $\mathcal{P}_{J,0}(\mathbb{D})$ coincide. Indeed, $\mathcal{P}_{I_m}(\mathbb{D})$ is exactly the set $\mathcal{S}_{m\times m}(\mathbb{D})$ of all $m \times m$ Schur functions in $\mathbb{D}$, i.e. the set of all matrix-valued functions $f : \mathbb{D} \to \mathbb{C}^{m\times m}$ which are holomorphic in $\mathbb{D}$ and the values of which are contractive complex $m \times m$ matrices.

Observe that the well-known concept of Potapov–Ginzburg transformation yields an interrelation between the classes $\mathcal{P}_J(\mathbb{D})$ and $\mathcal{S}_{m\times m}(\mathbb{D})$ on the one-hand side and between the strong J-Potapov class $\mathcal{P}'_J(\mathbb{D})$ and the strong Schur class $\mathcal{S}'_{m\times m}(\mathbb{D})$ of all $f \in \mathcal{S}_{m\times m}(\mathbb{D})$ for which the matrix $f(w)$ is strictly contractive for each $w \in \mathbb{D}$ on the other-hand side (see [8, Proposition 3.4]).

The sequences $(A_j)_{j=0}^{\infty}$ of Taylor coefficients of the matrix-valued functions which belong to the class $\mathcal{P}_{J,0}(\mathbb{D})$ can be characterized in a clear way. In order to recall this characterization we introduce some notations. Observe that, for each $m \times m$ signature matrix J and every nonnegative integer n, the complex $(n+1)m \times (n+1)m$ matrix

$$J_{[n]} := \operatorname{diag}(J, \dots, J) \tag{1.1}$$

is an $(n+1)m \times (n+1)m$ signature matrix. If $n \in \mathbb{N}_0$, then a sequence $(A_j)_{j=0}^{n}$ of complex $m \times m$ matrices is called a *J-Potapov sequence* (respectively, a *strict J-Potapov sequence*) if the block Toeplitz matrix

$$S_n := \begin{pmatrix} A_0 & 0_{m\times m} & \dots & 0_{m\times m} \\ A_1 & A_0 & \dots & 0_{m\times m} \\ \vdots & \vdots & & \vdots \\ A_n & A_{n-1} & \dots & A_0 \end{pmatrix} \tag{1.2}$$

is $J_{[n]}$-contractive (respectively, strictly $J_{[n]}$-contractive). For each $n \in \mathbb{N}_0$ we will use $\mathcal{P}^{\leqslant}_{J,n}$ (respectively, $\mathcal{P}^{<}_{J,n}$) to designate the set of all J-Potapov sequences (respectively, strict J-Potapov sequences) $(A_j)_{j=0}^{n}$. From [9, Lemma 3.2 (respectively, Lemma 3.3)] it follows that if $(A_j)_{j=0}^{n}$ belongs to $\mathcal{P}^{\leqslant}_{J,n}$ (respectively, $\mathcal{P}^{<}_{J,n}$), then $(A_j)_{j=0}^{k} \in \mathcal{P}^{\leqslant}_{J,k}$ (respectively, $(A_j)_{j=0}^{k} \in \mathcal{P}^{<}_{J,k}$) for each $k \in \mathbb{N}_{0,n}$. A sequence $(A_j)_{j=0}^{\infty}$ of complex $m \times m$ matrices is said to be a *J-Potapov sequence* (respectively, a *strict J-Potapov sequence*) if for each $n \in \mathbb{N}_0$ the sequence $(A_j)_{j=0}^{n}$ is a J-Potapov sequence (respectively, a strict J-Potapov sequence). We will write $\mathcal{P}^{\leqslant}_{J,\infty}$ for the set of all J-Potapov sequences $(A_j)_{j=0}^{\infty}$ and $\mathcal{P}^{<}_{J,\infty}$ for the set of all strict J-Potapov sequences $(A_j)_{j=0}^{\infty}$.

Now we can formulate the Taylor series characterization of the class $\mathcal{P}_{J,0}(\mathbb{D})$.

Theorem 1.1. *Let J be an $m \times m$ signature matrix. Then:*

(a) If $f \in \mathcal{P}_{J,0}(\mathbb{D})$ and if

$$f(w) = \sum_{j=0}^{\infty} A_j w^j \tag{1.3}$$

is the Taylor series representation of f in some neighborhood of 0, then $(A_j)_{j=0}^{\infty}$ is a J-Potapov sequence.

(b) If $(A_j)_{j=0}^{\infty}$ is a J-Potapov sequence, then there is a unique $f \in \mathcal{P}_{J,0}(\mathbb{D})$ such that (1.3) holds for all w belonging to some neighborhood of 0.

A proof of Theorem 1.1 is given in [8, Theorem 6.2].

Considering the special case $J = I_m$ one can see immediately that Theorem 1.1 is a generalization of a well-known characterization of the Taylor coefficients of $m \times m$ Schur functions defined on $\mathbb{D}$ (see, e.g., [2, Theorem 5.1.1]).

In [8] the following interpolation problem for functions belonging to the class $\mathcal{P}_{J,0}(\mathbb{D})$ is discussed:

Interpolation problem for Potapov functions (P): Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^{n}$ be a sequence of complex $m \times m$ matrices. Describe the set $\mathcal{P}_{J,0}[\mathbb{D}, (A_j)_{j=0}^{n}]$ of all matrix-valued functions $f \in \mathcal{P}_{J,0}(\mathbb{D})$ such that

$$\frac{f^{(j)}(0)}{j!} = A_j$$

for each $j \in \mathbb{N}_{0,n}$ where the notation $f^{(j)}$ stands for the jth derivative of f.

In [8, Theorem 7.2], it was proved that the set $\mathcal{P}_{J,0}\left[\mathbb{D}, (A_j)_{j=0}^{n}\right]$ is nonempty if and only if $(A_j)_{j=0}^{n}$ is a J-Potapov sequence.

We will now give some more notations that will be used throughout this paper. For each $n \in \mathbb{N}_0$, let the matrix polynomials $e_{n,m} : \mathbb{C} \to \mathbb{C}^{m\times(n+1)m}$ and $\varepsilon_{n,m} : \mathbb{C} \to \mathbb{C}^{(n+1)m\times m}$ be defined by

$$e_{n,m}(w) := (I_m, wI_m, \ldots, w^n I_m) \quad \text{and} \quad \varepsilon_{n,m}(w) := \left(\bar{w}^n I_m, \bar{w}^{n-1} I_m, \ldots, I_m\right)^*. \tag{1.4}$$

Let J be an $m \times m$ signature matrix, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(A_j)_{j=0}^{\kappa}$ be a sequence of complex $m \times m$ matrices, and let $n \in \mathbb{N}_{0,\kappa}$. Then we will continue to use the notation S_n given by (1.2). Further, we designate

$$P_{nJ} := J_{[n]} - S_n J_{[n]} S_n^* \quad \text{and} \quad Q_{nJ} := J_{[n]} - S_n^* J_{[n]} S_n. \tag{1.5}$$

In the case $n \in \mathbb{N}_{1,\kappa}$ we will use the block matrices

$$y_n := (A_1^*, A_2^*, \ldots, A_n^*)^* \quad \text{and} \quad z_n := (A_n, A_{n-1}, \ldots, A_1). \tag{1.6}$$

If, additionally, $s \in \mathbb{N}_{n+1,\kappa+1}$ (where, in the case $\kappa = +\infty$, we set $\kappa + 1 := +\infty$), then let

$$y_{s,n} := \left(A_{s-n}^*, A_{s-n+1}^*, \ldots, A_{s-1}^*\right)^* \quad \text{and} \quad z_{s,n} := (A_{s-1}, A_{s-2}, \ldots, A_{s-n}). \tag{1.7}$$

Moreover, for each $n \in \mathbb{N}_{0,\kappa}$ we will work with the matrices

$$M_{n+1J} := \begin{cases} 0_{m\times m}, & \text{if } n = 0, \\ -z_n J_{[n-1]} S_{n-1}^* P_{n-1J}^{+} y_n, & \text{if } n \in \mathbb{N}_{1,\kappa}, \end{cases} \tag{1.8}$$

$$L_{n+1J} := \begin{cases} J - A_0 J A_0^*, & \text{if } n = 0, \\ J - A_0 J A_0^* - z_n Q_{n-1J}^{+} z_n^*, & \text{if } n \in \mathbb{N}_{1,\kappa} \end{cases} \tag{1.9}$$

and

$$R_{n+1J} := \begin{cases} J - A_0^* J A_0, & \text{if } n = 0, \\ J - A_0^* J A_0 - y_n^* P_{n-1J}^{+} y_n, & \text{if } n \in \mathbb{N}_{1,\kappa}. \end{cases} \tag{1.10}$$

Observe that if $(A_j)_{j=0}^{\kappa}$ is a J-Potapov sequence, then for each $n \in \mathbb{N}_{0,\kappa}$ the matrices L_{n+1J} and R_{n+1J} are both nonnegative Hermitian (see [9, Lemma 3.7]).

Let us now recall the notion of a matrix ball. Let $p, q \in \mathbb{N}$. For each $C \in \mathbb{C}^{p\times q}$, each $A \in \mathbb{C}^{p\times p}$ and each $B \in \mathbb{C}^{q\times q}$, the set of all $X \in \mathbb{C}^{p\times q}$ which admit the representation $X = C + AKB$ with some contractive (respectively, strictly contractive) complex $p \times q$ matrix K is called the *matrix ball* (respectively, *open matrix ball*) with *center* C, *left semi-radius* A, and *right semi-radius* B and it will be denoted by $\mathcal{K}(C; A, B)$ (respectively, $\mathcal{K}^\circ(C; A, B)$).

The theory of matrix and operator balls was developed by Yu.L. Smuljan [12] (see also [2, Section 1.5]).

In [9, Theorem 3.9] there was shown the following result which enlightens the inner structure of J-Potapov sequences.

Theorem 1.2. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^{n+1}$ be a sequence of complex $m \times m$ matrices. Then:*

(*a*) *The following statements are equivalent:*
- (*i*) $(A_j)_{j=0}^{n+1}$ *is a J-Potapov sequence.*
- (*ii*) $(A_j)_{j=0}^{n}$ *is a J-Potapov sequence and A_{n+1} belongs to the matrix ball* $\mathcal{K}\left(M_{n+1J}; \sqrt{L_{n+1J}}, \sqrt{R_{n+1J}}\right)$.

(*b*) *The following statements are equivalent:*
- (*iii*) $(A_j)_{j=0}^{n+1}$ *is a strict J-Potapov sequence.*
- (*iv*) $(A_j)_{j=0}^{n}$ *is a strict J-Potapov sequence and A_{n+1} belongs to the open matrix ball* $\mathcal{K}^\circ\left(M_{n+1J}; \sqrt{L_{n+1J}}, \sqrt{R_{n+1J}}\right)$.

Considering the special choice $J = I_m$, we see that Theorem 1.2 is a generalization of a well-known result for $m \times m$ Schur sequences (see, e.g., [2, Theorem 3.5.1]).

In [9], so-called J-central sequences are studied. A sequence $(A_j)_{j=0}^{\infty}$ of complex $m \times m$ matrices is said to be a *J-central sequence* if there exists some $n \in \mathbb{N}$ such that $A_k = M_{kJ}$ for each integer k with $k \geqslant n$. If $n \in \mathbb{N}$, then we will say that $(A_j)_{j=0}^{\infty}$ is a *J-central sequence of order n* if $A_k = M_{kJ}$ for each integer k with $k \geqslant n$. In this case the smallest positive integer n for which $(A_j)_{j=0}^{\infty}$ is a J-central sequence of order n is called *the minimal order of the J-central sequence* $(A_j)_{j=0}^{\infty}$. Theorem 1.2 implies that a J-central sequence $(A_j)_{j=0}^{\infty}$ of order n is a J-Potapov sequence if and only if $(A_j)_{j=0}^{n-1}$ is a J-Potapov sequence.

If $n \in \mathbb{N}_0$, and if $(A_j)_{j=0}^{n}$ is a sequence of $m \times m$ matrices then the sequence $(A_j)_{j=0}^{\infty}$ defined recursively by $A_k := M_{kJ}$ for each $k \in \mathbb{N}_{n+1,\infty}$ is said to be *the J-central sequence corresponding to* $(A_j)_{j=0}^{n}$.

In view of Theorem 1.1, the concept of J-centrality can also be fomulated in terms of J-Potapov functions. Let $f \in \mathcal{P}_{J,0}(\mathbb{D})$, and let (1.3) be the Taylor series representation of f in some neighborhood of 0. Then f is called a *J-central J-Potapov function* if $(A_j)_{j=0}^{\infty}$ is a J-central sequence. If $n \in \mathbb{N}$, then f is called a *J-central J-Potapov function of order n* (respectively, *of minimal order n*) if $(A_j)_{j=0}^{\infty}$ is a J-central J-Potapov sequence of order n (respectively, of minimal order n). Furthermore, if $n \in \mathbb{N}_0$, if some J-Potapov sequence $(A_j)_{j=0}^{n}$ is given, and if $(A_j)_{j=0}^{\infty}$ is the J-central J-Potapov sequence corresponding to $(A_j)_{j=0}^{n}$, then the (uniquely determined) function $f \in \mathcal{P}_{J,0}(\mathbb{D})$ satisfying (1.3) in some neighborhood of 0 is said to be *the J-central J-Potapov function corresponding to* $(A_j)_{j=0}^{n}$.

2. Representations of J-central J-Potapov functions as left and right quotients of matrix polynomials

In this section, we will derive explicit representations of J-central J-Potapov functions corresponding to some J-Potapov sequence $(A_j)_{j=0}^{n}$ where n is a nonnegative integer. In the following remark the (simple) case $n = 0$ is treated.

Remark 2.1. Let J be an $m \times m$ signature matrix, and let $(A_j)_{j=0}^0$ be a J-Potapov sequence. Then it is readily checked that the constant function (defined on $\mathbb{D}$) with value A_0 is the J-central J-Potapov function corresponding to $(A_j)_{j=0}^0$.

Now we turn our attention to the case $n \in \mathbb{N}$. First we make some easy observations on rational matrix-valued functions and their series of Taylor coefficients.

Remark 2.2. Let $n, p, q \in \mathbb{N}$, let $(B_j)_{j=0}^n$ be a sequence of complex $p \times q$ matrices, and let $(C_j)_{j=1}^n$ be a sequence of complex $q \times q$ matrices. Let the matrix polynomials $g : \mathbb{C} \to \mathbb{C}^{p\times q}$ and $h : \mathbb{C} \to \mathbb{C}^{q\times q}$ be defined by

$$g(w) := \sum_{k=0}^{n} B_k w^k \quad \text{and} \quad h(w) := I_q + \sum_{k=1}^{n} C_k w^k. \tag{2.1}$$

Then the $p \times q$ matrix-valued function $f := gh^{-1}$ is meromorphic in $\mathbb{C}$ and holomorphic at the origin. Let $f(w) = \sum_{j=0}^{\infty} A_j w^j$ be the Taylor series representation of f in some neighborhood of 0. Then $A_0 = B_0$ holds. Moreover, it is readily checked that for each $k \in \mathbb{N}_{1,n}$ the equation $A_k = B_k - \sum_{j=1}^{k} A_{k-j}C_j$ is fulfilled and that for each $k \in \mathbb{N}_{n+1,\infty}$ the recursion formula

$$A_k = -\sum_{j=1}^{n} A_{k-j}C_j \tag{2.2}$$

holds true.

Lemma 2.3. *Let $n, p, q \in \mathbb{N}$, and let $(C_j)_{j=1}^n$ be a sequence of complex $q \times q$ matrices. Let $(A_j)_{j=0}^{\infty}$ be a sequence of complex $p \times q$ matrices such that for each $k \in \mathbb{N}_{n+1,\infty}$ the recursion formula (2.2) is satisfied. Let $B_0 := A_0$, and for each $k \in \mathbb{N}_{1,n}$ let*

$$B_k := A_k + \sum_{j=1}^{k} A_{k-j}C_j. \tag{2.3}$$

Let the matrix polynomials $g : \mathbb{C} \to \mathbb{C}^{p\times q}$ and $h : \mathbb{C} \to \mathbb{C}^{q\times q}$ be defined by (2.1), and let $f := gh^{-1}$. Then f admits the Taylor series representation $f(w) = \sum_{j=0}^{\infty} A_j w^j$ in some neighborhood of 0. Moreover, the matrix polynomials g and h can be represented via

$$g(w) = A_0 + we_{n-1,p}(w)(y_n + S_{n-1}V_n) \tag{2.4}$$

and

$$h(w) = I_q + we_{n-1,q}(w)V_n \tag{2.5}$$

for each $w \in \mathbb{C}$ where $V_n := (C_1^*, C_2^*, \ldots, C_n^*)^*$ and where y_n and S_{n-1} are given by (1.6) and (1.2).

Proof. It is readily checked that Eqs. (2.4) and (2.5) hold true. Furthermore, f is a (well-defined) $p \times q$ matrix-valued function which is meromorphic in $\mathbb{C}$ and holomorphic at 0. Let $f(w) = \sum_{j=0}^{\infty} \widetilde{A}_j w^j$ be the Taylor series representation of f in some neighborhood of 0. Then Remark 2.2 implies $\widetilde{A}_0 = B_0 = A_0$. Hence there is a positive integer s such that $\widetilde{A}_k = A_k$ holds for each $k \in \mathbb{N}_{0,s-1}$. If $s \leqslant n$, then Remark 2.2 and (2.3) yield $\widetilde{A}_s = B_s - \sum_{j=1}^{s} \widetilde{A}_{s-j}C_j = A_s$. Moreover, if $s \geqslant n+1$, then from Remark 2.2 and (2.2) we get $\widetilde{A}_s = -\sum_{j=1}^{n} \widetilde{A}_{s-j}C_j = A_s$. Consequently, $\widetilde{A}_k = A_k$ is true for each $k \in \mathbb{N}_0$. Thus, the proof is complete. $\square$

In [9, Theorem 5.8] certain recursion formulas for the elements of a J-central J-Potapov sequence $(A_j)_{j=0}^{\infty}$ were derived. Our next considerations are aimed at obtaining some slight generalizations of these formulas.

Lemma 2.4. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, let $k \in \mathbb{N}_{n+1,\infty}$, and let $(A_j)_{j=0}^{k-1}$ be a J-Potapov sequence. Then the following statements hold:*

(a) $P_{n-1,J}P_{n-1,J}^{+}S_{n-1}J_{[n-1]}z_{k,n}^{*} = S_{n-1}J_{[n-1]}z_{k,n}^{*}$ *and* $Q_{n-1,J}Q_{n-1,J}^{+}z_{k,n}^{*} = z_{k,n}^{*}$.
(b) $Q_{n-1,J}Q_{n-1,J}^{+}S_{n-1}^{*}J_{[n-1]}y_{k,n} = S_{n-1}^{*}J_{[n-1]}y_{k,n}$ *and* $P_{n-1,J}P_{n-1,J}^{+}y_{k,n} = y_{k,n}$.

Proof. Let

$$B_{kn} := \left(z_{n+1,n}^{*}, z_{n+2,n}^{*}, \ldots, z_{k,n}^{*}\right)^{*}. \tag{2.6}$$

Then using the block partitions

$$S_{k-1} = \begin{pmatrix} S_{n-1} & 0 \\ B_{kn} & S_{k-n-1} \end{pmatrix}$$

and $J_{[k-1]} = \operatorname{diag}(J_{[n-1]}, J_{[k-n-1]})$ we get

$$P_{k-1,J} = \begin{pmatrix} P_{n-1,J} & -S_{n-1}J_{[n-1]}B_{kn}^{*} \\ * & * \end{pmatrix}. \tag{2.7}$$

Since $(A_j)_{j=0}^{k-1}$ is a J-Potapov sequence, the matrix S_{k-1} is $J_{[k-1]}$-contractive. Hence S_{k-1}^{*} is $J_{[k-1]}$-contractive as well (see, e.g., [2, Theorem 1.3.3]), i.e., the matrix $P_{k-1,J}$ is nonnegative Hermitian. Thus, from (2.6), (2.7), and a well-known characterization of nonnegative Hermitian block matrices (see, e.g., [2, Lemma 1.1.9]) we obtain

$$\mathcal{R}\left(S_{n-1}J_{[n-1]}z_{k,n}^{*}\right) \subseteq \mathcal{R}\left(S_{n-1}J_{[n-1]}B_{kn}^{*}\right) \subseteq \mathcal{R}(P_{n-1,J}). \tag{2.8}$$

Hence the first equation stated in (a) holds true. According to [9, Lemma 3.5], the inclusions in (2.8) imply $\mathcal{R}\left(z_{k,n}^{*}\right) \subseteq \mathcal{R}(Q_{n-1,J})$. Thus, the second equation stated in (a) is satisfied as well. Part (b) can be shown analogously to part (a). □

In the sequel, we will use the following notations. Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, and let $(A_j)_{j=0}^{n}$ be a J-Potapov sequence. Then let

$$\mathcal{Y}_{nJ} := \left\{V \in \mathbb{C}^{nm\times m} : Q_{n-1,J}V = S_{n-1}^{*}J_{[n-1]}y_n\right\} \tag{2.9}$$

and

$$\mathcal{Z}_{nJ} := \left\{W \in \mathbb{C}^{m\times nm} : WP_{n-1,J} = z_nJ_{[n-1]}S_{n-1}^{*}\right\}. \tag{2.10}$$

Remark 2.5. Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, and let $(A_j)_{j=0}^{n}$ be a J-Potapov sequence. Furthermore, let

$$V_n^{\square} := Q_{n-1,J}^{+}S_{n-1}^{*}J_{[n-1]}y_n \quad \text{and} \quad W_n^{\square} := z_nJ_{[n-1]}S_{n-1}^{*}P_{n-1,J}^{+}. \tag{2.11}$$

Then Lemma 2.4 yields in particular $V_n^{\square} \in \mathcal{Y}_{nJ}$ and $W_n^{\square} \in \mathcal{Z}_{nJ}$.

The recursion formulas given in the following proposition will play a key role in our investigations.

Proposition 2.6. *Let $n \in \mathbb{N}$, and let $(A_j)_{j=0}^{\infty}$ be a J-Potapov sequence which is J-central of order $n+1$. Furthermore, let $V_n \in \mathcal{Y}_{nJ}$ and $W_n \in \mathcal{Z}_{nJ}$. For each integer k with $k \geqslant n+1$, then*

$$A_k = -z_{k,n}V_n \quad \text{and} \quad A_k = -W_ny_{k,n}. \tag{2.12}$$

Proof. Let $V_n^{\square}$ be given by (2.11). Then, according to [9, Theorem 5.8], we have $A_k = -z_{k,n}V_n^{\square}$ for each $k \in \mathbb{N}_{n+1,\infty}$. Thus, Lemma 2.4 yields $-z_{k,n}V_n = -z_{k,n}Q_{n-1,J}^{+}Q_{n-1,J}V_n = -z_{k,n}V_n^{\square} = A_k$ for each $k \in \mathbb{N}_{n+1,\infty}$. Analogously, the second equation stated in (2.12) can be verified. □

The next two theorems contain the desired representations of J-central J-Potapov functions as quotients of matrix polynomials. These quotient representations are a generalization of the corresponding result for $m \times m$ Schur functions, which was proved in [6].

Theorem 2.7. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^n$ be a J-Potapov sequence. Denote $f_{c,n}$ the J-central J-Potapov function corresponding to $(A_j)_{j=0}^n$. If $n \in \mathbb{N}$, then let $V_n \in \mathcal{Y}_{nJ}$. Let the matrix polynomials $\pi_{nJ} : \mathbb{C} \to \mathbb{C}^{m\times m}$ and $\rho_{nJ} : \mathbb{C} \to \mathbb{C}^{m\times m}$ be given by*

$$\pi_{nJ}(\zeta) := \begin{cases} A_0, & \text{if } n = 0, \\ A_0 + \zeta e_{n-1,m}(\zeta)(y_n + S_{n-1}V_n), & \text{if } n \in \mathbb{N} \end{cases} \tag{2.13}$$

and

$$\rho_{nJ}(\zeta) := \begin{cases} I_m, & \text{if } n = 0, \\ I_m + \zeta e_{n-1,m}(\zeta)V_n, & \text{if } n \in \mathbb{N}. \end{cases} \tag{2.14}$$

Then

$$f_{c,n} = \pi_{nJ,\mathbb{D}}\, \rho_{nJ,\mathbb{D}}^{-1}, \tag{2.15}$$

where $\pi_{nJ,\mathbb{D}}$ (respectively, $\rho_{nJ,\mathbb{D}}$) is the restriction of π_{nJ} (respectively, ρ_{nJ}) onto $\mathbb{D}$.

Proof. In the case $n = 0$ the assertion follows immediately from Remark 2.1. Now suppose $n \in \mathbb{N}$. Let $f_n := \pi_{nJ}\rho_{nJ}^{-1}$. Obviously, f_n is meromorphic in $\mathbb{C}$ and holomorphic at 0. Let $(A_j)_{j=0}^\infty$ be the J-central sequence corresponding to $(A_j)_{j=0}^n$. In view of Theorem 1.2, $(A_j)_{j=0}^\infty$ is a J-Potapov sequence, which is J-central of order $n + 1$. Hence Proposition 2.6 implies $A_k = -z_{k,n}V_n$ for each $k \in \mathbb{N}_{n+1,\infty}$. Thus, application of Lemma 2.3 yields that f_n admits the Taylor series representation $f_n(w) = \sum_{j=0}^\infty A_j w^j$ in some neighborhood of 0. Consequently, $f_{c,n}$ is the restriction of f_n onto $\mathbb{D} \cap \mathbb{H}_{f_n}$. This implies (2.15). □

The following theorem can be shown similarly.

Theorem 2.8. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^n$ be a J-Potapov sequence. Denote $f_{c,n}$ the J-central J-Potapov function corresponding to $(A_j)_{j=0}^n$. If $n \in \mathbb{N}$, then let $W_n \in \mathcal{Z}_{nJ}$. Let the matrix polynomials $\sigma_{nJ} : \mathbb{C} \to \mathbb{C}^{m\times m}$ and $\tau_{nJ} : \mathbb{C} \to \mathbb{C}^{m\times m}$ be given by*

$$\sigma_{nJ}(\zeta) := \begin{cases} A_0, & \text{if } n = 0, \\ (W_nS_{n-1} + z_n)\zeta \varepsilon_{n-1,m}(\zeta) + A_0, & \text{if } n \in \mathbb{N} \end{cases} \tag{2.16}$$

and

$$\tau_{nJ}(\zeta) := \begin{cases} I_m, & \text{if } n = 0, \\ W_n\zeta \varepsilon_{n-1,m}(\zeta) + I_m, & \text{if } n \in \mathbb{N}. \end{cases} \tag{2.17}$$

Then

$$f_{c,n} = \tau_{nJ,\mathbb{D}}^{-1}\, \sigma_{nJ,\mathbb{D}}\,, \tag{2.18}$$

where $\sigma_{nJ,\mathbb{D}}$ (respectively, $\tau_{nJ,\mathbb{D}}$) is the restriction of σ_{nJ} (respectively, τ_{nJ}) onto $\mathbb{D}$.

At this point it seems to be useful to present some more preliminaries on the matrices Q_{nJ} and P_{nJ} given by (1.5). Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, and let us consider an arbitrary sequence $(A_j)_{j=0}^n$ of complex $m \times m$ matrices. Then from the block representations

$$S_n = \begin{pmatrix} S_{n-1} & 0 \\ z_n & A_0 \end{pmatrix} \quad \text{and} \quad S_n = \begin{pmatrix} A_0 & 0 \\ y_n & S_{n-1} \end{pmatrix}$$

of S_n we see that Q_{nJ} and P_{nJ} can be represented via

$$Q_{nJ} = \begin{pmatrix} Q_{n-1J} - z_n^* J z_n & -z_n^* J A_0 \\ -A_0^* J z_n & R_{1J} \end{pmatrix}, \tag{2.19}$$

$$Q_{nJ} = \begin{pmatrix} R_{1J} - y_n^* J_{[n-1]} y_n & -y_n^* J_{[n-1]} S_{n-1} \\ -S_{n-1}^* J_{[n-1]} y_n & Q_{n-1J} \end{pmatrix}, \tag{2.20}$$

$$P_{nJ} = \begin{pmatrix} P_{n-1J} & -S_{n-1} J_{[n-1]} z_n^* \\ -z_n J_{[n-1]} S_{n-1}^* & L_{1J} - z_n J_{[n-1]} z_n^* \end{pmatrix} \tag{2.21}$$

and

$$P_{nJ} = \begin{pmatrix} L_{1J} & -A_0 J y_n^* \\ -y_n J A_0^* & P_{n-1J} - y_n J y_n^* \end{pmatrix}. \tag{2.22}$$

For each $s \in \mathbb{N}$ and each $B \in \mathbb{C}^{nm\times s}$, then (2.19) and (2.20) yield

$$\begin{pmatrix} B \\ 0_{m\times s} \end{pmatrix}^* Q_{nJ} \begin{pmatrix} B \\ 0_{m\times s} \end{pmatrix} = B^* Q_{n-1J} B - B^* z_n^* J z_n B \tag{2.23}$$

and

$$\begin{pmatrix} 0_{m\times s} \\ B \end{pmatrix}^* Q_{nJ} \begin{pmatrix} 0_{m\times s} \\ B \end{pmatrix} = B^* Q_{n-1J} B. \tag{2.24}$$

Similarly, for each $s \in \mathbb{N}$ and each $C \in \mathbb{C}^{nm\times s}$, from (2.21) and (2.22) we get

$$\begin{pmatrix} C \\ 0_{m\times s} \end{pmatrix}^* P_{nJ} \begin{pmatrix} C \\ 0_{m\times s} \end{pmatrix} = C^* P_{n-1J} C \tag{2.25}$$

and

$$\begin{pmatrix} 0_{m\times s} \\ C \end{pmatrix}^* P_{nJ} \begin{pmatrix} 0_{m\times s} \\ C \end{pmatrix} = C^* P_{n-1J} C - C^* y_n J y_n^* C. \tag{2.26}$$

Lemma 2.9. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, and let $(A_j)_{j=0}^n$ be a J-Potapov sequence. Then:*

(a) For each $V_n \in \mathcal{Y}_{nJ}$,

$$Q_{nJ} \begin{pmatrix} I_m \\ V_n \end{pmatrix} = \begin{pmatrix} R_{n+1J} \\ 0_{nm\times m} \end{pmatrix}.$$

(b) For each $W_n \in \mathcal{Z}_{nJ}$,

$$(W_n, I_m) P_{nJ} = (0_{m\times nm}, L_{n+1J}).$$

Proof. According to [9, Lemma 3.7], we have

$$R_{n+1J} = R_{1J} - y_n^* J_{[n-1]} y_n - y_n^* J_{[n-1]} S_{n-1} Q_{n-1J}^+ S_{n-1}^* J_{[n-1]} y_n. \tag{2.27}$$

Moreover, part (b) of Lemma 2.4 implies

$$Q_{n-1J} Q_{n-1J}^+ S_{n-1}^* J_{[n-1]} y_n = S_{n-1}^* J_{[n-1]} y_n. \tag{2.28}$$

Now let $V_n \in \mathcal{Y}_{nJ}$. Then

$$Q_{n-1J} V_n = S_{n-1}^* J_{[n-1]} y_n \tag{2.29}$$

holds. From (2.28) and (2.29) we get

$$\begin{aligned} y_n^* J_{[n-1]} S_{n-1} V_n &= y_n^* J_{[n-1]} S_{n-1} Q_{n-1J}^+ Q_{n-1J} V_n \\ &= y_n^* J_{[n-1]} S_{n-1} Q_{n-1J}^+ S_{n-1}^* J_{[n-1]} y_n. \end{aligned} \tag{2.30}$$

Thus, using (2.20), (2.30), (2.27), and (2.29) we obtain

$$Q_{nJ}\begin{pmatrix} I_m \\ V_n \end{pmatrix} = \begin{pmatrix} R_{1J} - y_n^* J_{[n-1]} y_n - y_n^* J_{[n-1]} S_{n-1} V_n \\ -S_{n-1}^* J_{[n-1]} y_n + Q_{n-1J} V_n \end{pmatrix} = \begin{pmatrix} R_{n+1J} \\ 0_{nm\times m} \end{pmatrix}.$$

Hence part (a) is verified. Part (b) can be checked analogously. □

Remark 2.10. Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, and let $(A_j)_{j=0}^n$ be a J-Potapov sequence. Further, let $V_n \in \mathcal{Y}_{nJ}$, let $W_n \in \mathcal{Z}_{nJ}$, and let

$$V_n^{\#} := \begin{pmatrix} I_m \\ V_n \end{pmatrix} \quad \text{and} \quad W_n^{\#} := (W_n, I_m). \tag{2.31}$$

Then it is readily checked that the matrix polynomials π_{nJ}, ρ_{nJ}, σ_{nJ}, and τ_{nJ} given by (2.13), (2.14), (2.16), and (2.17), respectively, satisfy the equations

$$\pi_{nJ} = e_{n,m} S_n V_n^{\#}, \quad \rho_{nJ} = e_{n,m} V_n^{\#}, \quad \sigma_{nJ} = W_n^{\#} S_n \varepsilon_{n,m}, \quad \text{and} \quad \tau_{nJ} = W_n^{\#} \varepsilon_{n,m}.$$

In the sequel, we will use the following notion. Let $p, q \in \mathbb{N}$, and let b be a $p \times q$ matrix polynomial, i.e., there are an $n \in \mathbb{N}_0$ and a matrix $B \in \mathbb{C}^{(n+1)p\times q}$ such that $b(\zeta) = e_{n,p}(\zeta)B$ holds for each $\zeta \in \mathbb{C}$. Then the *reciprocal matrix polynomial* $\tilde{b}^{[n]}$ *of* b *with respect to the unit circle* $\mathbb{T}$ *and the formal degree* n is given by $\tilde{b}^{[n]}(\zeta) := B^* \varepsilon_{n,p}(\zeta)$ for each $\zeta \in \mathbb{C}$.

Let $n \in \mathbb{N}_0$, let J be an $m \times m$ signature matrix, let $(A_j)_{j=0}^n$ be a J-Potapov sequence, and let the matrix polynomials π_{nJ}, ρ_{nJ}, σ_{nJ}, and τ_{nJ} be introduced as in Theorems 2.7 and 2.8. Then Theorems 2.7 and 2.8 imply immediately

$$\tau_{nJ}\pi_{nJ} = \sigma_{nJ}\rho_{nJ} \quad \text{and} \quad \tilde{\pi}_{nJ}^{[n]}\tilde{\tau}_{nJ}^{[n]} = \tilde{\rho}_{nJ}^{[n]}\tilde{\sigma}_{nJ}^{[n]}. \tag{2.32}$$

The following result contains further important facts on the interplay between these matrix polynomials. In the special case $J = I_m$ it was shown in [7]. We will follow the proof given there.

Proposition 2.11. *Let* $n \in \mathbb{N}_0$, *let* J *be an* $m \times m$ *signature matrix, and let* $(A_j)_{j=0}^n$ *be a* J*-Potapov sequence. If* $n \geqslant 1$, *then let* $V_n \in \mathcal{Y}_{nJ}$ *and* $W_n \in \mathcal{Z}_{nJ}$. *Let the matrix polynomials* π_{nJ}, ρ_{nJ}, σ_{nJ}, *and* τ_{nJ} *be given by* (2.13), (2.14), (2.16), *and* (2.17), *respectively. Then*:

(*a*) *For each* $z \in \mathbb{T}$ *the equations*

$$(\rho_{nJ}(z))^* J \rho_{nJ}(z) - (\pi_{nJ}(z))^* J \pi_{nJ}(z) = R_{n+1J} \tag{2.33}$$

and

$$\tau_{nJ}(z) J (\tau_{nJ}(z))^* - \sigma_{nJ}(z) J (\sigma_{nJ}(z))^* = L_{n+1J} \tag{2.34}$$

hold true.

(*b*) *For each* $w \in \mathbb{C}$ *the identities*

$$\tilde{\rho}_{nJ}^{[n]}(w) J \rho_{nJ}(w) - \tilde{\pi}_{nJ}^{[n]}(w) J \pi_{nJ}(w) = w^n R_{n+1J} \tag{2.35}$$

and

$$\tau_{nJ}(w) J \tilde{\tau}_{nJ}^{[n]}(w) - \sigma_{nJ}(w) J \tilde{\sigma}_{nJ}^{[n]}(w) = w^n L_{n+1J} \tag{2.36}$$

are fulfilled.

Proof. The case $n = 0$ is trivial. Now suppose $n \geqslant 1$. For each $w \in \mathbb{C}$, let

$$F_n(w) := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ wI_m & 0 & 0 & \cdots & 0 & 0 \\ w^2I_m & wI_m & 0 & \cdots & 0 & 0 \\ w^3I_m & w^2I_m & wI_m & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ w^nI_m & w^{n-1}I_m & w^{n-2}I_m & \cdots & wI_m & 0 \end{pmatrix} \in \mathbb{C}^{(n+1)m\times(n+1)m}. \tag{2.37}$$

Then it is readily checked that for each $w \in \mathbb{C}\setminus\{0\}$ the identity

$$\left(e_{n,m}\left(\frac{1}{\bar{w}}\right)\right)^* Je_{n,m}(w) = J_{[n]}F_n\left(\frac{1}{w}\right) + J_{[n]} + (F_n(\bar{w}))^*J_{[n]} \tag{2.38}$$

holds true. Moreover, an easy calculation (see, e.g., [2, Lemma 4.2.1]) yields $F_n(w)S_n = S_nF_n(w)$ for each $w \in \mathbb{C}$. Thus, from (2.38) we get

$$S_n^*\left(e_{n,m}\left(\frac{1}{\bar{w}}\right)\right)^* Je_{n,m}(w)S_n = S_n^*J_{[n]}S_nF_n\left(\frac{1}{w}\right) + S_n^*J_{[n]}S_n + (F_n(\bar{w}))^*S_n^*J_{[n]}S_n \tag{2.39}$$

for each $w \in \mathbb{C}\setminus\{0\}$. Now let $z \in \mathbb{T}$. Then from (2.38), (2.39), and (1.5) the equation

$$(e_{n,m}(z))^*Je_{n,m}(z) - S_n^*(e_{n,m}(z))^*Je_{n,m}(z)S_n = Q_{nJ}F_n(\bar{z}) + Q_{nJ} + (F_n(\bar{z}))^*Q_{nJ} \tag{2.40}$$

follows. Taking into account Lemma 2.9 and (2.37), we infer

$$\left(V_n^\#\right)^* Q_{nJ}F_n(\bar{z})V_n^\# = 0_{m\times m} \tag{2.41}$$

and

$$\left(V_n^\#\right)^* Q_{nJ}V_n^\# = R_{n+1J} \tag{2.42}$$

where $V_n^\#$ is given by (2.31). In view of Remark 2.10, (2.40), (2.41), and (2.42) we obtain

$$\begin{aligned}&(\rho_{nJ}(z))^*J\rho_{nJ}(z) - (\pi_{nJ}(z))^*J\pi_{nJ}(z)\\ &\quad= \left(V_n^\#\right)^*\left((e_{n,m}(z))^*Je_{n,m}(z) - S_n^*(e_{n,m}(z)^*Je_{n,m}(z)S_n\right)V_n^\#\\ &\quad= \left(V_n^\#\right)^* Q_{nJ}F_n(\bar{z})V_n^\# + \left(V_n^\#\right)^* Q_{nJ}V_n^\# + \left(V_n^\#\right)^*(F_n(\bar{z}))^*Q_{nJ}V_n^\# = R_{n+1J},\end{aligned}$$

i.e., (2.33) holds. Eq. (2.34) can be verified analogously. Thus part (a) is proved. Furthermore, it is readily checked (see, e.g., [2, Lemma 1.2.2]) that the identities

$$\begin{aligned}&\tilde{\rho}_{nJ}^{[n]}(z) = z^n(\rho_{nJ}(z))^*, \quad \tilde{\pi}_{nJ}^{[n]}(z) = z^n(\pi_{nJ}(z))^*,\\ &\tilde{\tau}_{nJ}^{[n]}(z) = z^n(\tau_{nJ}(z))^* \quad\text{and}\quad \tilde{\sigma}_{nJ}^{[n]}(z) = z^n(\sigma_{nJ}(z))^*\end{aligned}$$

hold. From part (a) it follows therefore that the equations stated in (b) are valid for $w = z$. The Identity Theorem for holomorphic functions yields that these equations are actually fulfilled for each $w \in \mathbb{C}$. □

In view of Theorems 2.7 and 2.8, it is desirable to construct such quotient representations of a J-central J-Potapov function f where the zeros of the determinant of the "denominator function" are exactly the poles of f. Our next goal is to derive such representations. For the special case $J = I_m$, it was shown in [4] that there is a quotient representation of a central $m \times m$ Schur function where the determinant of the "denominator function" vanishes nowhere in $\mathbb{D}$. The proof given there employs an idea which was used by Ellis and Gohberg [3] for matrix polynomials constructed from positive Hermitian block Toeplitz matrices. Our approach will be based on a slight modification of the method applied in [4].

Lemma 2.12. *Let $n, p, q \in \mathbb{N}$, let $A \in \mathbb{C}^{p\times q}$, let $V \in \mathbb{C}^{np\times q}$, and let the matrix polynomial $g : \mathbb{C} \to \mathbb{C}^{p\times q}$ be given by $g(w) := A + we_{n-1,p}(w)V$. Let $\zeta \in \mathbb{C}$ and let $x \in \mathcal{N}(g(\zeta))$. Then there exists a matrix $B \in \mathbb{C}^{np\times q}$ such that*

$$\begin{pmatrix} A \\ V \end{pmatrix} x = \begin{pmatrix} 0_{p\times q} \\ B \end{pmatrix} x - \zeta \begin{pmatrix} B \\ 0_{p\times q} \end{pmatrix} x. \tag{2.43}$$

Proof. Let $T_{1,p} := 0_{p\times p}$, and in the case $n \in \mathbb{N}_{2,\infty}$ let

$$T_{n,p} := \begin{pmatrix} 0_{(n-1)p\times p} & I_{(n-1)p} \\ 0_{p\times p} & 0_{p\times(n-1)p} \end{pmatrix}. \tag{2.44}$$

Then it is readily checked that

$$[we_{n-1,p}(w) - \zeta e_{n-1,p}(\zeta)](I - \zeta T_{n,p}) = (w - \zeta)e_{n-1,p}(w)$$

holds for each $w \in \mathbb{C}$. Let $B := (I - \zeta T_{n,p})^{-1} V$. Then, for each $w \in \mathbb{C}$, we get

$$g(w)x = g(w)x - g(\zeta)x = [we_{n-1,p}(w) - \zeta e_{n-1,p}(\zeta)]Vx = (w - \zeta)e_{n-1,p}(w)Bx.$$

Comparing coefficients of corresponding powers of w we obtain (2.43). □

Lemma 2.13. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, and let $(A_j)_{j=0}^n$ be a J-Potapov sequence. Let $V_n \in \mathcal{Y}_{nJ}$, and let $\rho_{nJ} : \mathbb{C} \to \mathbb{C}^{m\times m}$ be defined by (2.14). Let $\zeta \in \mathbb{C}$, and let $x \in \mathcal{N}(\rho_{nJ}(\zeta))$. Then there exists a matrix $B \in \mathbb{C}^{nm\times m}$ such that the equations*

$$\begin{pmatrix} I_m \\ V_n \end{pmatrix} x = \begin{pmatrix} 0_{m\times m} \\ B \end{pmatrix} x - \zeta \begin{pmatrix} B \\ 0_{m\times m} \end{pmatrix} x \tag{2.45}$$

and

$$(|\zeta|^2 - 1)x^*B^*Q_{n-1J}Bx = x^*R_{n+1J}x + |\zeta|^2 x^*B^*z_n^*Jz_nBx \tag{2.46}$$

are satisfied.

Proof. According to Lemma 2.12, there exists a $B \in \mathbb{C}^{nm\times m}$ such that (2.45) holds. Further, using Lemma 2.9, $R_{n+1J} = R_{n+1J}^*$, and (2.45) we get

$$\begin{aligned} \zeta x^* \begin{pmatrix} I_m \\ V_n \end{pmatrix}^* Q_{nJ} \begin{pmatrix} B \\ 0_{m\times m} \end{pmatrix} x &= \zeta x^*(R_{n+1J}, 0_{m\times nm}) \begin{pmatrix} B \\ 0_{m\times m} \end{pmatrix} x \\ &= x^*(R_{n+1J}, 0_{m\times nm}) \left[\begin{pmatrix} 0_{m\times m} \\ B \end{pmatrix} x - \begin{pmatrix} I_m \\ V_n \end{pmatrix} x \right] \\ &= -x^*R_{n+1J}x. \end{aligned} \tag{2.47}$$

From (2.24), (2.45), Lemma 2.9, (2.47), and (2.23) we then infer

$$\begin{aligned} &x^*B^*Q_{n-1J}Bx \\ &= x^* \begin{pmatrix} 0_{m\times m} \\ B \end{pmatrix}^* Q_{nJ} \begin{pmatrix} 0_{m\times m} \\ B \end{pmatrix} x \\ &= \left[\begin{pmatrix} I_m \\ V_n \end{pmatrix} x + \zeta \begin{pmatrix} B \\ 0_{m\times m} \end{pmatrix} x \right]^* Q_{nJ} \left[\begin{pmatrix} I_m \\ V_n \end{pmatrix} x + \zeta \begin{pmatrix} B \\ 0_{m\times m} \end{pmatrix} x \right] \end{aligned}$$

$$= x^* R_{n+1J} x - x^* R_{n+1J} x - (x^* R_{n+1J} x)^* + |\zeta|^2 x^* \left(B^* Q_{n-1J} B - B^* z_n^* J z_n B\right) x$$
$$= |\zeta|^2 x^* B^* Q_{n-1J} B x - |\zeta|^2 x^* B^* z_n^* J z_n B x - x^* R_{n+1J} x$$

and consequently (2.46). □

Lemma 2.14. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, and let $(A_j)_{j=0}^n$ be a J-Potapov sequence. Let $V_n \in \mathcal{Y}_{nJ}$, and let $\pi_{nJ} : \mathbb{C} \to \mathbb{C}^{m\times m}$ be defined by (2.13). Let $\zeta \in \mathbb{C}$, and let $x \in \mathcal{N}(\pi_{nJ}(\zeta))$. Then there exists a matrix $C \in \mathbb{C}^{nm\times m}$ such that*

$$S_n \begin{pmatrix} I_m \\ V_n \end{pmatrix} x = \begin{pmatrix} 0_{m\times m} \\ C \end{pmatrix} x - \zeta \begin{pmatrix} C \\ 0_{m\times m} \end{pmatrix} x. \tag{2.48}$$

Proof. Use

$$S_n \begin{pmatrix} I_m \\ V_n \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ y_n & S_{n-1} \end{pmatrix} \begin{pmatrix} I_m \\ V_n \end{pmatrix} = \begin{pmatrix} A_0 \\ y_n + S_{n-1} V_n \end{pmatrix}$$

and Lemma 2.12. □

Lemma 2.15. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, and let $(A_j)_{j=0}^n$ be a J-Potapov sequence. Let $V_n \in \mathcal{Y}_{nJ}$, and let $\rho_{nJ} : \mathbb{C} \to \mathbb{C}^{m\times m}$ and $\pi_{nJ} : \mathbb{C} \to \mathbb{C}^{m\times m}$ be defined by (2.14) and (2.13). Let $\zeta \in \mathbb{C}$, and let $x \in \mathcal{N}(\rho_{nJ}(\zeta)) \cap \mathcal{N}(\pi_{nJ}(\zeta))$. Then there exists a matrix $B \in \mathbb{C}^{nm\times m}$ such that the equations (2.45) and*

$$(|\zeta|^2 - 1) x^* B^* Q_{n-1J} B x = x^* R_{n+1J} x \tag{2.49}$$

are satisfied.

Proof. The case $x = 0_{m\times 1}$ is trivial. Suppose $x \neq 0_{m\times 1}$. According to Lemmas 2.13 and 2.14, there exist complex $nm \times m$ matrices B and C such that (2.45), (2.46), and (2.48) are satisfied. Let $T_{n+1,m}$ be given by (2.44). Then using (2.48) and (2.45) we obtain

$$\begin{aligned}
\left(T_{n+1,m}^* - \zeta I\right) \begin{pmatrix} C \\ 0_{m\times m} \end{pmatrix} x &= \begin{pmatrix} 0_{m\times m} \\ C \end{pmatrix} x - \zeta \begin{pmatrix} C \\ 0_{m\times m} \end{pmatrix} x = S_n \begin{pmatrix} I_m \\ V_n \end{pmatrix} x \\
&= S_n \begin{pmatrix} 0_{m\times m} \\ B \end{pmatrix} x - \zeta S_n \begin{pmatrix} B \\ 0_{m\times m} \end{pmatrix} x \\
&= \begin{pmatrix} A_0 & 0 \\ y_n & S_{n-1} \end{pmatrix} \begin{pmatrix} 0_{m\times m} \\ B \end{pmatrix} x - \zeta \begin{pmatrix} S_{n-1} & 0 \\ z_n & A_0 \end{pmatrix} \begin{pmatrix} B \\ 0_{m\times m} \end{pmatrix} x \\
&= \begin{pmatrix} 0_{m\times m} \\ S_{n-1} B \end{pmatrix} x - \zeta \begin{pmatrix} S_{n-1} B \\ z_n B \end{pmatrix} x = \left(T_{n+1,m}^* - \zeta I\right) \begin{pmatrix} S_{n-1} B \\ z_n B \end{pmatrix} x.
\end{aligned} \tag{2.50}$$

Because of $\mathcal{N}(\rho_{nJ}(0)) = \{0_{m\times 1}\}$ and $x \neq 0_{m\times 1}$ we have $\zeta \neq 0$. Hence $\det\left(T_{n+1,m}^* - \zeta I\right) \neq 0$ holds. Thus, (2.50) implies

$$\begin{pmatrix} C \\ 0_{m\times m} \end{pmatrix} x = \begin{pmatrix} S_{n-1} B \\ z_n B \end{pmatrix} x.$$

In particular, $z_n B x = 0$ is valid. Taking into account (2.46) we get (2.49). □

In a similar way one can check that the following lemma is true.

Lemma 2.16. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, and let $(A_j)_{j=0}^n$ be a J-Potapov sequence. Let $W_n \in \mathcal{Z}_{nJ}$, and let $\tau_{nJ} : \mathbb{C} \to \mathbb{C}^{m\times m}$ and $\sigma_{nJ} : \mathbb{C} \to \mathbb{C}^{m\times m}$ be defined by (2.17) and (2.16). Let $\zeta \in \mathbb{C}$, and let $x \in \mathcal{N}\left([\tau_{nJ}(\zeta)]^*\right) \cap \mathcal{N}\left([\sigma_{nJ}(\zeta)]^*\right)$. Then there exists a matrix $E \in \mathbb{C}^{m\times nm}$ such that the equations*

$$x^*(W_n, I_m) = x^*(E, 0_{m\times m}) - \zeta x^*(0_{m\times m}, E)$$

and

$$(|\zeta|^2 - 1)x^*EP_{n-1_J}E^*x = x^*L_{n+1_J}x$$

are satisfied.

Lemma 2.17. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^n$ be a J-Potapov sequence. If $n \in \mathbb{N}$, then let $V_n := Q_{n-1_J}^+ S_{n-1}^* J_{[n-1]} y_n$. Let $\rho_{n_J} : \mathbb{C} \to \mathbb{C}^{m\times m}$ and $\pi_{n_J} : \mathbb{C} \to \mathbb{C}^{m\times m}$ be defined by (2.14) and (2.13). Then*

$$\mathcal{N}(\rho_{n_J}(\zeta)) \cap \mathcal{N}(\pi_{n_J}(\zeta)) = \{0_{m\times 1}\} \tag{2.51}$$

holds for each $\zeta \in \mathbb{D}$. If $(A_j)_{j=0}^n$ is moreover a strict J-Potapov sequence, then (2.51) is valid for each $\zeta \in \mathbb{D} \cup \mathbb{T}$.

Proof. The case $n = 0$ is trivial. Let $n \in \mathbb{N}$, let $\zeta \in \mathbb{C}$, and let $x \in \mathcal{N}(\rho_{n_J}(\zeta)) \cap \mathcal{N}(\pi_{n_J}(\zeta))$. Then Remark 2.5 and Lemma 2.15 imply that there exists a matrix $B \in \mathbb{C}^{nm\times m}$ such that (2.45) and (2.49) are fulfilled. According to [9, Lemma 3.7], the matrix R_{n+1_J} is nonnegative Hermitian. Hence (2.49) yields

$$(|\zeta|^2 - 1)x^*B^*Q_{n-1_J}Bx \geqslant 0. \tag{2.52}$$

Now suppose $\zeta \in \mathbb{D}$. Since Q_{n-1_J} is nonnegative Hermitian, from (2.52) we get $x^*B^*Q_{n-1_J}Bx = 0$ and therefore $x^*B^*Q_{n-1_J}^+ = x^*B^*\sqrt{Q_{n-1_J}}\sqrt{Q_{n-1_J}}^+Q_{n-1_J}^+ = 0_{1\times nm}$. Thus, using (2.45) we obtain

$$\begin{aligned}
0 &= x^*B^*Q_{n-1_J}^+S_{n-1}^*J_{[n-1]}y_nx = x^*B^*V_nx = (0_{1\times m}, x^*B^*)\begin{pmatrix} I_m \\ V_n \end{pmatrix} x \\
&= (0_{1\times m}, x^*B^*)\cdot\left[\begin{pmatrix} 0_{m\times m} \\ B \end{pmatrix} x - \zeta \begin{pmatrix} B \\ 0_{m\times m} \end{pmatrix} x\right] \\
&= x^*B^*Bx - \zeta \begin{pmatrix} 0_{m\times 1} \\ Bx \end{pmatrix}^* \begin{pmatrix} Bx \\ 0_{m\times 1} \end{pmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
x^*B^*Bx = |x^*B^*Bx| &\leqslant |\zeta| \sqrt{\begin{pmatrix} 0_{m\times 1} \\ Bx \end{pmatrix}^* \begin{pmatrix} 0_{m\times 1} \\ Bx \end{pmatrix}} \sqrt{\begin{pmatrix} Bx \\ 0_{m\times 1} \end{pmatrix}^* \begin{pmatrix} Bx \\ 0_{m\times 1} \end{pmatrix}} \\
&= |\zeta| x^*B^*Bx
\end{aligned}$$

and therefore $x^*B^*Bx = 0$, i.e., $Bx = 0$ holds. Taking into account (2.45), we get $x = 0_{m\times 1}$. Thus, (2.51) is verified for each $\zeta \in \mathbb{D}$. Now suppose that $(A_j)_{j=0}^n$ is a strict J-Potapov sequence, and let $\zeta \in \mathbb{T}$. Then from [9, Lemmas 3.3 and 3.7] we know that R_{n+1_J} is positive Hermitian. Hence (2.49) yields immediately $x = 0_{m\times 1}$, i.e., (2.51) holds. Thus, the proof is complete. □

The following lemma can be shown analogously.

Lemma 2.18. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^n$ be a J-Potapov sequence. If $n \in \mathbb{N}$, then let $W_n := z_n J_{[n-1]} S_{n-1}^* P_{n-1_J}^+$. Let $\tau_{n_J} : \mathbb{C} \to \mathbb{C}^{m\times m}$ and $\sigma_{n_J} : \mathbb{C} \to \mathbb{C}^{m\times m}$ be defined by (2.17) and (2.16). Then*

$$\mathcal{N}\left([\tau_{n_J}(\zeta)]^*\right) \cap \mathcal{N}\left([\sigma_{n_J}(\zeta)]^*\right) = \{0_{m\times 1}\} \tag{2.53}$$

holds for each $\zeta \in \mathbb{D}$. If $(A_j)_{j=0}^n$ is moreover a strict J-Potapov sequence, then (2.53) is valid for each $\zeta \in \mathbb{D} \cup \mathbb{T}$.

Now let a positive integer n and a J-Potapov sequence $(A_j)_{j=0}^n$ be given, and let $f_{c,n}$ be the J-central J-Potapov function corresponding to $(A_j)_{j=0}^n$. According to Remark 2.5, the matrix $V_n^{\square} := Q_{n-1,J}^{+} S_{n-1}^{*} J_{[n-1]} y_n$ belongs to $\mathcal{Y}_{n,J}$. Now we will see that the particular choice $V_n := V_n^{\square}$ in Theorem 2.7 ensures that, in the quotient representation (2.15) of $f_{c,n}$, the zeros of the determinant of the "denominator function" $\rho_{nJ,\mathbb{D}}$ are exactly the poles of $f_{c,n}$.

Proposition 2.19. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^n$ be a J-Potapov sequence. Denote $f_{c,n}$ the J-central J-Potapov function corresponding to $(A_j)_{j=0}^n$. If $n \in \mathbb{N}$, then let $V_n := Q_{n-1,J}^{+} S_{n-1}^{*} J_{[n-1]} y_n$. Let $\rho_{nJ} : \mathbb{C} \to \mathbb{C}^{m\times m}$ be defined by (2.14). Then $\mathbb{H}_{f_{c,n}} = \{\zeta \in \mathbb{D} : \det \rho_{nJ}(\zeta) \neq 0\}$.*

Proof. From Remark 2.5 we know that, in the case $n \in \mathbb{N}$, $V_n \in \mathcal{Y}_{nJ}$ holds true. Thus, Theorem 2.7 yields immediately $\{\zeta \in \mathbb{D} : \det \rho_{nJ}(\zeta) \neq 0\} \subseteq \mathbb{H}_{f_{c,n}}$. Now we consider an arbitrary $\zeta \in \mathbb{H}_{f_{c,n}}$. Let $\pi_{nJ} : \mathbb{C} \to \mathbb{C}^{m\times m}$ be given by (2.13). Then Theorem 2.7 implies $\pi_{nJ}(\zeta) = f_{c,n}(\zeta)\rho_{nJ}(\zeta)$. In particular, $\mathcal{N}(\rho_{nJ}(\zeta)) \subseteq \mathcal{N}(\pi_{nJ}(\zeta))$ holds. Taking into account Lemma 2.17 we get $\mathcal{N}(\rho_{nJ}(\zeta)) = \{0_{m\times 1}\}$, i.e., $\det \rho_{nJ}(\zeta) \neq 0$. $\square$

The following result can be proved in an analogous way.

Proposition 2.20. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^n$ be a J-Potapov sequence. Denote $f_{c,n}$ the J-central J-Potapov function corresponding to $(A_j)_{j=0}^n$. If $n \in \mathbb{N}$, then let $W_n := z_n J_{[n-1]} S_{n-1}^{*} P_{n-1,J}^{+}$. Let $\tau_{nJ} : \mathbb{C} \to \mathbb{C}^{m\times m}$ be defined by (2.17). Then $\mathbb{H}_{f_{c,n}} = \{\zeta \in \mathbb{D} : \det \tau_{nJ}(\zeta) \neq 0\}$.*

3. Recursion formulas

Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, and let $(A_j)_{j=0}^n$ be a J-Potapov sequence. Denote $f_{c,n}$ the J-central J-Potapov function corresponding to $(A_j)_{j=0}^n$. The goal of this section is to derive a recursive procedure for the construction of quotient representations of $f_{c,n}$ of the types stated in Theorems 2.7 and 2.8. The results in this section generalize the corresponding results for $m \times m$ Schur sequences, which were obtained in [6].

In the sequel, we will use the following notation. If $n \in \mathbb{N}_0$ and if $(A_j)_{j=0}^{n+1}$ is a J-Potapov sequence, then we will work with the sets

$$\mathcal{L}_{n+1,J} := \left\{ t \in \mathbb{C}^{m\times m} : L_{n+1,J} t = A_{n+1} - M_{n+1,J} \right\} \tag{3.1}$$

and

$$\mathcal{R}_{n+1,J} := \left\{ u \in \mathbb{C}^{m\times m} : u R_{n+1,J} = A_{n+1} - M_{n+1,J} \right\}. \tag{3.2}$$

Remark 3.1. Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^{n+1}$ be a J-Potapov sequence. Let $t_{n+1} := L_{n+1,J}^{+}(A_{n+1} - M_{n+1,J})$ and $u_{n+1} := (A_{n+1} - M_{n+1,J}) R_{n+1,J}^{+}$. In view of Theorem 1.2, there is a contractive $m \times m$ matrix K such that $A_{n+1} - M_{n+1,J} = \sqrt{L_{n+1,J}} K \sqrt{R_{n+1,J}}$. Consequently, $t_{n+1} \in \mathcal{L}_{n+1,J}$ and $u_{n+1} \in \mathcal{R}_{n+1,J}$.

Remark 3.2. Let J be an $m \times m$ signature matrix, and let $(A_j)_{j=0}^{1}$ be a J-Potapov sequence. Let $t_1 \in \mathcal{L}_{1,J}$ and $u_1 \in \mathcal{R}_{1,J}$. Further, let

$$V_1 := J A_0^{*} t_1 \quad \text{and} \quad W_1 := u_1 A_0^{*} J. \tag{3.3}$$

Then it is readily checked that $V_1 \in \mathcal{Y}_{1,J}$ and $W_1 \in \mathcal{Z}_{1,J}$ hold.

Lemma 3.3. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, and let $(A_j)_{j=0}^{n+1}$ be a J-Potapov sequence. Further, let $V_n \in \mathcal{Y}_{nJ}$, let $W_n \in \mathcal{Z}_{nJ}$, and let $V_n^{\#}$ and $W_n^{\#}$ be defined by (2.31). Then:*

(a) For each $t_{n+1} \in \mathcal{L}_{n+1J}$ the matrix

$$V_{n+1} := \begin{pmatrix} V_n \\ 0_{m\times m} \end{pmatrix} + J_{[n]} \left(W_n^{\#} S_n\right)^* t_{n+1} \tag{3.4}$$

belongs to $\mathcal{Y}_{n+1J}$ and the matrix

$$V_{n+1}^{\#} := \begin{pmatrix} I_m \\ V_{n+1} \end{pmatrix} \tag{3.5}$$

satisfies the equations

$$V_{n+1}^{\#} = \begin{pmatrix} V_n^{\#} \\ 0_{m\times m} \end{pmatrix} + \begin{pmatrix} 0_{m\times m} \\ J_{[n]} \left(W_n^{\#} S_n\right)^* t_{n+1} \end{pmatrix} \tag{3.6}$$

and

$$S_{n+1} V_{n+1}^{\#} = \begin{pmatrix} S_n V_n^{\#} \\ 0_{m\times m} \end{pmatrix} + \begin{pmatrix} 0_{m\times m} \\ J_{[n]} \left(W_n^{\#}\right)^* t_{n+1} \end{pmatrix}. \tag{3.7}$$

(b) For each $u_{n+1} \in \mathcal{R}_{n+1J}$ the matrix

$$W_{n+1} := (0_{m\times m}, W_n) + u_{n+1} \left(S_n V_n^{\#}\right)^* J_{[n]} \tag{3.8}$$

belongs to $\mathcal{Z}_{n+1J}$ and the matrix

$$W_{n+1}^{\#} := (W_{n+1}, I_m) \tag{3.9}$$

satisfies the equations

$$W_{n+1}^{\#} = \left(0_{m\times m}, W_n^{\#}\right) + \left(u_{n+1} \left(S_n V_n^{\#}\right)^* J_{[n]}, 0_{m\times m}\right) \tag{3.10}$$

and

$$W_{n+1}^{\#} S_{n+1} = \left(0_{m\times m}, W_n^{\#} S_n\right) + \left(u_{n+1} \left(V_n^{\#}\right)^* J_{[n]}, 0_{m\times m}\right). \tag{3.11}$$

Proof. Let $t_{n+1} \in \mathcal{L}_{n+1J}$. Denote

$$G_{n+1} := (Q_{n-1J} - z_n^* J z_n) \left[V_n + J_{[n-1]} \left(S_{n-1}^* W_n^* + z_n^*\right) t_{n+1}\right] - z_n^* J A_0 J A_0^* t_{n+1}$$

and

$$H_{n+1} := -A_0^* J z_n \left[V_n + J_{[n-1]} \left(S_{n-1}^* W_n^* + z_n^*\right) t_{n+1}\right] + R_{1J} J A_0^* t_{n+1}.$$

Then, in view of

$$\begin{aligned} V_{n+1} &= \begin{pmatrix} V_n \\ 0_{m\times m} \end{pmatrix} + J_{[n]} \begin{pmatrix} S_{n-1} & 0 \\ z_n & A_0 \end{pmatrix}^* \begin{pmatrix} W_n^* \\ I_m \end{pmatrix} t_{n+1} \\ &= \begin{pmatrix} V_n + J_{[n-1]} \left(S_{n-1}^* W_n^* + z_n^*\right) t_{n+1} \\ J A_0^* t_{n+1} \end{pmatrix} \end{aligned}$$

and (2.19), we get

$$Q_{nJ}V_{n+1} = \begin{pmatrix} G_{n+1} \\ H_{n+1} \end{pmatrix}. \tag{3.12}$$

According to [9, Lemma 3.7], the matrix M_{n+1J} admits the representation $M_{n+1J} = -z_n Q^{+}_{n-1J} S^{*}_{n-1} J_{[n-1]} y_n$. Thus, using Lemma 2.4 and $V_n \in \mathcal{Y}_{nJ}$ we obtain

$$z_n V_n = z_n Q^{+}_{n-1J} Q_{n-1J} V_n = -M_{n+1J}. \tag{3.13}$$

Because of $W_n \in \mathcal{Z}_{nJ}$ we have

$$\begin{aligned} Q_{n-1J}J_{[n-1]}\left(S^*_{n-1}W^*_n + z^*_n\right) &= S^*_{n-1}J_{[n-1]}P_{n-1J}W^*_n + Q_{n-1J}J_{[n-1]}z^*_n \\ &= \left(S^*_{n-1}J_{[n-1]}S_{n-1} + Q_{n-1J}\right)J_{[n-1]}z^*_n = z^*_n. \end{aligned} \tag{3.14}$$

Thus, Lemma 2.4 implies

$$z_n J_{[n-1]}\left(S^*_{n-1}W^*_n + z^*_n\right) = z_n Q^{+}_{n-1J} Q_{n-1J} J_{[n-1]}\left(S^*_{n-1}W^*_n + z^*_n\right) = z_n Q^{+}_{n-1J} z^*_n. \tag{3.15}$$

Using $V_n \in \mathcal{Y}_{nJ}$, (3.13), (3.14), (3.15), (1.9), and $t_{n+1} \in \mathcal{L}_{n+1J}$ yields

$$\begin{aligned} G_{n+1} &= S^*_{n-1}J_{[n-1]}y_n + z^*_n J M_{n+1J} + \left(z^*_n - z^*_n J z_n Q^{+}_{n-1J} z^*_n\right) t_{n+1} - z^*_n J A_0 J A^*_0 t_{n+1} \\ &= S^*_{n-1}J_{[n-1]}y_n + z^*_n J M_{n+1J} + z^*_n J L_{n+1J} t_{n+1} \\ &= S^*_{n-1}J_{[n-1]}y_n + z^*_n J A_{n+1}, \end{aligned} \tag{3.16}$$

and in view of (3.13), (3.15), $R_{1J}JA^*_0 = A^*_0 J L_{1J}$, (1.9), and $t_{n+1} \in \mathcal{L}_{n+1J}$ we get

$$\begin{aligned} H_{n+1} &= A^*_0 J M_{n+1J} - A^*_0 J z_n Q^{+}_{n-1J} z^*_n t_{n+1} + R_{1J} J A^*_0 t_{n+1} \\ &= A^*_0 J (M_{n+1J} + L_{n+1J} t_{n+1}) = A^*_0 J A_{n+1}. \end{aligned} \tag{3.17}$$

Applying (3.12), (3.16), and (3.17) we obtain

$$Q_{nJ}V_{n+1} = \begin{pmatrix} S^*_{n-1} & z^*_n \\ 0 & A^*_0 \end{pmatrix} \operatorname{diag}(J_{[n-1]}, J) \begin{pmatrix} y_n \\ A_{n+1} \end{pmatrix} = S^*_n J_{[n]} y_{n+1},$$

i.e., V_{n+1} belongs to $\mathcal{Y}_{n+1J}$. Further, (3.6) follows immediately from (3.4). Moreover, (3.6) and (3.13) yield

$$\begin{aligned} S_{n+1}V^{\#}_{n+1} &= \begin{pmatrix} S_n & 0 \\ z_{n+1} & A_0 \end{pmatrix} \begin{pmatrix} V^{\#}_n \\ 0_{m\times m} \end{pmatrix} + \begin{pmatrix} A_0 & 0 \\ y_{n+1} & S_n \end{pmatrix} \begin{pmatrix} 0_{m\times m} \\ J_{[n]}\left(W^{\#}_n S_n\right)^* t_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} S_n V^{\#}_n \\ z_{n+1} V^{\#}_n \end{pmatrix} + \begin{pmatrix} 0 \\ S_n J_{[n]} S^*_n \left(W^{\#}_n\right)^* t_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} S_n V^{\#}_n \\ A_{n+1} - M_{n+1J} \end{pmatrix} + \begin{pmatrix} 0 \\ S_n J_{[n]} S^*_n \left(W^{\#}_n\right)^* t_{n+1} \end{pmatrix}. \end{aligned} \tag{3.18}$$

Lemma 2.9 and $t_{n+1} \in \mathcal{L}_{n+1J}$ imply

$$\begin{aligned} S_n J_{[n]} S^*_n (W^{\#}_n)^* t_{n+1} &= \left[W^{\#}_n (J_{[n]} - P_{nJ})\right]^* t_{n+1} = \left[W^{\#}_n J_{[n]} - (0, L_{n+1J})\right]^* t_{n+1} \\ &= J_{[n]}\left(W^{\#}_n\right)^* t_{n+1} - \begin{pmatrix} 0 \\ L_{n+1J} t_{n+1} \end{pmatrix} \\ &= J_{[n]}\left(W^{\#}_n\right)^* t_{n+1} - \begin{pmatrix} 0 \\ A_{n+1} - M_{n+1J} \end{pmatrix}. \end{aligned} \tag{3.19}$$

From (3.18) and (3.19) then (3.7) follows. Thus, part (a) is proved. Part (b) can be verified analogously. □

Proposition 3.4. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^{n+1}$ be a J-Potapov sequence. If $n \in \mathbb{N}$ then let $V_n \in \mathcal{Y}_{nJ}$ and let $W_n \in \mathcal{Z}_{nJ}$. For each $\zeta \in \mathbb{C}$, let π_{nJ}, ρ_{nJ}, σ_{nJ}, and τ_{nJ} be defined by (2.13), (2.14), (2.16), and (2.17), respectively. Then:*

(a) Let $t_{n+1} \in \mathcal{L}_{n+1J}$. In the case $n = 0$ let V_1 be given by (3.3). If $n \in \mathbb{N}$ then let V_{n+1} be defined by (2.31) and (3.4). Then the the matrix polynomials π_{n+1J} and ρ_{n+1J} defined for each $\zeta \in \mathbb{C}$ by

$$\pi_{n+1J}(\zeta) := \pi_{nJ}(\zeta) + \zeta J \tilde{\tau}_n^{[n]}(\zeta) t_{n+1} \tag{3.20}$$

and

$$\rho_{n+1J}(\zeta) := \rho_{nJ}(\zeta) + \zeta J \tilde{\sigma}_n^{[n]}(\zeta) t_{n+1} \tag{3.21}$$

admit the representations

$$\pi_{n+1J}(\zeta) = A_0 + \zeta e_{n,m}(\zeta)(y_{n+1} + S_n V_{n+1}) \tag{3.22}$$

and

$$\rho_{n+1J}(\zeta) = I_m + \zeta e_{n,m}(\zeta) V_{n+1}. \tag{3.23}$$

(b) Let $u_{n+1} \in \mathcal{R}_{n+1J}$. In the case $n = 0$ let W_1 be given by (3.3). If $n \in \mathbb{N}$ then let W_{n+1} be defined by (2.31) and (3.8). Then the matrix polynomials σ_{n+1J} and τ_{n+1J} defined for each $\zeta \in \mathbb{C}$ by

$$\sigma_{n+1J}(\zeta) := \sigma_{nJ}(\zeta) + u_{n+1} \zeta \tilde{\rho}_n^{[n]}(\zeta) J \tag{3.24}$$

and

$$\tau_{n+1J}(\zeta) := \tau_{nJ}(\zeta) + u_{n+1} \zeta \tilde{\pi}_n^{[n]}(\zeta) J \tag{3.25}$$

admit the representations

$$\sigma_{n+1J}(\zeta) = (W_{n+1} S_n + z_{n+1}) \zeta \varepsilon_{n,m}(\zeta) + A_0 \tag{3.26}$$

and

$$\tau_{n+1J}(\zeta) = W_{n+1} \zeta \varepsilon_{n,m}(\zeta) + I_m. \tag{3.27}$$

(c) Let $t_{n+1} \in \mathcal{L}_{n+1J}$ and let $u_{n+1} \in \mathcal{R}_{n+1J}$. For each $\zeta \in \mathbb{C}$, then the matrix polynomials π_{n+1J}, ρ_{n+1J}, σ_{n+1J}, and τ_{n+1J} given by (3.20), (3.21), (3.24), and (3.25) satisfy the equations

$$\begin{pmatrix} \zeta J \tilde{\tau}_{n+1J}^{[n+1]}(\zeta) & \pi_{n+1J}(\zeta) \\ \zeta J \tilde{\sigma}_{n+1J}^{[n+1]}(\zeta) & \rho_{n+1J}(\zeta) \end{pmatrix} = \begin{pmatrix} \zeta J \tilde{\tau}_{nJ}^{[n]}(\zeta) & \pi_{nJ}(\zeta) \\ \zeta J \tilde{\sigma}_{nJ}^{[n]}(\zeta) & \rho_{nJ}(\zeta) \end{pmatrix} \begin{pmatrix} \zeta I_m & t_{n+1} \\ \zeta u_{n+1}^* & I_m \end{pmatrix} \tag{3.28}$$

and

$$\begin{pmatrix} \zeta \tilde{\rho}_{n+1J}^{[n+1]}(\zeta) J & \zeta \tilde{\pi}_{n+1J}^{[n+1]}(\zeta) J \\ \sigma_{n+1J}(\zeta) & \tau_{n+1J}(\zeta) \end{pmatrix} = \begin{pmatrix} \zeta I_m & \zeta t_{n+1}^* \\ u_{n+1} & I_m \end{pmatrix} \begin{pmatrix} \zeta \tilde{\rho}_{nJ}^{[n]}(\zeta) J & \zeta \tilde{\pi}_{nJ}^{[n]}(\zeta) J \\ \sigma_{nJ}(\zeta) & \tau_{nJ}(\zeta) \end{pmatrix}. \tag{3.29}$$

Proof. Let $t_{n+1} \in \mathcal{L}_{n+1J}$ and let $u_{n+1} \in \mathcal{R}_{n+1J}$. If $n = 0$, then a straightforward calculation yields all the assertions. Now let $n \geqslant 1$. Using Remark 2.10 and Lemma 3.3, we get then

$$\begin{aligned}\pi_{n+1J}(\zeta) &= e_{n,m}(\zeta)S_nV_n^{\#} + \zeta J e_{n,m}(\zeta)(W_n^{\#})^* t_{n+1}\\ &= e_{n+1,m}(\zeta)\begin{pmatrix} S_nV_n^{\#} \\ 0_{m\times m}\end{pmatrix} + e_{n+1,m}(\zeta)\begin{pmatrix} 0_{m\times m} \\ J_{[n]}\left(W_n^{\#}\right)^* t_{n+1}\end{pmatrix}\\ &= e_{n+1,m}(\zeta)S_{n+1}V_{n+1}^{\#} = (I_m, \zeta e_{n,m}(\zeta))\begin{pmatrix} A_0 & 0 \\ y_{n+1} & S_n\end{pmatrix}\begin{pmatrix} I_m \\ V_{n+1}\end{pmatrix}\\ &= A_0 + \zeta e_{n,m}(\zeta)(y_{n+1} + S_nV_{n+1})\end{aligned}$$

for each $\zeta \in \mathbb{C}$, i.e., (3.22) holds true. Equations (3.23), (3.26), and (3.27) can be checked similarly. Further, (3.28) and (3.29) follow by a straightforward calculation. □

Proposition 3.5. *Let J be an $m \times m$ signature matrix, let $k \in \mathbb{N}$, and let $(A_j)_{j=0}^k$ be a J-Potapov sequence. Let π_{0J}, ρ_{0J}, σ_{0J}, and τ_{0J} be the constant matrix polynomials given in (2.13), (2.14), (2.16), and (2.17), respectively. For each $n \in \mathbb{N}_{0,k-1}$, let $t_{n+1} \in \mathcal{L}_{n+1J}$ and let $u_{n+1J} \in \mathcal{R}_{n+1J}$, and let the matrix polynomials π_{n+1J}, ρ_{n+1J}, σ_{n+1J}, and τ_{n+1J} be recursively defined by (3.20), (3.21), (3.24), and (3.25), respectively. Then for each $n \in \mathbb{N}_{0,k}$, the J-central J-Potapov function $f_{c,n}$ corresponding to $(A_j)_{j=0}^n$ admits the representations*

$$f_{c,n} = \pi_{nJ,\mathbb{D}}\rho_{nJ,\mathbb{D}}^{-1} \quad \text{and} \quad f_{c,n} = \tau_{nJ,\mathbb{D}}^{-1}\sigma_{nJ,\mathbb{D}}$$

where $\pi_{nJ,\mathbb{D}}$ (respectively, $\rho_{nJ,\mathbb{D}}, \tau_{nJ,\mathbb{D}}, \sigma_{nJ,\mathbb{D}}$) is the restriction of π_{nJ} (respectively, $\rho_{nJ}, \tau_{nJ}, \sigma_{nJ}$) onto $\mathbb{D}$.

Proof. Apply Proposition 3.4, Remark 3.2, Lemma 3.3, and Theorems 2.7 and 2.8. □

Our next considerations are aimed at deriving a recurrent construction of such quotient representations of a J-central J-Potapov function $f_{c,n}$ where the zeros of the determinant of the "denominator function" are exactly the poles of $f_{c,n}$.

Proposition 3.6. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^{n+1}$ be a J-Potapov sequence. If $n \in \mathbb{N}$, then let $V_n \in \mathcal{Y}_{nJ}$ and let $W_n \in \mathcal{Z}_{nJ}$. For each $\zeta \in \mathbb{C}$, let π_{nJ}, ρ_{nJ}, σ_{nJ}, and τ_{nJ} be defined by (2.13), (2.14), (2.16), and (2.17), respectively. Let $t_{n+1} := L_{n+1J}^{+}(A_{n+1} - M_{n+1J})$, let $u_{n+1} := (A_{n+1} - M_{n+1J})R_{n+1J}^{+}$, and let the matrix polynomials π_{n+1J}, ρ_{n+1J}, σ_{n+1J}, and τ_{n+1J} be given for all $\zeta \in \mathbb{C}$ by (3.20), (3.21), (3.24), and (3.25), respectively. For each $\zeta \in \mathbb{D}$, then the following statements hold:*

(a) *If $\mathcal{N}(\rho_{nJ}(\zeta)) \cap \mathcal{N}(\pi_{nJ}(\zeta)) = \{0_{m\times 1}\}$ then $\mathcal{N}(\rho_{n+1J}(\zeta)) \cap \mathcal{N}(\pi_{n+1J}(\zeta)) = \{0_{m\times 1}\}$.*
(b) *If $\mathcal{N}\left([\tau_{nJ}(\zeta)]^*\right) \cap \mathcal{N}\left([\sigma_{nJ}(\zeta)]^*\right) = \{0_{m\times 1}\}$ then $\mathcal{N}\left([\tau_{n+1J}(\zeta)]^*\right) \cap \mathcal{N}\left([\sigma_{n+1J}(\zeta)]^*\right) = \{0_{m\times 1}\}$.*

Proof. Let $\zeta \in \mathbb{D}$. Suppose $\mathcal{N}(\rho_{nJ}(\zeta)) \cap \mathcal{N}(\pi_{nJ}(\zeta)) = \{0_{m\times 1}\}$, and let $x \in \mathcal{N}(\rho_{n+1J}(\zeta)) \cap \mathcal{N}(\pi_{n+1J}(\zeta))$. If $n = 0$ then let V_1 be given by (3.3). In the case $n \geqslant 1$, let V_{n+1} be defined by (2.31) and (3.4). Then Remark 3.2 and Lemma 3.3 yield $V_{n+1} \in \mathcal{Y}_{n+1J}$. Thus, Remark 3.1, Proposition 3.4, and Lemma 2.15 imply that there exists a matrix $B \in \mathbb{C}^{(n+1)m\times m}$ such that the equations

$$\begin{pmatrix} I_m \\ V_{n+1}\end{pmatrix} x = \begin{pmatrix} 0_{m\times m} \\ B\end{pmatrix} x - \zeta\begin{pmatrix} B \\ 0_{m\times m}\end{pmatrix} x \tag{3.30}$$

and

$$(|\zeta|^2 - 1)x^*B^*Q_{nJ}Bx = x^*R_{n+2J}x \tag{3.31}$$

are satisfied. According to [9, Lemma 3.7], the matrix R_{n+2J} is nonnegative Hermitian. Thus, in view of $\zeta \in \mathbb{D}$ and $Q_{nJ} \geq 0_{m\times m}$, equation (3.31) yields $x^*B^*Q_{nJ}Bx = 0$ and therefore $\sqrt{Q_{nJ}}Bx = 0$, i.e., $Q_{nJ}Bx = 0$. From (3.30) we obtain in particular $x = -\zeta(I_m, 0)Bx$. Hence, using Lemma 2.9 we get

$$R_{n+1}x = -\zeta R_{n+1J}(I_m, 0)Bx = -\zeta \left(I_m, V_n^*\right) Q_{nJ}Bx = 0.$$

Because of Remark 3.1 this implies

$$t_{n+1}x = L^+_{n+1J}(A_{n+1} - M_{n+1J})R^+_{n+1}R_{n+1}x = 0$$

and, in view of (3.20) and (3.21), consequently $\pi_n(\zeta)x = 0$ and $\rho_n(\zeta)x = 0$. Consequently, we have $x = 0_{m\times 1}$, i.e., $\mathcal{N}(\rho_{n+1J}(\zeta)) \cap \mathcal{N}(\pi_{n+1J}(\zeta)) = \{0_{m\times 1}\}$ holds. Thus, part (a) is proved. Part (b) can be verified analogously. □

Proposition 3.7. *Let J be an $m \times m$ signature matrix, let $k \in \mathbb{N}$, and let $(A_j)_{j=0}^k$ be a J-Potapov sequence. Let $\pi_{0J}, \rho_{0J}, \sigma_{0J}$, and τ_{0J} be the constant matrix polynomials given in (2.13), (2.14), (2.16), and (2.17), respectively. For each $n \in \mathbb{N}_{0,k-1}$, let $t_{n+1} := L^+_{n+1J}(A_{n+1} - M_{n+1J})$ and let $u_{n+1J} := (A_{n+1} - M_{n+1J})R^+_{n+1J}$, and let the matrix polynomials π_{n+1J}, ρ_{n+1J}, σ_{n+1J}, and τ_{n+1J} be recursively defined by (3.20), (3.21), (3.24), and (3.25), respectively. Then for each $n \in \mathbb{N}_{0,k}$ and for each $\zeta \in \mathbb{D}$, $\det \rho_{nJ}(\zeta) \neq 0$ holds if and only if $\det \tau_{nJ}(\zeta) \neq 0$. Moreover, the J-central J-Potapov function $f_{c,n}$ corresponding to $(A_j)_{j=0}^n$ admits for each $\zeta \in \mathbb{D}$ with $\det \rho_{nJ}(\zeta) \neq 0$ the representations*

$$f_{c,n}(\zeta) = \pi_{nJ}(\zeta)(\rho_{nJ}(\zeta))^{-1} \quad \text{and} \quad f_{c,n}(\zeta) = (\tau_{nJ}(\zeta))^{-1}\sigma_{nJ}(\zeta). \tag{3.32}$$

Further, $\mathbb{H}_{f_{c,n}} = \{\zeta \in \mathbb{D} : \det \rho_{nJ}(\zeta) \neq 0\}$ holds true.

Proof. Let $n \in \mathbb{N}_{0,k}$. Taking into account Remark 3.1 and Proposition 3.5 we get $\{\zeta \in \mathbb{D} : \det \rho_{nJ}(\zeta) \neq 0\} \subseteq \mathbb{H}_{f_{c,n}}$ and $\{\zeta \in \mathbb{D} : \det \tau_{nJ}(\zeta) \neq 0\} \subseteq \mathbb{H}_{f_{c,n}}$. Now let $\zeta \in \mathbb{H}_{f_{c,n}}$. Then Remark 3.1 and Proposition 3.5 yield

$$\pi_{nJ}(\zeta) = f_{c,n}(\zeta)\rho_{nJ}(\zeta) \quad \text{and} \quad \sigma_{nJ}(\zeta) = \tau_{nJ}(\zeta)f_{c,n}(\zeta). \tag{3.33}$$

Obviously, $\mathcal{N}(\rho_{0J}(\zeta)) \cap \mathcal{N}(\pi_{0J}(\zeta)) = \{0_{m\times 1}\}$ and $\mathcal{N}\left([\tau_{0J}(\zeta)]^*\right) \cap \mathcal{N}\left([\sigma_{0J}(\zeta)]^*\right) = \{0_{m\times 1}\}$ hold. Thus, using Proposition 3.6, Remark 3.1, Proposition 3.4, Remark 3.2, and Lemma 3.3 we get $\mathcal{N}(\rho_{nJ}(\zeta)) \cap \mathcal{N}(\pi_{nJ}(\zeta)) = \{0_{m\times 1}\}$ and $\mathcal{N}\left([\tau_{nJ}(\zeta)]^*\right) \cap \mathcal{N}\left([\sigma_{nJ}(\zeta)]^*\right) = \{0_{m\times 1}\}$. In view of (3.33) we obtain therefore $\det \rho_{nJ}(\zeta) \neq 0$ and $\det \tau_{nJ}(\zeta) \neq 0$. Hence $\mathbb{H}_{f_{c,n}} = \{\zeta \in \mathbb{D} : \det \rho_{nJ}(\zeta) \neq 0\} = \{\zeta \in \mathbb{D} : \det \tau_{nJ}(\zeta) \neq 0\}$ holds true. Consequently, application of Remark 3.1 and Proposition 3.5 yields (3.32) for each $\zeta \in \mathbb{D}$ with $\det \rho_{nJ}(\zeta) \neq 0$. Thus, the proof is complete. □

4. Recursion formulas for the Arov–Krein resolvent matrices

In this section we generalize some results for $p \times q$ Schur sequences obtained in [7, Section 4] to the case of J-Potapov sequences.

Lemma 4.1. *Let $n \in \mathbb{N}_0$, let J be an $m \times m$ signature matrix, and let $(A_j)_{j=0}^n$ be a J-Potapov sequence. If $n \geqslant 1$, then let $V_n \in \mathcal{Y}_{nJ}$ and $W_n \in \mathcal{Z}_{nJ}$. Let the matrix polynomials π_{nJ}, ρ_{nJ}, σ_{nJ}, and τ_{nJ} be given by (2.13), (2.14), (2.16), and (2.17), respectively. Let the matrix-valued functions $\mathfrak{C}_{nJ} : \mathbb{C} \to \mathbb{C}^{2m\times 2m}$ and $\mathfrak{D}_{nJ} : \mathbb{C} \to \mathbb{C}^{2m\times 2m}$ be defined by*

$$\mathfrak{C}_{nJ}(\zeta) := \begin{pmatrix} \zeta J\tilde{\tau}^{[n]}_{nJ}(\zeta) & \pi_{nJ}(\zeta) \\ \zeta J\tilde{\sigma}^{[n]}_{nJ}(\zeta) & \rho_{nJ}(\zeta) \end{pmatrix} \begin{pmatrix} \sqrt{L_{n+1J}}^+ & 0_{m\times m} \\ 0_{m\times m} & \sqrt{R_{n+1J}}^+ \end{pmatrix} \tag{4.1}$$

and

$$\mathfrak{D}_{nJ}(\zeta) := \begin{pmatrix} \sqrt{R_{n+1J}}^+ & 0_{m\times m} \\ 0_{m\times m} & \sqrt{L_{n+1J}}^+ \end{pmatrix} \begin{pmatrix} \zeta\,\tilde{\rho}^{[n]}_{nJ}(\zeta)J & \zeta\,\tilde{\pi}^{[n]}_{nJ}(\zeta)J \\ \sigma_{nJ}(\zeta) & \tau_{nJ}(\zeta) \end{pmatrix}. \tag{4.2}$$

Then, for each $\zeta \in \mathbb{C}$,

$$\mathfrak{D}_{nJ}(\zeta)\begin{pmatrix} 0_{m\times m} & I_m \\ -I_m & 0_{m\times m} \end{pmatrix}\mathfrak{C}_{nJ}(\zeta) = \zeta^{n+1}\begin{pmatrix} 0_{m\times m} & R_{n+1J}R^+_{n+1J} \\ -L_{n+1J}L^+_{n+1J} & 0_{m\times m} \end{pmatrix}.$$

Proof. Taking into account (2.32), Proposition 2.11, and the identities $\sqrt{R_{n+1J}}^{+} R_{n+1J} \sqrt{R_{n+1J}}^{+} = R_{n+1J} R_{n+1J}^{+}$ and $\sqrt{L_{n+1J}}^{+} L_{n+1J} \sqrt{L_{n+1J}}^{+} = L_{n+1J} L_{n+1J}^{+}$, the assertion follows by a straightforward calculation. □

In the sequel, if a nonnegative integer n, an $m \times m$ signature matrix J, and a J-Potapov sequence $(A_j)_{j=0}^{n}$ are given, then let $\boldsymbol{\pi}_{0J}$, $\boldsymbol{\rho}_{0J}$, $\boldsymbol{\sigma}_{0J}$, and $\boldsymbol{\tau}_{0J}$ be the constant matrix-valued functions defined by

$$\boldsymbol{\pi}_{0J}(\zeta) := A_0, \quad \boldsymbol{\rho}_{0J} := I_m, \quad \boldsymbol{\sigma}_{0J} := A_0, \quad \text{and} \quad \boldsymbol{\tau}_{0J} := I_m. \tag{4.3}$$

Furthermore, for each $k \in \mathbb{N}_{0,n-1}$, let $t_{k+1} := L_{k+1J}^{+}(A_{k+1} - M_{k+1J})$ and $u_{k+1} := (A_{k+1} - M_{k+1J})R_{k+1J}^{+}$, and let the matrix polynomials $\boldsymbol{\pi}_{k+1J}$, $\boldsymbol{\rho}_{k+1J}$, $\boldsymbol{\sigma}_{k+1J}$, and $\boldsymbol{\tau}_{k+1J}$ for each $\zeta \in \mathbb{C}$ be recursively defined by

$$\boldsymbol{\pi}_{k+1J}(\zeta) := \boldsymbol{\pi}_{kJ}(\zeta) + \zeta J \tilde{\boldsymbol{\tau}}_{k}^{[k]}(\zeta) t_{k+1}, \tag{4.4}$$

$$\boldsymbol{\rho}_{k+1J}(\zeta) := \boldsymbol{\rho}_{kJ}(\zeta) + \zeta J \tilde{\boldsymbol{\sigma}}_{k}^{[k]}(\zeta) t_{k+1}, \tag{4.5}$$

$$\boldsymbol{\sigma}_{k+1J}(\zeta) := \boldsymbol{\sigma}_{kJ}(\zeta) + u_{k+1} \zeta \tilde{\boldsymbol{\rho}}_{k}^{[k]}(\zeta) J, \tag{4.6}$$

and

$$\boldsymbol{\tau}_{k+1J}(\zeta) := \boldsymbol{\tau}_{kJ}(\zeta) + u_{k+1} \zeta \tilde{\boldsymbol{\pi}}_{k}^{[k]}(\zeta) J. \tag{4.7}$$

Remark 4.2. Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, let $(A_j)_{j=0}^{n}$ be a J-Potapov sequence, and let $k \in \mathbb{N}_{0,n-1}$. Then Remark 3.1, Proposition 3.4, Remark 3.2, and Lemma 3.3 imply that there are matrices $\mathbf{V}_{k+1} \in \mathcal{Y}_{k+1J}$ and $\mathbf{W}_{k+1} \in \mathcal{Z}_{k+1J}$ such that the matrix polynomials $\boldsymbol{\pi}_{k+1J}$, $\boldsymbol{\rho}_{k+1J}$, $\boldsymbol{\sigma}_{k+1J}$, and $\boldsymbol{\tau}_{k+1J}$ can be represented for each $\zeta \in \mathbb{C}$ via

$$\boldsymbol{\pi}_{k+1J}(\zeta) = A_0 + \zeta e_{k,m}(\zeta)(y_{k+1} + S_k \mathbf{V}_{k+1}),$$
$$\boldsymbol{\rho}_{k+1J}(\zeta) = I_m + \zeta e_{k,m}(\zeta) \mathbf{V}_{k+1},$$
$$\boldsymbol{\sigma}_{k+1J}(\zeta) = (\mathbf{W}_{k+1} S_k + z_{k+1}) \zeta \varepsilon_{k,m}(\zeta) + A_0,$$

and

$$\boldsymbol{\tau}_{k+1J}(\zeta) = \mathbf{W}_{k+1} \zeta \varepsilon_{k,m}(\zeta) + I_m.$$

In the following, we will use the $2m \times 2m$ signature matrix

$$j_{mm} := \operatorname{diag}(I_m, -I_m). \tag{4.8}$$

Moreover, if J is an $m \times m$ signature matrix, then we will work with the $2m \times 2m$ signature matrix

$$J^{\square} := \operatorname{diag}(J, -J). \tag{4.9}$$

The following result generalizes some parts of Proposition 4.7 in [7] to the case of finite J-Potapov sequences.

Proposition 4.3. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, and let $(A_j)_{j=0}^{n}$ be a J-Potapov sequence. For each $k \in \mathbb{N}_{0,n}$, let the matrix-valued functions $\mathfrak{C}_{kJ}^{\bullet} : \mathbb{C} \to \mathbb{C}^{2m \times 2m}$ and $\mathfrak{D}_{kJ}^{\bullet} : \mathbb{C} \to \mathbb{C}^{2m \times 2m}$ be defined by*

$$\mathfrak{C}_{kJ}^{\bullet}(\zeta) := \begin{pmatrix} \zeta J \tilde{\boldsymbol{\tau}}_{kJ}^{[k]}(\zeta) & \boldsymbol{\pi}_{kJ}(\zeta) \\ \zeta J \tilde{\boldsymbol{\sigma}}_{kJ}^{[k]}(\zeta) & \boldsymbol{\rho}_{kJ}(\zeta) \end{pmatrix} \begin{pmatrix} \sqrt{L_{k+1J}}^{+} & 0_{m \times m} \\ 0_{m \times m} & \sqrt{R_{k+1J}}^{+} \end{pmatrix} \tag{4.10}$$

and

$$\mathfrak{D}_{kJ}^{\bullet}(\zeta) := \begin{pmatrix} \sqrt{R_{k+1J}}^{+} & 0_{m \times m} \\ 0_{m \times m} & \sqrt{L_{k+1J}}^{+} \end{pmatrix} \begin{pmatrix} \zeta \tilde{\boldsymbol{\rho}}_{kJ}^{[k]}(\zeta) J & \zeta \tilde{\boldsymbol{\pi}}_{kJ}^{[k]}(\zeta) J \\ \boldsymbol{\sigma}_{kJ}(\zeta) & \boldsymbol{\tau}_{kJ}(\zeta) \end{pmatrix}. \tag{4.11}$$

Moreover, for each $k \in \mathbb{N}_{0,n-1}$, let $K_{k+1J} := \sqrt{L_{k+1J}}^{+}(A_{k+1} - M_{k+1J})\sqrt{R_{k+1J}}^{+}$, and let the matrix-valued functions $G_{k+1J} : \mathbb{C} \to \mathbb{C}^{2m\times 2m}$ and $H_{k+1J} : \mathbb{C} \to \mathbb{C}^{2m\times 2m}$ be defined by

$$G_{k+1J}(\zeta) := \begin{pmatrix} I_m & K_{k+1J} \\ K^*_{k+1J} & I_m \end{pmatrix} \begin{pmatrix} \zeta\sqrt{L_{k+1J}}\sqrt{L_{k+2J}}^{+} & 0_{m\times m} \\ 0_{m\times m} & \sqrt{R_{k+1J}}\sqrt{R_{k+2J}}^{+} \end{pmatrix}$$

and

$$H_{k+1J}(\zeta) := \begin{pmatrix} \zeta\sqrt{R_{k+2J}}^{+}\sqrt{R_{k+1J}} & 0_{m\times m} \\ 0_{m\times m} & \sqrt{L_{k+2J}}^{+}\sqrt{L_{k+1J}} \end{pmatrix} \begin{pmatrix} I_m & K^*_{k+1J} \\ K_{k+1J} & I_m \end{pmatrix}.$$

Then, for each $k \in \mathbb{N}_{0,n-1}$, the following statements hold:

(a) $\mathfrak{C}^{\bullet}_{k+1J} = \mathfrak{C}^{\bullet}_{kJ} G_{k+1J}$ and $\mathfrak{D}^{\bullet}_{k+1J} = H_{k+1J}\mathfrak{D}^{\bullet}_{kJ}$.
(b) $\mathfrak{C}^{\bullet}_{k+1J} = \mathfrak{C}^{\bullet}_{0J} G_{1J} G_{2J}\cdot \ldots \cdot G_{k+1J}$ and $\mathfrak{D}^{\bullet}_{k+1J} = H_{k+1J}H_{kJ}\cdot \ldots \cdot H_{1J}\mathfrak{D}^{\bullet}_{0J}$.
(c) For each $\zeta \in \mathbb{C}$,

$$\begin{aligned} &\operatorname{diag}\left(L_{k+2J}L^{+}_{k+2J}, -R_{k+2J}R^{+}_{k+2J}\right) - (G_{k+1J}(\zeta))^* j_{mm} G_{k+1J}(\zeta) \\ &\quad = \operatorname{diag}\left((1-|\zeta|^2)L_{k+2J}L^{+}_{k+2J}, 0_{m\times m}\right), \end{aligned} \tag{4.12}$$

$$\begin{aligned} &(G_{k+1J}(\zeta))^* \cdot \operatorname{diag}\left(L_{k+1J}L^{+}_{k+1J}, -R_{k+1J}R^{+}_{k+1J}\right) \cdot G_{k+1J}(\zeta) \\ &\quad = (G_{k+1J}(\zeta))^* j_{mm} G_{k+1J}(\zeta), \end{aligned} \tag{4.13}$$

$$\begin{aligned} &\operatorname{diag}\left(R_{k+2J}R^{+}_{k+2J}, -L_{k+2J}L^{+}_{k+2J}\right) - H_{k+1J}(\zeta) j_{mm} (H_{k+1J}(\zeta))^* \\ &\quad = \operatorname{diag}\left((1-|\zeta|^2)R_{k+2J}R^{+}_{k+2J},\ 0_{m\times m}\right), \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} &H_{k+1J}(\zeta) \cdot \operatorname{diag}\left(R_{k+1J}R^{+}_{k+1J},\ -L_{k+1J}L^{+}_{k+1J}\right) \cdot (H_{k+1J}(\zeta))^* \\ &\quad = H_{k+1J}(\zeta) j_{mm} (H_{k+1J}(\zeta))^*. \end{aligned} \tag{4.15}$$

Proof. Let $k \in \mathbb{N}_{0,n-1}$, let $t_{k+1} := L^{+}_{k+1J}(A_{k+1} - M_{k+1J})$, and let $u_{k+1} := (A_{k+1} - M_{k+1J})R^{+}_{k+1J}$. Then Remark 4.2, Remark 3.1, and Proposition 3.4 yield

$$\begin{pmatrix} \zeta J\tilde{\boldsymbol{\tau}}^{[k+1]}_{k+1J}(\zeta) & \boldsymbol{\pi}_{k+1J}(\zeta) \\ \zeta J\tilde{\boldsymbol{\sigma}}^{[k+1]}_{k+1J}(\zeta) & \boldsymbol{\rho}_{k+1J}(\zeta) \end{pmatrix} = \begin{pmatrix} \zeta J\tilde{\boldsymbol{\tau}}^{[k]}_{kJ}(\zeta) & \boldsymbol{\pi}_{kJ}(\zeta) \\ \zeta J\tilde{\boldsymbol{\sigma}}^{[k]}_{kJ}(\zeta) & \boldsymbol{\rho}_{kJ}(\zeta) \end{pmatrix} \begin{pmatrix} \zeta I_m & t_{k+1} \\ \zeta u^*_{k+1} & I_m \end{pmatrix} \tag{4.16}$$

for each $\zeta \in \mathbb{C}$. Furthermore, from [9, Lemma 3.7 and Proposition 3.8] we know that $L_{k+2J} \leqslant L_{k+1J}$ and $R_{k+2J} \leqslant R_{k+1J}$ are valid. In particular,

$$\sqrt{L_{k+1J}}^{+}\sqrt{L_{k+1J}}\sqrt{L_{k+2J}}^{+} = \sqrt{L_{k+2J}}^{+} \tag{4.17}$$

and

$$\sqrt{R_{k+1J}}^{+}\sqrt{R_{k+1J}}\sqrt{R_{k+2J}}^{+} = \sqrt{R_{k+2J}}^{+} \tag{4.18}$$

hold true. Taking into account (4.16), (4.17), and (4.18), the first equation stated in (a) follows by straightforward calculation. The second equation of part (a) can be checked analogously. Moreover, part (b) is an immediate consequence of part (a). Now we are going to verify part (c). From [9, Proposition 4.1] the equations

$$L_{k+2J} = \sqrt{L_{k+1J}}\left(I - K_{k+1J}K^*_{k+1J}\right)\sqrt{L_{k+1J}} \tag{4.19}$$

and

$$R_{k+2J} = \sqrt{R_{k+1J}}\left(I - K_{k+1J}^{*}K_{k+1J}\right)\sqrt{R_{k+1J}} \tag{4.20}$$

follow. Thus, in view of

$$\begin{pmatrix} I_m & K_{k+1J} \\ K_{k+1J}^{*} & I_m \end{pmatrix}^{*} j_{mm} \begin{pmatrix} I_m & K_{k+1J} \\ K_{k+1J}^{*} & I_m \end{pmatrix}$$
$$= \begin{pmatrix} I_m - K_{k+1J}K_{k+1J}^{*} & 0 \\ 0 & -\left(I_m - K_{k+1J}^{*}K_{k+1J}\right) \end{pmatrix},$$

$\sqrt{L_{k+2J}}^{+}L_{k+2J}\sqrt{L_{k+2J}}^{+} = L_{k+2J}L_{k+2J}^{+}$, and $\sqrt{R_{k+2J}}^{+}R_{k+2J}\sqrt{R_{k+2J}}^{+} = R_{k+2J}R_{k+2J}^{+}$, we get (4.12). Further, taking into account $L_{k+1J}L_{k+1J}^{+}K_{k+1J} = K_{k+1J}$ and $K_{k+1J}R_{k+1J}R_{k+1J}^{+} = K_{k+1J}$, we obtain

$$\begin{pmatrix} I_m & K_{k+1J} \\ K_{k+1J}^{*} & I_m \end{pmatrix}^{*} \begin{pmatrix} L_{k+1J}L_{k+1J}^{+} & 0 \\ 0 & -R_{k+1J}R_{k+1J}^{+} \end{pmatrix} \begin{pmatrix} I_m & K_{k+1J} \\ K_{k+1J}^{*} & I_m \end{pmatrix}$$
$$= \begin{pmatrix} L_{k+1J}L_{k+1J}^{+} - K_{k+1J}K_{k+1J}^{*} & 0 \\ 0 & -\left(R_{k+1J}R_{k+1J}^{+} - K_{k+1J}^{*}K_{k+1J}\right) \end{pmatrix}. \tag{4.21}$$

Furthermore, (4.19) and (4.20) imply immediately

$$L_{k+2J} = \sqrt{L_{k+1J}}\left(L_{k+1J}L_{k+1J}^{+} - K_{k+1J}K_{k+1J}^{*}\right)\sqrt{L_{k+1J}} \tag{4.22}$$

and

$$R_{k+2J} = \sqrt{R_{k+1J}}\left(R_{k+1J}R_{k+1J}^{+} - K_{k+1J}^{*}K_{k+1J}\right)\sqrt{R_{k+1J}}. \tag{4.23}$$

Thus, using (4.21), (4.22), (4.23), and (4.12) we obtain (4.13). Eqs. (4.14) and (4.15) can be checked analogously. □

Corollary 4.4. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^{n}$ be a J-Potapov sequence. Let the matrix-valued functions $\mathfrak{C}_{nJ}^{\bullet}$ and $\mathfrak{D}_{nJ}^{\bullet}$ be defined as in Proposition 4.3. Further, let*

$$\mathfrak{E}_{nJ}(\zeta) := \operatorname{diag}\left(L_{n+1J}L_{n+1J}^{+}, -R_{n+1J}R_{n+1J}^{+}\right) - \left(\mathfrak{C}_{nJ}^{\bullet}(\zeta)\right)^{*} J^{\square}\mathfrak{C}_{nJ}^{\bullet}(\zeta)$$

and

$$\mathfrak{F}_{nJ}(\zeta) := \operatorname{diag}\left(R_{n+1J}R_{n+1J}^{+}, -L_{n+1J}L_{n+1J}^{+}\right) - \mathfrak{D}_{nJ}^{\bullet}(\zeta)J^{\square}\left(\mathfrak{D}_{nJ}^{\bullet}(\zeta)\right)^{*}$$

for each $\zeta \in \mathbb{C}$. Then:

(a) For each $\zeta \in \mathbb{D}$, the inequalities $\mathfrak{E}_{nJ}(\zeta) \geqslant 0_{2m\times 2m}$ and $\mathfrak{F}_{nJ}(\zeta) \geqslant 0_{2m\times 2m}$ hold.
(b) For each $\zeta \in \mathbb{T}$, the identities $\mathfrak{E}_{nJ}(\zeta) = 0_{2m\times 2m}$ and $\mathfrak{F}_{nJ}(\zeta) = 0_{2m\times 2m}$ are satisfied.
(c) For each $\zeta \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$, the inequalities $-\mathfrak{E}_{nJ}(\zeta) \geqslant 0_{2m\times 2m}$ and $-\mathfrak{F}_{nJ}(\zeta) \geqslant 0_{2m\times 2m}$ hold.

Proof. Taking into account (4.3), (4.10), (4.11), (1.9), (1.10), and the relations $\sqrt{L_{1J}}^{+}L_{1J}\sqrt{L_{1J}}^{+} = L_{1J}L_{1J}^{+}$ and $\sqrt{R_{1J}}^{+}R_{1J}\sqrt{R_{1J}}^{+} = R_{1J}R_{1J}^{+}$, it is readily checked that $\mathfrak{E}_{0J}(\zeta) = \operatorname{diag}\left((1-|\zeta|^2)L_{1J}L_{1J}^{+}, 0_{m\times m}\right)$ and $\mathfrak{F}_{0J}(\zeta) = \operatorname{diag}\left((1-|\zeta|^2)R_{1J}R_{1J}^{+}, 0_{m\times m}\right)$ hold true for each $\zeta \in \mathbb{C}$. Thus, in the case $n = 0$ all the assertions follow. Now let $n \geqslant 1$. Then, for each $\zeta \in \mathbb{C}$, from Proposition 4.3 we obtain

$$\begin{aligned}\mathfrak{E}_{nJ}(\zeta) &= \operatorname{diag}\left((1-|\zeta|^2)L_{n+1J}L_{n+1J}^{+}, 0_{m\times m}\right) \\ &\quad + (G_{nJ}(\zeta))^{*} j_{mm} G_{nJ}(\zeta) - (G_{nJ}(\zeta))^{*}\left(\mathfrak{C}_{n-1J}^{\bullet}(\zeta)\right)^{*} J^{\square}\mathfrak{C}_{n-1J}^{\bullet}(\zeta)G_{nJ}(\zeta) \\ &= \operatorname{diag}\left((1-|\zeta|^2)L_{n+1J}L_{n+1J}^{+}, 0_{m\times m}\right) + (G_{nJ}(\zeta))^{*}\mathfrak{E}_{n-1J}(\zeta)G_{nJ}(\zeta)\end{aligned}$$

and, similarly,

$$\mathfrak{F}_{nJ}(\zeta) = \operatorname{diag}\left((1-|\zeta|^2)R_{n+1,J}R_{n+1,J}^+, 0_{m\times m}\right) + H_{nJ}(\zeta)\mathfrak{F}_{n-1,J}(\zeta)(H_{nJ}(\zeta))^*.$$

Hence, in view of $L_{n+1,J}L_{n+1,J}^+ \geqslant 0_{m\times m}$ and $R_{n+1,J}R_{n+1,J}^+ \geqslant 0_{m\times m}$ the assertions follow by induction. □

5. The nondegenerate case

In this section, we extend larger parts of results obtained in Section 3 of [6] for finite $p \times q$ Schur sequences.

Let J be an $m \times m$ signature matrix and let $n \in \mathbb{N}_0$. Throughout this section, we consider a strict J-Potapov sequence $(A_j)_{j=0}^n$. Observe that in this case the matrices P_{kJ} and Q_{kJ} are nonsingular for each $k \in \mathbb{N}_{0,n}$. Moreover, from [9, Lemmata 3.3 and 3.7] we know that, for each $k \in \mathbb{N}_{0,n}$, the matrices L_{k+1J} and R_{k+1J} are nonsingular as well.

Remark 5.1. Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, and let $(A_j)_{j=0}^n$ be a strict J-Potapov sequence. For each $k \in \mathbb{N}_{1,n}$, then the sets $\mathcal{Y}_{kJ}$, $\mathcal{Z}_{kJ}$, $\mathcal{L}_{kJ}$, and $\mathcal{R}_{kJ}$ defined by (2.9), (2.10), (3.1), and (3.2), respectively, fulfill $\mathcal{Y}_{kJ} = \{V_k^\square\}$, $\mathcal{Z}_{kJ} = \{W_k^\square\}$, $\mathcal{L}_{kJ} = \{t_k^\square\}$, and $\mathcal{R}_{kJ} = \{u_k^\square\}$ where

$$V_k^\square := Q_{k-1,J}^{-1}S_{k-1}^*J_{[k-1]}y_k, \quad W_k^\square := z_kJ_{[k-1]}S_{k-1}^*P_{k-1,J}^{-1}, \tag{5.1}$$

$$t_k^\square := L_{kJ}^{-1}(A_k - M_{kJ}), \quad \text{and} \quad u_k^\square := (A_k - M_{kJ})R_{kJ}^{-1}. \tag{5.2}$$

Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^n$ be a strict J-Potapov sequence. In the sequel, for each $k \in \mathbb{N}_{0,n}$ and each $\zeta \in \mathbb{C}$, let the matrix polynomials π_{kJ}, ρ_{kJ}, σ_{kJ}, and τ_{kJ} be defined by

$$\pi_{kJ}(\zeta) := \begin{cases} A_0, & \text{if } k = 0, \\ A_0 + \zeta e_{k-1,m}(\zeta)J_{[k-1]}P_{k-1,J}^{-1}y_k, & \text{if } k \in \mathbb{N}, \end{cases} \tag{5.3}$$

$$\rho_{kJ}(\zeta) := \begin{cases} I_m, & \text{if } k = 0, \\ I_m + \zeta e_{k-1,m}(\zeta)J_{[k-1]}S_{k-1}^*P_{k-1,J}^{-1}y_k, & \text{if } k \in \mathbb{N}, \end{cases} \tag{5.4}$$

$$\sigma_{kJ}(\zeta) := \begin{cases} A_0, & \text{if } k = 0, \\ z_kQ_{k-1,J}^{-1}J_{[k-1]}\zeta\varepsilon_{k-1,m}(\zeta) + A_0, & \text{if } k \in \mathbb{N}, \end{cases} \tag{5.5}$$

and

$$\tau_{kJ}(\zeta) := \begin{cases} I_m, & \text{if } k = 0, \\ z_kQ_{k-1,J}^{-1}S_{k-1}^*J_{[k-1]}\zeta\varepsilon_{k-1,m}(\zeta) + I_m, & \text{if } k \in \mathbb{N}. \end{cases} \tag{5.6}$$

Remark 5.2. Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}$, and let $(A_j)_{j=0}^n$ be a strict J-Potapov sequence. Let $k \in \mathbb{N}_{1,n}$. Then it is readily checked that the equations

$$J_{[k-1]}S_{k-1}^*P_{k-1,J}^{-1} = Q_{k-1,J}^{-1}S_{k-1}^*J_{[k-1]},$$

$$I + S_{k-1}Q_{k-1,J}^{-1}S_{k-1}^*J_{[k-1]} = J_{[k-1]}P_{k-1,J}^{-1},$$

and

$$J_{[k-1]}S_{k-1}^*P_{k-1,J}^{-1}S_{k-1} + I = Q_{k-1,J}^{-1}J_{[k-1]}$$

hold. In particular, the matrix polynomials π_{kJ}, ρ_{kJ}, σ_{kJ}, and τ_{kJ} defined by (5.3), (5.4), (5.5), and (5.6), respectively, fulfill

$$\pi_{kJ}(\zeta) = A_0 + \zeta e_{k-1,m}(\zeta)(y_k + S_{k-1}V_k^{\square}),$$
$$\rho_{kJ}(\zeta) = I_m + \zeta e_{k-1,m}(\zeta)V_k^{\square},$$
$$\sigma_{kJ}(\zeta) = \left(W_k^{\square}S_{k-1} + z_k\right)\zeta\varepsilon_{k-1,m}(\zeta) + A_0$$

and

$$\tau_{kJ}(\zeta) = W_k^{\square}\zeta\varepsilon_{k-1,m}(\zeta) + I_m$$

for each $\zeta \in \mathbb{C}$ where $V_k^{\square}$ and $W_k^{\square}$ are given by (5.1).

Proposition 5.3. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^n$ be a strict J-Potapov sequence. For every $k \in \mathbb{N}_{0,n}$, let the matrix polynomials π_{kJ}, ρ_{kJ}, σ_{kJ}, and τ_{kJ} be defined by (5.3), (5.4), (5.5), and (5.6), respectively. Then the following statements hold:*

(a) Let $k \in \mathbb{N}_{0,n}$. Then the J-central J-Potapov function $f_{c,k}$ corresponding to $(A_j)_{j=0}^k$ admits the representations

$$f_{c,k} = \pi_{kJ,\mathbb{D}}\,\rho_{kJ,\mathbb{D}}^{-1} \quad \text{and} \quad f_{c,k} = \tau_{kJ,\mathbb{D}}^{-1}\,\sigma_{kJ,\mathbb{D}} \tag{5.7}$$

where $\pi_{kJ,\mathbb{D}}$ (respectively, $\rho_{kJ,\mathbb{D}}, \sigma_{kJ,\mathbb{D}}$, and $\tau_{kJ,\mathbb{D}}$) denotes the restriction of π_{kJ} (respectively, ρ_{kJ}, σ_{kJ}, and τ_{kJ}) onto $\mathbb{D}$. Moreover, for the set $\mathbb{H}_{f_{c,k}}$ of all points of $\mathbb{D}$ at which $f_{c,k}$ is holomorphic the equations

$$\mathbb{H}_{f_{c,k}} = \{\zeta \in \mathbb{D} : \det \rho_{kJ}(\zeta) \neq 0\} = \{\zeta \in \mathbb{D} : \det \tau_{kJ}(\zeta) \neq 0\}$$

hold true.

(b) Let $n \in \mathbb{N}$, let $k \in \mathbb{N}_{0,n-1}$, and let $t_{k+1}^{\square}$ and $u_{k+1}^{\square}$ be given by (5.2). Then, for each $\zeta \in \mathbb{C}$, the recursion formulas

$$\pi_{k+1J}(\zeta) = \pi_{kJ}(\zeta) + \zeta J\tilde{\tau}_k^{[k]}(\zeta)t_{k+1}^{\square}, \tag{5.8}$$
$$\rho_{k+1J}(\zeta) = \rho_{kJ}(\zeta) + \zeta J\tilde{\sigma}_k^{[k]}(\zeta)t_{k+1}^{\square}, \tag{5.9}$$
$$\sigma_{k+1J}(\zeta) = \sigma_{kJ}(\zeta) + u_{k+1}^{\square}\zeta\tilde{\rho}_k^{[k]}(\zeta)J, \tag{5.10}$$

and

$$\tau_{k+1J}(\zeta) = \tau_{kJ}(\zeta) + u_{k+1}^{\square}\zeta\tilde{\pi}_k^{[k]}(\zeta)J \tag{5.11}$$

are fulfilled.

Proof. Use Remarks 5.1 and 5.2, Theorems 2.7 and 2.8, Propositions 2.19 and 2.20, Remark 3.2, Lemma 3.3, and Proposition 3.4. □

Lemma 5.4. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^n$ be a strict J-Potapov sequence. For each $k \in \mathbb{N}_{0,n}$ and each $\zeta \in \mathbb{C}$, let the matrix polynomials π_{kJ}, ρ_{kJ}, σ_{kJ}, and τ_{kJ} be given by (5.3), (5.4), (5.5), and (5.6), respectively. For each $k \in \mathbb{N}_{0,n}$, let the matrix-valued functions $\mathfrak{C}_{kJ} : \mathbb{C} \to \mathbb{C}^{2m\times 2m}$ and $\mathfrak{D}_{kJ} : \mathbb{C} \to \mathbb{C}^{2m\times 2m}$ be defined by*

$$\mathfrak{C}_{kJ}(\zeta) := \begin{pmatrix} \zeta J\tilde{\tau}_{kJ}^{[k]}(\zeta) & \pi_{kJ}(\zeta) \\ \zeta J\tilde{\sigma}_{kJ}^{[k]}(\zeta) & \rho_{kJ}(\zeta) \end{pmatrix} \begin{pmatrix} \sqrt{L_{k+1J}}^{-1} & 0_{m\times m} \\ 0_{m\times m} & \sqrt{R_{k+1J}}^{-1} \end{pmatrix} \tag{5.12}$$

and

$$\mathfrak{D}_{kJ}(\zeta) := \begin{pmatrix} \sqrt{R_{k+1J}}^{-1} & 0_{m\times m} \\ 0_{m\times m} & \sqrt{L_{k+1J}}^{-1} \end{pmatrix} \begin{pmatrix} \zeta\tilde{\rho}_{kJ}^{[k]}(\zeta)J & \zeta\tilde{\pi}_{kJ}^{[k]}(\zeta)J \\ \sigma_{kJ}(\zeta) & \tau_{kJ}(\zeta) \end{pmatrix}. \tag{5.13}$$

Then

$$\mathfrak{D}_{nJ}(\zeta)U_{mm}\mathfrak{C}_{nJ}(\zeta) = \zeta^{n+1}U_{mm}$$

holds for each $\zeta \in \mathbb{C}$ *where*

$$U_{mm} := \begin{pmatrix} 0_{m\times m} & I_m \\ -I_m & 0_{m\times m} \end{pmatrix}.$$

Proof. Use Remarks 5.1 and 5.2 as well as Lemma 4.1. □

Let us now recall the notion of linear fractional transformations of matrices (see Potapov [11]). Let A and B be complex $2m \times 2m$ matrices and let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

be the $m \times m$ block representations of A and B. If the set

$$\mathcal{Q}_{(c,d)} := \left\{x \in \mathbb{C}^{m\times m} : \det(cx + d) \neq 0\right\}$$

is nonempty, then let $\mathcal{S}_A^{(m,m)} : \mathcal{Q}_{(c,d)} \to \mathbb{C}^{m\times m}$ be defined by

$$\mathcal{S}_A^{(m,m)}(x) := (ax + b)(cx + d)^{-1}.$$

If the set

$$\mathcal{R}_{\binom{\gamma}{\delta}} := \{x \in \mathbb{C}^{m\times m} : \det(x\gamma + \delta) \neq 0\}$$

is nonempty, then let $\mathcal{T}_A^{(m,m)} : \mathcal{R}_{\binom{\gamma}{\delta}} \to \mathbb{C}^{m\times m}$ be defined by

$$\mathcal{T}_A^{(m,m)}(x) := (x\gamma + \delta)^{-1}(x\alpha + \beta).$$

Observe that $\mathcal{Q}_{(c,d)} \neq \emptyset$ if and only if $\operatorname{rank}(c, d) = m$. Moreover, $\mathcal{R}_{\binom{\gamma}{\delta}} \neq \emptyset$ if and only if $\operatorname{rank}\binom{\gamma}{\delta} = m$.

Remark 5.5. Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^n$ be a strict J-Potapov sequence. Let the matrix-valued functions $\mathfrak{C}_{nJ} : \mathbb{C} \to \mathbb{C}^{2m\times 2m}$ and $\mathfrak{D}_{nJ} : \mathbb{C} \to \mathbb{C}^{2m\times 2m}$ be defined as in Lemma 5.4. Taking into account Lemma 5.4 and a well-known result on linear fractional transformations of matrices (see, e.g., [2, Lemma 1.6.2 and Proposition 1.6.1]), we infer that for each $\zeta \in \mathbb{C} \setminus \{0\}$ the mappings $\mathcal{S}^{(m,m)}_{\mathfrak{C}_{nJ}(\zeta)}$ and $\mathcal{T}^{(m,m)}_{\mathfrak{D}_{nJ}(\zeta)}$ are well-defined and fulfill

$$\mathcal{S}^{(m,m)}_{\mathfrak{C}_{nJ}(\zeta)} = \mathcal{T}^{(m,m)}_{\mathfrak{D}_{nJ}(\zeta)}. \tag{5.14}$$

Further, one can easily see that $\mathcal{S}^{(m,m)}_{\mathfrak{C}_{nJ}(0)}$ and $\mathcal{T}^{(m,m)}_{\mathfrak{D}_{nJ}(0)}$ are well-defined and also coincide, i.e., (5.14) is valid for all $\zeta \in \mathbb{C}$.

The following result generalizes parts of Proposition 3.2 in [6].

Proposition 5.6. *Let* J *be an* $m \times m$ *signature matrix, let* $n \in \mathbb{N}$, *and let* $(A_j)_{j=0}^n$ *be a strict* J*-Potapov sequence. For each* $k \in \mathbb{N}_{0,n}$, *let the matrix-valued functions* $\mathfrak{C}_{kJ} : \mathbb{C} \to \mathbb{C}^{2m\times 2m}$ *and* $\mathfrak{D}_{kJ} : \mathbb{C} \to \mathbb{C}^{2m\times 2m}$ *be defined as in Lemma 5.4. Further, for each* $k \in \mathbb{N}_{1,n}$, *let the matrix* K_{kJ} *be given by* $K_{kJ} := \sqrt{L_{kJ}}^{-1}(A_k - M_{kJ})\sqrt{R_{kJ}}^{-1}$, *and let the matrix-valued functions* $G_{kJ} : \mathbb{C} \to \mathbb{C}^{2m\times 2m}$ *and* $H_{kJ} : \mathbb{C} \to \mathbb{C}^{2m\times 2m}$ *be defined by*

$$G_{kJ}(\zeta) := \begin{pmatrix} I_m & K_{kJ} \\ K_{kJ}^* & I_m \end{pmatrix} \begin{pmatrix} \zeta\sqrt{L_{kJ}}\sqrt{L_{k+1J}}^{-1} & 0_{m\times m} \\ 0_{m\times m} & \sqrt{R_{kJ}}\sqrt{R_{k+1J}}^{-1} \end{pmatrix}$$

and

$$H_{kJ}(\zeta) := \begin{pmatrix} \zeta\sqrt{R_{k+1,J}}^{-1}\sqrt{R_{kJ}} & 0_{m\times m} \\ 0_{m\times m} & \sqrt{L_{k+1,J}}^{-1}\sqrt{L_{kJ}} \end{pmatrix} \begin{pmatrix} I_m & K_{kJ}^* \\ K_{kJ} & I_m \end{pmatrix}.$$

Then for each $k \in \mathbb{N}_{1,n}$ the equations

$$\mathfrak{C}_{kJ} = \mathfrak{C}_{0J} G_{1J} G_{2J} \cdot \ldots \cdot G_{kJ} \quad \text{and} \quad \mathfrak{D}_{kJ} = H_{kJ} H_{k-1,J} \cdot \ldots \cdot H_{1J} \mathfrak{D}_{0J}$$

are satisfied. Moreover, for each $k \in \mathbb{N}_{1,n}$ and each $\zeta \in \mathbb{C}$, the relations

$$j_{mm} - (G_{kJ}(\zeta))^* j_{mm} G_{kJ}(\zeta) = \operatorname{diag}\left((1-|\zeta|^2)I_m, 0_{m\times m}\right)$$

and

$$j_{mm} - H_{kJ}(\zeta) j_{mm} (H_{kJ}(\zeta))^* = \operatorname{diag}\left((1-|\zeta|^2)I_m, 0_{m\times m}\right)$$

hold.

Proof. Use part (b) of Proposition 5.3 and Proposition 4.3. □

Proposition 5.7. *Let J be an $m \times m$ signature matrix, let $n \in \mathbb{N}_0$, and let $(A_j)_{j=0}^n$ be a strict J-Potapov sequence. Let the matrix-valued functions $\mathfrak{C}_{nJ}$ and $\mathfrak{D}_{nJ}$ be defined as in Lemma 5.4. Further, let*

$$\mathfrak{E}_{nJ}(\zeta) := j_{mm} - (\mathfrak{C}_{nJ}(\zeta))^* J^{\square} \mathfrak{C}_{nJ}(\zeta) \text{ and } \mathfrak{F}_{nJ}(\zeta) := j_{mm} - \mathfrak{D}_{nJ}(\zeta) J^{\square} (\mathfrak{D}_{nJ}(\zeta))^*$$

for each $\zeta \in \mathbb{C}$. Then:

(a) *For each $\zeta \in \mathbb{D}$, the inequalities $\mathfrak{E}_{nJ}(\zeta) \geqslant 0_{2m\times 2m}$ and $\mathfrak{F}_{nJ}(\zeta) \geqslant 0_{2m\times 2m}$ hold.*
(b) *For each $\zeta \in \mathbb{T}$, the identities $\mathfrak{E}_{nJ}(\zeta) = 0_{2m\times 2m}$ and $\mathfrak{F}_{nJ}(\zeta) = 0_{2m\times 2m}$ are satisfied.*
(c) *For each $\zeta \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$, the inequalities $-\mathfrak{E}_{nJ}(\zeta) \geqslant 0_{2m\times 2m}$ and $-\mathfrak{F}_{nJ}(\zeta) \geqslant 0_{2m\times 2m}$ hold.*

Proof. Use part (b) of Proposition 5.3 and Corollary 4.4. □

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