Fast algorithms for computing Jones polynomials of certain links

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Abstract

We give a fast algorithm for computing Jones polynomials of 2-bridge links. Given the Tait graph with \(n\) edges of a 2-bridge diagram, this algorithm runs with \(O(n)\) additions and multiplications in polynomials of degree \(O(n)\), namely in \(O(n^2 \log n)\) time. We also propose an algorithm that, given the Tait graph with \(n\) edges of a closed 3-braid diagram, computes the Jones polynomial of the closed 3-braid link in \(O(n^2 \log n)\) time.

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1. Introduction

Knot theory is a subfield of topology. A knot is a simple (nonself-intersecting) closed curve embedded in \(\mathbb{R}^3\). More generally, one may study links. A link is a finite collection of disjointly embedded knots. Works on knot theory have led to many important advances in other areas of topology, biology, chemistry and physics [1].

One of the oldest and most fundamental problems in knot theory is that of determining whether a knot is unknotted (i.e. whether it is capable of being continuously deformed without self-intersection so that it lies in a plane). We call this problem UNKNOTTING. Haken [8] showed that UNKNOTTING is decidable and gave a procedure for this problem. Hass, Lagarias and Pippenger [11] presented an explicit complexity bound of Haken’s decision procedure and showed that UNKNOTTING is in \(\text{NP}\). One interesting question is whether UNKNOTTING is in \(\text{co-NP}\). Hara, Tani and Yamamoto [10] showed that UNKNOTTING is in \(\text{AM} \cap \text{co-AM}\). A more general problem is that of determining whether two knots are equivalent (i.e. whether we can deform the one knot to the other knot without self-intersection). We call this problem KNOT\_EQUIVALENCE. Waldhausen [23] showed that KNOT\_EQUIVALENCE is decidable. Haken also outlined an approach to decide KNOT\_EQUIVALENCE. The final step in this program was completed by Hemion [12], with a correction by Matveev [18]. There appears to be no explicit complexity bounds

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of KNOT EQUIVALENCE, either upper or lower. However, we currently have no algorithm for determining whether two links are equivalent links.

For classifying and characterizing links, various invariants have been defined and well studied in knot theory. Alexander [3] defined an invariant of a link that can be computed in polynomial time. It turns out that many nontrivial knots have the same Alexander polynomial of the trivial knot. The Jones polynomial [14] is more powerful for distinguishing link types. Kauffman [15] gave a combinatorial method for calculating the Jones polynomial by means of the Kauffman bracket polynomial. We denote that the number of the edges of the Tait graph of a link diagram \( \tilde{L} \) is \( c(\tilde{L}) \). It takes \( O\left(2^{c(\tilde{L})}\right) \) arithmetic operations in polynomials of degree \( O(c(\tilde{L})) \) to compute a Jones polynomial by Kauffman’s method. Actually, Jaeger, Vertigan and Welsh showed that computing the Jones polynomial is generally \#P-hard [13,24]. It is expected to require exponential time in the worst case. Sekine, Imai and Imai [21] designed an algorithm for computing Jones polynomials in \( O\left(2^{O(\sqrt{c(L)})}\right) \) time.

Recently, it has been recognized that it is important to compute Jones polynomials for links with reasonable restrictions. Makowsky [17] showed that Jones polynomials are computed from the Tait graph \( G \) of a link diagram \( \tilde{L} \) in polynomial time if the treewidth of \( G \) is bounded by a constant. Mighton [19] showed that Jones polynomials are computed from the Tait graph \( G \) of a link diagram \( \tilde{L} \) with \( O(c(\tilde{L})^4) \) operations in polynomials of degree \( O(c(\tilde{L})) \) if the treewidth of \( G \) is at most 2. Hara, Tani and Yamamoto [9] showed that Jones polynomials of 2-bridge links are computed from the Tait graph of a link diagram \( \tilde{L} \) with \( O(c(\tilde{L})^2) \) operations in polynomials of degree \( O(c(\tilde{L})) \), and Jones polynomials of closed 3-braid links and arborescent links are computed from the Tait graph of a link diagram \( \tilde{L} \) with \( O(c(\tilde{L})^5) \) operations in polynomials of degree \( O(c(\tilde{L})) \). Uktsuri and Imai [22] showed that Jones polynomials of pretzel links are computed from the Tait graph of a link diagram \( \tilde{L} \) in \( O(c(\tilde{L})^2) \) time.

In this paper, we propose fast algorithms for computing Jones polynomials of 2-bridge links and closed 3-braid links. The class of 2-bridge links and the class of closed 3-braid links are basic classes and have been well studied in knot theory. Our algorithms compute Jones polynomials of 2-bridge links (respectively closed 3-braid links) from the Tait graph of a 2-bridge diagram (respectively the Tait graph of a closed 3-braid diagram) \( \tilde{L} \) with \( O(c(\tilde{L})) \) additions and multiplications in polynomials of degree \( O(c(\tilde{L})) \), namely in \( O(c(\tilde{L})^2 \log c(\tilde{L})) \) time because polynomials of degree \( O(n) \) can be multiplied in \( O(n \log n) \) time [7]. Treewidths of the Tait graphs of 2-bridge links are 2 at most. Treewidths of the Tait graphs of closed 3-braid links are 3 at most. Our algorithm for 2-bridge links is faster than Mighton’s algorithm and our algorithm for closed 3-braid links is faster than Makowsky’s algorithm. It is well-known that both 2-bridge links and closed 3-braid links can be represented by integer sequences [6]. For designing our algorithms, we show that Jones polynomials of 2-bridge links (respectively closed 3-braid links) can be computed from their integer sequence that represents a 2-bridge diagram (their integer sequence that represents a closed 3-braid diagram) \( \tilde{L} \) with \( O(c(\tilde{L})) \) additions and multiplications in polynomials of degree \( O(c(\tilde{L})) \) by using simple recurrence formulas. We also show that the integer sequences of 2-bridge links (respectively closed 3-braid links) are able to be constructed from their Tait graph that represents a 2-bridge diagram (respectively their Tait graph that represents a closed 3-braid diagram) \( \tilde{L} \) in \( O(c(\tilde{L})) \) time. It is known that we can determine whether two integer sequences of 2-bridge diagrams \( \tilde{L} \) and \( \tilde{L}' \) represent the same 2-bridge link in \( O(n \log^2 n) \) time where \( n = c(\tilde{L}) + c(\tilde{L}') \) [6]. Hence, we can determine whether two given Tait graphs of 2-bridge diagrams \( \tilde{L} \) and \( \tilde{L}' \) represent the same 2-bridge link in \( O(n \log^2 n) \) time where \( n = c(\tilde{L}) + c(\tilde{L}') \).

The paper is organized in the following way. Section 2 contains some basic notations and definitions of knot theory and a key recurrence formula. In Section 3, we provide algorithms for 2-bridge links. Section 4 deals with algorithms for closed 3-braid links.

2. Preliminaries

In this section, we give some basic notations and definitions of knot theory. For details, see Adams [1], Burde and Zieschang [6] and Lickorish [16].

A link of \( n \) components is \( n \) simple closed curves in \( \mathbb{R}^3 \) that are mutually disjoint. A link of one component is a knot. An image of a link by an orthogonal projection from \( \mathbb{R}^3 \) to a plane is regular if it contains only finitely many multiple points, all multiple points are double points and these are traverse points. A regular image of a link is called a link diagram if the overcrossing/undercrossing information is marked at every double point in the image (see Fig. 1). Furthermore, the double points are called crossings. For any link diagram \( \tilde{L} \), we denote the number of the crossings.
A trivial link diagram is a link diagram without a crossing. A link is trivial if the link has a trivial link diagram. A link is oriented if each of its components is given an orientation.

A continuous bijection \( f \) from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) is called homeomorphism if \( f \) has a continuous inverse mapping. Let \( I \) be the closed interval \([0, 1]\). A Link \( L \) is equivalent or ambient isotopic to a Link \( L' \) if there exists a homeomorphism \( h_t \) from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) for \( 0 \leq t \leq 1 \) satisfying the following:

(i) \( h_0 \) is the identity.
(ii) \( h_1(L) = L' \).
(iii) There exists a continuous mapping \( H \) from \( \mathbb{R}^3 \times I \) to \( \mathbb{R}^3 \) satisfying that for any pair of \( x \in \mathbb{R}^3 \) and \( t \in I \), \( H(x, t) = h_t(x) \).

Definition 1. The Kauffman bracket polynomial is a function from link diagrams to the Laurent polynomial ring \( \mathbb{Z}[A^{\pm 1}] \) with integer coefficients in an indeterminate \( A \). It maps a link diagram \( \tilde{L} \) to \( \langle \tilde{L} \rangle \in \mathbb{Z}[A^{\pm 1}] \) and is characterized by

(i) \( \langle \bigcirc \rangle = 1 \),
(ii) \( \langle \tilde{L} \sqcup \bigcirc \rangle = (-A^{-2} - A^2) \langle \tilde{L} \rangle \) and
(iii) \( \langle \bigtriangledown \rangle = A^{-1} \langle \bigtriangleup \rangle + A^{-1} \langle \downdeptriangle \rangle \).

Here, \( \bigcirc \) is the trivial knot diagram and \( \tilde{L} \sqcup \bigcirc \) is the disjoint sum of \( \tilde{L} \) and \( \bigcirc \). In (iii), the formula refers to three link diagrams that are exactly the same except near a point where they differ in the way indicated.

Note that for any link diagram \( \tilde{L} \), the degree of \( \langle \tilde{L} \rangle \) is \( O(c(\tilde{L})) \) and the coefficients of \( \langle \tilde{L} \rangle \) are \( O(2^{c(\tilde{L})}) \).

Proposition 2.

\[
\begin{align*}
\langle \bigcirc \bigcirc \rangle &= -A^{-3} \langle \bigcirc \rangle, \\
\langle \bigtriangleup \bigcirc \rangle &= -A^3 \langle \bigcirc \rangle.
\end{align*}
\]

Proof of the proposition is provided in Appendix.

The writhe \( w(\tilde{L}) \) of an oriented link diagram \( \tilde{L} \) is the sum of the signs of the crossings of \( \tilde{L} \), where each crossing has sign +1 or −1 as defined (by convention) in Fig. 2.
The *Jones polynomial* $V(L)$ of an oriented link $L$ is defined by

$$V(L) = (-A)^{-3\omega(\bar{L})} \langle \bar{L} \rangle \bigg|_{t^{1/2} = A^{-2}},$$

where $\bar{L}$ is an oriented link diagram of $L$. It is known that $V(L)$ is well-defined and $V(L) \in \mathbb{Z}[t^{\pm1/2}]$.

A *tangle* is a portion of a link diagram from which there emerge just four arcs pointing in the compass directions NW, NE, SW, SE (see Fig. 3). The tangle consisting of two vertical strings without a crossing is called *0-tangle*. The 0-tangle twisted $k$ times is called *$k$-tangle* and is denoted by $I_k$. They are called *integer tangles* (see Fig. 4).

**Proposition 3.** For any integer $k$,

$$I_k = A I_{k-1} + A^{-1} I_{k-1}$$

holds.

Proof of the proposition is provided in Appendix.
For a link diagram $\tilde{L}$, a link diagram $\tilde{L}#(k)$ is shorthand for a link diagram twisted it $k$ times as shown in Fig. 5. For convenience, $\tilde{L}(0)$ denotes $\tilde{L}$ itself. We have the following proposition by Proposition 2.

**Proposition 4.** For any link diagram $\tilde{L}$ and any integer $k$,

$$\langle \tilde{L}(k) \rangle = (-A^{-3})^k \langle \tilde{L} \rangle$$

holds.

Given any link diagram $\tilde{L}$, we can colour the faces black and white in such a way that no two faces with a common edge are the same colour. We colour the unique unbounded face white. Such a colouring is called the Tait coloring of $\tilde{L}$. As shown in Fig. 6, we can get an edge-labelled planar graph $G$ of $\tilde{L}$. Its vertices are the black faces of the Tait colouring and two vertices are joined by a labelled edge if they share a crossing. The label of the edge is $+1$ or $-1$ according to the (conventional) rule shown in Fig. 7. We may call the label the sign. We call $G$ the Tait graph of $\tilde{L}$. Note that the number of the edges of $G$ is $c(\tilde{L})$. A Tait graph $G$ is isomorphic to a Tait graph $G'$ if there exists a bijection $f$ from the vertex set of $G$ to the vertex set of $G'$ satisfying the following:

(i) For any pair of vertices $u$ and $v$ of $G$, the number of the edges in $G$ that joins $u$ and $v$ and are labelled “$+1$” is equal to the number of the edges in $G'$ that joins $f(u)$ and $f(v)$ and are labelled “$+1$”.

(ii) For any pair of vertices $u$ and $v$ of $G$, the number of the edges in $G$ that joins $u$ and $v$ and are labelled “$-1$” is equal to the number of the edges in $G'$ that joins $f(u)$ and $f(v)$ and are labelled “$-1$”.

Such a function $f$ is called an isomorphism from $G$ to $G'$.

Let $G = (V, E, l)$ be a Tait graph, where $V$ is the vertex set of $G$, $E$ is the edge set of $G$ and $l$ is the edge-labelling function from $E$ to $\{-1, +1\}$. For any vertex $v \in V$, $\deg_G(v)$ denotes the degree of $v$ in $G$, $N_G(v)$ denotes the set of the neighbours of $v$ in $G$ and $G - v$ denotes the subgraph of $G$ induced by $V - \{v\}$. We define a function $\text{edge}_{\text{sign}}_G$ from $V \times V$ to $\mathbb{Z}$ such that for any pair of vertices $u$ and $v$ of $G$, $\text{edge}_{\text{sign}}_G(u, v)$ is the sum of the signs of the edges...
of $G$ that join $u$ and $v$. For a set $S$, we denote the size of $S$ by $|S|$. For a positive integer $n$, let $\mathbb{N}_{\leq n} = \{1, 2, \ldots, n\}$. For an integer $n$, \(\text{sign}(n)\) is defined by the following:

\[
\text{sign}(n) = \begin{cases}
  +1 & \text{if } n > 0, \\
  0 & \text{if } n = 0, \\
  -1 & \text{if } n < 0.
\end{cases}
\]

**Proposition 5.** For any Tait graph $G = (V, E, l)$, the following hold.

(i) All of the degrees of the vertices in $G$ are computed in $O(|E|)$ time.

(ii) For any vertex $v \in V$, $G - v$ is constructed in $O(|E|)$ time.

(iii) All of the neighbour sets of the vertices in $G$ are computed in $O(|E|)$ time if $\max\{|N_G(v) : v \in V\}$ has a constant bound.

(iv) Whether $G$ is a cycle or not is determined in $O(|E|)$ time.

(v) All of the values in \{edge sign$_G(u, v) : u, v \in V$ are adjacent\} are computed in $O(|E|)$ time.

Proof of the proposition is provided in Appendix.

We prove a key recurrence formula. The key recurrence formula is of use to prove recurrence formulas for designing our algorithms in Lemmas 11 and 18. For an integer $n$, we set

\[
Q_n(X) = \frac{1 - X^n}{1 - X} = \begin{cases}
  1 + X + \cdots + X^{n-1} & \text{if } n > 0, \\
  0 & \text{if } n = 0, \\
  -X^{-1} - X^{-2} - \cdots - X^n & \text{if } n < 0.
\end{cases}
\]

Note that

\[
XQ_n(X) + 1 = Q_{n+1}(X) \quad \text{(1)}
\]

and

\[
X^{-1}Q_{n+1}(X) - X^{-1} = Q_n(X). \quad \text{(2)}
\]

**Lemma 6.** Let $\tilde{L}$ be a link diagram, $k$ an integer and \{\(b_n\)\}_{n \in \mathbb{Z}} a sequence of polynomials in $\mathbb{Z}[A^{\pm 1}]$. Suppose that for any integer $n$,

\[
b_n = Ab_{n-1} + A^{-1}\langle \tilde{L}\#(n + k) \rangle.
\]

Then,

\[
b_n = A^n b_0 - (-A)^{-3(n+k)-1}\langle \tilde{L}\rangle Q_n(-A^4).
\]

**Proof.** We prove by induction on $|n|$. By Proposition 4, we have

\[
b_n = Ab_{n-1} + A^{-1}\langle \tilde{L}\#(n + k) \rangle
\]

\[
= Ab_{n-1} + A^{-1}(-A^{-3}A^{n+k})\langle \tilde{L} \rangle
\]

\[
= Ab_{n-1} - (-A)^{-3(n+k)-1}\langle \tilde{L} \rangle.
\]
3. Algorithms for 2-bridge links

The class of 2-bridge links are an important and very well understood class in knot theory which were classified by H. Schubert [20]. Often, a property that is suspected to hold for all links is first proved to hold for this particular class.

For any link diagram, an overpass is a subarc of the link diagram that goes over at least one crossing but never goes under a crossing. A maximal overpass is an overpass that could not be made any longer (see Fig. 8).

The bridge number of a link diagram is the number of maximal overpasses in the link diagram. The bridge number of a link is the least bridge number of all of the link diagrams of the link. A 2-bridge link is a link whose bridge number is 2.

First, we show that an integer sequence of a 2-bridge link is able to be constructed from its Tait graph (where the Tait graph represents the link diagram $\widetilde{L}$) in $O(c(\widetilde{L}))$ time. Next, we also show that Jones polynomials of 2-bridge links can be computed from an integer sequence that represents a link diagram $\widetilde{L}$ with $O(c(\widetilde{L}))$ additions and multiplications in polynomials of degree $O(c(\widetilde{L}))$ by using simple recurrence formulas.

We denote the link diagram consisting of integer tangles $I_{a_1}, \ldots, I_{a_m}$ as shown in Fig. 9 by $\widetilde{R}(a_1, \ldots, a_m)$, where $a_1, \ldots, a_m$ are integers. The Tait graph of $\widetilde{R}(a_1, \ldots, a_m)$ is denoted by $GR(a_1, \ldots, a_m)$. Note that for any non-zero integers $a_1, \ldots, a_m$, the Tait graph $G = GR(a_1, \ldots, a_m) = (V, E, I)$ has a vertex $v \in V$ such that $G - v$ is a path (see Fig. 10). We call a link diagram $\widetilde{L}$ a 2-bridge diagram if the Tait graph $G = (V, E, I)$ of $\widetilde{L}$ satisfies the following:

(i) There exists a vertex $v \in V$ such that
   (a) $G - v$ is a path,
   (b) both of endvertices of the path $G - v$ are adjacent to $v$ in $G$ and
   (c) for any vertex $u \in N_G(v)$, all of the edges that join $u$ and $v$ have the same sign.

(ii) For any vertex $u \in V$, if $\deg_G(u) = 2$, then the two edges incident to $u$ have the same sign.

(iii) $G$ has no loop edge.

It is clear that for any 2-bridge diagram $\widetilde{L}$, there exists an integer sequence $(a_1, \ldots, a_m)$ such that $\widetilde{R}(a_1, \ldots, a_m) = \widetilde{L}$, namely $GR(a_1, \ldots, a_m)$ is the Tait graph of $\widetilde{L}$. We remark that the expression $GR(a_1, \ldots, a_m)$ is not unique. It is well-known that every 2-bridge link has a 2-bridge diagram [6].

If $n = 0$, then $b_0 = A^0b_0 - (-A)^{-3k-1}\langle\tilde{L}\rangle Q_0(-A^4)$ since $Q_0(-A^4) = 0$.

If $n > 0$, then, by induction hypothesis and the Eq. (1), we have

$$b_n = Ab_{n-1} - (-A)^{-3(n+k)-1}\langle\tilde{L}\rangle$$
$$= A\left(A^{n-1}b_0 - (-A)^{-3(n+k-1)-1}\langle\tilde{L}\rangle Q_{n-1}(-A^4)\right) - (-A)^{-3(n+k)-1}\langle\tilde{L}\rangle$$
$$= A^n b_0 - (-A)^{-3(n+k)-1}\langle\tilde{L}\rangle \left((-A^4)Q_{n-1}(-A^4) + 1\right)$$
$$= A^n b_0 - (-A)^{-3(n+k)-1}\langle\tilde{L}\rangle Q_n(-A^4).$$

If $n < 0$, then, by $b_{n+1} = Ab_n - (-A)^{-3(n+k+1)-1}\langle\tilde{L}\rangle$, induction hypothesis and the Eq. (2), we have

$$b_n = A^{-1}b_{n+1} + A^{-1}(-A)^{-3(n+k+1)-1}\langle\tilde{L}\rangle$$
$$= A^{-1}\left(A^{n+1}b_0 - (-A)^{-3(n+k+1)-1}\langle\tilde{L}\rangle Q_{n+1}(-A^4)\right) - A^{-4}(-A)^{-3(n+k)-1}\langle\tilde{L}\rangle$$
$$= A^n b_0 - (-A)^{-3(n+k)-1}\langle\tilde{L}\rangle \left((-A^4)Q_{n+1}(-A^4) - (-A^4)\right)$$
$$= A^n b_0 - (-A)^{-3(n+k)-1}\langle\tilde{L}\rangle Q_n(-A^4).$$

Fig. 8. An overpass and a maximal overpass.
Proposition 7. Let $m$ be an odd number and $a_1, \ldots, a_m$ non-zero integers. Let $G$ be the Tait graph of a 2-bridge diagram. $G_R(a_1, \ldots, a_m)$ is isomorphic to $G$ if and only if there exists a bijection from the vertex set of $G_R(a_1, \ldots, a_m)$ to the vertex set of $G$ satisfying the following:

(i) If $u$ and $v$ are adjacent vertices in $G_R(a_1, \ldots, a_m)$, then the number of the edges that join $u$ and $v$ in $G_R(a_1, \ldots, a_m)$ is equal to the number of the edges that join $f(u)$ and $f(v)$ in $G$ and an edge that joins $u$ and $v$ in $G_R(a_1, \ldots, a_m)$ and an edge that joins $f(u)$ and $f(v)$ in $G$ have the same label.

(ii) If $u$ and $v$ are not adjacent vertices in $G_R(a_1, \ldots, a_m)$, then $f(u)$ and $f(v)$ are not adjacent vertices in $G$.

Proof. For any pair of adjacent vertices $u$ and $v$ in $G_R(a_1, \ldots, a_m)$, all of the edges that join $u$ and $v$ in $G_R(a_1, \ldots, a_m)$ have the same sign because $m$ is an odd number and $a_1, \ldots, a_m$ are non-zero integers. For any pair of adjacent vertices $u$ and $v$ in $G$, all of the edges that join $u$ and $v$ in $G$ have the same sign by the definition of 2-bridge diagram. □
Lemma 8. Let $G = (V, E, l)$ be the Tait graph of a 2-bridge diagram and $v \in V$ a vertex satisfying that $G - v$ is a path and both of endvertices of the path $G - v$ are adjacent to $v$ in $G$. Suppose that an integer sequence $(a_1, \ldots, a_{|N_G(v)| - 1})$ is obtained by the following way (see Fig. 11):

(i) Choose an endvertex of the path $G - v$, index the vertex $v_1$ and go along the path $G - v$ from $v_1$ to the other endvertex.

(ii) Index the vertices of $N_G(v) - \{v_1\}$ in the way which $v_{i+1}$ is the vertex which we visit $i$-th in $N_G(v) - \{v_1\}$ for $i = 1, \ldots, |N_G(v)| - 1$.

(iii) Let $a_{2i-1} = -\text{edge-\text{sign}}_G(v, v_i)$ for $i = 1, \ldots, |N_G(v)|$.

(iv) Let $a_{2j}$ be the sum of the signs of the edges between $v_j$ and $v_{j+1}$ in the path $G - v$ for $j = 1, \ldots, |N_G(v)| - 1$.

Then, $G_R(a_1, \ldots, a_{|N_G(v)| - 1})$ is isomorphic to $G$.

Proof of the lemma is provided in Appendix.

Lemma 9. Let $G = (V, E, l)$ be the Tait graph of a 2-bridge diagram. A vertex $v \in V$ has the maximum degree of $G$ if and only if $G - v$ is a path and both of endvertices of the path $G - v$ are adjacent to $v$ in $G$.

Proof. ($\Leftrightarrow$) If $|V| = 2$, then it is clear that $v$ has the maximum degree of $G$. We assume that $|V| > 2$. Let $u \in V - \{v\}$ be a vertex of $G - v$ and $p$ the number of the edges that join $u$ and $v$ in $G$.

We consider the case where $u$ is an endvertex of the path $G - v$ (see Fig. 12). Then, $\deg_G(u) = p + 1$. We have $\deg_G(v) \geq p + 1$ because the other endvertex of the path $G - v$ is adjacent to $v$. These imply that $\deg_G(u) \leq \deg_G(v)$.

We consider the case where $u$ is not an endvertex of the path $G - v$ (see Fig. 13). Then, $\deg_G(u) = p + 2$. We have $\deg_G(v) \geq p + 2$ since both of endvertices of $G - v$ are adjacent to $v$ in $G$. Therefore, $\deg_G(u) \leq \deg_G(v)$.

($\Rightarrow$) Let $v' \in V$ be a vertex such that $G - v'$ is a path and both of endvertices of the path $G - v'$ are adjacent to $v'$ in $G$. We already showed that $v'$ has the maximum degree of $G$. If the number of the vertices that have the maximum degree of $G$ is 1, then it is clear that the lemma holds. We assume that the number of the vertices that have the maximum degree of $G$ is more than 1 and $v$ and $v'$ are different vertices. Let $p$ be the number of the edges that join $v$ and $v'$ in $G$. Then, $\deg_G(v) \leq p + 2$ because $v$ is a vertex of the path $G - v'$. Therefore, $p \leq \deg_G(v) = \deg_G(v') \leq p + 2$.

We consider the case where $\deg_G(v) = p$ (see Fig. 14). Then, $V = \{v, v'\}$ and $v$ and $v'$ are adjacent in $G$. Therefore, $G - v$ is a path and both of endvertices of the path $G - v$ are adjacent to $v$ in $G$.

We consider the case where $\deg_G(v) = p + 1$ (see Fig. 15). Then, $v$ is an endvertex of the path $G - v'$ and $v$ and the other endvertex of the path $G - v'$ are adjacent to $v'$ in $G$ by the assumption. Therefore, $G - v$ is a path and both of endvertices of the path $G - v$ are adjacent to $v$ in $G$ because $\deg_G(v') = p + 1$.

We consider the case where $\deg_G(v) = p + 2$ (see Fig. 16). Then, $v$ is not an endvertex of the path $G - v'$ and both of endvertex of the path $G - v'$ are adjacent to $v'$ in $G$ by the assumption. Therefore, $G - v$ is a path and both of endvertices of the path $G - v$ are adjacent to $v$ in $G$ since $\deg_G(v') = p + 2$. □
Fig. 12. The case where $u$ is an endvertex of the path $G - v$.

Fig. 13. The case where $u$ is not an endvertex of the path $G - v$.

Fig. 14. The case where $\deg_G(v) = p$.

Given the Tait graph $G$ of a 2-bridge diagram, Procedure seq_{2-bridge} constructs an integer sequence $(a_1, \ldots, a_m)$ such that $G_R(a_1, \ldots, a_m)$ is isomorphic to $G$.

Procedure seq_{2-bridge}
Input: The Tait graph $G = (V, E, l)$ of a 2-bridge diagram.
Output: An integer sequence $(a_1, \ldots, a_m)$ such that $G_R(a_1, \ldots, a_m)$ is isomorphic to $G$.
Index a vertex $u_0 \in V$ in the way which $\deg_G(u_0)$ is the maximum degree of $G$;
Compute $N_{G-u_0}(v)$ for all vertices $v \in V - \{u_0\}$;
Index a vertex $u_1 \in V - \{u_0\}$ in the way which $u_1$ is an endvertex of the path $G - u_0$;
for $i := 2$ to $|V| - 1$ do
  Index a vertex $u_i \in N_{G-u_0}(u_{i-1})$ in the way which $u_i$ is not $u_{i-2}$;
  Compute \text{edge\_sign}_G(u_0, u_i)$ for $i = 1, \ldots, |V| - 1$;
  Compute \text{edge\_sign}_G(u_i, u_{i+1})$ for $i = 1, \ldots, |V| - 2$;
  Initialize $i$ as “1” and $k$ as “1”;
  while $i < |V| - 1$ do begin
    { $k$ is an odd number } 
    $a_k := -\text{edge\_sign}_G(u_0, u_i)$;
    Increment $k$;
    { $k$ is an even number } 
    Initialize $a_k$ as “0”;
    repeat 
      $a_k := a_k + \text{edge\_sign}_G(u_i, u_{i+1})$;
      Increment $i$;
      until \text{deg}_G(u_i) \neq 2 or $i = |V| - 1$;
    Increment $k$;
  end;
  $a_k := -\text{edge\_sign}_G(u_0, u_{|V| - 1})$;

\textbf{Theorem 10.} Procedure \texttt{seq\_2-bridge} constructs an integer sequence $(a_1, \ldots, a_m)$ such that $G_R(a_1, \ldots, a_m)$ is isomorphic to $G$ in $O(|E|)$ time.
Lemma 6. by applying Proposition 3 to $G$

Proof. $G - u_0$ is a path and both of endvertices of the path $G - u_0$ are adjacent to $v$ in $G$ by Lemma 9. Then, the procedure constructs an integer sequence $(a_1, \ldots, a_m)$ such that $G_{\tilde{R}}(a_1, \ldots, a_m)$ is isomorphic to $G$ by Lemma 8.

We estimate the running time of the procedure. We can index a vertex $u_0 \in V$ in the way which $\deg_G(u_0)$ is the maximum degree of $G$ in $O(|E|)$ time by Proposition 5(i). We can compute $N_{G - u_0}(v)$ for all vertices $v \in V - \{u_0\}$ in $O(|E|)$ time by Proposition 5(ii) and (iii) because $G - u_0$ is a path and $|N_{G - u_0}(v)| \leq 2$. We can index a vertex $u_1 \in V - \{u_0\}$ in the way which $u_1$ is an endvertex of the path $G - u_0$ in $O(|E|)$ time by Proposition 5(i) since $\deg_{G - u_0}(u_1) \leq 1$. For $i = 2, \ldots, |V| - 1$, we can index a vertex $u_i \in N_{G - u_0}(u_{i-1})$ in the way which $u_i$ is not $u_{i-2}$ in constant time because $|N_{G - u_0}(u_{i-1})| \leq 2$. We can compute edge_{sign}(u_0, u_i) for $i = 1, \ldots, |V| - 1$ and edge_{sign}(u_j, u_{j+1}) for $j = 1, \ldots, |V| - 2$ in $O(|E|)$ time by Proposition 5(v). The while loop finishes in $O(|V|)$ time because $i$ is incremented once at least each iteration in the repeat until loop. Therefore, the procedure finishes in $O(|E|)$ time. □

Then, we compute the Kauffman bracket polynomial $\langle \tilde{R}(a_1, \ldots, a_m) \rangle$ from the integer sequence $(a_1, \ldots, a_m)$.

Lemma 11. For any integer sequence $(a_1, \ldots, a_m)$, the following recurrence formula holds.

$$\langle \tilde{R}(a_1, \ldots, a_m) \rangle = \begin{cases} A^{a_1}(-A^{-2} - A^2) - (-A)^{-3a_1+2}Q_{a_1}(-A^4) & \text{if } m = 1, \\ A^{a_2}(-A^{-3}a_1 - (-A)^{-3a_2+2}\langle \tilde{R}(a_1) \rangle Q_{a_2}(-A^4) & \text{if } m = 2, \\ A^{a_m}(-A^{-3}a_{m-1}\langle \tilde{R}(a_1, \ldots, a_{m-2}) \rangle -(-A)^{-3a_m+2}\langle \tilde{R}(a_1, \ldots, a_{m-1}) \rangle Q_{a_m}(-A^4) & \text{if } m \geq 3. \end{cases}$$

Proof. We consider the case where $m = 1$. We have:

$$\langle \tilde{R}(a_1) \rangle = A\langle \tilde{R}(a_1 - 1) \rangle + A^{-1}\langle \#(a_1 - 1) \rangle$$

by applying Proposition 3 to $I_{a_1}$ of $\tilde{R}(a_1)$. We also have

$$\langle \tilde{R}(0) \rangle = -A^{-2} - A^2$$

by Definition 1(i) and (ii) because $\tilde{R}(0)$ is $\bigcirc \sqcup \bigcirc$ (see Fig. 17). Hence, the Eqs. (3) and (4) imply the case where $m = 1$ by Lemma 6.

We consider the case where $m = 2$. We have:

$$\langle \tilde{R}(a_1, a_2) \rangle = A\langle \tilde{R}(a_1, a_2 - 1) \rangle + A^{-1}\langle \#(a_2 - 1) \rangle$$

by applying Proposition 3 to $I_{a_2}$ of $\tilde{R}(a_1, a_2)$. We also have

$$\langle \tilde{R}(a_1, 0) \rangle = (-A^{-3})^{a_1}\langle \bigcirc \rangle$$

by Proposition 4 since $\tilde{R}(a_1, 0) = \bigcirc \#(a_1)$ (see Fig. 18). Hence, the Eqs. (5) and (6) imply the case where $m = 2$ by Lemma 6.

We consider the case where $m \geq 3$. We have:

$$\langle \tilde{R}(a_1, \ldots, a_m) \rangle = A\langle \tilde{R}(a_1, \ldots, a_{m-1}, a_m - 1) \rangle + A^{-1}\langle \tilde{R}(a_1, \ldots, a_{m-1}) \#(a_m - 1) \rangle$$

(7)
by applying Proposition 3 to $I_{am}$ of $\tilde{R}(a_1, \ldots, a_m)$. We also have

$$\langle \tilde{R}(a_1, \ldots, a_{m-1}, 0) \rangle = (-A^{-3})^{a_{m-1}} \langle \tilde{R}(a_1, \ldots, a_{m-2}) \rangle$$

(8)

by Proposition 4 because $\tilde{R}(a_1, \ldots, a_{m-1})$ is $\tilde{R}(a_1, \ldots, a_{m-2}) \# (a_{m-1})$ (see Figs. 19 and 20). Hence, the Eqs. (7) and (8) imply the case where $m \geq 3$ by Lemma 6. □

Given an integer sequence $(a_1, \ldots, a_m)$, Procedure bra_2-bridge computes the Kauffman bracket polynomial $\langle \tilde{R}(a_1, \ldots, a_m) \rangle$ by using the recurrence formula in Lemma 11. While the procedure is running, every Kauffman bracket polynomial is computed once at most.

Procedure bra_2-bridge

Input: An integer sequence $(a_1, \ldots, a_m)$.
Output: The Kauffman bracket polynomial $\langle \tilde{R}(a_1, \ldots, a_m) \rangle$.

Compute $\langle \tilde{R}(a_1) \rangle$ from $Q_{a_1}(-A^4)$;
if $m \geq 2$ then Compute $\langle \tilde{R}(a_1, a_2) \rangle$ from $\langle \tilde{R}(a_1) \rangle$ and $Q_{a_2}(-A^4)$;
for $i := 3$ to $m$ do
Compute $\langle \tilde{R}(a_1, \ldots, a_i) \rangle$ from $\langle \tilde{R}(a_1, \ldots, a_{i-2}) \rangle$, $\langle \tilde{R}(a_1, \ldots, a_{i-1}) \rangle$ and $Q_{a_i}(-A^4)$;
Theorem 12. Procedure $\text{bra.2-bridge}$ computes the Kauffman bracket polynomial $\langle \tilde{R}(a_1, \ldots, a_m) \rangle$ with $O(c(\tilde{R}(a_1, \ldots, a_m)))$ additions and multiplications in polynomials of degree $O(c(\tilde{R}(a_1, \ldots, a_m)))$, namely in $O(c(\tilde{R}(a_1, \ldots, a_m))^2 \log c(\tilde{R}(a_1, \ldots, a_m)))$ time.

Proof. It is clear that the procedure computes the Kauffman bracket polynomial $\langle \tilde{R}(a_1, \ldots, a_m) \rangle$ from the integer sequence $(a_1, \ldots, a_m)$ by Lemma 11. We estimate the running time of the procedure. For $i = 1, \ldots, m$, we can compute $Q_{a_i}(-A^4)$ in $O(|a_i|)$ time. We can compute $\langle \tilde{R}(a_1) \rangle$ with $O(1)$ operations in polynomials of degree $O(c(\tilde{R}(a_1, \ldots, a_m)))$ from $Q_{a_1}(-A^4)$ by Lemma 11. We can compute $\langle \tilde{R}(a_1, a_2) \rangle$ with $O(1)$ operations in polynomials of degree $O(c(\tilde{R}(a_1, \ldots, a_m)))$ from $\langle \tilde{R}(a_1) \rangle$ and $Q_{a_2}(-A^4)$ by Lemma 11. For $i = 3, \ldots, m$, we can compute $\langle \tilde{R}(a_1, \ldots, a_i) \rangle$ with $O(1)$ operations in polynomials of degree $O(c(\tilde{R}(a_1, \ldots, a_m)))$ from $\langle \tilde{R}(a_1, \ldots, a_{i-2}) \rangle$, $\langle \tilde{R}(a_1, \ldots, a_{i-1}) \rangle$ and $Q_{a_i}(-A^4)$ by Lemma 11. Therefore, the procedure finishes with $O(c(\tilde{R}(a_1, \ldots, a_m)))$ operations in polynomials of degree $O(c(\tilde{R}(a_1, \ldots, a_m)))$ since $c(\tilde{R}(a_1, \ldots, a_m)) = |a_1| + \cdots + |a_m|$. □

Corollary 13. The Jones polynomial of a 2-bridge link is computed from the Tait graph of a 2-bridge diagram $\tilde{L}$ with $O(c(\tilde{L}))$ additions and multiplications in polynomials of degree $O(c(\tilde{L}))$, namely in $O(c(\tilde{L})^2 \log c(\tilde{L}))$ time.

4. Algorithms for closed 3-braid links


A braid is a set of mutually disjoint strings, all of which are attached to a horizontal bar at the top and at the bottom and each string intersects any horizontal plane between the two bars exactly once. Given any braid, its ends on the bottom edge may be joined to those on the top edge to produce the closure of the braid. A closed braid link is the closure of a braid. A braid with three strings is a 3-braid and the closure of a 3-braid is a closed 3-braid link (see Fig. 21).

Braids are not a particular type of links. However, Alexander [2] showed that every link is a closed braid link. The 3-braid links is the Tait graph of a closed 3-braid link. The closed 3-braid links is a basic class of links and has been well studied.

First, we show that an integer sequence of a closed 3-braid link is able to be constructed from its Tait graph (where the Tait graph represents the link diagram $\tilde{L}$) in $O(c(\tilde{L}))$ time. Next, we also show that Jones polynomials of closed 3-braid links can be computed from an integer sequence that represents a link diagram $\tilde{L}$ with $O(c(\tilde{L}))$ additions and multiplications in polynomials of degree $O(c(\tilde{L}))$ by using simple recurrence formulas.

We denote the link diagram consisting of integer tangles $I_{a_1}, \ldots, I_{a_m}$ as shown in Fig. 22 by $\tilde{B}(a_1, \ldots, a_m)$, where $a_1, \ldots, a_m$ are integers. The Tait graph of $B(a_1, \ldots, a_m)$ is denoted by $G_B(a_1, \ldots, a_m)$. Note that for any non-zero integers $a_1, \ldots, a_m$, the Tait graph $G = G_B(a_1, \ldots, a_m) = (V, E, l)$ has a vertex $v \in V$ such that $G - v$ is a cycle or a graph consisting of one vertex with no edge (see Fig. 23). We call a link diagram $\tilde{L}$ a closed 3-braid diagram if the Tait graph $G = (V, E, l)$ of $\tilde{L}$ consists of two vertices with no edge or has a vertex $v \in V$ satisfying the following:

(i) $G - v$ is a cycle.
(ii) For any vertex $u \in N_G(v)$, all of the edges that join $u$ and $v$ have the same sign.
(iii) For any vertex $u \in V - \{v\}$, if $\deg_G(u) = 2$, then the two edges incident to $u$ have the same sign.
(iv) $v$ has no loop edge.

We call $v$ a closed 3-braid vertex of $G$. If $G$ consists of two vertices with no edge, then we call the vertices closed 3-braid vertices of $G$. It is clear that for any closed 3–bridge diagram $\tilde{L}$, there exists an integer sequence $(a_1, \ldots, a_m)$ such that $\tilde{B}(a_1, \ldots, a_m) = \tilde{L}$, namely $G_B(a_1, \ldots, a_m)$ is the Tait graph of $\tilde{L}$. We remark that the expression $G_B(a_1, \ldots, a_m)$ is not unique. Any closed 3-braid link has a closed 3-braid diagram.

Proposition 14. Let $a_1, \ldots, a_m$ are non-zero integers. Let $G$ be the Tait graph of a closed 3-braid diagram and the size of the vertex set of $G$ more than 3. $G_B(a_1, \ldots, a_m)$ is isomorphic to $G$ if and only if there exists a bijection from the vertex set of $G_B(a_1, \ldots, a_m)$ to the vertex set of $G$ satisfying the following:

(i) If $u$ and $v$ are adjacent vertices in $G_B(a_1, \ldots, a_m)$, then the number of the edges that join $u$ and $v$ in $G_B(a_1, \ldots, a_m)$ is equal to the number of the edges that join $\tilde{f}(u)$ and $\tilde{f}(v)$ in $G$ and an edge that joins $u$ and $v$ in $G_B(a_1, \ldots, a_m)$ and an edge that joins $\tilde{f}(u)$ and $\tilde{f}(v)$ in $G$ have the same label.
(ii) If $u$ and $v$ are not adjacent vertices in $G_B(a_1, \ldots, a_m)$, then $\tilde{f}(u)$ and $\tilde{f}(v)$ are not adjacent vertices in $G$.  

Proof. For any pair of adjacent vertices $u$ and $v$ in $G_B(a_1, \ldots, a_m)$, all of the edges that join $u$ and $v$ in $G_B(a_1, \ldots, a_m)$ have the same sign because $a_1, \ldots, a_m$ are non-zero integers and the size of the vertex set of $G_B(a_1, \ldots, a_m)$ is more than 3. For any pair of adjacent vertices $u$ and $v$ in $G$, all of the edges that join $u$ and $v$ in $G$ have the same sign since the size of the vertex set of $G$ is more than 3. □

Lemma 15. Let $G = (V, E, l)$ be the Tait graph of a closed 3-braid diagram and $v \in V$ a closed 3-braid vertex of $G$. Suppose that an integer sequence $(a_1, \ldots, a_m)$ is obtained by the following way:

(i) If $|N_G(v)| > 0$ and $|V| \neq 3$, then the integer sequence $(a_1, \ldots, a_m)$ is obtained by the following way (see Fig. 24):

(a) Let $m = 2|N_G(v)|$.

(b) Choose a vertex in $N_G(v)$, index the vertex $v_1$ and go around the cycle $G - v$ from $v_1$ to $v_1$.

(c) Index the vertices of $N_G(v) - \{v_1\}$ in the way which $v_i + 1$ is the vertex we visit $i$-th in $N_G(v) - \{v_1\}$ for $i = 1, \ldots, |N_G(v)| - 1$.

(d) Let $a_1$ be the sum of the signs of the edges between $v_i|N_G(v)|$ and $v_1$ in the cycle $G - v$ and $a_{2i+1}$ the sum of the signs of the edges between $v_i$ and $v_{i+1}$ in the cycle $G - v$ for $i = 1, \ldots, |N_G(v)| - 1$.

(e) Let $a_{2j} = -\text{edge sign}_G(v, v_j)$ for $j = 1, \ldots, |N_G(v)|$.

(ii) If $|V| = 3$, then the integer sequence $(a_1, \ldots, a_m)$ is obtained by the following way (see Fig. 25):

(a) Let $m = 4$.

(b) Let $V = \{v, v_1, v_2\}$.

(c) Let $a_2 = -\text{edge sign}_G(v, v_1)$ and $a_4 = -\text{edge sign}_G(v, v_2)$.
Let $G = (V, E, l)$ be the Tait graph of a closed 3-braid diagram and $|V| > 3$. Then, the following hold.

(i) If there exists a vertex $v \in V$ such that $|N_G(v)| \geq 4$ or $|N_G(v)| \leq 1$, then $v$ is a closed 3-braid vertex of $G$.

(ii) Let $v \in V$ be a closed 3-braid vertex of $G$. Suppose that for any vertex $u \in V$, $2 \leq |N_G(u)| \leq 3$. Then, $1 \leq ||u \in V : |N_G(u)| = 3|| \leq 4$ and the following hold.

(a) $\cap \{u \in V : |N_G(v)| = 3\} N_G(u) = \emptyset$ if and only if $|N_G(v)| = 3$.

(b) $\cap \{u \in V : |N_G(v)| = 3\} N_G(u) \neq \emptyset$ if and only if $v \in \cap \{u \in V : |N_G(v)| = 3\} N_G(u)$.

Proof. (i) For any vertex $u \in V - \{v\}$, $G - u$ is not a cycle because $|N_{G-u}(v)| \neq 2$. Then, $v$ is the only vertex such that $G - v$ is a cycle. Therefore, the lemma holds.

(ii) Suppose that $|N_G(u)| = 2$ for any vertex $u \in V$. Let $N_G(v) = \{v_1, v_2\}$. Then, $|N_{G-v}(v_1)| = |N_{G-v}(v_2)| = 1$. Therefore, $V = \{v, v_1, v_2\}$ because $G - v$ is a cycle. Then, there exists a vertex $u \in V$ such that $|N_G(u)| = 3$ since $|V| > 3$ and for any vertex $u \in V$, $2 \leq |N_G(u)| \leq 3$. If $|\{u \in V : |N_G(u)| = 3\}| \geq 5$, then there exists a vertex $w \in V$ such that $|N_G(w)| \geq 4$. Hence, $1 \leq ||u \in V : |N_G(u)| = 3|| \leq 4$.

Given the Tait graph $G$ of a closed 3-braid diagram, Procedure seq\_3-braid constructs an integer sequence $(a_1, \ldots, a_m)$ such that $G_B(a_1, \ldots, a_m)$ is isomorphic to $G$.

**Procedure seq\_3-braid**

**Input:** The Tait graph $G = (V, E, l)$ of a closed 3-braid diagram.

**Output:** An integer sequence $(a_1, \ldots, a_m)$ such that $G_B(a_1, \ldots, a_m)$ is isomorphic to $G$.

Index a vertex $u_0 \in V$ in the way which $u_0$ is a closed 3-braid vertex of $G$.
Compute $N_{G-u_0}(v)$ for all vertices $v \in V - \{u_0\}$;
if $|N_G(u_0)| > 0$ then Index a vertex $v \in N_G(u_0)$ in the way which $u_1$ is $v$;
else Index a vertex $v \in V - \{u_0\}$ in the way which $u_1$ is $v$;
for $i := 2$ to $|V| - 1$ do
    Index a vertex $u_i \in N_{G-u_0}(u_{i-1})$ in the way which $u_i$ is not $u_{i-2}$;
Compute edge_sign($u_0, u_i$) for $i = 1, \ldots, |V| - 1$;
Compute edge_sign($u_{|V|-1}, u_1$) and edge_sign($u_j, u_{j+1}$) for $j = 1, \ldots, |V| - 2$;
Initialize $i$ as “1” and $k$ as “1”; 
while $i < |V|$ do begin 
  $\{ k$ is an odd number $\}$ 
  Initialize $a_k$ as “0”; 
  repeat 
    if $i < |V| - 1$ then $a_k := a_k + \text{edge}_G(u_i, u_{i+1})$
    else $a_k := a_k + \text{edge}_G(u_{|V| - 1}, u_1)$;
    Increment $i$;
  until $i = |V|$ or $\deg_G(u_i) \neq 2$;
  Increment $k$; 
  $\{ k$ is an even number $\}$
  if $i < |V|$ then $a_k := -\text{edge}_G(u_0, u_i)$
  else $a_k := -\text{edge}_G(u_0, u_1)$;
  Increment $k$;
end;

if for any $v \in V$, $|N_G(v)| = 2$ then begin 
  if $\text{edge}_G(u_1, u_2) = 2$ then begin $a_1 := 1$; $a_3 := 1$; end;
  if $\text{edge}_G(u_1, u_2) = 0$ then begin $a_1 := 1$; $a_3 := -1$; end;
  if $\text{edge}_G(u_1, u_2) = -2$ then begin $a_1 := -1$; $a_3 := -1$; end;
end;

Theorem 17. Procedure seq_3-braid constructs an integer sequence $(a_1, \ldots, a_m)$ such that $G_B(a_1, \ldots, a_m)$ is isomorphic to $G$ in $O(|E|)$ time.

Proof. The procedure constructs an integer sequence $(a_1, \ldots, a_m)$ such that $G_B(a_1, \ldots, a_m)$ is isomorphic to $G$ by Lemma 15. We estimate the running time of the procedure.

If $|V| \leq 3$, then we can clearly find a closed 3-braid vertex of $G$ in $O(|E|)$ time by Proposition 5(ii) and (iv). We assume that $|V| > 3$ and show that we can find a closed 3-braid vertex of $G$ in $O(|E|)$ time by Lemma 16. We can determine whether there exists a vertex $v \in V$ such that $|N_G(v)| \geq 4$ or $|N_G(v)| \leq 1$ in $O(|E|)$ time because it is sufficient to count the numbers of their neighbours until four at most. We assume that for any vertex $v \in V$, $2 \leq |N_G(v)| \leq 3$. We can compute $\bigcap_{u \in |u|} N_G(v) \leq 3$ in $O(|E|)$ time since $|u \in V : |N_G(u)| = 1| \leq 4$. Then, we can find a closed 3-braid vertex of $G$ in $O(|E|)$ time by Proposition 5(ii) and (iv) because $|u \in V : |N_G(u)| = 3| \leq 4$ and $|\bigcap_{u \in |u|} N_G(v) \leq 3|$. We can compute $N_G(u_0) \in V$ of $G$ in $O(|E|)$ time by Proposition 5(ii) and (iii) because $G \neq u_0$ is a cycle or a graph consisting of one vertex and $|N_G(u_0)| \leq 2$. For $i = 2, \ldots, |V| - 1$, we can index a vertex $u_i \in N_G(u_{i-1})$ in the way which $u_i$ is not $u_{i-2}$ in constant time because $|N_G(u_{i-1})| \leq 2$. We can compute $\text{edge}_G(u_0, u_i)$ for $i = 1, \ldots, |V| - 1$, $\text{edge}_G(u_{|V| - 1}, u_0)$ and $\text{edge}_G(u_j, u_{j+1})$ for $j = 1, \ldots, |V| - 2$ in $O(|E|)$ time by Proposition 5(v). The while loop finishes in $O(|V|)$ time because $i$ is incremented at least as many times as the repeat until loop. Therefore, the procedure finishes in $O(|E|)$ time. \[ \square \]

Then, we compute the Kauffman bracket polynomial $\langle B(a_1, \ldots, a_m) \rangle$ from the integer sequence $(a_1, \ldots, a_m)$.

Lemma 18. For any integer sequence $(a_1, \ldots, a_m)$, the following recurrence formula holds.

$$
\langle \tilde{B}(a_1, \ldots, a_m) \rangle = \begin{cases} 
(A^2 - A^{-2})(\tilde{R}(a_1)) & \text{if } m = 1, \\
A^{a_m}(\tilde{B}(a_1, \ldots, a_{m-1})) - (A^{-2})^{-3a_m+2} & \text{if } m \geq 2 \text{ and } m \text{ is an even number,} \\
A^{a_m}(\tilde{B}(a_1, \ldots, a_{m-1})) - (A^{-2})^{-3(a_1+a_m)+2} & \text{if } m \geq 3 \text{ and } m \text{ is an odd number.}
\end{cases}
$$

Proof. We consider the case where $m = 1$. We have

$$
\langle \tilde{B}(a_1) \rangle = \langle \tilde{R}(a_1) \cup \emptyset \rangle
$$

(9)
Let \( \tilde{B}(a_1) \) be a braid computed once at most. Therefore, the Eqs. (9) and (10) imply the case where \( m \) is an even number by Lemma 6.

We consider the case where \( m \) is an odd number by Lemma 6. We have:

\[
\tilde{B}(a_1, \ldots, a_m) = A(\tilde{B}(a_1, \ldots, a_m - 1)) + A^{-1}(\tilde{R}(a_1, \ldots, a_m)\#(a_m - 1))
\]

by applying Proposition 3 to \( \tilde{B}(a_1, \ldots, a_m) \). We also have:

\[
\tilde{B}(a_1, \ldots, a_m, 0) = \hat{\tilde{B}}(a_1, \ldots, a_m - 1)
\]

since \( \tilde{B}(a_1, \ldots, a_m, 0) \) is \( \tilde{B}(a_1, \ldots, a_m - 1) \) (see Fig. 28). Hence, the Eqs. (11) and (12) imply the case where \( m \geq 2 \) and \( m \) is an odd number by Lemma 6.

We consider the case where \( m \geq 3 \) and \( m \) is an odd number, we get

\[
\tilde{B}(a_1, \ldots, a_m) = A(\tilde{B}(a_1, \ldots, a_m - 1)) + A^{-1}(\tilde{R}(a_1, \ldots, a_m)\#(a_1 + a_m - 1))
\]

by applying Proposition 3 to \( \tilde{B}(a_1, \ldots, a_m) \). We also have

\[
\tilde{B}(a_1, \ldots, a_m, 0) = \hat{\tilde{B}}(a_1, \ldots, a_m - 1)
\]

because \( \tilde{B}(a_1, \ldots, a_m, 0) \) is \( \tilde{B}(a_1, \ldots, a_m - 1) \) (see Fig. 29). Hence, the Eqs. (13) and (14) imply the case where \( m \geq 3 \) and \( m \) is an odd number by Lemma 6.

Given an integer sequence \((a_1, \ldots, a_m)\), Procedure bra_3-braid computes the Kauffman bracket polynomial \( \langle \tilde{B}(a_1, \ldots, a_m) \rangle \) by using the recurrence formulas in Lemmas 11 and 18. While the procedure is running, every Kauffman bracket polynomial is computed once at most.

**Procedure bra_3-braid**

**Input:** An integer sequence \((a_1, \ldots, a_m)\).

**Output:** The Kauffman bracket polynomial \( \langle \tilde{B}(a_1, \ldots, a_m) \rangle \).

1. Compute \( \langle \tilde{R}(a_1) \rangle \) from \( Q_{a_1}(-A^4) \);
2. Compute \( \langle \tilde{B}(a_1) \rangle \) from \( \langle \tilde{R}(a_1) \rangle \);
3. if \( m \geq 2 \) then begin
   1. Compute \( \langle \tilde{B}(a_1, a_2) \rangle \) from \( \langle \tilde{B}(a_1) \rangle, \langle \tilde{R}(a_1) \rangle \) and \( Q_{a_2}(-A^4) \);
   2. Compute \( \langle \tilde{R}(a_1, a_2) \rangle \) from \( \langle \tilde{R}(a_1) \rangle \) and \( Q_{a_2}(-A^4) \);
Compute \(\tilde{B}(a_2)\) from \(Q_{a_2}(-A^4)\); end; if \(m \geq 3\) then begin
  Compute \(\tilde{B}(a_1, a_2, a_3)\) from \(\tilde{B}(a_1, a_2), \tilde{B}(a_2)\) and \(Q_{a_3}(-A^4)\);
  Compute \(\tilde{R}(a_1, a_2, a_3)\) from \(\tilde{R}(a_1, a_2), \tilde{R}(a_1)\) and \(Q_{a_3}(-A^4)\);
  Compute \(\tilde{R}(a_2, a_3)\) from \(\tilde{R}(a_2)\) and \(Q_{a_3}(-A^4)\);
end;
for \(i := 4\) to \(m\) do begin
  if \(i\) is an even number then
    Compute \(\tilde{B}(a_1, \ldots, a_i)\) from \(\tilde{B}(a_1, \ldots, a_{i-1}), \tilde{R}(a_1, \ldots, a_{i-1})\) and \(Q_{a_i}(-A^4)\)
  else
    Compute \(\tilde{B}(a_1, \ldots, a_i)\) from \(\tilde{B}(a_1, \ldots, a_{i-1}), \tilde{R}(a_2, \ldots, a_{i-1})\) and \(Q_{a_i}(-A^4)\);
    Compute \(\tilde{R}(a_1, \ldots, a_i)\) from \(\tilde{B}(a_1, \ldots, a_{i-1}), \tilde{R}(a_1, \ldots, a_{i-1})\) and \(Q_{a_i}(-A^4)\);
    Compute \(\tilde{R}(a_2, \ldots, a_i)\) from \(\tilde{R}(a_2, \ldots, a_{i-1})\) and \(Q_{a_i}(-A^4)\);
end;

Theorem 19. Procedure bra_3-braid computes the Kauffman bracket polynomial \(\tilde{B}(a_1, \ldots, a_m)\) with \(O(c(\tilde{B}(a_1, \ldots, a_m))\) additions and multiplications in polynomials of degree \(O(c(\tilde{B}(a_1, \ldots, a_m)))\), namely in \(O(c(\tilde{B}(a_1, \ldots, a_m))^2 \log c(\tilde{B}(a_1, \ldots, a_m)))\) time.

Proof. It is clear that the procedure computes the Kauffman bracket polynomial \(\tilde{B}(a_1, \ldots, a_m)\) from the integer sequence \((a_1, \ldots, a_m)\) by Lemmas 11 and 18. We estimate the running time of the procedure. For \(i = 1, \ldots, m\), we can compute \(Q_{a_i}(-A^4)\) in \(O(|a_i|)\) time. We can compute \(\tilde{R}(a_1)\) with \(O(1)\) operations in polynomials of degree \(O(c(\tilde{B}(a_1, \ldots, a_m)))\) from \(Q_{a_1}(-A^4)\) by Lemma 11. We can compute \(\tilde{B}(a_1)\) with \(O(1)\) operations in polynomials of degree \(O(c(\tilde{B}(a_1, \ldots, a_m)))\) from \(\tilde{R}(a_1)\) by Lemma 18. We can compute \(\tilde{B}(a_1, a_2)\) with \(O(1)\) operations in polynomials of degree \(O(c(\tilde{B}(a_1, \ldots, a_m)))\) from \(\tilde{B}(a_1)\), \(\tilde{R}(a_2)\) and \(Q_{a_2}(-A^4)\) by Lemma 18. We can compute \(\tilde{R}(a_1, a_2)\) with \(O(1)\) operations in polynomials of degree \(O(c(\tilde{R}(a_1, \ldots, a_m)))\) from \(\tilde{R}(a_1)\) and \(Q_{a_2}(-A^4)\) by Lemma 11. We can compute \(\tilde{R}(a_2)\) with \(O(1)\) operations in polynomials of degree \(O(c(\tilde{R}(a_1, \ldots, a_m)))\) from \(Q_{a_2}(-A^4)\) by Lemma 11. We can compute \(\tilde{B}(a_1, a_2, a_3)\) with \(O(1)\) operations in polynomials of degree \(O(c(\tilde{B}(a_1, \ldots, a_m)))\) from \(\tilde{B}(a_1, a_2), \tilde{R}(a_2)\) and \(Q_{a_3}(-A^4)\) by Lemma 18. We can compute \(\tilde{R}(a_1, a_2, a_3)\) with \(O(1)\) operations in polynomials of degree \(O(c(\tilde{R}(a_1, \ldots, a_m)))\) from \(\tilde{R}(a_1, a_2), \tilde{R}(a_1)\) and \(Q_{a_3}(-A^4)\) by Lemma 11. We can compute \(\tilde{R}(a_2, a_3)\) with \(O(1)\) operations in polynomials of degree \(O(c(\tilde{R}(a_1, \ldots, a_m)))\) from \(\tilde{R}(a_2)\) and \(Q_{a_3}(-A^4)\) by Lemma 11. For \(i = 4, \ldots, m\), if \(i\) is an even number, then we can compute \(\tilde{B}(a_1, \ldots, a_i)\) with \(O(1)\) operations in polynomials of degree \(O(c(\tilde{B}(a_1, \ldots, a_m)))\) from \(\tilde{B}(a_1, \ldots, a_{i-1}), \tilde{R}(a_1, \ldots, a_{i-1})\) and \(Q_{a_i}(-A^4)\) by Lemma 18. For \(i = 4, \ldots, m\), if \(i\) is an odd number, then we can compute \(\tilde{B}(a_1, \ldots, a_i)\) with \(O(1)\) operations in polynomials of degree \(O(c(\tilde{B}(a_1, \ldots, a_m)))\) from \(\tilde{B}(a_1, \ldots, a_{i-1}), \tilde{R}(a_2, \ldots, a_{i-1})\) and \(Q_{a_i}(-A^4)\) by Lemma 18. For \(i = 4, \ldots, m\), we can compute \(\tilde{R}(a_1, \ldots, a_i)\) with \(O(1)\)
Corollary 20. The Jones polynomial of a closed 3-braid link is computed from the Tait graph of a closed 3-braid diagram \( \tilde{L} \) with \( O(c(\tilde{L})) \) additions and multiplications in polynomials of degree \( O(c(\tilde{L})) \), namely in \( O((\tilde{L})^2 \log c(\tilde{L})) \) time.

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Appendix

Proof of Proposition 2. By Definition 1(ii) and (iii),

\[
\begin{align*}
\left( \begin{array}{c} 0 \\ \end{array} \right) &= A \left( \begin{array}{c} 0 \\ \end{array} \right) + A^{-1} \left( \begin{array}{c} 0 \\ \end{array} \right) \\
&= A(\hat{\ve}) + A^{-1}(-A^{-2} - A^2)(\hat{\ve}) \\
&= -A^{-3}(\hat{\ve}).
\end{align*}
\]

By Definition 1(ii) and (iii),

\[
\begin{align*}
\left( \begin{array}{c} 0 \\ \end{array} \right) &= A \left( \begin{array}{c} 0 \\ \end{array} \right) + A^{-1} \left( \begin{array}{c} 0 \\ \end{array} \right) \\
&= A(-A^{-2} - A^2)(\hat{\ve}) + A^{-1}(\hat{\ve}) \\
&= -A^3(\hat{\ve}). \quad \Box
\end{align*}
\]

Proof of Proposition 3. If \( k > 0 \), then it is clear that the lemma holds by Definition 1(iii). If \( k \leq 0 \), then, by Definition 1(iii) and Proposition 2,

\[
A \left( \begin{array}{c} l_{-1} \\ \end{array} \right) = A^2 \left( \begin{array}{c} l_{-1} \\ \end{array} \right) + \left( \begin{array}{c} l_k \\ \end{array} \right) \\
= A^2(-A^{-3}) \left( \begin{array}{c} l_{k-1} \\ \end{array} \right) + \left( \begin{array}{c} l_k \\ \end{array} \right) \\
= -A^{-1} \left( \begin{array}{c} l_{k-1} \\ \end{array} \right) + \left( \begin{array}{c} l_k \\ \end{array} \right). \quad \Box
\]

Proof of Proposition 5. (i) By traversing all of the endvertices of the edges in \( G \), all of the degrees of the vertices of \( G \) are computed in \( O(|E|) \) time.

(ii) Let \( G - v = (V', E', l') \), \( V' = V - \{v\} \) and \( E' = \{ e \in E : e \text{ is not incident to } v \} \). Therefore, \( G - v \) is computed in \( O(|E|) \) time.
(iii) By traversing all of the endvertices of the edges in $G$, all of the neighbour sets of the vertices of $G$ are computed in $O(|E|)$ time because $\max(|NG(v)|: v \in V)$ has a constant bound.

(iv) By traversing all of the endvertices of the edges in $G$, whether all of the numbers of the neighbours of the vertices in $G$ are two or not is determined in $O(|E|)$ time because it is sufficient to count the numbers of their neighbors until two at most. If all of the number of the neighbours of the vertices are two, then whether $G$ is connected or not is determined in $O(|E|)$ time.

(v) By traversing all of the endvertices and the labels of the edges in $G$, all of the values in $\{\text{edge_sign}_G(u, v) : u, v \in V \text{ are adjacent} \}$ are computed in $O(|E|)$ time. $\square$

**Proof of Lemma 8.** Let $G' = (V', E', l')$ be $G_R(a_1, \ldots, a_{2n}|G(v)|−1)$. We show that $|V| = |V'|$ and there exists an isomorphism from $G'$ to $G$.

Let $n = |NG(v)|$. The size of the edge set of the path $G - v$ is equal to $\sum_{k=1}^{n-1} |a_{2k}|$ by the way of constructing the integer sequence $(a_1, \ldots, a_{2n-1})$. Then, $|V| = 2 + \sum_{k=1}^{n-1} |a_{2k}|$. Furthermore, $|V'| = 2 + \sum_{k=1}^{n-1} |a_{2k}|$ by the definition of $G_R(a_1, \ldots, a_{2n-1})$. Therefore, $|V| = |V'|$.

To define an isomorphism from $G'$ to $G$, we index the vertices of $V$ and $V'$ with the following way (see Figs. 11 and 30). We index the vertices of $V$ in the way which $u_0 = v$, $u_{i+\sum_{k=1}^{i-1} |a_{2k}|} = v_i$ for $i = 1, \ldots, n$ and $u_j$ and $u_{j+1}$ are adjacent in the path $G - v$ for $j = 1, \ldots, |V| - 2$. Let $E'_k$ be the set of edges of $E'$ corresponding to the crossings of $u_{a_k}$ in $\tilde{K}(a_1, \ldots, a_{2n-1})$ and $V'_k$ the set of the endvertices of the edges of $E'_k$ for $k = 1, \ldots, 2n - 1$. Note that $\bigcap_{k \in N \leq a} V'_{2i-1} \neq \emptyset$ and $|V'_{2k-1}| = 2$ for $k = 1, \ldots, n$. Let $v' \in V'$ be a vertex of $\bigcap_{k \in N \leq a} V'_{2i-1}$ and $v'_k \in V'$ the other vertex of $V'_{2k-1}$ for $k = 1, \ldots, n$. Then, $G' - v'$ is a path. We index the vertices of $V'$ in the way which $u_0' = v'$, $u_{i+\sum_{k=1}^{i-1} |a_{2k}|} = v_i'$ for $i = 1, \ldots, n$ and $u_j'$ and $u_{j+1}'$ are adjacent in the path $G' - v'$ for $j = 1, \ldots, |V| - 2$. Note that if a pair of vertices consists of two adjacent vertices in $G'$, then either the pair consists of $v'$ and $v'_i$ for $i = 1, \ldots, n$ or $u_j'$ and $u_{j+1}'$ for $j = 1, \ldots, |V| - 2$.

Let $f$ be a bijection from $V'$ to $V$ such that $f(u_i') = u_i$ for $i = 0, \ldots, |V| - 1$. Note that $f(v') = v$ and $f(v_i') = v_i$ for $i = 1, \ldots, n$. Then, if $x$ and $y$ are not adjacent vertices in $G'$, then $f(x)$ and $f(y)$ are not adjacent vertices in $G$. We show that $f$ is an isomorphism from $G'$ to $G$ by Proposition 7.

We consider the pairs of $v'$ and $v'_i$ for $i = 1, \ldots, n$. The number of the edges that join $v'$ and $v'_i$ in $G'$ is equal to $|a_{2i-1}|$ for $i = 1, \ldots, n$ by the definition of $G_R(a_1, \ldots, a_{2n-1})$. The number of the edges that join $v$ and $v_i$ in $G$ is equal to $|a_{2i-1}|$ for $i = 1, \ldots, n$ by the way of constructing the integer sequence $(a_1, \ldots, a_{2n-1})$. The label of an
edge that joins \( v' \) and \( v_i' \) in \( G' \) is \( -\text{sign}(a_{2i-1}) \) for \( i = 1, \ldots, n \) by the definition of \( G_R(a_1, \ldots, a_{2n-1}) \). The label of an edge that joins \( v \) and \( v_i \) in \( G \) is \( -\text{sign}(a_{2i-1}) \) for \( i = 1, \ldots, n \) by the way of constructing the integer sequence \( (a_1, \ldots, a_{2n-1}) \).

We consider the pairs of \( u_j' \) and \( u_{j+1}' \) for \( j = 1, \ldots, |V| - 2 \). The number of the edges that join \( u_j' \) and \( u_{j+1}' \) in \( G' \) is equal to 1 for \( j = 1, \ldots, |V| - 2 \) because \( u_j' \) and \( u_{j+1}' \) are adjacent vertices in the path \( G' - v' \). The number of the edges that join \( u_j \) and \( u_{j+1} \) in \( G \) is equal to 1 for \( j = 1, \ldots, |V| - 2 \) since \( u_j \) and \( u_{j+1} \) are adjacent vertices in the path \( G - v \). There exists the unique integer \( k_j \) such that \( 1 + \sum_{k=1}^{j-1} |a_{2k}| \leq j < 1 + \sum_{k=1}^{j} |a_{2k}| \) for \( j = 1, \ldots, |V| - 2 \). The label of the edge that joins \( u_j' \) and \( u_{j+1}' \) in \( G' \) is \( \text{sign}(a_{2k_j}) \) for \( j = 1, \ldots, |V| - 2 \) by the definition of \( G_R(a_1, \ldots, a_{2n-1}) \). The label of the edge that joins \( u_j \) and \( u_{j+1} \) in \( G \) is \( \text{sign}(a_{2k_j}) \) for \( j = 1, \ldots, |V| - 2 \) by the way of constructing the integer sequence \( (a_1, \ldots, a_{2n-1}) \).

Proof of Lemma 15. It is clear that the lemma holds if \( |N_G(v)| = 0 \) or \( |V| \leq 3 \). We consider the case where \( |N_G(v)| > 0 \) and \( |V| > 3 \). Let \( G' = (V', E', l') \) be \( G_R(a_1, \ldots, a_{2|N_G(v)|}) \). We show that \( |V| = |V'| \) and there exists an isomorphism from \( G \) to \( G' \).

Let \( n = |N_G(v)| \). The size of the edge set of the cycle \( G - v \) is equal to \( \sum_{k=1}^{n} |a_{2k-1}| \) by the way of constructing the integer sequence \( (a_1, \ldots, a_{2n}) \). Then, \( |V| = 1 + \sum_{k=1}^{n} |a_{2k-1}| \). Furthermore, \( |V'| = 1 + \sum_{k=1}^{n} |a_{2k-1}| \) by the definition of \( G_R(a_1, \ldots, a_{2n}) \). Therefore, \( |V| = |V'| \).

To define an isomorphism from \( G' \) to \( G \), we index the vertices of \( V \) and \( V' \) in the following way (see Figs. 24 and 31). We index the vertices of \( V \) in the way which \( u_0 = v, u_1 + \sum_{k=1}^{1} |a_{2k-1}| = v_i \) for \( i = 1, \ldots, n, u_j \) and \( u_{j+1} \) are adjacent in the cycle \( G - v \) for \( j = 1, \ldots, |V| - 2 \), and \( u_{|V|-1} \) and \( u_1 \) are adjacent in the cycle \( G - v \). Note that if a pair of vertices consists of two adjacent vertices in \( G \), then either the pair consists of \( v \) and \( v_i \) for \( i = 1, \ldots, n, u_j \) and \( u_{j+1} \) for \( j = 1, \ldots, |V| - 2 \), or \( u_{|V|-1} \) and \( u_1 \). Let \( E'_k \) be the set of edges of \( E' \) corresponding to the crossings of \( I_{ak} \) in \( B(a_1, \ldots, a_{2n}) \) and \( V_k' \) the set of the endvertices of the edges of \( E'_k \) for \( k = 1, \ldots, 2n \). Note that \( \cap_{i \in [n]} V'_{2k} \neq \emptyset \) and \( |V'_k| = 2 \) for \( k = 1, \ldots, n \). Let \( v' \in V \) be a vertex of \( \cap_{i \in [n]} V'_{2k} \) satisfying that \( G' - v' \) is a cycle and \( v'_k \in V' \) the other vertex of \( V'_k \) for \( k = 1, \ldots, n \). We index the vertices of \( V' \) in the way which \( u'_0 = v', u'_1 + \sum_{k=1}^{1} |a_{2k-1}| = v'_i \) for \( i = 1, \ldots, n, u'_j \) and \( u'_{j+1} \) are adjacent in the cycle \( G' - v' \) for \( j = 1, \ldots, |V| - 2 \), and \( u'_{|V|-1} \) and \( u'_1 \) are adjacent in the cycle \( G' - v' \). Note that if a pair of vertices consists of two adjacent vertices in \( G' \), then either the pair consists of \( v' \) and \( v'_i \) for \( i = 1, \ldots, n, u'_j \) and \( u'_{j+1} \) for \( j = 1, \ldots, |V| - 2 \), or \( u'_{|V|-1} \) and \( u'_1 \).
Let $f$ be a bijection from $V'$ to $V$ such that $f(u'_i) = u_i$ for $i = 0, \ldots, |V| - 1$. Note that $f(v'_i) = v_i$ for $i = 1, \ldots, n$. Then, if $x$ and $y$ are not adjacent vertices in $G'$, then $f(x)$ and $f(y)$ are not adjacent vertices in $G$. We show that $f$ is an isomorphism from $G'$ to $G$ by Proposition 14.

We consider the pairs of $v'_i$ and $v'_j$ for $i, j = 1, \ldots, n$. The number of the edges that join $v'_i$ and $v'_j$ in $G'$ is equal to $|a_{2i}|$ for $i = 1, \ldots, n$ by the definition of $G_B(a_1, \ldots, a_{2n})$. The number of the edges that join $v_i$ and $v_j$ in $G$ is equal to $|a_{2i}|$ for $i = 1, \ldots, n$ by the way of constructing the integer sequence $(a_1, \ldots, a_{2n})$. The label of an edge that joins $v'_i$ and $v'_j$ in $G'$ is $\text{sign}(a_{2i})$ for $i = 1, \ldots, n$ by the definition of $G_B(a_1, \ldots, a_{2n})$. The label of an edge that joins $v_i$ and $v_j$ in $G$ is $\text{sign}(a_{2i})$ for $i = 1, \ldots, n$ by the way of constructing the integer sequence $(a_1, \ldots, a_{2n})$.

We consider the pairs of $u'_j$ and $u'_{j+1}$ for $j = 1, \ldots, |V| - 2$ and the pair of $u'_0$ and $u'_1$. The number of the edges that join $u'_j$ and $u'_{j+1}$ in $G'$ for $j = 1, \ldots, |V| - 2$ and the number of the edges that join $u'_0$ and $u'_1$ in $G'$ are equal to 1 because $u'_j$ and $u'_{j+1}$ are adjacent vertices in the cycle $G' - v$ and $u'_0$ and $u'_{|V| - 1}$ and $u'_1$ are adjacent vertices in the cycle $G' - v$. The number of the edges that join $u_j$ and $u_{j+1}$ in $G$ for $j = 1, \ldots, |V| - 2$ and the number of the edges that join $u_0$ and $u_1$ in $G$ are equal to 1 since $u_j$ and $u_{j+1}$ are adjacent vertices in the cycle $G - v$ and $u_0$ and $u_1$ are adjacent vertices in the cycle $G - v$. There exists the unique integer $k_j$ such that $1 + \sum_{k_j=1}^{|a_{2j+1}|} |a_{2j+1}| \leq j < 1 + \sum_{k_j=1}^{|a_{2j+1}|} |a_{2j+1}|$ for $j = 1, \ldots, |V| - 2$ where $a_{2n+1} = a_1$. The label of the edge that joins $u'_j$ and $u'_{j+1}$ in $G'$ is $\text{sign}(a_{2j+1})$ for $j = 1, \ldots, |V| - 2$ and the label of the edge that joins $u'_0$ and $u'_1$ in $G'$ is $\text{sign}(a_1)$ by the definition of $G_B(a_1, \ldots, a_{2n})$ where $a_{2n+1} = a_1$. The label of the edge that joins $u_j$ and $u_{j+1}$ in $G$ is $\text{sign}(a_{2j+1})$ for $j = 1, \ldots, |V| - 2$ and the label of the edge that joins $u_0$ and $u_1$ in $G$ is $\text{sign}(a_1)$ by the way of constructing the integer sequence $(a_1, \ldots, a_{2n})$ where $a_{2n+1} = a_1$. \hfill \square

References


