Bipartite almost distance-hereditary graphs

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Abstract

The notion of distance-heredity in graphs has been extended to construct the class of almost distance-hereditary graphs (an increase of the distance by one unit is allowed by induced subgraphs). These graphs have been characterized in terms of forbidden induced subgraphs [M. Aïder, Almost distance-hereditary graphs, Discrete Math. 242 (1–3) (2002) 1–16]. Since the distance in bipartite graphs cannot increase exactly by one unit, we have to adapt this notion to the bipartite case.

In this paper, we define the class of bipartite almost distance-hereditary graphs (an increase of the distance by two is allowed by induced subgraphs) and obtain a characterization in terms of forbidden induced subgraphs.

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1. Introduction

We only consider finite undirected connected graphs with no loops or multiple edges $G = (V, E)$, where $V$ is the vertex set and $E$ is the edge set. The distance $d_G(u, v)$ (or simply $d(u, v)$ if no ambiguity arises) between two vertices $u$ and $v$ of a connected graph $G$ is the length of a shortest $\{u, v\}$-path of $G$. For convenience, the $(x, y)$ section of an induced path $P$ (respectively, an induced cycle $C$) is denoted by $P(x, y)$ (respectively, $C(x, y)$). Sometimes, if $P$ is an induced path and $u$ and $v$ are vertices in $P$, the notation $|P(u, v)|$ will be used to designate the number of edges in $P(u, v)$ (that is, $|P(u, v)| = d_P(u, v)$). A chord of a cycle is an edge connecting two non-consecutive vertices of the cycle.

A graph $G$ is distance hereditary if each connected induced subgraph $H$ of $G$ has the property that $d_H(u, v) = d_G(u, v)$ for every pair of vertices $u$ and $v$ in $H$. Distance hereditary graphs were introduced by Howorka [6] and characterized in terms of forbidden subgraphs in [2,4–6]. In the bipartite case, we have the following characterization.

Theorem 1 (Bandelt and Mulder [2]). For a connected graph $G$, the following assertions are equivalent.

(i) $G$ is bipartite and distance-hereditary.
(ii) $G$ does contain neither triangles (cycles of length 3), nor induced cycles of length at least 5, nor $C_6^*$, the graph in Fig. 1.
The notion of distance heredity has been extended to define the class of \((k, +)-distance\) hereditary graphs where an increase of the distance by at most a constant integer \(k\) is permissible in any connected induced subgraph \([1,3]\) and characterizations in terms of forbidden induced subgraphs are obtained in case \(k = 1\) (the case \(k = 1\) corresponds to the class of almost distance-hereditary graphs defined in \([1]\)).

**Theorem 2** (Aïder \([1]\)). A graph \(G\) is almost distance-hereditary if and only if \(G\) neither contains a \(C_n\)-configuration (see Fig. 2), for \(n \geq 6\), nor a \(2C_5\)-configuration (see Fig. 3), as induced subgraphs.

In Fig. 2 (and all our graphs), a solid line is used for an edge, a dashed line for a possible edge which can be included in the graph and a dotted line for an induced path, possibly trivial (if the ends of the path coincide).

Observe that if a graph \(G\) is bipartite almost distance-hereditary then \(G\) is distance-hereditary. Thus in the bipartite case, the previous notion of almost distance-heredity presents no interest. Moreover, any variation of the distance between every two vertices of a bipartite graph is even and is greater than or equal to two. Therefore, we have to introduce and use the following definition.
A bipartite graph $G = (V, E)$ is bipartite almost distance-hereditary if for all connected induced subgraphs $H = (Y, F)$ of $G$, we have

$$ \forall u, v \in Y, \quad d_H(u, v) \leq d_G(u, v) + 2. $$

It is easy to see that equivalently, a bipartite graph $G = (V, E)$ is bipartite almost distance-hereditary if and only if for every pair of non-adjacent vertices $u$ and $v$, the length of each induced $\{u, v\}$-path either is equal to $d_G(u, v)$ or to $d_G(u, v) + 2$.

In the following section, containing our main results, we show that a bipartite graph is bipartite almost distance-hereditary if and only if it contains neither a $2C_6$-biconfiguration nor a $C_{2q}$-configuration ($q \geq 4$) as induced subgraphs and give an other formulation of this characterization.

2. Characterization of bipartite almost distance hereditary graphs

Let us define a $C_{2q}$-biconfiguration to be a bipartite (even) cycle of length $2q$ without chords or with chords which are all incident to a single vertex (see Fig. 4). It is not difficult to see that $C_{2q}$-biconfigurations are not bipartite almost distance hereditary if $2q \geq 8$, because each of these graphs can be seen as the union of two disjoint induced $\{u, v\}$-paths, respectively, of length two and $2q - 2$.

Thus, a necessary condition for a bipartite graph $G$ to be bipartite almost distance hereditary is that every cycle on eight or more vertices has (even) chords which are not all incident to a single vertex.

Now, let us define a $2C_6$-biconfiguration of type (a) to be a graph obtained by combining two $C_6$-biconfigurations by connecting by an induced path two vertices which are not endpoints of chords of a cycle (Fig. 5(a)) (the induced path can be empty, and the related vertices are then identified) and a $2C_6$-biconfiguration of type (b) to be the graph isomorphic to Fig. 5(b).

One can easily see that each of these $2C_6$-biconfigurations satisfies the necessary condition above (they do not contain a cycle on eight or more vertices with all (even) chords incident to a single vertex) but each of them contains two induced $\{u, v\}$-paths whose lengths differ by 4, and, consequently, cannot be contained in a bipartite almost distance hereditary graph.

Let us observe that the only cycles allowed in a bipartite almost distance hereditary graph but not in a bipartite distance hereditary graph are those of length 6.

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**Fig. 4.** $C_{2q}$-biconfiguration.

**Fig. 5.** $2C_6$-biconfigurations.
Moreover, the $C_6$-biconfigurations ($C_6$ and $C_6^*$) can be contained in a bipartite almost distance hereditary graph and constitute the smallest examples of bipartite almost distance hereditary graphs that are not distance hereditary.

However, a $2C_6$-biconfiguration of type (b) is not bipartite almost distance hereditary. In fact, we have the following:

**Lemma 3.** A necessary condition for a graph $G$ to be bipartite almost distance hereditary is that it neither contains $2C_6$-configurations nor $C_{2q}$-biconfigurations as induced subgraphs, for $q \geq 4$.

**Proof.** If a graph $G$ contains a $2C_6$-configuration (Fig. 5), then if we delete the vertices $x$ and $y$, the distance between the vertices $u$ and $v$ increases by 4. An increase by at least 4 of the distance between two vertices at distance 2 in a cycle on eight or more vertices is also generated if a $G$ contains such a cycle.

**Theorem 4.** Let $G$ be a connected bipartite graph. Then $G$ is bipartite almost distance hereditary if and only if $G$ contains neither a $2C_6$-biconfiguration nor a $C_{2q}$-configuration as induced subgraphs, for $q \geq 4$.

**Proof.** For the ‘only if’, we have just to use Lemma 1. For the ‘if part’, let $G$ be a connected bipartite graph and assume that $G$ is not a bipartite almost distance hereditary graph. Therefore, $G$ contains a pair $\{u, v\}$ of vertices linked by an induced path $Q$ of length at least $d(u, v) + 4$. Note that since the vertices $u$ and $v$ are not adjacent, $d(u, v) \geq 2$.

Choose such a pair of vertices $u$ and $v$ such that $d(u, v)$ is as small as possible and then consider all shortest $\{u, v\}$-paths and $\{u, v\}$-paths of length at least $d(u, v) + 4$. Let $P$ be a shortest $\{u, v\}$-path and $Q$ be an induced $\{u, v\}$-path of length at least $d(u, v) + 4$ such that the total number of vertices in these two paths is as small as possible. Henceforth, it will be referred to this choice of the vertices $u$ and $v$ and the paths $P$ and $Q$ as the minimality condition.

Thus, either $u$ and $v$ are contained in a cycle, induced by $P \cup Q$, say $C(u, v)$, of length at least $2d(u, v) + 4$ or the paths $P$ and $Q$ contain two common vertices $x$ and $y$ (not necessarily distinct), such that

$$d_Q(u, x) \geq d_P(u, x) + 2 \quad \text{and} \quad d_Q(y, v) \geq d_P(y, v) + 2$$

(1)

By the minimality condition above, we then have

$$d_Q(u, x) = d_P(u, x) + 2 \quad \text{and} \quad d_Q(y, v) = d_P(y, v) + 2.$$  

(2)

Two cases are possible according to whether the vertices $u$ and $v$ are contained in a cycle $C(u, v)$ or not.

**Case 1:** $u$ and $v$ are contained in a cycle $C(u, v)$ of length at least $2d(u, v) + 4$.

Since $d(u, v) \geq 2$, therefore the length of the cycle $C(u, v)$ is at least 8, and must contain non adjacent chords (otherwise, a $C_{2q}$-configuration, with $q \geq 4$ will be induced in $G$). Let $x$ (respectively, $y$) be the vertex of $P \setminus \{u, v\}$ closest to $u$ (respectively, $v$) having neighbors in $Q$, and let $x'$ (respectively, $y'$) the neighbor of $x$ on $Q$ closest to $v$ (respectively, $u$) (Fig. 6).

If $d_P(u, x) + d_Q(u, x') + 1 \geq 8$, then the length of the cycle induced by $P(u, x) \cup Q(u, x') \cup \{u, v\}$ is greater than or equal to 8 and all its chords are adjacent with $x$, a contradiction with the choice of $x'$. Therefore, $d_Q(u, x') < 7 - d_P(u, x)$. If $d_Q(u, x') \leq d_P(u, x) + 2$, then one can substitute the vertex $x$ by $x'$ without violating the distance conditions, contradicting the minimality condition on $d(u, v)$.

Hence $d_Q(u, x') > d_P(u, x) + 2$, yielding to $d_P(u, x) = 1$ and $d_Q(u, x') = 4$ or 5.

From the bipartition property of $G$, it follows that $d_Q(u, x') = 4$. By symmetry we get $d_P(v, y) = 1$ and $d_Q(v, y') = 4$.

First assume $d_Q(u, x') < d_Q(u, y')$. 

![Fig. 6.](image-url)
By our assumption, we have

\[ |Q(u, v)| = d_Q(u, x') + d_Q(x', y') + d_Q(y', v) = d_Q(x', y') + 8 \geq |P(u, v)| + 4 \]

and therefore

\[ d_Q(x', y') \geq d_P(x, y) - 2. \]

According to the minimality condition of \( d(u, v) \), we must have

\[ d_Q(u, y') + 1 \leq d_P(u, y) + 2, \]

yielding \( d_Q(x', y') = d_P(x, y) - 2 \).

If \( x' \) and \( y' \) are not adjacent, then together with the \( \{x', y'\} \)-paths \( P' = Q(x', y') \) and \( Q' = \{x', x\} \cup P(x, y) \cup \{y, y'\} \) these vertices fulfill our assumption, contradicting the minimality condition of \( d(u, v) \) together with the \( \{u, v\} \)-paths \( P \) and \( Q \).

Hence, \( x' \) and \( y' \) either coincide or are adjacent, implying \( d_P(x, y) = d_Q(x', y') + 2 \geq 2 \).

By removing the inner vertices of \( P(x, y) \) we obtain a 2\( C_6 \)-configuration of type (a) and we are done in this case.

Therefore, \( d_Q(u, x') > d_Q(u, y') \).

It follows that

\[ |Q(u, v)| = |Q(u, y')| + |Q(y', v)| = d_Q(u, y') + d_Q(y', v) < d_Q(u, x') + d_Q(y', v) = 4 + 4 = 8. \]  

(4)

On the other hand, by assumption, we have

\[ |Q(u, v)| \geq |P(u, v)| + 4 \geq 7. \]

Hence, \( |P(u, v)| = 3 \) and \( |Q(u, v)| = 7 \) and the configuration pointed out in Fig. 7 implying a 2\( C_6 \)-configuration of type (b) and we are also done in this case.

**Case 2:** a shortest \( \{u, v\} \)-path \( P \) contains two vertices \( x \) and \( y \) such that

\[ d_Q(u, x) \geq d_P(u, x) + 2 \quad \text{and} \quad d_Q(y, v) \geq d_P(y, v) + 2 \]

(5)

Let \( x \) (respectively, \( y \)) be the first (respectively, the last) vertex belonging both to \( P \) and to \( Q \) (and then satisfying the two inequalities above). Hence, we can assume that \( P(u, x) \cup Q(u, x) \) and \( P(y, v) \cup Q(y, v) \) induces two elementary cycles which we denote by \( C(u, x) \) and \( C(y, v) \), respectively, (Fig. 8).

If the length of both \( C(u, x) \) and \( C(y, v) \) is equal to 6, a 2\( C_6 \)-biconfiguration of type (a) is induced and we are done.

Let us consider the cycle \( C(u, x) \) (the same argument applies for \( C(v, y) \)), and let us assume that its length is greater than or equal to 8. \( C(u, x) \) must contain at least two chords, each one linking a vertex in \( P \) to a vertex in \( Q \). Moreover, these chords do not have a common endpoint in \( P \).

Let \( z \) and \( z' \) be the two vertices of \( P(u, x) \) having neighbors in \( Q(u, x) \) closest to \( x \) and let \( z' \) be the neighbor of \( z \) in \( Q(u, x) \) closest to \( x \) (Fig. 9).
Let us assume that $d_P(z, x) < d_Q(z', x) - 1$. Since $z$ and $z'$ are adjacent, the parity of the distances $d_P(z, x)$ and $d_Q(z', x)$ is not the same. It follows that $d_P(z, x) \leq d_Q(z', x) - 3$. Let $C$ be the cycle induced by $P(z, x) \cup \{z, z'\} \cup Q(z', x)$. Its length is greater than or equal to 8 (according to the fact that $d_P(z, x) \geq 2$) and all its chords are incident to $\beta$. This is a contradiction with the hypothesis that $G$ does not contain any such cycle.

If $d_P(z, x) = d_Q(z', x) - 1$, let us consider the two $\{z, v\}$-paths $P'$ and $Q'$ defined as follows:

$$P' = P(z, v) \quad \text{and} \quad Q' = \{z, z'\} \cup Q(z', x).$$

$P'$ and $Q'$ fulfill our assumption. Moreover, we have $|P' \cup Q'| < |P \cup Q|$, a contradiction to the hypothesis of minimality condition.

If $d_P(z, x) > d_Q(z', x) - 1$ (therefore $d_P(z, x) \geq d_Q(z', x) + 1$), then $d_Q(u, z') \geq d_P(u, z) + 3$ and put $P' = P(u, z) \cup \{z, z'\} \cup Q(z', x) \cup P(x, v)$. The paths $P'$ and $Q$ satisfy the hypotheses and $|P' \cup Q| < |P \cup Q|$, a contradiction with the minimality condition.

We can obtain another characterization of bipartite almost distance hereditary graphs solely in terms of forbidden induced subgraphs.

Let us define a minimal forbidden connected subgraph to be a bipartite connected graph, which is not bipartite almost distance hereditary, such that every connected induced subgraph is bipartite almost distance hereditary.

**Theorem 5.** The following graphs are minimal forbidden connected subgraphs and the only minimal forbidden connected subgraphs:

(a) chordless cycles of length greater than or equal to 8,
(b) $2C_6$-configurations,
(c) the graphs in Fig. 10.

**Proof.** Note that all these graphs are obviously minimal forbidden connected subgraphs. To achieve the proof, note that it is sufficient to prove that a bipartite graph $G$ contains none of the configurations in Fig. 10 as induced subgraphs only if $G$ does not contain chordless cycles on 8 or more vertices in which all the chords are incident to a same vertex.

Observe that for cycles on 8 vertices or more, there is nothing to prove, since such cycles either induce one of the configurations in Fig. 10, or all their chords are incident to a same vertex. An even cycle on 10 vertices or more, having at least one chord either contains a cycle of length 8, or one of the configurations in Fig. 10 is induced. □
3. Some open questions

Bandelt and Mulder [2] have completely characterized the class of bipartite distance-hereditary graphs (indeed, they obtained combinatorial, algorithmic and metric characterizations). For bipartite almost distance-hereditary graphs, we have just given a combinatorial characterization. It will be interesting to study algorithmic and metric aspects.

Note that the characterization in Theorem 4 can be easily adapted to the larger class of bipartite graphs $G = (V, E)$ such that for all connected induced subgraphs $H = (Y, F)$, we have

$$\forall u, v \in Y, \quad d_H(u, v) \leq d_G(u, v) + 2r \quad (\text{where } r \geq 2).$$

The forbidden configurations can be defined similarly.

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