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Asymptotic oscillations

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Abstract

Given a function $f: X \to \mathbb{R}$ defined on the support of a ballean, we introduce the notion of slow oscillation in direction of a filter on X. We show that there exists a filter on X responsible for the rate of slow oscillation of f at infinity. We apply this result to the Stone–Čech compactifications of discrete groups.

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Let (X, d) be a metric space. A function f is called *slowly oscillating* if, for any $\varepsilon > 0$ and $r > 0$, there exists a bounded subset V of X such that

$$\operatorname{diam}\bigl(f B_d(x, r)\bigr) < \varepsilon$$

for every $x \in X \setminus V$, where

$$B_d(x, r) = \bigl\{y \in X\colon d(x, y) < r\bigr\} \quad \text{and} \quad \operatorname{diam} A = \sup\bigl\{|x - y|\colon x, y \in A\bigr\}.$$

This notion was introduced by N. Higson (see for example [7, p. 29]) in the context of index theory. For applications of slowly oscillating functions see [1,3,7].

In this note we study the following question: given an arbitrary function $f: X \to \mathbb{R}$, how far is it from being slowly oscillating? To answer this question we introduce the notion of slow oscillation in the direction of a filter on X. For one special case this notion appeared in [3]. We show that there exists a maximal filter $\operatorname{so}(f)$ on X such that f is slowly oscillating in the direction of $\operatorname{so}(f)$. This filter can be considered as a measure of the slow oscillation of f at infinity.

Following [2,6], we base our exposition on the concept of balleans, the asymptotic counterparts of the uniform topological spaces.

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1. Preliminaries

A *ball structure* is a triplet $\mathcal{B} = (X, P, B)$, where X, P are nonempty sets and, for any $x \in X$ and $\alpha \in P$, $B(x,\alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x,\alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of B, P is called the set of radiuses.

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we put

$$B^*(x,\alpha) = \{y \in X\colon\ x \in B(y,\alpha)\}, \qquad B(A,\alpha) = \bigcup_{a\in A} B(a,\alpha).$$

A ball structure $\mathcal{B}$ is called a *ballean* if

- for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x,\alpha) \subseteq B^*(x,\alpha'), \qquad B^*(x,\alpha) \subseteq B(x,\beta');$$

- for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B\big(B(x,\alpha),\beta\big) \subseteq B(x,\gamma).$$

Replacing every ball $B(x,\alpha)$ by $B(x,\alpha) \cup B^*(x,\alpha)$, we get an equivalent ballean [6], so in what follows we suppose that $B(x,\alpha) = B^*(x,\alpha)$ for all $x \in X$, $\alpha \in P$.

Every metric space (X,d) determines a *metric ballean* $(X, \mathbb{R}^+, B_d)$.

A ballean $\mathcal{B}$ is called *connected* if, for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x,\alpha)$.

A subset $Y \subseteq X$ is called *bounded* if there exist $x \in X$ and $\alpha \in P$ such that $Y \subseteq B(x,\alpha)$. We say that $\mathcal{B}$ is *bounded* if its support is bounded. A subset $Z \subseteq X$ is called *cobounded* if $X \setminus Z$ is bounded.

We say that a filter φ on X is *going to infinity* if $X \setminus V \in \varphi$ for every bounded subset V of X.

We use also a preordering $\leqslant$ on the set P: $\alpha \leqslant \beta$ if and only if $B(x,\alpha) \subseteq B(x,\beta)$ for every $x \in X$. A subset $P' \subseteq P$ is called *cofinal* if, for every $\alpha \in P$, there exists $\beta \in P'$ such that $\alpha \leqslant \beta$. The *cofinality* $\operatorname{cf}\mathcal{B}$ is the minimal cardinality of cofinal subsets of P.

We endow X with the discrete topology and use the Stone–Čech compactification βX of X. We take the points of βX to be the ultrafilters on X with the points of X identified with the principal ultrafilters. For every subset $A \subseteq X$, we put $\overline{A} = \{q \in \beta G\colon\ A \in q\}$. The topology of βG can be defined by stating that the family $\{\overline{A}\colon\ A \subseteq X\}$ is a base for the open sets. For every filter φ on X, the subset $\overline{\varphi} = \bigcap\{\overline{A}\colon\ A \in \varphi\}$ is clearly closed in βX, and for every nonempty closed subset K of βX, there exists a filter φ on X such that $K = \overline{\varphi}$.

Given a ballean $\mathcal{B} = (X, P, B)$, we denote by $X^\#$ the set of all ultrafilters on X going to infinity. We say that two ultrafilters $r, q \in X^\#$ are *parallel* (and write $r \parallel q$) if there exists $\alpha \in P$ such that, for every $R \in r$, we have $B(R,\alpha) \in q$. We denote by $\sim$ the minimal (by inclusion) closed (in $X^\# \times X^\#$) equivalence on $X^\#$ such that $\parallel \subseteq \sim$. For the equivalence $\sim$ and its relationship with slowly oscillating functions see [6].

2. Oscillation in direction of filter

In what follows all balleans are supposed to be connected and unbounded.

Let $\mathcal{B} = (X, P, B)$ be a ballean, φ be a filter on X going to infinity. We say that a function $f : X \to \mathbb{R}$ is *slowly oscillating in direction of* φ if, for any $\varepsilon > 0$ and $\alpha \in P$, there exists $F \in \varphi$ such that

$$\operatorname{diam} f\big(B(x,\alpha)\big) < \varepsilon$$

for every $x \in F$. We say that f is *slowly oscillating* if f is slowly oscillating in direction of the filter of cobounded subsets of X.

We say that f is *oscillating* on a subset $Y \subseteq X$ if there exist $\varepsilon > 0$ and $\alpha \in P$ such that

$$\operatorname{diam} f\big(B(x,\alpha)\big) \geqslant \varepsilon$$

for every $x \in Y$. In the case $Y = X$, we say that f is *very oscillating*.

For any $\varepsilon > 0$ and $\alpha \in P$, we put

$$X(f,\varepsilon,\alpha) = \big\{x \in X\colon\ \operatorname{diam} f\big(B(x,\alpha)\big) < \varepsilon\big\}.$$

Clearly, f is very oscillating if and only if there exist $\varepsilon > 0$ and $\alpha \in P$ such that $X(f,\varepsilon,\alpha) = \emptyset$. Assume that $X(f,\varepsilon,\alpha)$ is bounded for some ε and α. We fix an arbitrary point $y \notin X(f,\varepsilon,\alpha)$ and choose $\beta \in P$ such that $B(y,\alpha) \subseteq B(x,\beta)$ for every $x \in X(f,\varepsilon,\alpha)$. Then $X(f,\varepsilon,\beta) = \emptyset$. Hence, a function f is not very oscillating if and only if every subset $X(f,\varepsilon,\alpha)$ is unbounded.

In what follows we suppose that f is not very oscillating so every subset $X(f,\varepsilon,\alpha)$ is unbounded. If $\varepsilon < \varepsilon_1$, $\varepsilon < \varepsilon_2$ and $\alpha > \alpha_1$, $\alpha > \alpha_2$, then $X(f,\varepsilon,\alpha) \subseteq X(f,\varepsilon_1,\alpha_1) \cap X(f,\varepsilon_2,\alpha_2)$. This means that the family

$$\{X(f,\varepsilon,\alpha)\setminus V\colon\ \varepsilon > 0,\ \alpha \in P,\ V \text{ is bounded}\}$$

forms a base of some filter $\mathrm{so}(f)$ on X. In the following theorem we fix some basic properties of $\mathrm{so}(f)$.

Theorem 1. *The filter* $\mathrm{so}(f)$ *has the following properties*:

(i) $\mathrm{so}(f)$ *is going to infinity*;
(ii) f *is slowly oscillating in direction of* $\mathrm{so}(f)$;
(iii) f *is oscillating on every subset* $X \setminus F$, $F \in \mathrm{so}(f)$, $F \neq X$.

If a filter φ *on* X *satisfies* (i), (ii), (iii), *then* $\varphi = \mathrm{so}(f)$.

Given a ballean $\mathcal{B} = (X, P, B)$, we say that a filter φ on X is *thick* if

(a) φ is going to infinity;
(b) for any $F \in \varphi$ and $\alpha \in P$, there exists $H \in \varphi$ such that $B(H,\alpha) \subseteq F$;
(c) φ has a base of cardinality $\leqslant \mathrm{cf}\,\mathcal{B}$.

The adjective “thick” in this definition is related to condition (b).

Theorem 2. *The filter* $\mathrm{so}(f)$ *is thick.*

Proof. We need only to check (b). Fix an arbitrary $\varepsilon > 0$, $\alpha \in P$ and a bounded subset V of X. Choose $\beta \in P$ such that $B(B(x,\alpha),\alpha) \subseteq B(x,\beta)$ for every $x \in X$. Then

$$B\bigl(X(f,\varepsilon,\beta) \setminus B(V,\alpha),\alpha\bigr) \subseteq X(f,\varepsilon,\alpha) \setminus V. \qquad \square$$

In the next section we study the question whether, given a thick filter φ on X, there exists a function $f : X \to \mathbb{R}$ such that $\varphi = \mathrm{so}(f)$.

Theorem 3. *A function* $f : X \to \mathbb{R}$ *is slowly oscillating in direction of a filter* φ *if and only if* f *is slowly oscillating in direction of every ultrafilter* q *on* X *such that* $\varphi \subseteq q$.

Proof. If f is slowly oscillating in direction of φ, then f is slowly oscillating in direction of every filter ψ such that $\varphi \subseteq \psi$.

Assume that f is slowly oscillating in direction of every ultrafilter $q \in \overline{\varphi}$ (this is the same as $\varphi \subseteq q$). Fix an arbitrary $\varepsilon > 0$ and $\alpha \in P$. For every $q \in \overline{\varphi}$, we pick $A_q \in q$ such that

$$\operatorname{diam} f\bigl(B(x,\alpha)\bigr) < \varepsilon$$

for every $x \in A_q$. Then we consider the open covering $\{\overline{A_q}\colon\ q \in \overline{\varphi}\}$ of the compact space $\overline{\varphi}$ and choose some finite subcovering $\{\overline{A_{q_1}}, \dots, \overline{A_{q_n}}\}$. Then $A = A_{q_1} \cup \dots \cup A_{q_n} \in \varphi$, for otherwise, $\varphi \cup \{X \setminus A\}$ has the finite intersection property, and so it is contained in some ultrafilter p. This is clearly not possible since $\overline{\varphi} \subseteq \overline{A}$ means that every ultrafilter containing φ must contains A as well. Since

$$\operatorname{diam} f\bigl(B(x,\alpha)\bigr) < \varepsilon$$

for every $x \in A_{q_1} \cup \dots \cup A_{q_n} \in \varphi$, we see that f is slowly oscillating in direction of φ. $\square$

Corollary. *If $q \in \overline{\mathrm{so}(f)}$, then $q' \in \overline{\mathrm{so}(f)}$ for every ultrafilter q' such that $q \sim q'$.*

Proof. By Theorem 1, we have

$$\overline{\mathrm{so}(f)} = \{q \in X^{\#}\colon\ f \text{ is slowly oscillating in direction of } q\}.$$

If $r \in X^{\#}$ and $r \parallel q$, then by Theorem 2, $r \in \overline{\mathrm{so}(f)}$, and so $q' \in \overline{\mathrm{so}(f)}$ whenever $q \sim q'$. □

3. Inverse problem

Let $\mathcal{B} = (X, P, B)$ be a ballean, φ be a thick filter on X. Does there exist a function $f : X \to \mathbb{R}$ such that $\varphi = \mathrm{so}(f)$? We begin with a negative answer to the question in this general form, and then give a positive solution of the inverse problem in some special case.

Example. Let $X = \{2^n\colon n \in \omega\}$, $d(x, y) = |x - y|$ for all $x, y \in X$, so we have a metric ballean (X, d). It is easy to see that every function $f : X \to \mathbb{R}$ is slowly oscillating, so $\mathrm{so}(f)$ is the filter of cofinite subsets of X. On the other hand, every filter φ on X with a countable base is thick provided that $\bigcap \varphi = \emptyset$.

Theorem 4. *Let (X, d) be an unbounded metric space, φ be a thick filter on X. Assume that there exists $r > 0$ such that $|B(x, r)| > 1$ for every $x \in X$. Then there exists a function $f : X \to \mathbb{R}$ such that $\varphi = \mathrm{so}(f)$.*

Proof. Since φ is thick it has a countable base. We fix a base $\{F_n\colon n \in \omega\}$ of φ such that $F_0 = X$, $B(F_{n+1}, r + n) \subseteq F_n$ for every $n \in \omega$. At the first step we choose a subset $A_0 \subseteq F_0 \setminus F_1$ such that the family $\{B(x, r)\colon x \in A_0\}$ is maximal disjoint. Put

$$B_0 = \bigcup \{B(x, r)\colon x \in A_0\}, \qquad C_0 = (F_0 \setminus F_1) \cup B_0,$$

and $f(x) = 1$ for every $x \in A_0$, $f(x) = \frac{1}{2}$ for every $x \in C_0 \setminus A_0$.

At the second step we choose a subset $A_1 \subseteq F_1 \setminus (F_2 \cup C_0)$ such that the family $\{B(x, r)\colon x \in A_1\}$ is maximal disjoint. Put

$$B_1 = \bigcup \{B(x, r)\colon x \in A_1\} \setminus C_0, \qquad C_1 = \big(F_1 \setminus (F_2 \cup C_0)\big) \cup (B_1 \setminus C_0),$$

and $f(x) = \frac{1}{2}$ for every $x \in A_1$, $f(x) = \frac{1}{3}$ for every $x \in C_1 \setminus A_1$.

At the third step we choose a subset $A_2 \subseteq F_2 \setminus (F_3 \cup C_1)$ such that the family $\{B(x, r)\colon x \in A_2\}$ is maximal disjoint. Put

$$B_2 = \bigcup \{B(x, r)\colon x \in A_2\} \setminus C_1, \qquad C_2 = \big(F_2 \setminus (F_3 \cup C_1)\big) \cup (B_2 \setminus C_1),$$

and $f(x) = \frac{1}{3}$ for every $x \in A_2$, $f(x) = \frac{1}{4}$ for every $x \in C_2 \setminus A_2$.

After ω steps, the function $f : X \to \mathbb{R}$ defined in such a way is slowly oscillating in direction of φ and f is oscillating on every subset $X \setminus F_{n+1}$, $n \in \omega$. By Theorem 1, $\varphi = \mathrm{so}(f)$. □

Let $\mathcal{B} = (X, P, B)$ be a ballean, Y be an unbounded subset of X. We say that a function $f : X \to \mathbb{R}$ is *slowly oscillating on* Y if, for any $\alpha \in P$ and $\varepsilon > 0$, there exists a bounded subset V of X such that

$$\operatorname{diam} f\big(B(y, \alpha)\big) < \varepsilon$$

for every $y \in Y \setminus V$, equivalently, f is slowly oscillating in direction of the filter on X with base $\{Y \setminus V\colon V \text{ is bounded}\}$.

The subsets Y, Z of X are called *asymptotically disjoint* if, for every $\alpha \in P$, there exists a bounded subset V of X such that

$$B(Y \setminus V, \alpha) \cap B(Z \setminus V, \alpha) = \emptyset.$$

Theorem 5. *Let (X, d) be a metric space, Y and Z be unbounded subsets of X. Assume that there exists $r > 0$ such that $|B(x, r)| > 1$ for every $x \in X$. Then the following statements are equivalent*:

(i) *Y, Z are asymptotically disjoint*;
(ii) *there exists a function $f : X \to \mathbb{R}$ such that f is slowly oscillating on Y and f is oscillating on Z.*

Proof. (i) $\Longrightarrow$ (ii) Fix some point $x_0 \in X$. Since Y, Z are asymptotically disjoint, we can choose an increasing sequence of natural numbers $(k_n)_{n\in\omega}$ such that

$$B\bigl(Y \setminus B(x_0, k_n), n\bigr) \cap Z = \emptyset$$

for every $n \in \omega$. For every $m \in \omega$, we put

$$F_m = \bigcup_{n\geqslant m} B\bigl(Y \setminus B(x_0, k_n), n - m\bigr),$$

and denote by φ the filter on X with base $\{F_m\colon m \in \omega\}$. By construction, φ is thick and $X \setminus Z \in \varphi$. By Theorem 4, there exists a function $f : X \to \mathbb{R}$ such that $\varphi = \mathrm{so}(f)$. By Theorem 1, f is slowly oscillating in direction of φ and f is oscillating on Z. By construction of φ, f is slowly oscillating on Y.

(ii) $\Longrightarrow$ (i). We pick $\varepsilon > 0$ and $k \in \omega$ such that $\operatorname{diam} f(B(z, k)) \geqslant \varepsilon$ for every $z \in Z$. Fix an arbitrary $m \in \omega$ and choose a bounded subset V of X such that $\operatorname{diam} f(B(y, m)) < \varepsilon$ for every $y \in Y \setminus V$. Then $Z \cap B(Y \setminus V, m) = \emptyset$. It follows that Y and Z are asymptotically disjoint. □

Question. Let $\mathcal{B} = (X, P, B)$ be a ballean and let $f : X \to \mathbb{R}$ be a function which is not very oscillating. Does there exist an unbounded subset Y of X such that f is slowly oscillating on Y? This is so if $\mathcal{B}$ is a metric ballean.

4. Application to βG

Let G be an infinite discrete group, βG be the Stone–Čech compactification of G, $G^* = \beta G \setminus G$. Using the universal property of the Stone–Čech compactification, the group multiplication on G can be extended to βG in such a way that, for every $r \in \beta G$, the right shift $x \mapsto xr$ is continuous, and for every $g \in G$, the left shift $x \mapsto gx$ is continuous. Formally, the product rq of the ultrafilters $r, q \in \beta G$ is defined by the rule: given any subset A of G,

$$A \in rq \iff \bigl\{g \in G\colon g^{-1}A \in q\bigr\} \in r.$$

For more information about the compact right topological semigroup βG and its combinatorial applications see [4].

For an infinite discrete group G with identity e, we denote by $\mathcal{F}_e(G)$ the family of all finite subsets of G containing e, and consider the ballean $\mathcal{B}_r(G) = (G, \mathcal{F}_e(G), B_r)$, where $B_r(x, F) = Fx$. A subset $V \subseteq G$ is bounded in $\mathcal{B}_r(G)$ if and only if V is finite, so $G^\# = G^*$. We note also that the ultrafilters $r, q \in G^*$ are parallel if and only if $r = xq$ for some $x \in G$.

Theorem 6. *Let G be a countable discrete group, φ be a filter on G with a countable base, $\bigcap \varphi = \emptyset$. Then the following statements are equivalent*

(i) *$\overline{\varphi}$ is a left ideal of βG*;
(ii) *there exists a function $f : G \to \mathbb{R}$ such that $\varphi = \mathrm{so}(f)$.*

Proof. (i) $\Longrightarrow$ (ii). We show that φ is thick. Fix an arbitrary $F \in \varphi$, $x \in G$. For every $q \in \overline{\varphi}$, we have $xq \in \overline{\varphi}$, so there is $A_q \in q$ such that $xA_q \subseteq F$. We consider an open covering $\{\overline{A_q}\colon q \in \overline{\varphi}\}$ of $\overline{\varphi}$ and choose some open subcovering $\overline{A_{q_1}}, \dots, \overline{A_{q_n}}$. Then

$$A_{q_1} \cup \dots \cup A_{q_n} \in \varphi, \qquad x(A_{q_1} \cup \dots \cup A_{q_n}) \subseteq F.$$

Since G is countable, $\mathcal{B}_r(G)$ is metrizable [5]. If $H \in F_l(G)$, then $|B_r(x, H)| = |H|$, so we can apply Theorem 4.

(ii) $\Longrightarrow$ (i). Let $q \in \overline{\mathrm{so}(f)}$. By Theorem 2, $\overline{\mathrm{so}(f)}$ contains all ultrafilters $r \in G^*$ such that $r \parallel q$. It means that $xq \in \overline{\mathrm{so}(f)}$ for every $x \in G$. Since βG is right topological semigroup and G is dense in βG, we have $pq \in \overline{\mathrm{so}(f)}$ for every $p \in \beta G$. Hence, $\mathrm{so}(f)$ is a left ideal. □

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