Measuring the coupled risks: A copula-based CVaR model

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Abstract

Integrated risk management for financial institutions requires an approach for aggregating risk types (such as market and credit) whose distributional shapes vary considerably. The financial institutions often ignore risks' coupling influence so as to underestimate the financial risks. We constructed a copula-based Conditional Value-at-Risk (CVaR) model for market and credit risks. This technique allows us to incorporate realistic marginal distributions that capture essential empirical features of these risks, such as skewness and fat-tails while allowing for a rich dependence structure. Finally, the numerical simulation method is used to implement the model. Our results indicate that the coupled risks for the listed company’s stock maybe are undervalued if credit risk is ignored, especially for the listed company with bad credit quality.

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1. Introduction

The measurement of financial risks has been one of the main preoccupations of actuaries and insurance practitioners for a very long time. They require more risk types to be considered and achieve the goal of integrated risk management. The difficulty of the goal is how to aggregate different risk types and hence risk distributions. It requires modeling the multivariate dependence between the various risk types. In practice, some kind of heuristics, based on strong assumptions, are often used to merge the economic capital numbers for the various risk types into one overall capital number. For example, it is frequently assumed that the loss distributions resulting from the different risk types are multivariate normal distribution. However, this is certainly not true for credit or operational losses. Ane and Kharoubi [1] and Embrechts et al. [2] argue that this assumption is frequently unsatisfactory because large changes are observed more frequently than predicted under the normality assumption. Market risk typically generates portfolio value distributions that are nearly symmetric and often approximated as normal. Credit risk especially operational risk, generates more skewed distributions because of occasional extreme loss. Kuritzkes et al. [3] present stylized pictures of a very broad range of risk types that are faced by a financial conglomerate. Since deviations from normality, modeling based on a copula parameterized by non-normal margins, is an attractive alternative [4].

A copula is essentially a function that links univariate marginal distributions to the joint multivariate distribution function, which is simply a convenient way to describe joint distributions of two or more random variables. Copulas allow the aggregation of diverse marginal distributions and capture some of the essential features found in risk management (e.g. skewness and fat tails). They were introduced in the seminal paper by Sklar [5], who showed that all finite dimensional probability laws have an associated copula function that describes the dependency of their marginal distributions. Embrechts et al. [6]...
are among the first to introduce this methodology to the finance literature. Li [7] provides an application to credit risk and credit derivatives, and uses the Gaussian copula to simulate the dependent structure of the time-until-default. Others [8–11] expanded the original model of Li introducing different types of copula, beyond the Gaussian copula, to model the joint default dependency. Breymann et al. [10] showed that the empirical fit of the Student’s $t$-copula is generally superior to that of the Gaussian copula, the dependent structure of the normal multivariate distribution. One reason for this is the ability of the Student’s $t$-copula to capture better phenomenon of dependent extreme values, which is often observed in financial data. In fact, the Student’s $t$-copula has two salient features that the term “beyond correlation” alludes to. First, unlike the Gaussian copula where dependence is captured only via correlation, the Student’s $t$-copula retains the use of correlation while introducing a single additional parameter, the degrees of freedom, which determines the potential for extreme co-movements. In particular, increasing the value of the degrees of freedom parameter decreases the tendency of underlying to exhibit extreme co-movements. Moreover, the Gaussian copula is a particular case of the Student’s $t$-copula, when the number of degrees of freedom tends to infinity. Second, the Student’s $t$-copula supports extreme co-movements regardless of the marginal behavior of the individual assets (e.g., this extreme behavior is manifestly present even if the individual assets follow “light” tail distributions). Therefore, Di Clemente and Romano [11] study four different types of dependence structure for the credit assets in portfolio: the Gaussian copula, the Student’s $t$-copula, the grouped $t$-copula and the Clayton $n$-copula. They find that the copulas with tail dependence produce tail risk measurements greater than those obtained from the Gaussian copula. Other applications of copulas in finance econometrics include multivariate option pricing [12–14], asset allocation [15,16], integrated risk management [17–21], nonlinear autoregressive dependence [22–26], contagion [27,28], and so on.

Based on copula method, the techniques of integrated risk management can be divided into top-down and bottom-up categories [29]. In the top-down approaches, the separately determined marginal distributions of losses resulting from different risk types are linked by adequate copula functions. Combined with the copulas, the top-down approaches [18,19,30–32] are widely used in finance. In contrast, the bottom-up approaches model the complex interactions already described above on the level of the risk factors which makes them far more exact. Since possible stochastic dependencies between risk factors can be taken into account, there is no need for a later aggregation of the portfolio by copulas. The bottom-up approaches need to identify the common risk factors, which is more difficult than the top-down approaches. A few researchers developed this approaches using copulas. Medova and Smith [33] use Monte Carlo simulations to allow for a varying exposure of the credit portfolio (employing a structural credit risk model). Aas et al. [34] use a bottom-up approach to aggregate market, credit and ownership risk. For aggregating operational and business risk, they use a top-down approach (employing a Gaussian copula).

In the paper, we adopt the bottom-up approach to aggregate market and credit risk for the portfolio of the companies’ stocks. The market risk is easily measured from a company’s stock, but the credit risk of the company is more difficult which is researched using the structural approach and the reduced-form approach. In order to gain the market and credit risks’ coupling influence, we have to simulate the default events of the companies based on the notion of time-until-default and combine it with the stock return of company. For the portfolio of the companies’ stocks, the market risk and the default correlation can be described by the copula function. Technically, the dependent structure of the time-until-default are simulated following a copula-based approach first illustrated in Li [7]. In fact, the copula approach takes as given the marginal default probabilities of the different companies, and plugs them into a copula function which provides the model with the dependence structure to generate joint default probabilities. Our study follows the research contributions of Di Clemente and Romano [11] who simulate the time-until-default using the hazard rate and calibrates the hazard rate using credit spread. Since it is difficult to gain the credit spread of the listed companies in China, we develop an approach to gain the probability of default using the market price of stock. So we use the probability of default instead of credit spread curve to calibrate the hazard rate. In order to improve the precision of the default probability, we describe marginal distributions of stock’s returns using Generalized Error Distribution (GED) that preserves important properties of the individual risk, such as fat tail. At the same time, market and credit risk distributions are known to be time varying, reflecting the effects of stochastic volatility. With this in mind, we estimate dynamic models for these two risks using a multivariate GARCH approach. Moreover, we consider two parametric families of copulas (the elliptical copulas and Archimedean copulas) in order to compare the effect of the dependence structure for aggregated risks. One of the main reasons why Archimedean copulas are of interest is that they are not elliptical copulas and allow modeling a big variety of different dependence structures. For example, Clayton copula can be used to model lower tail dependence.

After generating the scenarios on the loss of capital by Monte Carlo method, we adopt a Conditional Value-at-Risk (CVaR) criterion [35,36] to measure the coupled risks of the market and credit risks. CVaR is a coherent risk measure that estimates the mean of the beyond-VaR tail region, which may be widely used in the optimization of the portfolio [35,37–40]. One of the main reasons why the CVaR is of interest is that it can capture the tail risk which is important to credit risk. The CVaR of portfolio is usually calculated using linear programming method. However, if the size of portfolio becomes larger, it costs more time to calculate the CVaR of the portfolio. So we introduce the order statistics (OS) estimation method to calculate the CVaR of the portfolio, which is simple and fast.

The rest of the paper proceeds as follows. In the second section, we describe how to gain the probability of default using Distance-to-Default, which is implied by Moody’s KMV approach. The KMV approach implements the Vasicek–Kealhofer model [41,42], which in turn is an extension of Merton’s [43]. The advantage of this approach is that it relies solely on stock market data. Section 3 introduces the mathematical forms of the five copulas (the Gaussian copula, the Student’s $t$-copula,
the Clayton copula, the Gumbel copula and the Frank copula). We choose these copulas to research their effect for integrated risk measure and describe how to calibrate and simulate these copulas. In Section 4, we measure the coupled risks of the market and credit risks with CVaR criterion. In particular, we simulate the scenarios on the loss of capital by Monte Carlo method, and gain the CVaR using order statistics (OS) estimation. In Section 5, we apply the whole described methodology to two different types of samples, which come from Chinese security market. Finally, Section 6 concludes the paper.

2. Assessing the probability of default using market prices

At time $t$, the return of a stock is marked as $r_t$, then

$$
E(r_t) = r_t + \varepsilon_t,
$$

where $E(r_t)$ is the expected return of the stock, the mean of random error $\varepsilon_t$ is 0, its variance is $\sigma_t^2$. So the stock price $S_t$ and its expected price can be written as

$$
S_t = S_{t-1} (1 + E(r_t) + \varepsilon_t),
$$

$$
E(S_t) = S_{t-1} (1 + E(r_t)).
$$

In general, the value of a company’s assets $V_t$ always equals the sum of its liabilities $B_t$ and all shareholders equity $NP_t$, as

$$
V_t = NP_t + B_t.
$$

If a company is unable to pay debts with their assets, the company is default, namely $NP_t < 0$. If we assume that all shareholders equity are priced efficiently by the market, then $NP_t = N_i S_i$, where $N_i$ is the number of all stocks. The probability of default on the time interval $[t-1, t]$ is given by

$$
\Pr(NP_t \leq 0 | NP_{t-1} > 0) = \Pr(N_i S_i \leq 0 | N_i S_{i-1} > 0)
= \Pr(N_i S_{i-1} (1 + E(r_t) + \varepsilon_t) \leq 0 | N_i S_{i-1} > 0)
= \Pr(1 + E(r_t) + \varepsilon_t \leq 0 | N_i S_{i-1} > 0)
= \Pr \left( \frac{\varepsilon_t}{\sigma_t} \leq -\frac{(1 + E(r_t))}{\sigma_t} | N_i S_{i-1} > 0 \right).
$$

According to KMV model, Distance-to-Default (DD) of the company is written as:

$$
DD = \frac{E(NP_t) - 0}{\sqrt{\text{VAR}(NP_t)}} = \frac{E(N_i S_i) - 0}{\sqrt{\text{VAR}(N_i S_i)}} = \frac{N_i S_{i-1} (1 + E(r_t))}{N_i S_{i-1} \sigma_t}
= \frac{1 + E(r_t)}{\sigma_t}.
$$

Substituting into Eq. (5), we obtain

$$
\Pr(NP_t \leq 0 | NP_{t-1} > 0) = \Pr \left( \frac{\varepsilon_t}{\sigma_t} \leq -DD | N_i S_{i-1} > 0 \right).
$$

Hence, if we owned the distribution of the random variable $\varepsilon_t/\sigma_t$ and the DD, the probability of default on the time interval $[t-1, t]$ is gained from Eq. (6) (as illustrated in Fig. 1). The precision of the probability of default depends on the estimation of $E(S_i)$ and $\sigma_t$.

According to CAPM model, we have

$$
E(r_t) - r_f = \beta(E(r_m) - r_f),
$$

Fig. 1. The probability of default of a listed company.
where \( r_f \) is the risk-free interest rate and \( E(r_{mt}) \) is the expected return of the market portfolio at time \( t \), \( \beta \) is the sensitivity of the asset return to the market portfolio return.

The market price of risk is defined as

\[
\lambda = \frac{E(r_{mt}) - r_f}{\sigma_{mt}}.
\]

The return of the market portfolio at time \( t \) is

\[
\begin{align*}
    r_{mt} & = E(r_{mt}) + \varepsilon_{mt} \\
    & = r_f + \lambda \sigma_{mt} + \varepsilon_{mt}.
\end{align*}
\]

(7)

Observers of financial markets have long noted that the volatility of financial price movements varies stochastically. The well-established ARCH and GARCH models of Engle [44] and Bollerslev [45] and the plethora of descendants provide a very convenient framework for empirical modeling of volatility dynamics. A practical problem encountered in fitting ARCH(\( p \)) models to financial data is that in order to obtain a good fitting model, the order \( p \) needs to be fairly large, e.g. often in excess of 8–10 or more. Bollerslev expended ARCH model and established the class of generalized autoregressive conditionally heteroscedastic (GARCH) models, such as

\[
\begin{align*}
    \varepsilon_t & = \sigma_t \xi_t \\
    \sigma_t^2 & = \kappa + \sum_{i=1}^p \gamma_i \sigma_{t-i}^2 + \sum_{i=1}^q \eta_i \varepsilon_{t-i}^2,
\end{align*}
\]

(8)

where \( \varepsilon_t \) is identically independently distributed (i.i.d) and \( \varepsilon_t \sim N(0, 1) \), \( \kappa > 0 \), denoted as \( \varepsilon_t \sim GARCH(p, q) \). The unconditional distribution of \( \varepsilon_t \) in GARCH isn’t a normal distribution, its tail is “fatter” than normal distribution’s. But a lot of researches show that the conditional distribution of \( \varepsilon_t \) is sometimes non-normal. So, we suppose that the conditional distribution of the market return is described by normal distribution for it contains market risk only, and the tail of stock return’s conditional distribution is “fatter” than normal distribution’s because stock return includes market risk and special risk of the company. Therefore, we adopt Generalized Error Distribution (GED) to describe the conditional distribution of stock return. The density of a GED random variable normalized to have a mean and variance of zero and one, respectively, is given by

\[
f(x) = \frac{\nu \cdot \exp(-|x|^{\nu}/2)}{\nu \cdot 2^{(\nu+1)/\nu} \Gamma(1/\nu)},
\]

where \(-\infty < x < \infty, 0 < \nu < \infty, \Gamma(*)\) is the gamma function, \( \nu \) is defined as

\[
\nu = \left( \frac{2^{-2/\nu} \Gamma(1/\nu)}{\Gamma(3/\nu)} \right)^{1/2},
\]

and the degrees of freedom \( \nu \) is a tail-thickness parameter. When \( \nu = 2 \), \( x \) has a standard normal distribution. When \( \nu < 2 \), the distribution of \( x \) has thicker tails than a normal distribution.

If GARCH(1, 1) model is selected, we get a multivariate GARCH-M model, such as

\[
\begin{align*}
    r_t - r_f & = \alpha + \beta (E(r_{mt}) - r_f) + \varepsilon_t \\
    \sigma_t^2 & = \omega + \eta \varepsilon_{t-1}^2 + \gamma \sigma_{t-1}^2 \\
    r_{mt} - r_f & = \theta + \lambda \sigma_{mt} + \varepsilon_{mt} \\
    \sigma_{mt}^2 & = \omega_m + \eta_m \varepsilon_{m(t-1)}^2 + \gamma_m \sigma_{m(t-1)}^2,
\end{align*}
\]

(9)

where \( \omega > 0, \eta \geq 0, \gamma \geq 0, \eta + \gamma < 1, \omega_m > 0, \eta_m \geq 0, \gamma_m \geq 0, \eta_m + \gamma_m < 1 \), the distribution of \( \varepsilon_t \) is the GED, the distribution of \( \varepsilon_{mt} \) is the normal distribution.

Eq. (9) gives us the daily estimates of stock return and variance. Then we can gain the daily probability of default. But the daily estimates contain too much noise and are too frequent for most applications, the quarterly or yearly estimates are more reasonable. To create a probability of default during \( T \) day from the daily estimates, the stock variance during \( T \) days is

\[
\sigma_{[t:t+T]}^2 = \sum_{i=0}^{T-1} \sigma_{[t+i:t+i+1]}^2.
\]

(10)

Then the expect value of the shareholders equity after \( T \) day is

\[
E(NP_{t+T-1}) = E(N_t S_{t+T-1}) = N_t S_{t-1} \prod_{i=0}^{T-1} (1 + E(r_{t+i})).
\]

(11)
Distance-to-Default of the company during \(T \) day is

\[
DD = \frac{\prod_{i=0}^{T-1} (1 + E(t_{i+1}))}{\sqrt{\sum_{i=0}^{T-1} \sigma_{t_i}^2}}.
\]  

Employing Eq. (12), we can estimate the probability of default during \(T \) day.

3. Measuring the co-dependency of risks: The copula methodology

As we have discussed above, the market portfolio return contains the information of market risk, however the stock’s return includes market risk and other special information of the company, such as credit risk of the company. The two kinds of financial data are dependent, that is to say \(\varepsilon_m \) and \(\varepsilon_t \) are dependent. We describe the dependent structure using the copula function, which is based on a separate statistical treatment of dependence and marginal behavior. The copula function has for a long time been recognized as a powerful tool for modeling dependence between random variables. The use of copula methodology in financial applications is a relatively new and fast-growing field. From a practical point of view, the advantage of the copula-based approach to modeling is that appropriate marginal distributions for the components of a multivariate system can be selected freely, and then linked through a suitable copula. That is, copula functions allow one to model the dependent structure of the marginal distributions independently. Any multivariate distribution function can serve as a copula.

The definition of a \(d \)-dimensional copula is a multivariate distribution, \(C \), with uniformly distributed margins \(U(0, 1) \) on \([0, 1] \). Sklar’s theorem states that every multivariate distribution \(F \) with continuous margins \(F_1, F_2, \ldots, F_d \) can be written as

\[
F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))
\]

for some copula \(C \). If let \(X = (X_1, \ldots, X_d)^T \) be a random vector with multivariate distribution \(F \) and continuous margins \(F_1, F_2, \ldots, F_d \), the copula \(C \) of \(X \) may be extracted from Eq. (13):

\[
C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)),
\]

where \(F_i^{-1} \) is the generalized inverse of \(F_i \).

There are a lot of copulas. In our survey we concentrate on two parametric families of copulas: the elliptical copulas and the Archimedean copulas. The Gaussian copula is a very popular elliptical copula, but cannot fit to possess tail dependence, which has neither upper nor lower tail dependence. The Student’s \(t \)-copula is generally superior to the Gaussian one on this condition, which allows for joint fat tails and an increased probability of joint extreme events compared with the Gaussian copula. The Student’s \(t \)-copula introduces an additional parameter, namely the degrees of freedom \(v \). Increasing the value of \(v \) decreases the tendency to exhibit extreme co-movements.

**Multivariate Gaussian copula.** The multivariate Gaussian copula is the copula of the multivariate normal distribution. It can be written as

\[
C^G(u_1, \ldots, u_d) = \Phi(u_1, \ldots, u_d),
\]

where \(\Phi^{-1} \), as usual, is the inverse of the standard univariate normal distribution function \(\Phi \). Its density is

\[
C^G(u_1, \ldots, u_d) = \frac{1}{|R|^{1/2}} \exp \left( -\frac{1}{2} \omega^T (R^{-1} - I) \omega \right),
\]

where \(\omega = (\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d))^T \), \(R \) is the correlation matrix.

**Multivariate Student’s \(t \)-copula.** The multivariate Student’s \(t \)-copula is the copula of the multivariate Student’s \(t \)-distribution, which is given by

\[
C^{t}_{v,R}(u_1, \ldots, u_d) = \Phi^t(u_1, \ldots, u_d)
\]

where \(\Phi^t \) is the standardized multivariate Student’s \(t \) distribution with correlation matrix \(R \) and \(v \) degrees of freedom, \(\Phi^{-1} \) is the inverse of the univariate cumulative distribution function of Student’s \(t \) with \(v \) degrees of freedom.

The density of the Student’s \(t \)-copula is

\[
c^{t}_{v,R}(u_1, \ldots, u_d) = \frac{\Gamma((v + d)/2) [\Gamma(v/2)]^d}{|R|^{1/2} \Gamma(v/2) \prod_{i=1}^d \Gamma((v + 1)/2)} \left(1 + \omega^T R^{-1} \omega\right)^{-(v+d)/2} 
\]

where \(\omega = (\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d))^T \), \(\Gamma(*) \) is the gamma function.

The Student’s \(t \)-copula allows for joint extreme events, but not for asymmetries. If one believes in the asymmetries in equity return dependence structures reported by, for instance, Longin and Solnik [46] and Ang and Chen [47], the Student’s
t-copula may also be too restrictive to provide a reasonable fit. The Archimedean copulas might be a better choice, which include the Clayton copula, the Gumbel copula, the Frank copula, and so on. The Archimedean copulas have a simple closed form instead of the integral form, but their densities perhaps aren’t the simple formula for multivariate copula except for the Clayton copula, which offers a recursive formula.

**Multivariate Clayton copula.** The Clayton copula, which is an asymmetric copula, exhibits greater dependence in the negative tail than in the positive. The multivariate Clayton copula can be written as

\[ C_{\alpha}(u_1, \ldots, u_d) = \left[ \sum_{i=1}^{d} u_i^{-\alpha} - d + 1 \right]^{-1/\alpha}, \tag{19} \]

where \( \alpha > 0 \) is a parameter controlling the dependence. Perfect dependence is obtained if \( \alpha \to \infty \), while \( \alpha = 0 \) implies independence.

The Clayton copula density is given by:

\[ c_{\alpha}(u_1, \ldots, u_d) = \alpha^d \frac{\prod_{i=1}^{d} u_i^{-\alpha-1}}{\Gamma(\frac{1}{\alpha})} \left( \sum_{i=1}^{d} u_i^{-\alpha} - d + 1 \right)^{-\frac{1}{\alpha}}, \tag{20} \]

where \( \Gamma(\cdot) \) is the gamma function.

**Multivariate Gumbel copula.** The Gumbel copula is also an asymmetric copula, but it is exhibiting greater dependence in the positive tail than in the negative. The multivariate Gumbel copula is given by

\[ C_{\alpha}(u_1, \ldots, u_d) = \exp \left\{ - \left[ \sum_{i=1}^{d} (\ln u_i)^\alpha \right]^{1/\alpha} \right\}, \tag{21} \]

where \( \alpha \geq 1 \) is a parameter controlling the dependence. Perfect dependence is obtained if \( \alpha \to \infty \), while \( \alpha = 1 \) implies independence.

**Multivariate Frank copula.** The Frank copula is a symmetric copula. The multivariate Frank copula is given by

\[ C_{\alpha}(u_1, \ldots, u_d) = -\frac{1}{\alpha} \ln \left\{ \frac{1}{\prod_{i=1}^{d} (e^{-\alpha u_i} - 1)^{d+1}} \right\}, \tag{22} \]

where \( \alpha > 0 \) is a parameter controlling the dependence.

In our paper, we choose the five copulas to research their effect for integrated risk measure. The different copulas parameters are estimated by the maximum likelihood method, which need calculate the log-likelihood function. However, for some multivariate copulas such as the Gumbel copula and the Frank copula, the copula density is not a general form for any dimensions. We induce the \( d \)-dimensional density according to their copula function using MATLAB tools. See Appendix A for more details with regard to the estimation of the copula parameters. The algorithm for simulating Monte Carlo scenarios from the copula is in Appendix B.

For aggregating market and credit risks, we can gain their datum from Eq. (9), given the marginal distribution functions of \( \varepsilon_m \) and \( \varepsilon_c \). Then the copula parameters are estimated. Finally, the random scenarios with copula structure are generated using Monte Carlo simulation.

### 4. Measuring the coupled risks: A copula-based CVaR method

In order to describe the company’s special information, we suppose the stock’s return obey the distribution of GED instead of normal distribution. So the traditional mean-variance framework cannot be traded with this case. Then we adopt a copula-based Conditional Value-at-Risk (CVaR) method to measure the coupled risks.

Let \((\Omega, A, P)\) be a probability space such that \( \Omega \) is the sample space, \( A \) is the \( \sigma \)-field of events, and \( P \) is the probability measure. For a measurable real-valued random variable \( X \) on this probability space, the probability distribution of \( X \) is defined and denoted by \( F_X(x) = P(X \leq x) \). \( X \) represents a loss random variable such that for \( \omega \in \Omega \), the real number \( X(\omega) \) is the realization of a loss-and-profit function with \( X(\omega) \geq 0 \) for a loss and \( X(\omega) < 0 \) for a profit. Following the paper of Rockafellar and Uryasev [36], we give notions of VaR and CVaR such as:

**Definition 4.1.** Given \( X \), for the confidence level \( \alpha \in (0, 1) \) we define the following:

(i) The \( \alpha \)-Value-at-Risk (\( \alpha \)-VaR)

\[ \text{VaR}_\alpha(X) = \min(x : F_X(x) \geq \alpha). \tag{23} \]

(ii) The "upper" \( \alpha \)-Conditional Value-at-Risk (\( \alpha \)-CVaR)

\[ \text{CVaR}_\alpha^+(X) = E(X | X > \text{VaR}_\alpha(X)). \tag{24} \]
(iii) The α-CVaR
\[ \text{CVaR}_\alpha(X) = E(X^\alpha), \]
where the α -tail transform \( X^\alpha \) of \( X \) with the distribution function
\[ F_{X^\alpha}(x) = \begin{cases} 0, & x < \text{VaR}_\alpha(X) \\ \frac{F_X(x) - \alpha}{1 - \alpha}, & x \geq \text{VaR}_\alpha(X). \end{cases} \]  

(26)

The CVaR is a coherent risk measure, but the CVaR isn’t one except for some special conditions, for example, in the case of continuous distributions. Rockafellar and Uryasev [36] showed that \( \alpha \)-CVaR can be presented as a convex combination of \( \alpha \)-VaR and \( \alpha \)-CVaR,
\[ \text{CVaR}_\alpha(X) = \lambda_\alpha(X) \text{VaR}_\alpha(X) + [1 - \lambda_\alpha(X)] \text{CVaR}_\alpha(X), \]  

(27)

where
\[ \lambda_\alpha(X) = \frac{F_X(\text{VaR}_\alpha(X)) - \alpha}{1 - \alpha}, \quad 0 \leq \lambda_\alpha(X) \leq 1. \]

(28)

Accordingly, if the distribution of the portfolio’s loss under the coupled risks condition is known, we can estimate the \( \alpha \)-CVaR. Based on copula method, the loss scenarios are generated by Monte Carlo simulation. In the next section, we describe a scenario-based procedure for assessing the \( \alpha \)-CVaR under the coupled risks condition.

4.1. Simulating the loss scenarios with the market and credit risk

Assumed that the stock price of the company \( i \) is \( V_0 \) at the initial time \( t_0 = 0 \). We simulate \( N \) scenarios after \( T \) day. The stock’s simulative value of the company \( i \) at the ending time \( t = T \) in scenario \( j \) is denoted as \( V_j \), which should include the influence of the market and credit risks. Then the loss of the company \( i \) in the scenario \( j \) is given as
\[ L_j = -\frac{V_j - V_0}{V_0} \quad (j = 1, \ldots, N). \]  

(29)

Since \( V_j \) includes two parts of information: market and credit risk, we must simulate the two types of risks. The market risk can be forecasted by GARCH model (see Eq. (9)), in that the dependent structure of the \( \varepsilon_{mt} \) and \( \varepsilon_t \) is the chosen copula. For credit risk we simplify the default events in two situations: default or non-default. Then,
\[ V_j = \begin{cases} V_j^* \quad \text{non-default} \\ 0 \quad \text{default.} \end{cases} \]  

(30)

where \( V_j^* \) is the stock’s simulative value of the company \( i \) in scenario \( j \) if no default has happened for the given time horizon \( T \) day, \( \delta \) is the recovery rate of the stock associated with company \( i \).

First, we process the simulation of the stock value \( V_j^* \) after \( T \) day. According to the Monte Carlo simulation algorithm of copula, we can get the random vector \( (u_1, \ldots, u_\delta)^T \) with copula dependent structure. If the vector \( (u_1, \ldots, u_\delta)^T \) is translated by the probability density function of the \( \varepsilon_{mt} \) and \( \varepsilon_t \) marginal distributions, we get a simulative value in one scenario, namely \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_\delta)^T = (F_{\varepsilon_{mt}}^{-1}(u_1), \ldots, F_{\varepsilon_t}^{-1}(u_\delta))^T \).

Furthermore the time-until-default of company \( i \) must be simulated in order to gain the credit risk. We follow the copula approach, as described in Li [7], for modeling the default correlation between the companies. Let \( \tau \) be the time-until-default of the company \( i \). The hazard rate function of the company \( i \) is denoted as \( h_i(t) \). The probability that the company \( i \) does not default until time \( t \) is
\[ p(t) = \Pr(\tau > t) = \exp \left( -\int_0^t h_i(x) \, dx \right). \]  

(31)

The cumulative distribution function (c.d.f.) of \( \tau \) is given as
\[ F(t) = \Pr(\tau \leq t) = 1 - \exp \left( -\int_0^t h_i(x) \, dx \right). \]  

(32)

The hazard rate function completely characterizes the distribution of the random variable \( \tau \). Therefore, the calibration of \( h_i(t) \) from real data is important. Li [7] provide two ways to calibrate \( h_i(t) \). One is using the credit spread of the company \( i \). Another is using the probability of default.

Since it is difficult to gain the credit spread of the company \( i \) in China, we choose the probability of default. For the simplification, assume that the time structure of the hazard rate function is flat, i.e. \( h_i(t) = h_i \) for each \( t \). Then, we may calibrate \( h_i \) using the probability of default obtained from Eq. (6). If \( q(0, t) \) is the average cumulative default rate over the time horizon \((0, t)\), then
\[ F(t) = 1 - e^{-h_i t} = q(0,t) \Rightarrow h_i = \ln(1 - q(0,t))/t. \]  

(33)
If the hazard rate is calibrated, we can generate the scenarios about the vector of the time-until-default \( \tau = (\tau_1, \ldots, \tau_d)^T \) of the \( d \) companies using the following algorithm:

1. Generate the vector \((u_1, \ldots, u_d)\) from the selected copula using the algorithms in Appendix B.
2. Determine the time-until-default, \( \tau_i = F_i^{-1}(u_i), i = 1, \ldots, d \), \( F_i(\cdot) \) is the c.d.f. of \( \tau \) in Eq. (31).

For company \( i \), if the time-until-default \( \tau_i \) isn’t bigger than \( T \) day, a default event may happen for the given time horizon \( T \) day. Consequently in scenario \( j \), the simulations of the \( e_{mt} \) and \( \varepsilon_j \) can be generated using copulas simulation Algorithm and their marginal distributions, then \( r_j \) can be determined employing Eq. (9), \( V_j^* = V_0 (1 + r_j) \). Finally the simulations \( V_j \) of company \( i \) can be gained from Eq. (29) according to the simulating default status of company \( i \).

To reduce the error of this designed model and avoid the complex linear programming technique, we estimate the CVaR using order statistics (OS). An order statistic is the \( k \)th highest (or alternatively, lowest) observation in a sample. The theory of order statistics is well developed in the statistical literature. The estimating technology of CVaR order statistics is developed from VaR order statistics [48].

4.2. Estimating CVaR using order statistics method

Supposed that the simulative scenarios \( L_j (j = 0, 1, 2, \ldots, N) \), with the probability \( 1/N \), is selected from the population with its distributing function \( F(x) \). Sequencing the scenarios by size gives OS the sample:

\[
L_1 \leq L_2 \leq \cdots \leq L_N.
\]

Denote \( L_r \) as \( r \)th order statistic. The probability that \( j \) of the simulative scenarios do not exceed \( x \) obeys the binomial distribution,

\[
\Pr[j \text{ scenarios} \leq x] = \binom{N}{j} [F(x)]^j [1 - F(x)]^{N-j}.
\]

Hence, the probability that at least \( r \) scenarios in the sample do not exceed \( x \) is also a binomial distribution,

\[
G_r(x) = \sum_{j=r}^{N} \binom{N}{j} [F(x)]^j [1 - F(x)]^{N-j}.
\]

Then, \( G_r(x) \) is the distribution function of the order statistic. Moreover, the VaR is an order statistic, which is written as

\[
\text{VaR}_\alpha(X) = L_k,
\]

where \( \frac{k-1}{N} < \alpha \leq \frac{k}{N} \). Then, the \( \alpha \)-CVaR is written as

\[
\text{CVaR}_\alpha(X) = \text{VaR}_\alpha(X) + \frac{1}{N - k^*} \sum_{j=k^*+1}^{N} L_j / \left( N - k^* \right),
\]

where \( L_{k^*} \leq L_k < L_{k^*+1} \). According to Eq. (27), the estimator of \( \alpha \)-CVaR is

\[
\text{CVaR}_\alpha^*(X) = \frac{k^*/N - \alpha}{1 - \alpha} \text{VaR}_\alpha(X) + \frac{1 - k^*/N}{1 - \alpha} \text{CVaR}_\alpha^*(X)
\]

\[
= \frac{1}{N(1 - \alpha)} \left[ (k^* - N\alpha) L_k + \sum_{j=k^*+1}^{N} L_j \right].
\]

Now suppose that we wish to estimate the 1-\( \beta \) confidence interval for \( \alpha \)-VaR. Applying Eq. (33), we can get that

\[
\begin{align*}
G_r(x^{\text{low}}) &= \beta/2 \\
G_r(x^{\text{upper}}) &= 1 - \beta/2.
\end{align*}
\]

Then, we (numerically) solve Eq. (36) for the two implied values of \( F(x^{\text{low}}) \) and \( F(x^{\text{upper}}) \). If we know the \( F(x) \), which might be any type of the (parametric or nonparametric) distribution function, the \( \alpha \)-VaR confidence interval is gained. So, the calculation process has two main steps: the first, the \( F(x^{\text{low}}) \) and \( F(x^{\text{upper}}) \) is solved according to Eq. (36); the second, we regard these \( F(x) \) values as confidence levels and obtain the corresponding VaRs.

For the \( \alpha \)-CVaR, we can also use the calculation process to estimate the 1-\( \beta \) confidence interval. First, we can get the \( F(x^{\text{low}}) \) and \( F(x^{\text{upper}}) \) according to Eq. (36). Then, applying Eq. (35), and the value of the CVaR for \( \alpha = F(x^{\text{low}}) \) is the lower bound of 1-\( \beta \) confidence interval. If let \( \alpha = F(x^{\text{upper}}) \), we can gain the upper bound of 1-\( \beta \) confidence interval too.
Table 1  
The information of listed companies

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<td>Yantai Hualian Development Group CO., LTD.</td>
</tr>
<tr>
<td>600385</td>
<td>Shandong Jintai Group CO., LTD.</td>
</tr>
<tr>
<td>600576</td>
<td>Zhejiang Whwh Industry CO., LTD.</td>
</tr>
<tr>
<td>600076</td>
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</tr>
<tr>
<td>600225</td>
<td>Weifang Yaxing Chemical CO., LTD.</td>
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<td>Ustc Chuangxin CO., LTD.</td>
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Table 2  
The parameters estimations of GARCH-M for SSE180 Index

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Table 3  
The parameters estimations of GARCH for listed companies

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5. Instance analysis

Now, we implement the model of portfolio coupled risks measurement (described in the previous sections) to Chinese security market. In order to study the coupled risks’ influence to stock total risks, we choose two groups of samples: one is composed of bad companies which have higher credit risk; another includes some general companies and good companies, which have lower credit risk than bad ones. We compare two situations: consider market risk only or coupled risks and analyze their difference. Especially supposed that if default event has happened, stock price \(V_0\) would drop to \(\delta V_0\), where \(\delta\) is the recovery rate of the stock. Let \(\delta = 0\) be simplification of computation.

According to Chinese regulations, the stock will be labeled as “ST” for Special Treatment when a listed company experiences two consecutive annual net losses or when the net assets per share are lower than the stock’s par value in the current year. So the ST stock includes higher credit risk because of company’s financial predicament. Then we choose 180 Index of the Shanghai Stock Exchange (SSE 180 Index) as the market portfolio and 12 companies listed in the Shanghai Stock Exchange (SSE) in which there are 5 ST companies. Our sample data contains 147 weekly stock prices for the period during April 22, 2004 and April 19, 2007, which come from Wind Financial Database of the Wind Info. The information of the companies is shown in Table 1.

(1) Estimating the parameters of marginal distribution and copulas

According to the stock price, the parameters of GARCH model in Eq. (9) are estimated using the maximum likelihood estimates (shown in Tables 2–3). For ST companies, their parameters \(\nu\) of GED are far smaller than 2 in Table 3, which means that the conditional distributions of their stock returns have thicker tails than a normal distribution. However, the situation is different for non ST companies. Some companies have “fat” tails, another haven’t. The reason is that ST companies have higher credit risk and non ST companies maybe have or not. So the GED-GARCH model is more effective to capture the tail risk of listed companies.

In order to verify the hypothesis of the marginal distribution, we choose the Kolmogorov–Smirnov test which is a non-parameter test and make no assumption about the distribution of data. The null hypothesis of K–S test is that the data follow a specified distribution. We calculate the K–S statistics and \(P\)-values which are shown in Table 4. The lower the \(P\)-value is the less likely the two distributions are similar. Conversely, the higher or more close to 1 the \(P\)-value is the more similar the
two distributions are. So reject the null hypothesis if \( P \)-value is "small". As we see in Table 4, all of the \( P \)-values are bigger than 35%. Therefore, the hypothesis of the marginal distribution is right at 5% significance level.

In our paper, we study the default probability of listed companies. We can get the variance and the expected price serial of listed companies' stocks from Eq. \((9)\), then estimate the default probability of listed companies during \( T \) year from Eq. \((6)\), which are shown in Table 5. We calculate the default probability for different time horizons \( (T = 0.5 \text{ year and } T = 1 \text{ year}) \). The results show that the ST companies have bigger default probability, in other words these companies have serious credit risk; the non-ST companies need to be taken into account separately. Some have good credit quality because of very smaller default probability, whose credit risk can be ignored. The other companies have credit risk, which may not be ignored. Furthermore, the default probability for a half year is smaller than one for one year, which means that more values of the hazard rates is possibly produced for one year. So we choose one year as the time horizon in the followed research, which helps to the robustness of the procedures and calculations.

(2) Analyzing the effect of copulas for the coupled risks

In order to study the coupled risks of listed companies, we need simulate the scenarios with the coupled risks using copula method and measure these risks using CVaR criterion to judge their importance. At first, we need to estimate the parameters of the selected copulas. The parameters of the selected copulas have been fitted from a historical time series, by using the procedures described in Section 3. The parameter correlation matrix \( R \) of the Gaussian copula is summarized in Table 6. For the Student's \( t \)-copula, we have assessed the degrees of freedom \( \hat{v} = 18.740 \), while the correlation matrix \( R \) is reported in Table 7. The second, we estimate the parameters for the Archimedean copulas. The controlling parameters equal to 0.5015, 1.3308 and 2.7285 separately corresponding to the Clayton copula, the Gumbel copula and the Frank copula.

After estimating the copula parameters, we generate 100,000 scenarios of listed companies' stock price in the following one year using the Monte Carlo method, according to five different copulas. The dependence structure of this multivariate distribution is given by the chosen copula, while the margins are represented by GED with the degrees of freedom which are shown in Table 3. The scenarios are regarded as the market risk.

Successively, we simulate 100,000 Monte Carlo scenarios of the time-until-default \( \tau_i \) of each listed company with the copula dependence structure according to Eq. \((31)\). We know that company \( i \) defaults in scenario \( j \) if the determination of time-until-default \( \tau_i \leq T = 1 \text{ year} \). According to Eq. \((29)\), we can obtain 100,000 Monte Carlo scenarios for the portfolio loss \((28)\) with coupled risks.

According to the loss scenarios, the CVaR of listed companies can be computed based on CVaR OS estimating technology. Then we calculate the corresponding VaR. The results are shown in Table 8. In spite of the CVaR and VaR, the effect of the
In general, the CVaR considering the coupled risks isn't give us a direct impression: for the companies with good credit quality, it is acceptable to ignore coupled risks, because the credit risk is very smaller than those only considering market risk, which proves that the total risks of the capitals are underestimated if only considering market risk, denoted as CVaR*, which are shown in Table 9.

Notes: Denotes the confidence level as the 1% level.

Finally, we analyze the effect of the coupled risks in different companies. We give the CVaR for only considering the market risk, denoted as CVaR*, which are shown in Table 9. In general, the CVaR considering the coupled risks isn't smaller than those only considering market risk, which proves that the total risks of the capitals are underestimated if only considering market risk. So the loss aroused by credit risk break the old preventive measure that the company cannot prevent from the total risks. The relative errors between the CVaR and CVaR* in Table 9 give us a direct impression: for the companies with bad credit quality, it is incorrect to ignore coupled risks, because the relative errors are bigger, even up to 22.45%; for the companies with good credit quality, it is acceptable to ignore coupled risks, because the credit risk is very smaller, some up to 0; for the companies with medium credit risk, we should measure coupled risks, which help to improve the capability of defending financial risks. Further, we can find that the errors of the Gaussian copula are the smallest among different copulas for coupled risks is smaller than 10%. For non ST companies, the values of the elliptical copulas are bigger than those of the Archimedean copulas, which is not right for ST companies. Moreover, the CVaR is bigger than the VaR since the CVaR captures more information about the tail of the distribution.
those of the chosen copula. The reason is that Gaussian copula is the absence of tail dependence and cannot capture the tail risk. But the difference between these copulas is small since the extreme events are very rare for the stock data.

6. Conclusions

In this paper, we adopt the bottom-up approach to aggregate market and credit risk for the portfolio of the companies' stocks, and apply a simple Monte Carlo method to simulate the scenarios of the coupled risks. We generate scenarios for market and credit risk according to copula technology, and measure the coupled risks using the CVaR Order Statistics.

From our results, we find the difference is small between the Gaussian copula, the Student's t-copula, the Clayton copula, the Gumbel copula and the Frank copula. Since we choose the stock of the listed companies to study the coupled risks, the market and credit risk according to copula technology, and measure the coupled risks using the CVaR Order Statistics.

In conclusion, in terms of our results, we consider that the copula-based CVaR model can exactly measure the coupled risks in financial market. The model is effective at helping to prevent total risks of financial capital and ensures that the investor takes wise action.

Acknowledgements

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Appendix A. Estimation of copula parameters

Suppose that the marginal distributions of random vector $X = (x_1, \ldots, x_d)$ are denoted as $F_1(x_1), \ldots, F_d(x_d)$, $N$ observational vectors are $X^j$, where $j = 1, \ldots, N$.

The procedure used for estimating the parameter $R$ of the multivariate Gaussian copula is the following:

1) Translate the data $X^j$ into the uniform distribution $u^j = F_i(x^j)$.
2) Let $z_i^j = \Phi^{-1}(u_i^j)$, where $\Phi$ is the univariate standardized normal c.d.f.
3) $\tilde{R} = \text{cov}(z_i^j(z_d^j)^T)$, where $z^j = (z_1^j, \ldots, z_d^j)^T$.

Mashal and Naldi [9] introduce a procedure to estimate the multivariate Student's $t$-copula parameters, $\nu$ and $\tilde{R}$, which is followed as:

Procedure A: (the estimation of parameter $\hat{\nu}$)

1) Translate the data $X^j$ into the uniform distribution $u^j = F_i(x^j)$.
2) Estimate the degrees of freedom $\hat{\nu}$ for the Student $t$-copula by maximum likelihood estimation:

$$(2.1) \text{For each } \nu, \text{ estimate the correlation matrix } R \text{ using Procedure B.}$$

$$(2.2) \text{Calculate } \hat{\nu} = \arg \max \sum_{j=1}^N \ln \left( c(u_1^j, \ldots, u_d^j; \nu) \right), \text{ where the density of copula } c \text{ is given in Eq. (18).}$$

Procedure B: (the estimation of parameter $\hat{R}$)

1) Translate the data $X^j$ into the uniform distribution $u^j = F_i(x^j)$.
2) Let $z_i^j = \Phi^{-1}(u_i^j)$, where $\Phi$ is the univariate standardized normal c.d.f.
3) Initialize $\hat{R}_0 = \frac{1}{R} \sum_{j=1}^N z_i^j(z_d^j)^T$, where $z^j = (z_1^j, \ldots, z_d^j)^T$. 

Table 9

The comparison of the listed companies' risk under two situations

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<td>18.36</td>
<td>0.7478</td>
<td>17.84</td>
<td>0.7508</td>
<td>17.63</td>
</tr>
<tr>
<td>12</td>
<td>600576</td>
<td>0.7258</td>
<td>20.92</td>
<td>0.7274</td>
<td>21.05</td>
<td>0.7238</td>
<td>21.15</td>
<td>0.7323</td>
<td>20.28</td>
<td>0.7285</td>
<td>21.00</td>
</tr>
<tr>
<td>13</td>
<td>600225</td>
<td>0.7421</td>
<td>21.74</td>
<td>0.7404</td>
<td>22.01</td>
<td>0.7361</td>
<td>22.45</td>
<td>0.7459</td>
<td>21.61</td>
<td>0.7428</td>
<td>22.11</td>
</tr>
</tbody>
</table>

Notes: CVaR* is only considering the market risk. $\text{Error} = (\text{CVaR} - \text{CVaR}^* )/\text{CVaR}$. Denotes the confidence level as the 1% level.
are repeated until convergence, namely (22) and (15).

Table 10
The information about the Archimedean copulas

<table>
<thead>
<tr>
<th>Name</th>
<th>Clayton</th>
<th>Gumbel</th>
<th>Frank</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generator</td>
<td>$\psi(u) = e^{-\alpha} - 1$</td>
<td>$\psi(u) = (-\ln u)^\alpha$</td>
<td>$\psi(u) = -\ln e^{(-\alpha u) - 1}$</td>
</tr>
<tr>
<td>Inverse Generator</td>
<td>$\psi^{-1}(t) = (t + 1)^{-\frac{1}{\alpha}}$</td>
<td>$\psi^{-1}(t) = \exp(-t^\alpha)$</td>
<td>$\psi^{-1}(t) = -\frac{1}{\alpha} \ln(1 + e^{-t(e^{-\alpha t} - 1)})$</td>
</tr>
<tr>
<td>Parameter</td>
<td>$\alpha &gt; 0$</td>
<td>$\alpha &gt; 1$</td>
<td>$\alpha &gt; 0$</td>
</tr>
<tr>
<td>Y-Distribution</td>
<td>Gamma $\left(\frac{1}{\alpha}, 1\right)$</td>
<td>Stable $\left(\frac{1}{\alpha}, 1, \left[\cos(\frac{\pi}{2\alpha})\right]^\alpha, 0\right)$</td>
<td>Logarithmic series on $\mathbb{N}^+$ with $\theta = 1 - e^{-\alpha}$</td>
</tr>
<tr>
<td>Density of $Y$</td>
<td>$\frac{1}{\Gamma(\frac{1}{\alpha})} e^{-\alpha y}$</td>
<td>No closed form is known</td>
<td>$p(Y = k) = \frac{\theta^k}{\ln(1 - \alpha)^k}$</td>
</tr>
</tbody>
</table>

(4) Calculate $\hat{R}_{k+1} = \sum_{j=1}^d \frac{e^{t_j^T \frac{1}{\alpha} \hat{R}_{k+1} \hat{R}_{k+1}^T}}{\sum_{d=1}^d e^{t_j^T \frac{1}{\alpha} \hat{R}_{k+1} \hat{R}_{k+1}^T}}$, where $k = 1, 2, \ldots$, and $t_j = (t_j^1, \ldots, t_j^d)^T$.

(5) Rescale in order to obtain unit diagonal entries:

$$\tilde{R}_{k+1} = \frac{\hat{R}_{k+1}}{\sqrt{\hat{R}_{k+1}^T \hat{R}_{k+1}}}.$$

(6) The previous two steps (4) and (5) are repeated until convergence, namely $\hat{R}_{k+1} = \tilde{R}_{k}$.

In order to estimate the parameter $\alpha$ of the multivariate Clayton copula, Gumbel copula and Frank copula, we adopt inference for the margins or IFM method. The method estimates the parameters in two steps: the first is to estimate the parameters of the marginal distributions; the second is to estimate the copula parameters. As we suppose the marginal distributions are known, the procedure is as follows:

(1) Translate the data $x_i$ into the uniform distribution $u_i = F_i(x_i)$.

(2) Estimate the copula parameter $\alpha$,

$$\hat{\alpha} = \text{Arg Max}_{\alpha} \sum_{i=1}^n \ln[c_u(u_1^i, \ldots, u_d^i)].$$

where $c_u$ is the density function of copula. For the multivariate Clayton copula, $c_u$ is described as Eq. (20). For the multivariate Gumbel copula and Frank copula, we must calculate $c_u$ according to their copula function (Eqs. (21) and (22)) using MATLAB tools.

Appendix B. Simulation of the copulas

To generate a random vector from the Gaussian copula (15), we can use the following procedure.

(1) Find the Cholesky decomposition $A$ of the correlation matrix $R$: $R = AA^T$.

(2) Simulate $d$ independent standard normal random vector $z = (z_1, \ldots, z_d)^T$.

(3) Determine the vector $x = Az$.

(4) Determine the components $u_i = \Phi(x_i)$, $i = 1, \ldots, d$, the resultant vector is $(u_1, \ldots, u_d)^T \sim C_{\alpha}^d$.

In order to simulate a random vector with $C_{\alpha,\theta}$ dependent structure, the algorithm is followed as:

(1) Find the Cholesky decomposition $A$ of the correlation matrix $R$: $R = AA^T$.

(2) Simulate $d$ independent standard normal random vector $z = (z_1, \ldots, z_d)^T$.

(3) Simulate a random variate, $\xi$, from $\chi^2_{\alpha}$ distribution, independent of $z$.

(4) Determine the vector $y = Az$.

(5) Set $x = \frac{\xi}{\sqrt{\nu}}$.

(6) Determine the components $u_i = t_i(x_i)$ for $i = 1, \ldots, d$, the resultant vector is $(u_1, \ldots, u_d)^T \sim C_{\alpha,\theta}$.

For any multivariate Archimedean copula, the simulation is not easy. We adopt the Marshall and Olkin’s method [49] which is a construction method of copulas involving the Laplace transform and its inverse function. The $d$-dimensional Archimedean copulas may be written as:

$$C(u_1, \ldots, u_d) = \psi^{-1}(\psi(u_1), \ldots, \psi(u_d)),$$

where $\psi$ is a decreasing function known as the generator of the copula and $\psi^{-1}$ denotes the inverse of the generator.  

Algorithm A:

(1) Simulate $d$ independent uniform variable $x_i$ for $i = 1, \ldots, d$.

(2) Simulate a variable $Y$ with distribution function $G$ such that the Laplace transform of $G$ is the inverse of the generator.

(3) Define $z_i = -\frac{\ln(u_i)}{\alpha}$ for $i = 1, \ldots, d$.

(4) Define $u_i = \psi^{-1}(z_i)$ for $i = 1, \ldots, d$.

Then $(u_1, \ldots, u_d)$ have Archimedean copula dependence structure. The generators of the Archimedean copulas and the distribution function $G$ of $Y$ are shown in Table 10.
Though the density of \( \text{Stable}(\alpha, \beta, \gamma, \delta) \) distribution’s closed form is not known, the following simulation algorithm for generating random variables \( \text{Stable}(\alpha, \beta, \gamma, \delta) \) distributed:

Algorithm B:
1. Simulate an uniform variable \( \theta \sim U(-\frac{\pi}{2}, \frac{\pi}{2}) \).
2. Simulate an exponentially distributed variable \( W \) with mean 1 independently of \( \theta \).
3. Set \( \theta_0 = \arctan(\beta \tan(\pi \theta / 2)) / \alpha \).
4. Compute \( Z \sim \text{Stable}(\alpha, \beta, 1, 0) \)
   \[
   Z = \frac{\sin \alpha(\theta_0 + \theta)}{(\cos \alpha \theta_0 \cos \theta)^{1/\alpha}} \left[ \cos(\alpha \theta_0 + (\alpha - 1) \theta) \right]^{1/\alpha} \quad \alpha \neq 1
   \]
   \[
   Z = 2 \left[ \left( \frac{2}{\pi} + \beta \theta \right) \tan \theta - \beta \ln \left( \frac{2}{\pi} W \cos \theta \right) \right] \quad \alpha = 1.
   \]
5. Compute \( X \sim \text{Stable}(\alpha, \beta, \gamma, \delta) \)
   \[
   X = \gamma Z + \delta \quad \alpha \neq 1
   \]
   \[
   X = \gamma Z + \delta + \frac{2}{\pi} \gamma \ln(\gamma) \quad \alpha = 1.
   \]

Below, we give the algorithm for generating Logarithmic Distribution random variables:

Algorithm C:
1. Set \( c = \ln(1 - \theta) \).
2. Simulate an uniform variable \( V[0, 1] \).
3. IF \( V > \theta \) then set \( X = 1 \)
   ELSE
   Simulate an uniform variable \( U[0, 1] \).
   Set \( q = 1 - e^{cu} \)
   CASE:
   \[
   V \leq q^2: \text{set } X = \text{int}[1 + \ln(V) / \ln(q)].
   \]
   \[
   q^2 < V \leq q: \text{set } X = 1.
   \]
   \[
   V > q: \text{set } X = 2.
   \]
   ENDIF
   \( X \) is Logarithmic Series distributed.

References