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On the structure of the endomorphism ring of a certain local cohomology module

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ABSTRACT

Let $(R, \mathfrak{m})$ denote an n-dimensional Gorenstein ring. For an ideal $I \subset R$ of height c we are interested in the endomorphism ring $B = \operatorname{Hom}_R(H_I^c(R), H_I^c(R))$. It turns out that B is a commutative ring. In the case of $(R, \mathfrak{m})$ a regular local ring containing a field B is a Cohen–Macaulay ring. Its properties are related to the highest Lyubeznik number $l = \dim_k \operatorname{Ext}_R^d(k, H_I^c(R))$, $d = \dim R/I$. In particular $R \simeq B$ if and only if $l = 1$. Moreover, we show that the natural homomorphism $\operatorname{Ext}_R^d(k, H_I^c(R)) \to k$ is non-zero.

1. Introduction

Let $(R, \mathfrak{m}, k)$ denote a local Noetherian ring. For an ideal $I \subset R$ let $H_I^i(R)$, $i \in \mathbb{Z}$, denote the local cohomology modules of R with respect to I (see [3] for the definition). They carry several information about I and R. Their Bass numbers $\dim_k \operatorname{Ext}_R^j(k, H_I^i(R))$, $i, j \in \mathbb{Z}$, are in various directions important invariants (see for instance [14,15,12,13,20] and others). In the case of a regular local ring they are investigated by Lyubeznik (see [14]) known also as Lyubeznik invariants.

On the other hand, in recent research there are sufficient conditions when the endomorphism ring of $H_I^c(R)$, $c = \operatorname{height} I$, is isomorphic to R (see for instance [8] and also [19]). Note that it is not clear whether it is a commutative ring in general. The endomorphism ring $B := \operatorname{Hom}_R(H_I^c(R), H_I^c(R))$ is the main subject of our investigations here, in particular when $(R, \mathfrak{m})$ is a Gorenstein ring. It carries a lot of interesting properties. For an ideal $I \subset R$ let I_d denote the intersection of the highest dimensional primary components of I.

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Theorem 1.1. *Let $(R, \mathfrak{m})$ denote an n-dimensional Gorenstein ring. Let $I \subset R$ be an ideal with $c = \operatorname{height} I$ and $d = \dim R/I$. Then:*

(a) *The endomorphism ring $B := \operatorname{Hom}_R(H_I^c(R), H_I^c(R))$ is commutative. There is a natural isomorphism*

$$B = \operatorname{Hom}_R\bigl(H_I^c(R), H_I^c(R)\bigr) \simeq \operatorname{Ext}_R^c\bigl(H_I^c(R), R\bigr).$$

Moreover B is $\mathfrak{m}$-adically complete provided R is a complete local ring.

Suppose that $(R, \mathfrak{m})$ is a complete regular local ring containing a field. Assume that $d > 0$. Then:

(b) *The ring extension $R \subset B$ is module-finite and B is a Noetherian ring.*
(c) *B is a free R-module of rank $\dim_k \operatorname{Ext}_R^c(k, H_I^c(R))$ and therefore B is a Cohen–Macaulay module.*
(d) *B is a local Noetherian ring if and only if $V(I_d)$ is connected in codimension one.*
(e) *The natural homomorphism $R \to \operatorname{Hom}_R(H_I^c(R), H_I^c(R)) = B$ is an isomorphism if and only if the completion of the strict Henselization of R/I_d is connected in codimension one.*

By the aid of the result shown by Huneke and Lyubeznik (see [11, Theorem 2.9]) about the second Vanishing Theorem of Hartshorne (see [6, Theorem 7.5]) we get as a consequence of Theorem 1.1:

Corollary 1.2. *Let $(R, \mathfrak{m})$ be a complete n-dimensional regular local ring containing a field. Let $I \subset R$ denote an ideal and $d = \dim R/I \geqslant 2$. Then the following conditions are equivalent:*

(i) *The natural homomorphism $R \to \operatorname{Hom}_R(H_I^c(R), H_I^c(R)) = B$ is an isomorphism.*
(ii) *$H_I^i(R) = 0$ for all $i \geqslant n - 1$.*

In the general case of a local Gorenstein ring $(R, \mathfrak{m})$ it is not true (see Example 3.6) that B is a module finite extension, while – in this example – it is a Noetherian ring. We conjecture that B is always a Noetherian ring (see Example 3.6).

Another feature of our considerations here is the natural map

$$\phi : \operatorname{Ext}_R^d\bigl(k, H_I^c(R)\bigr) \to k, \quad k = R/\mathfrak{m},$$

where $I \subset R$ is an ideal of $\operatorname{height} I = c$ and $d = \dim R/I$. The homomorphism ϕ occurs as an edge homomorphism of a certain spectral sequence resp. as a homomorphism in the construction of the truncation complex (see Definition 5.2 and Lemma 6.1). In [8, Conjecture 2.7] Hellus and the author conjectured that it is always non-zero. Here we show that ϕ is related to the natural homomorphism $\lambda : \operatorname{Ext}_R^d(k, H_I^c(R)) \to H_{\mathfrak{m}}^d(H_I^c(R))$. As an application of our techniques there is the following statement:

Theorem 1.3. *Let $(R, \mathfrak{m})$ be an n-dimensional Gorenstein ring. Let $I \subset R$ denote an ideal with $c = \operatorname{height} I$ and $d = \dim R/I$. Then:*

(a) *If $\phi : \operatorname{Ext}_R^d(k, H_I^c(R)) \to k$ is non-zero, then $\lambda : \operatorname{Ext}_R^d(k, H_I^c(R)) \to H_{\mathfrak{m}}^d(H_I^c(R))$ is non-zero.*

Suppose that $(R, \mathfrak{m})$ is a complete regular local ring containing a field. Then:

(b) *The homomorphism $\phi : \operatorname{Ext}_R^d(k, H_I^c(R)) \to k$ is non-zero.*

In a certain sense (see Remark 6.3) one might consider the homomorphism λ as the limit version of the homomorphism $\operatorname{Ext}_R^d(k, K(R/I^\alpha)) \to H_{\mathfrak{m}}^d(K(R/I^\alpha))$ as it was studied by Hochster (see [9, Section 4]). Here $K(R/I^\alpha)$ denotes the canonical module of R/I^α, $\alpha \in \mathbb{N}$. For the proof of Theorem 1.3 we refer to Section 6 and Theorem 6.2.

As another application we prove a slight sharpening of Blickle's result (cf. [2, Theorem 1.1]) about a certain duality for the Lyubeznik numbers. In Section 2 of the paper there are preliminaries and auxiliary results needed in the sequel. In Section 3 we investigate the endomorphism ring of $H_I^c(R), c = \operatorname{height} I$, in the case of $(R, \mathfrak{m})$ a local Gorenstein ring and in Section 4 in the case of R a regular ring containing a field. To this end me make use of the results of Huneke and Sharp (see [12]) in the case of prime characteristic $p > 0$ and of Lyubeznik (see [14]) in characteristic zero. In Section 5 there are some additional results about the so-called Lyubeznik numbers. In Section 6 we study the natural homomorphism $\phi : \operatorname{Ext}_R^d(k, H_I^c(R)) \to k$. In Section 7 we discuss some additional examples.

With our notation we follow Matsumura's textbook [16]. In the context of the paper Gorenstein ring means always a local Gorenstein ring. A local ring $(A, \mathfrak{m})$ is called complete whenever it is $\mathfrak{m}$-adically complete.

2. Preliminaries and auxiliary results

In this section we will summarize a few auxiliary results. They are needed for further constructions related to local cohomology modules and their asymptotic behavior. Here let R denote a commutative Noetherian ring.

Let $\{M_\alpha\}_{\alpha \in \mathbb{N}}$ denote a family of R-modules with homomorphisms $\psi_{\alpha+1} : M_{\alpha+1} \to M_\alpha$. The inverse limit $\varprojlim M_\alpha$ is given by

$$\varprojlim M_\alpha \simeq \bigl\{(m_\alpha)_{\alpha \in \mathbb{N}}\colon\, m_\alpha \in M_\alpha,\, \psi_{\alpha+1}(m_{\alpha+1}) = m_\alpha \text{ for all } \alpha \in \mathbb{N}\bigr\}.$$

For an application we have to know whether the inverse limit of an inverse system of rings is a quasi-local ring. Here a commutative ring is called quasi-local provided there is a unique maximal ideal.

Lemma 2.1. *Let $\{(A_\alpha, \mathfrak{n}_\alpha)\}_{\alpha \in \mathbb{N}}$ denote an inverse system of local rings such that*

$$\phi_{\alpha+1} : (A_{\alpha+1}, \mathfrak{n}_{\alpha+1}) \to (A_\alpha, \mathfrak{n}_\alpha), \quad \alpha \in \mathbb{N},$$

is a local homomorphism. Then

$$A := \varprojlim (A_\alpha, \mathfrak{n}_\alpha) \simeq \bigl\{(b_\alpha)_{\alpha \in \mathbb{N}}\colon\, b_\alpha \in A_\alpha, \phi_{\alpha+1}(b_{\alpha+1}) = b_\alpha \textit{ for all } \alpha \in \mathbb{N}\bigr\}$$

is a quasi-local ring.

Proof. It is easily seen that A admits the structure of a commutative ring with identity element $(1)_{\alpha \in \mathbb{N}} \in A$. In order to show the claim it will be enough to show that the set of non-units forms an ideal.

Let $(b_\alpha)_{\alpha \in \mathbb{N}} \in A$ denote a unit. By definition there exists an element $(a_\alpha)_{\alpha \in \mathbb{N}} \in A$ such that

$$(a_\alpha b_\alpha)_{\alpha \in \mathbb{N}} = (1)_{\alpha \in \mathbb{N}}.$$

This means $a_\alpha b_\alpha = 1$ for all $i \in \mathbb{N}$, so that b_α is a unit in $(A_\alpha, \mathfrak{n}_\alpha)$ for all $i \in \mathbb{N}$. On the other hand let $(b_\alpha)_{\alpha \in \mathbb{N}} \in A$ denote an element such that for all $\alpha \in \mathbb{N}$ the element $b_\alpha \in A_\alpha$ is a unit. Then there is a sequence $a_\alpha \in A_\alpha, \alpha \in \mathbb{N}$, of elements such that $a_\alpha b_\alpha = 1$. We claim that $(a_\alpha)_{\alpha \in \mathbb{N}} \in A$. To this end we have to show that $\phi_{\alpha+1}(a_{\alpha+1}) = a_\alpha$ for all $\alpha \in \mathbb{N}$. Because $\phi_{\alpha+1}$ is a homomorphism of rings

$$1 = \phi_{\alpha+1}(a_{\alpha+1})\phi_{\alpha+1}(b_{\alpha+1}) = \phi_{\alpha+1}(a_{\alpha+1}) b_\alpha \quad \text{and} \quad 1 = a_\alpha b_\alpha,$$

so that $\phi_{\alpha+1}(a_{\alpha+1}) = a_\alpha$ because b_α is a unit in A_α.

Now let $(b_\alpha)_{\alpha\in\mathbb{N}} \in A$ be an element such that $b_\beta \in A_\beta$ is not a unit for some $\beta \in \mathbb{N}$. Then $b_\beta \in \mathfrak{n}_\beta$ and $b_\alpha \in \mathfrak{n}_\alpha$ for all $\alpha \in \mathbb{N}$ as easily seen since ϕ_α is a local homomorphism of local rings for all $\beta \in \mathbb{N}$. Therefore

$$A \setminus A^\star = \{(b_\alpha)_{\alpha\in\mathbb{N}} \in A\colon\ b_\alpha \in \mathfrak{n}_\alpha \text{ for all } \alpha \in \mathbb{N}\},$$

where $A^\star$ denotes the set of units of A. This is an ideal of A. □

Now let $(R, \mathfrak{m})$ denote a Gorenstein ring with $n = \dim R$. Let M be a finitely generated R-module and $d = \dim M$. Define

$$K^i(M) := \operatorname{Ext}_R^{n-i}(M, R), \quad i \in \mathbb{Z},$$

the i-th module of deficiency. For $i = d$ let

$$K(M) = K^d(M) = \operatorname{Ext}_R^c(M, R), \quad c = n - d,$$

denote the canonical module of M. Let $H^i_{\mathfrak{m}}(\cdot)$ denote the i-th local cohomology functor with support in $\mathfrak{m}$. By the Local Duality Theorem (see e.g. [3] or [17, Theorem 1.8]) there are the functorial isomorphisms

$$H^i_{\mathfrak{m}}(M) \simeq \operatorname{Hom}_R(K^i(M), E), \quad i \in \mathbb{Z},$$

where $E = E_R(R/\mathfrak{m})$ denotes the injective hull of $k = R/\mathfrak{m}$, the residue field.

For some basic properties about the modules of deficiency we refer to [3] and [18, Lemma 1.9]. In particular, $\operatorname{Ann}_R K(M) = (\operatorname{Ann}_R M)_d$, the intersection of all the $\mathfrak{p}$-primary components of $\operatorname{Ann}_R M$ such that $\dim R/\mathfrak{p} = d$. Moreover $K(M)$ satisfies Serre's condition S_2.

For an R-module M we will consider $K(K(M))$. To this end we investigate the following construction which is slightly more general than needed.

Proposition 2.2. *Let $(R, \mathfrak{m})$ denote a Gorenstein ring. Then there is a canonical isomorphism*

$$K(K(M) \otimes_R M) \simeq \operatorname{Hom}_R(K(M), K(M)) \simeq \operatorname{Hom}_R(M, K(K(M)))$$

for a finitely generated R-modules M with $d = \dim M$.

Proof. Let $R \to E^\cdot$ denote a minimal injective resolution of R and $c = \dim R - \dim M$. Let $X = \ker(E^c \to E^{c+1})$ in that resolution. Because $\operatorname{Hom}_R(Y, E^i) = 0$ for all $i < c$ for each R-module Y with $\operatorname{Supp}_R Y \subseteq \operatorname{Supp}_R X$, it follows that we can make an identification $K(Y) = \operatorname{Hom}_R(Y, X)$ for any finitely generated R-module Y with $\operatorname{Supp}_R Y \subseteq \operatorname{Supp}_R X$. This yields

$$\operatorname{Hom}_R(K(M), K(M)) = \operatorname{Hom}_R(K(M), \operatorname{Hom}_R(M, X)).$$

By the adjunction formula and the definition it implies the statements of the proposition. □

Of a particular interest of Proposition 2.2 is the case of $R/I = A$ for an ideal $I \subset R$.

Proposition 2.3. *Let I denote an ideal in the Gorenstein ring R. For $A = R/I$ there are the following results*:

(a) $K(K(A)) \simeq \operatorname{Ext}_R^c(\operatorname{Ext}_R^c(A, R), R)$, $c = \dim R - \dim A$.
(b) *$K(K(A))$ is isomorphic to the endomorphism ring* $\operatorname{Hom}_R(K(A), K(A))$ *which is commutative.*

(c) *There is an injection* $A/0_d \hookrightarrow \operatorname{Hom}_R(K(A), K(A))$.
(d) $\operatorname{Hom}_R(K(A), K(A))$ *is the* S_2*-ification of* $A/0_d$.

For the proof we refer to [1,10,18] and the previous Proposition 2.2. Next we recall a definition introduced by Hochster and Huneke (see [10, (3.4)]).

Definition 2.4. Let R denote a commutative Noetherian ring with finite dimension. We denote by $\mathbb{G}_R$ the undirected graph whose vertices are primes $\mathfrak{p}$ of R such that $\dim R = \dim R/\mathfrak{p}$, and two distinct vertices $\mathfrak{p}$, $\mathfrak{q}$ are joined by an edge if and only if $(\mathfrak{p}, \mathfrak{q})$ is an ideal of height one.

In view of Definition 2.4 observe the following: For a commutative ring R with $\dim R < \infty$ the variety $V(0_d)$ is connected in codimension one if and only if $\mathbb{G}_R$ is connected. For the notion of connectedness in codimension one we refer to Hartshorne's paper [4, Proposition 1.1 and Definition].

As an application here we describe when the endomorphism ring $K(K(A)) \simeq \operatorname{Hom}_R(K(A), K(A))$ of the canonical module $K(A)$ is a local ring, see [10, 3.6].

Lemma 2.5. *With the notation of Proposition* 2.3 *and assuming that* R *is complete the following conditions are equivalent*:

(i) $K(A)$ *is indecomposable.*
(ii) $V(0_d)$ *is connected in codimension one.*
(iii) $K(K(A))$ *is a local ring.*

As an additional tool on homological algebra we need a result on the behavior of inverse limits and Ext-modules. It seems to the author that this is not well known. Here R denotes an arbitrary commutative ring. Moreover $\varprojlim^1$ denotes the right derived functor of $\varprojlim$.

Lemma 2.6. *Let* $\{M_\alpha\}_{\alpha \in \mathbb{N}}$ *be a direct system of* R*-modules and* $M = \varinjlim M_\alpha$. *Let* N *denote an arbitrary* R*-module. Then there is a short exact sequence*

$$0 \to \varprojlim{}^1 \operatorname{Ext}_R^{i-1}(M_\alpha, N) \to \operatorname{Ext}_R^i(M, N) \to \varprojlim \operatorname{Ext}_R^i(M_\alpha, N) \to 0$$

for all $i \in \mathbb{Z}$. *In particular* $\operatorname{Hom}_R(M, N) \simeq \varprojlim \operatorname{Hom}_R(M_\alpha, N)$.

Proof. By the definition of the direct limit there is a short exact sequence of R-modules $0 \to \bigoplus M_\alpha \to \bigoplus M_\alpha \to M \to 0$. It induces a long exact cohomology sequence

$$\cdots \to \prod \operatorname{Ext}_R^{i-1}(M_\alpha, N) \xrightarrow{f} \prod \operatorname{Ext}_R^{i-1}(M_\alpha, N) \to \operatorname{Ext}_R^i(M, N)$$
$$\to \prod \operatorname{Ext}_R^i(M_\alpha, N) \xrightarrow{g} \operatorname{Ext}_R^i(M_\alpha, N) \to \cdots.$$

To this end recall that Ext transforms direct sums into direct products in the first variable (see [21]). Now it is known (see [21]) that

$$\operatorname{coker} f \simeq \varprojlim{}^1 \operatorname{Ext}_R^{i-1}(M_\alpha, N) \quad \text{and} \quad \ker g \simeq \varprojlim \operatorname{Ext}_R^i(M_\alpha, N),$$

which proves the claim. □

3. On a formal ring extension

For an arbitrary R-module there is the natural map $R \to \operatorname{Hom}_R(M, M)$. In general it is neither injective nor surjective. Let $(R, \mathfrak{m})$ again denote an n-dimensional Gorenstein ring.

Definition 3.1. Let $I \subset R$ denote an ideal of height c. Then we define

$$B := \operatorname{Hom}_R\big(H_I^c(R), H_I^c(R)\big),$$

the endomorphism ring of the local cohomology module $H_I^c(R)$ (see also the Introduction).

The structure of B as well as the properties of the natural homomorphism $R \to B$ are the main subject of our investigations. Moreover it is well known (see e.g. [5]) that there is a natural homomorphism

$$M \to \operatorname{Ext}_R^c\big(\operatorname{Ext}_R^c(M, R), R\big), \quad c = n - \dim M,$$

for any finitely generated R-module M. Let $I \subset R$ denote an ideal of R. So there is a natural homomorphism

$$R/I^\alpha \to \operatorname{Ext}_R^c\big(\operatorname{Ext}_R^c\big(R/I^\alpha, R\big), R\big), \quad c = n - d.$$

It is well known (see e.g. [1]) that the kernel coincides with the ideal $(I^\alpha)_d$, that is the intersection of all primary components of I^α whose dimension is $d = \dim R/I$.

Now let $R/I^{\alpha+1} \to R/I^\alpha$, $\alpha \in \mathbb{N}$, denote the natural homomorphism. Then there is a commutative diagram

$$\begin{array}{ccc} R/I^{\alpha+1} & \longrightarrow & \operatorname{Ext}_R^c(\operatorname{Ext}_R^c(R/I^{\alpha+1}, R), R) \\ \downarrow & & \downarrow \\ R/I^\alpha & \longrightarrow & \operatorname{Ext}_R^c(\operatorname{Ext}_R^c(R/I^\alpha, R), R). \end{array}$$

In the sequel we investigate the limits of the inverse systems in this commutative diagram. That is we investigate the homomorphism

$$\phi : \hat{R}^I \to \varprojlim \operatorname{Ext}_R^c\big(\operatorname{Ext}_R^c\big(R/I^\alpha, R\big), R\big),$$

where $\hat{R}^I$ denotes the I-adic completion of R.

Theorem 3.2. *Let $(R, \mathfrak{m})$ denote an n-dimensional Gorenstein ring. Let $I \subset R$ be an ideal with $c = \operatorname{height} I$ and $d = \dim R/I$.*

(a) *The endomorphism ring B (see Definition 3.1) admits the structure of a commutative ring.*
(b) *There is a natural isomorphism $B \simeq \varprojlim \operatorname{Ext}_R^c(\operatorname{Ext}_R^c(R/I^\alpha, R), R)$, with the inverse maps as defined above.*
(c) *The composition of the natural homomorphism $R \to \hat{R}^I$ with $\phi : \hat{R}^I \to B$ induces a non-zero homomorphism $k \otimes_R R \to k \otimes_R B$. In particular $B \neq 0$.*
(d) $\ker \phi = \bigcap_{\alpha \in \mathbb{N}} (I^\alpha)_d = 0_S$, *where $S = \bigcap_{\mathfrak{p} \in \operatorname{Assh} R/I} R \setminus \mathfrak{p}$ and* Assh *denotes the set of associated prime ideals of R/I that have dimension equal to* $\dim R/I$.

Proof. Let $\alpha \in \mathbb{N}$, the set of positive integers. Let $R \to E^{\cdot}$ denote a minimal injective resolution of R. Let $X = \ker(E^c \to E^{c+1})$ in that resolution. As at the beginning of the proof of Proposition 2.2 we can make identifications

$$K\big(R/I^\alpha\big) = \operatorname{Ext}_R^c\big(R/I^\alpha, R\big) = 0 :_X I^\alpha \quad \text{and} \quad H_I^c(R) = H_I^0(X).$$

Now put $B_\alpha = \operatorname{Hom}_R(K(R/I^\alpha), K(R/I^\alpha))$. Because $K(R/I^\alpha)$ can be viewed as an R-submodule of $K(R/I^{\alpha+1})$ the embedding $K(R/I^\alpha) \subseteq K(R/I^{\alpha+1})$ induces – as easily seen – equalities

$$B_\alpha = \operatorname{Hom}_R\big(K\big(R/I^\alpha\big), K\big(R/I^{\alpha+1}\big)\big) \quad \text{and} \quad B_\alpha = \operatorname{Hom}_R\big(K\big(R/I^\alpha\big), H_I^c(R)\big).$$

Therefore, the inclusion map $K(R/I^\alpha) \subseteq K(R/I^{\alpha+1})$ induces the restriction homomorphism

$$B_{\alpha+1} \to \operatorname{Hom}_R\big(K\big(R/I^\alpha\big), K\big(R/I^{\alpha+1}\big)\big) = B_\alpha,$$

which maps an endomorphism of $K(R/I^{\alpha+1})$ to its restriction to $K(R/I^\alpha)$. It follows that the family $\{B_\alpha\}_{\alpha\in\mathbb{N}}$ forms an inverse system of commutative semi-local Noetherian rings (see e.g. [1]). Because $H_I^c(R) = \bigcup_{\alpha\in\mathbb{N}} K(R/I^\alpha)$, the inverse limit can be naturally identified with $B = \operatorname{Hom}_R(H_I^c(R), H_I^c(R))$ in such a way that the resulting natural projection map

$$\operatorname{Hom}_R\big(H_I^c(R), H_I^c(R)\big) \to \operatorname{Hom}_R\big(K\big(R/I^\alpha\big), K\big(R/I^\alpha\big)\big)$$

is just given by the restriction of endomorphisms. Thus B is – as an inverse limit of commutative rings – a commutative ring.

Because $\operatorname{Supp}_R K(R/I^\alpha) \subseteq V(I)$, we can identify $B_\alpha = \operatorname{Hom}_R(K(R/I^\alpha), X)$, with

$$\operatorname{Ext}_R^c\big(K\big(R/I^\alpha\big), R\big) = \operatorname{Ext}_R^c\big(\operatorname{Ext}_R^c\big(R/I^\alpha, R\big), R\big).$$

These identifications are such that the natural homomorphisms

$$\operatorname{Ext}_R^c\big(\operatorname{Ext}_R^c\big(R/I^{\alpha+1}, R\big), R\big) \to \operatorname{Ext}_R^c\big(\operatorname{Ext}_R^c\big(R/I^\alpha, R\big), R\big)$$

induced by the natural epimorphism $R/I^{\alpha+1} \to R/I^\alpha$ corresponds to the restriction homomorphism $B_{\alpha+1} \to B_\alpha$. This finishes the proof (b).

For the proof of (c) notice that the homomorphisms $R \to \hat{R}^I$ as well as $\phi : \hat{R}^I \to B$ are ring homomorphisms. That is, they respect the identity. Therefore, the residue class $1 + \mathfrak{m}$ does not map to zero.

For the proof of (d) recall that the kernel of $R/I^\alpha \to \operatorname{Hom}_R(K(R/I^\alpha), K(R/I^\alpha))$ is equal to $(I^\alpha)_d$ (see Proposition 2.3). Then $\ker\phi = \bigcap_{\alpha\in\mathbb{N}} (I^\alpha)_d$, as follows by elementary properties of the inverse limit. Moreover $\ker\phi = 0_S$ by the Krull Intersection Theorem. □

Now let us relate the structure of ring B (isomorphic to $\varprojlim \operatorname{Ext}_R^c(\operatorname{Ext}_R^c(R/I^\alpha, R), R)$) to the local cohomology of R with respect to I. Surprisingly the Matlis dual of $H_{\mathfrak{m}}^d(H_I^c(R))$ admits the structure of a commutative ring. Here let $\hat{R}$ denote the ($\mathfrak{m}$-adic completion) of R.

Theorem 3.3. *Let $(R, \mathfrak{m}, k)$ denote an n-dimensional Gorenstein ring. Let $I \subset R$ be an ideal with $d = \dim R/I$ and $c = \operatorname{height} I$.*

(a) *There is a natural isomorphism*

$$\varprojlim \operatorname{Ext}^c_{\hat{R}}\big(\operatorname{Ext}^c_{\hat{R}}\big(\hat{R}/I^\alpha \hat{R}, \hat{R}\big), \hat{R}\big) \simeq \operatorname{Hom}_R\big(H^d_{\mathfrak{m}}\big(H^c_I(R)\big), E\big).$$

(b) *If R is in addition complete, then the endomorphism ring* $\operatorname{Hom}_R(H^c_I(R), H^c_I(R))$ *is* $\mathfrak{m}$*-adically complete.*
(c) *If $V(I_d)$ is connected in codimension one, then $(B, \mathfrak{n})$ is a quasi-local ring.*

Proof. By virtue of the arguments given in the proof of Theorem 3.2 and Proposition 2.3 we have isomorphisms

$$\operatorname{Ext}^c_R\big(K\big(R/I^\alpha\big), R\big) = \operatorname{Ext}^c_R\big(\operatorname{Ext}^c_R\big(R/I^\alpha, R\big), R\big), \quad \alpha \in \mathbb{N},$$

that commute with the restriction maps of the inverse systems. So there is an isomorphism

$$\operatorname{Ext}^c_R\big(H^c_I(R), R\big) \simeq \varprojlim \operatorname{Ext}^c_R\big(\operatorname{Ext}^c_R\big(R/I^\alpha, R\big), R\big).$$

This follows since $H^c_I(R) \simeq \varinjlim \operatorname{Ext}^c_R(R/I^\alpha, R)$ and Lemma 2.6 because $\operatorname{Ext}^{c-1}_R(X, R) = 0$ for any R-module X with $\operatorname{Supp}_R X \subset V(I)$.

Next we recall that $E = E_R(k)$ admits a unique structure of an $\hat{R}$-module such that the natural map $E \otimes_R \hat{R} \to E$ is an isomorphism. Then it is easily seen that there is an isomorphism

$$\operatorname{Hom}_{\hat{R}}\big(H^d_{\hat{\mathfrak{m}}}\big(H^c_{I\hat{R}}(\hat{R})\big), E\big) \simeq \operatorname{Hom}_R\big(H^d_{\mathfrak{m}}\big(H^c_I(R)\big), E\big).$$

That is, for the proof of (a) we may assume that $R = \hat{R}$. So, by the definition of $H^c_I(R)$ there is an isomorphism

$$\operatorname{Hom}_R\big(H^d_{\mathfrak{m}}\big(H^c_I(R)\big), E\big) \simeq \varprojlim \operatorname{Hom}_R\big(H^d_{\mathfrak{m}}\big(\operatorname{Ext}^c_R\big(R/I^\alpha, R\big)\big), E\big)$$

and the Local Duality Theorem implies the statement in (a).

By the definition of B there is the natural isomorphism

$$\varprojlim B/\mathfrak{m}^\alpha B \simeq \varprojlim\big(R/\mathfrak{m}^\alpha \otimes_R \operatorname{Hom}_R\big(H^d_{\mathfrak{m}}\big(H^c_I(R)\big), E\big)\big).$$

Since E is an injective R-module there are the following natural isomorphisms

$$R/\mathfrak{m}^\alpha \otimes_R \operatorname{Hom}_R\big(H^d_{\mathfrak{m}}\big(H^c_I(R)\big), E\big) \simeq \operatorname{Hom}_R\big(\operatorname{Hom}_R\big(R/\mathfrak{m}^\alpha, H^d_{\mathfrak{m}}\big(H^c_I(R)\big)\big), E\big)$$

for all $\alpha \in \mathbb{N}$. As a consequence there is the isomorphism

$$\varprojlim B/\mathfrak{m}^\alpha B \simeq \operatorname{Hom}_R\big(H^0_{\mathfrak{m}}\big(H^d_{\mathfrak{m}}\big(H^c_I(R)\big)\big), E\big).$$

Now $H^d_{\mathfrak{m}}(H^c_I(R))$ is a module whose support is contained in $V(\mathfrak{m})$ so that

$$H^0_{\mathfrak{m}}\big(H^d_{\mathfrak{m}}\big(H^c_I(R)\big)\big) \simeq H^d_{\mathfrak{m}}\big(H^c_I(R)\big).$$

But this implies $\varprojlim B/\mathfrak{m}^\alpha B \simeq B$. So, (b) is true.

For the proof of (c) let $\alpha \in \mathbb{N}$ be an integer. As before let $B_\alpha = \operatorname{Hom}_R(K(R/I^\alpha), K(R/I^\alpha))$ the endomorphism ring of the canonical module of R/I^α. Since $V(I) = V(I^\alpha)$, $\alpha \in \mathbb{N}$, is connected in codimension one we know (see Lemma 2.5) that $(B_\alpha, \mathfrak{n}_\alpha)$, $\alpha \in \mathbb{N}$, is a local ring.

Now we prove that $(B_{\alpha+1}, \mathfrak{n}_{\alpha+1}) \to (B_\alpha, n_\alpha)$ is a local homomorphism for all $\alpha \in \mathbb{N}$. This follows by contracting the maximal ideal $\mathfrak{n}_\alpha$ along the commutative diagram before Theorem 3.2. On the left it implies an injection $k = k \hookrightarrow B_\alpha/\mathfrak{n}_\alpha$. Therefore, on the right it yields $k \hookrightarrow B_{\alpha+1}/\mathfrak{n}_\alpha \cap B_{\alpha+1}$. Because $B_{\alpha+1}$ is finitely generated (in particular integral) over $R/I^{\alpha+1}$ it follows that $\mathfrak{n}_\alpha \cap B_{\alpha+1} = \mathfrak{n}_{\alpha+1}$. By virtue of Lemma 2.1 it follows that $(B, \mathfrak{n}) \simeq \varprojlim(B_\alpha, \mathfrak{n}_\alpha)$ is a quasi-local ring. □

Problem 3.4. It is a natural question to ask whether the commutative ring B constructed in Theorem 3.2 is a Noetherian ring. We do not know an answer in general. The stronger question whether B is a finitely generated R-module is not true.

Lemma 3.5. *Fix the notation of Theorem* 3.3. *Then the following conditions are equivalent*:

(i) $\operatorname{Ext}^c_{\hat{R}}(H^c_{I\hat{R}}(\hat{R}), \hat{R})$ *is a finitely generated* $\hat{R}$*-module.*
(ii) $\dim_k \operatorname{Hom}_R(k, H^d_{\mathfrak{m}}(H^c_I(R))) < \infty$.
(iii) $H^d_{\mathfrak{m}}(H^c_I(R))$ *is an Artinian* R*-module.*

Proof. By Theorem 3.3 it follows that $B \simeq \operatorname{Hom}_R(H^d_{\mathfrak{m}}(H^c_I(R)), E)$. There are isomorphisms (for the second use that E is an injective R-module)

$$B/\mathfrak{m}B \simeq k \otimes \operatorname{Hom}_R\big(H^d_{\mathfrak{m}}\big(H^c_I(R)\big), E\big) \simeq \operatorname{Hom}_R\big(\operatorname{Hom}_R\big(k, H^d_{\mathfrak{m}}\big(H^c_I(R)\big)\big), E\big).$$

By virtue of [16, Theorem 8.4] B is a finitely generated $\hat{R}$-module if and only if $\dim_k B/\mathfrak{m}B < \infty$. By Matlis duality the finiteness of this dimension is therefore equivalent to the finiteness of the socle dimension $\dim_k \operatorname{Hom}_R(k, H^d_{\mathfrak{m}}(H^c_I(R)))$. So we have the equivalence of the first two statements. Because of $\operatorname{Supp} H^d_{\mathfrak{m}}(H^c_I(R)) \subseteq V(\mathfrak{m})$ it follows that (ii) is equivalent to (iii). □

In the following there is an example of an ideal $I \subset R$ in a local complete Gorenstein ring R such that B is not a finitely generated R-module.

Example 3.6. Let k be a field and let $A = k[|u, v, x, y|]$ be the formal power series ring in four variables. Put $R = A/fA$, where $f = xv - yu$. Let $I = (x, y)R$. We will show that

$$B = \varprojlim \operatorname{Ext}^1_R\big(\operatorname{Ext}^1_R\big(R/I^\alpha, R\big), R\big)$$

is not a finitely generated R-module, while it admits the structure of a Noetherian ring.

To this end put $A_\alpha = R/I^\alpha \simeq A/((x, y)^\alpha, f)$ and $B_\alpha = k[|u, v|][a]/(a^\alpha)$, where a denotes a variable over $k[|u, v|]$. Consider the ring homomorphism $A \to B_\alpha$ induced by $x \mapsto ua$, $y \mapsto va$. As it is easily seen it induces an injection $A_\alpha \to B_\alpha, \alpha \in \mathbb{N}$. Clearly B_α, $\alpha \in \mathbb{N}$, is a two-dimensional Cohen–Macaulay ring. The cokernel of this embedding is

$$k\big[|u, v|\big][a]/\big(k\big[|u, v|\big][ua, va] + a^\alpha k\big[|u, v|\big][a]\big),$$

which is a finite dimensional k-vector space. Whence the dimension of B_α/A_α as an R-module is zero. Therefore B_α is the S_2-ification of A_α, that is $B_\alpha \simeq \operatorname{Ext}^1_R(\operatorname{Ext}^1_R(R/I^\alpha, R), R)$ (see Proposition 2.3).

So there are short exact sequences

$$0 \to A_\alpha \to B_\alpha \to H^1_{\mathfrak{m}}(A_\alpha) \to 0$$

for all $\alpha \in \mathbb{N}$. By passing to the inverse limit it induces a short exact sequence

$$0 \to R \to B \to \varprojlim H^1_{\mathfrak{m}}\big(R/I^\alpha\big) \to 0.$$

Moreover it yields that $B \simeq k[|u, v, a|]$, which is clearly a Noetherian ring. Moreover B is not a finitely generated R-module as easily seen.

Furthermore, by the local duality theorem there is the isomorphism

$$\varprojlim H^1_{\mathfrak{m}}\big(R/I^{\alpha}\big) \simeq \operatorname{Hom}_R\big(H^2_I(R), E\big).$$

Therefore, $\operatorname{Hom}_R(H^2_I(R), E)$ is not a finitely generated R-module. By Matlis duality it follows that $H^2_I(R)$ is not an Artinian R-module and therefore the socle $\operatorname{Hom}_R(k, H^2_I(R))$ is not of finite dimension. Originally this was shown by Hartshorne (see [7, Section 3]). In fact, the analysis of Hartshorne's example inspired the above construction.

In the following sections we shall discuss some particular cases in which B is a finitely generated R-module.

Proof of Theorem 1.1(a). The statement (a) follows by Theorem 3.2(a) and Theorem 3.3(b). □

4. The case of regular local rings

In this section let $(R, \mathfrak{m})$ denote a regular local ring. Let $I \subset R$ be an ideal of R. Huneke and Sharp (cf. [12]) in the case of prime characteristic $p > 0$ resp. Lyubeznik (see [14]) in the case of characteristic zero proved the following result:

Theorem 4.1. *(See [12] and [14].) Let $(R, \mathfrak{m})$ be a regular local ring containing a field. Let $I \subset R$ be an ideal. For all $i, j \in \mathbb{Z}$ the following hold:*

(a) *$H^j_{\mathfrak{m}}(H^i_I(R))$ is an injective R-module.*
(b) *$\operatorname{injdim}_R H^i_I(R) \leqslant \dim_R H^i_I(R) \leqslant \dim R - i$.*
(c) *$\operatorname{Ext}^j_R(k, H^i_I(R)) \simeq \operatorname{Hom}_R(k, H^j_{\mathfrak{m}}(H^i_I(R)))$ and $\dim_k \operatorname{Ext}^j_R(k, H^i_I(R)) < \infty$.*

As we shall see the previous result applies in an essential way in order to describe properties of the ring

$$B = \varprojlim \operatorname{Ext}^c_R\big(\operatorname{Ext}^c_R\big(R/I^i, R\big), R\big), \quad c = \dim R - \dim R/I,$$

introduced in Definition 3.1. But before let us recall the Bass numbers in Theorem 4.1(c) were introduced by Lyubeznik (cf. [14, 4.1]). In fact Lyubeznik has shown that they only depend upon R/I. With the above results in mind we shall describe the structure of the ring B in case $(R, \mathfrak{m})$ is a complete regular local ring containing a field.

Lemma 4.2. *With the notation as above assume that $(R, \mathfrak{m})$ is a complete regular local ring containing a field. There is an isomorphism*

$$B = \varprojlim \operatorname{Ext}^c_R\big(\operatorname{Ext}^c_R\big(R/I^{\alpha}, R\big), R\big) \simeq R^l,$$

where $l = \dim \operatorname{Ext}^d_R(k, H^c_I(R))$, $d = \dim R/I$, $c = \dim R - \dim R/I$.

Proof. By virtue of Theorem 4.1 it turns out that l is a finite number. As a consequence of Lemma 3.5 it follows that B is a finitely generated R-module. Moreover, by virtue of Theorem 4.1(a) and the definition of B as the Matlis dual of $H^d_{\mathfrak{m}}(H^c_I(R))$ (see Theorem 3.3) we see that B is flat as an R-module. Therefore B is a free R-module and $B \simeq R^l$, the direct sum of l copies of R. □

In the following we will give an interpretation of the rank l in topological terms. To this end we use results of Lyubeznik (see [15]) and Zhang (see [22]).

Theorem 4.3. *(See [15] and [22].) Let $(R, \mathfrak{m})$ be an n-dimensional regular local ring containing a field. Let $I \subset R$ denote an ideal with $c = \operatorname{height} I$ and $d = \dim R/I$. Let A denote the completion of the strict Henselization of the completion of R/I. Let t denote the number of connected components of the graph $\mathbb{G}_A$. Then*

$$\dim_k \operatorname{Hom}_R\bigl(k, H^d_{\mathfrak{m}}\bigl(H^c_I(R)\bigr)\bigr) = \dim_k \operatorname{Ext}^d_R\bigl(k, H^c_I(R)\bigr) = t.$$

It was pointed out by Lyubeznik (see [15]) that the graph $\mathbb{G}_A$, where A is the completion of the strict Henselization of the completion of R/I, is realized by a smaller ring. Namely, let $k \subset \hat{A}$ denote a coefficient field. It follows (see [11, Theorem 4.2]) that there exists a finite separable field extension $k \subset K$ such that the graphs $\mathbb{G}_{K \otimes_k \widehat{R/I}}$ and $\mathbb{G}_A$ are isomorphic.

Here we want to give another description of the invariant $t = \dim_k \operatorname{Ext}^d_R(k, H^c_I(R))$. As a first step in this direction there is the following result:

Theorem 4.4. *Let $(R, \mathfrak{m})$ denote an n-dimensional complete regular ring containing a field. Let I be an ideal of R and $c = \operatorname{height} I$. Then the following are equivalent*:

(i) *$V(I_d)$ is connected in codimension one.*
(ii) *$B = \operatorname{Hom}_R(H^c_I(R), H^c_I(R))$ is a local ring.*

Proof. Because $H^c_I(R) \simeq H^c_{I_d}(R)$ we may replace I by I_d. The implication (i) $\Rightarrow$ (ii) follows by the results shown in Theorem 3.3(c) and Lemma 4.2.

In order to prove (ii) $\Rightarrow$ (i) suppose that $V(I)$ is not connected in codimension one. Let $\mathbb{G}_1, \dots, \mathbb{G}_t$, $t > 1$, denote the connected components of $\mathbb{G}_{R/I}$. Moreover, let I_i, $i = 1, \dots, t$, denote the intersection of all minimal primes of $V(I)$ that are vertices of $\mathbb{G}_i$. Then

$$H^c_I(R) \simeq \bigoplus_{i=1}^t H^c_{I_i}(R)$$

as is a consequence of the Mayer–Vietoris sequence for local cohomology (see [15, Proposition 2.1] for the details). Clearly $c = \operatorname{height} I_i$, $i = 1, \dots, t$, and $H^c_{I_i}(R) \neq 0$. Moreover

$$\operatorname{Ext}^c_R\bigl(H^c_I(R), R\bigr) \simeq \bigoplus_{i=1}^t \operatorname{Ext}^c_R\bigl(H^c_{I_i}(R), R\bigr)$$

and therefore $B \simeq B_1 \times \dots \times B_t$, where

$$B_i = \operatorname{Ext}^c_R\bigl(H^c_{I_i}(R), R\bigr) \simeq \operatorname{Hom}_R\bigl(H^c_{I_i}, H^c_{I_i}(R)\bigr), \quad i = 1, \dots, t,$$

are local rings as shown by (i) $\Rightarrow$ (ii). That $B \simeq B_1 \times \dots \times B_t$ is indeed a decomposition as a direct product of rings follows e.g. since

$$\operatorname{Hom}_R\bigl(H^c_{I_i}(R), H^c_{I_j}(R)\bigr) \simeq \varprojlim \operatorname{Hom}\bigl(\operatorname{Ext}^c_R\bigl(R/I_i^\alpha, R\bigr), H^c_{I_j}(R)\bigr) = 0 \quad \text{for all } i \neq j.$$

This holds because $\operatorname{Hom}(K(R/I_i^\alpha), H^c_{I_j}(R)) = 0$ since $\operatorname{Ass}_R K(R/I_i^\alpha) \cap V(I_j) = \emptyset$ for $i \neq j$ as follows by the definition of I_i, $i = 1, \dots, t$. Because $t > 1$ it yields that B is not a local ring. $\square$

As a corollary of the previous statement we are able to describe the number of connected components of $\mathbb{G}_{R/I}$ in terms of the ring structure of $B = \operatorname{Ext}_R^c(H_I^c(R), R)$.

Corollary 4.5. *With the notation of Theorem 4.4 the ring B is a semi-local ring. The number of maximal ideals of B is equal to the number of connected components of $\mathbb{G}_{R/I}$.*

Proof. First B is a semi-local ring as follows because it is a finitely generated R-module. If $\mathbb{G}_{R/I}$ is connected in codimension one, then B is a local ring (see Theorem 4.4). Then the claim follows as in the proof of Theorem 4.4 by $H_I^c(R) \simeq \bigoplus_{i=1}^t H_{I_i}^c(R)$. Here I_i, $i = 1, \dots, t$, denotes the intersection of all minimal primes of $V(I)$ that are vertices of $\mathbb{G}_i$, the connected components of $\mathbb{G}_{R/I}$. □

Proof of Theorem 1.1. The proof of the statements (b) and (c) of Theorem 1.1 follow by Lemma 4.2 and Theorem 4.3. The claim (d) of Theorem 1.1 is shown in Theorem 4.4. Finally (e) is a consequence of the results by Lyubeznik (see [15]) and Zhang (see [22]). □

5. On Lyubeznik numbers

In this section let $(R, \mathfrak{m})$ be a regular local ring containing a field. Let $I \subset R$ be an ideal of R. We will add a few results concerning the Lyubeznik numbers

$$\dim_k \operatorname{Hom}_R\bigl(k, H_{\mathfrak{m}}^j\bigl(H_I^i(R)\bigr)\bigr) = \dim_k \operatorname{Ext}_R^j\bigl(k, H_I^i(R)\bigr)$$

in general. That means, we are interested in them for all pairs (j, i) not necessary for $(j, i) = (d, c)$ where d is the dimension of R/I and c is the height of I. As a first step towards this direction we improve the estimate in Theorem 4.1(b). To this end we use the Hartshorne–Lichtenbaum Vanishing Theorem. It yields that $H_I^n(R) = 0$, $n = \dim R$, whenever $(R, \mathfrak{m})$ is a complete local domain and I is an ideal with $\dim R/I > 0$ (see [6, Theorem 3.1] or [17, Theorem 2.20]).

Lemma 5.1. *Let $(R, \mathfrak{m})$ denote a regular local ring with $\dim R = n$. Let I be an ideal of R and pure height $c < n$. Then*

$$\dim H_I^i(R) \leqslant n - i - 1$$

for all $c < i \leqslant n$ and $\dim H_I^c(R) = n - c$.

Proof. First we prove the second statement $\dim H_I^c(R) = n - c$. Let $\mathfrak{p} \in V(I)$ denote a prime ideal such that $\dim R_\mathfrak{p} = c$. Then

$$0 \neq H_{IR_\mathfrak{p}}^c(R_\mathfrak{p}) \simeq H_I^c(R) \otimes_R R_\mathfrak{p}$$

because $c = \dim R_\mathfrak{p}$ and $\operatorname{Rad} IR_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. Recall that $\mathfrak{p} \in V(I)$ is a minimal prime ideal. Therefore $\dim H_I^c(R) \geqslant \dim R/\mathfrak{p} = n - c$. The equality is true since $\operatorname{Supp} H_I^c(R) \subseteq V(I)$.

Now let $c < i \leqslant n$. First of all note that $H_I^n(R) = 0$ as follows by the Hartshorne–Lichtenbaum Vanishing Theorem (cf. [6] or [17, Theorem 2.20]). Now suppose the contrary to the claim. Then there is a minimal prime ideal $\mathfrak{p} \in \operatorname{Supp} H_I^i(R)$ such that $\dim H_I^i(R) = \dim R/\mathfrak{p} \geqslant n - i$, and therefore $i \geqslant \dim R_\mathfrak{p}$. But $0 \neq H_I^i(R) \otimes R_\mathfrak{p} \simeq H_{IR_\mathfrak{p}}^i(R_\mathfrak{p})$, and therefore $i = \dim R_\mathfrak{p}$ by the Grothendieck Vanishing Theorem. Moreover height $IR_\mathfrak{p} = c$ because I is of pure height c. Therefore $H_{IR_\mathfrak{p}}^i(R_\mathfrak{p}) \neq 0$ for height $IR_\mathfrak{p} < i = \dim R_\mathfrak{p}$, and this is a contradiction to the Hartshorne–Lichtenbaum Vanishing Theorem. □

For a better understanding of the Lyubeznik numbers we need an auxiliary construction of a certain complex $C^{\cdot}(I)$ for an ideal I of R. Let $R \xrightarrow{\sim} E^{\cdot}$ denote a minimal injective resolution of R. Because $\Gamma_I(E^{\cdot})^i = 0$ for all $i < c = \operatorname{height} I < n$ there is a homomorphism of complexes

$$0 \to H_I^c(R)[-c] \to \Gamma_I(E^{\cdot}),$$

where $H_I^c(R)$ is considered as a complex concentrated in homological degree zero.

Definition 5.2. The cokernel of the embedding $H_I^c(R)[-c] \to \Gamma_I(E^{\cdot})$ is defined as $C_R^{\cdot}(I)$, the truncation complex with respect to I. So there is a short exact sequence of complexes of R-modules

$$0 \to H_I^c(R)[-c] \to \Gamma_I\big(E^{\cdot}\big) \to C_R^{\cdot}(I) \to 0.$$

We observe that $H^i(C_R^{\cdot}(I)) \simeq H_I^i(R)$ for all $i \neq c$ while $H^c(C_R^{\cdot}(I)) = 0$.

By applying the derived functor $\mathrm{R}\Gamma_{\mathfrak{m}}$ of the section functor it induces (in the derived category) an exact triple of complexes

$$\mathrm{R}\Gamma_{\mathfrak{m}}\big(H_I^c(R)\big)[-c] \to E[-n] \to \mathrm{R}\Gamma_{\mathfrak{m}}\big(C^{\cdot}(I)\big) \to \mathrm{R}\Gamma_{\mathfrak{m}}\big(H_I^c(R)\big)[-c][+1].$$

Recall that $\mathrm{R}\Gamma_{\mathfrak{m}}(\Gamma_I(E^{\cdot})) \simeq \Gamma_{\mathfrak{m}}(\Gamma_I(E^{\cdot})) \simeq \Gamma_{\mathfrak{m}}(E^{\cdot}) \simeq E[-n]$, where $E = E_R(R/\mathfrak{m})$ denotes the injective hull of the residue field.

In order to compute the hyper cohomology $H_{\mathfrak{m}}^i(C^{\cdot}(I))$ there is the following E_2-term spectral sequence (see [21] for the details)

$$E_2^{p,q} = H_{\mathfrak{m}}^p\big(H^q\big(C^{\cdot}(I)\big)\big) \quad \Rightarrow \quad E_{\infty}^{p+q} = H_{\mathfrak{m}}^{p+q}\big(C^{\cdot}(I)\big).$$

Now recall that $H^q(C^{\cdot}(I)) = H_I^q(R)$ for $c < q < n$, and $H^q(C^{\cdot}(I)) = 0$ for $q \leqslant c$ resp. $q \geqslant n$. We notice that $H_I^n(R) = 0$ as a consequence of the Hartshorne–Lichtenbaum Vanishing Theorem.

Proposition 5.3. *With notation of Lemma* 5.1 *and Definition* 5.2 *there is a short exact sequence*

$$0 \to H_{\mathfrak{m}}^{n-1}\big(C^{\cdot}(I)\big) \to H_{\mathfrak{m}}^d\big(H_I^c(R)\big) \to E \to 0$$

and isomorphisms $H_{\mathfrak{m}}^{j-1}(C^{\cdot}(I)) \simeq H_{\mathfrak{m}}^{j-c}(H_I^c(R))$ *for all* $j < n$ *and all* $j > n = \dim R$.

Proof. The proof follows by the long exact cohomology sequence induced by the above exact triple of complexes in the derived category. The only claim we have to show is the vanishing of $H_{\mathfrak{m}}^n(C^{\cdot}(I))$. To this end we apply the previous spectral sequence. By virtue of Lemma 5.1 we know that $\dim H_I^q(R) < n - q$ for all $q \neq c$. Therefore $E_2^{n-q,q} = H_{\mathfrak{m}}^{n-q}(H_I^q(R)) = 0$ for all $q \neq c$. That is, in the above spectral sequence all the initial terms of level n vanish. Therefore the limit term vanishes also, that is $H_{\mathfrak{m}}^n(C^{\cdot}(I)) = 0$, as required. □

As an application of our investigations we prove a slight improvement of a duality result shown by Blickle (cf. [2, Theorem 1.1]).

Corollary 5.4. *Let* $I \subset R$ *denote an ideal of pure height* c *of a regular local ring* $(R, \mathfrak{m})$ *containing a field. Suppose that* $c < n$ *and* $\operatorname{Supp} H_I^i(R) \subseteq V(\mathfrak{m})$ *for all* $i \neq c$. *Then the following are true*:

(a) *There is a short exact sequence*

$$0 \to H_I^{n-1}(R) \to H_{\mathfrak{m}}^d\big(H_I^c(R)\big) \to E \to 0.$$

(b) *For $j < n$ there are isomorphisms $H_I^{j-1}(R) \simeq H_{\mathfrak{m}}^{j-c}(H_I^c(R))$.*

Proof. Because $\operatorname{Supp} H_I^i(R) \subseteq V(\mathfrak{m})$ for all $i \neq c$ it follows that $H_{\mathfrak{m}}^p(H^q(C^\cdot(I))) = 0$ for all $p \neq 0$. So the previous spectral sequence degenerates to isomorphisms

$$H_{\mathfrak{m}}^q\big(C^\cdot(I)\big) \simeq H_{\mathfrak{m}}^0\big(H^q\big(C^\cdot(I)\big)\big)$$

for all $q \in \mathbb{Z}$. Then the claim is a consequence of the statements in Proposition 5.3. Recall that $H_{\mathfrak{m}}^0(H^q(C^\cdot(I))) \simeq H^q(C^\cdot(I))$ for all $q \in \mathbb{Z}$ because $\operatorname{Supp} H_I^i(R) \subseteq V(\mathfrak{m})$ for all $i \neq c$ by the assumption. □

Under the assumption of Corollary 5.4 Blickle (cf. [2, Theorem 1.1]) proved the following equalities

$$\dim_k \operatorname{Hom}_R\big(k, H_{\mathfrak{m}}^0\big(H_I^{j-1}(R)\big)\big) = \dim_k \operatorname{Hom}_R\big(k, H_{\mathfrak{m}}^{j-c}\big(H_I^c(R)\big)\big) - \delta_{j-c,d}$$

for all $j \in \mathbb{N}$. In fact, this is a consequence of the present Corollary 5.4. The assumption $\operatorname{Supp} H_I^i(R) \subseteq V(\mathfrak{m})$ for all $i \neq c$ is fulfilled for instance whenever $\operatorname{cd} IR_{\mathfrak{p}} = \operatorname{height} I$ for all $\mathfrak{p} \in V(I) \setminus \{\mathfrak{m}\}$.

Corollary 5.5. *With the notation of Corollary* 5.4 *let I denote an ideal of R. Suppose there is an integer $c < a < n$ such that $H_I^i(R) = 0$ for all $i \neq c, a$.*

(a) *There is a short exact sequence*

$$0 \to H_{\mathfrak{m}}^{n-a-1}\big(H_I^a(R)\big) \to H_{\mathfrak{m}}^d\big(H_I^c(R)\big) \to E \to 0.$$

(b) *For $j < n$ there are isomorphisms $H_{\mathfrak{m}}^{j-a-1}(H_I^a(R)) \simeq H_{\mathfrak{m}}^{j-c}(H_I^c(R))$.*

Proof. By the assumption on the vanishing of $H_I^i(R)$ it follows that $C^\cdot(I) \xrightarrow{\sim} H_I^a(R)[-a]$ in the derived category. Thus, the statement is an immediate consequence of Proposition 5.3. □

It would be of some interest to get an understanding of the Lyubeznik numbers in general.

6. On a trace map

Let $(R, \mathfrak{m})$ denote a Gorenstein ring and $n = \dim R$. Let $I \subset R$ denote an ideal of $\operatorname{height} I = c$ and $\dim R/I = d$. Moreover we remark that $H_I^c(R) \simeq H_{I_1}^c(R)$, where I_1 is the intersection of the primary components of I of pure height c. This is an easy consequence of the Mayer–Vietoris sequence. Therefore, we may always assume that I is of pure height c.

Lemma 6.1. *With the previous notation there is a natural homomorphism*

$$\phi : \operatorname{Ext}_R^d\big(k, H_I^c(R)\big) \to k, \quad k = R/\mathfrak{m}.$$

Proof. Apply the derived functor $\mathrm{R}\,\mathrm{Hom}_R(k,\cdot)$ to the short exact sequence as it is considered in the definition of the truncation complex (see Definition 5.2). The resulting exact triple in the derived category induces a long exact cohomology sequence

$$\cdots \to \mathrm{Ext}_R^{i-c}\big(k, H_I^c(R)\big) \to \mathrm{Ext}_R^i\big(k, \Gamma_{\mathfrak{m}}\big(E^{\cdot}\big)\big) \to \mathrm{Ext}_R^i\big(k, C_R^{\cdot}(I)\big) \to \cdots.$$

Now consider the complex $\mathrm{R}\,\mathrm{Hom}_R(k, \Gamma_{\mathfrak{m}}(E^{\cdot}))$. Since $\Gamma_{\mathfrak{m}}(E^{\cdot})$ is a complex of injective modules the following isomorphic complexes

$$\mathrm{Hom}_R\big(k, \Gamma_{\mathfrak{m}}\big(E^{\cdot}\big)\big) \simeq \mathrm{Hom}_R\big(k, E^{\cdot}\big) \simeq k[-n],$$

represent $\mathrm{R}\,\mathrm{Hom}_R(k, \Gamma_{\mathfrak{m}}(E^{\cdot}))$ in the derived category. By virtue of the long exact cohomology sequence it yields the natural homomorphism of the statement. □

In [8, Conjecture 2.7] Hellus and the author conjectured that the homomorphism in Lemma 6.1 is in general non-zero. In the next we shall confirm this question in the case of $(R, \mathfrak{m})$ a regular local ring containing a field. To this end we need a few auxiliary constructions.

By virtue of Proposition 5.3 and Lemma 6.1 there is the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}_R^d(k, H_I^c(R)) & \xrightarrow{\ \phi\ } & k \\ \Big\downarrow{\scriptstyle \lambda} & & \Big\downarrow \\ H_{\mathfrak{m}}^d(H_I^c(R)) & \xrightarrow{\ \psi\ } & E. \end{array}$$

Here the vertical homomorphism $k \to E$ is – by construction – the natural inclusion. Therefore, λ is not zero, provided ϕ is not zero.

Theorem 6.2. *Let $(R, \mathfrak{m})$ denote a regular local ring containing a field with* $\dim R = n$. *Let $I \subset R$ denote an ideal of* height $I = c$ *and* $\dim R/I = d$. *Then the homomorphism*

$$\phi : \mathrm{Ext}_R^d\big(k, H_I^c(R)\big) \to k$$

is non-zero.

Proof. We may assume that R is a complete local ring. By applying $\mathrm{Hom}_R(k,\cdot)$ to the above diagram it implies the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}_R^d(k, H_I^c(R)) & \xrightarrow{\ \phi\ } & k \\ \Big\downarrow{\scriptstyle \bar{\lambda}} & & \Big\| \\ \mathrm{Hom}_R(k, H_{\mathfrak{m}}^d(H_I^c(R))) & \xrightarrow{\ \bar{\psi}\ } & k. \end{array}$$

By virtue of Theorem 4.1 the vertical homomorphism $\bar{\lambda}$ is an isomorphism. Therefore it will be enough to show that $\bar{\psi}$ is not zero. By Matlis duality it follows that

$$B = \mathrm{Ext}_R^c\big(H_I^c(R), R\big) \simeq \mathrm{Hom}_R\big(H_{\mathfrak{m}}^d\big(H_I^c(R)\big), E\big).$$

So, $\bar{\psi}$ is the Matlis dual of the natural homomorphism $k \otimes_R R \to k \otimes_R B$ which is non-zero as shown in Theorem 3.2(c). This proves that $\bar{\psi}$ is non-zero. By the previous observation it follows that ϕ is non-zero too. □

Remark 6.3. Let $(R, \mathfrak{m})$ be a local ring that is the factor ring of a Gorenstein ring. Let M be a finitely generated R-module with $d = \dim_R M$. In connection to his canonical element conjecture (see [9, Section 4]) Hochster has studied the natural homomorphism $\operatorname{Ext}_R^d(k, K(M)) \to H_{\mathfrak{m}}^d(K(M))$, where $K(M)$ denotes the canonical module of M. In particular, he considered the problem whether this map is non-zero.

In our situation here the natural homomorphism $\lambda : \operatorname{Ext}_R^d(k, H_I^c(R)) \to H_{\mathfrak{m}}^d(H_I^c(R))$ is the direct limit of the natural homomorphisms $\lambda_\alpha : \operatorname{Ext}_R^d(k, K(R/I^\alpha)) \to H_{\mathfrak{m}}^d(K(R/I^\alpha))$, $\alpha \in \mathbb{N}$. Recall that $H_I^c(R) \simeq \varinjlim K(R/I^\alpha)$ and that local cohomology commutes with direct limits. So in a certain sense, λ is the stable value of all of the λ_α. It would be of some interest to relate the non-vanishing of λ to other problems in commutative algebra.

7. Examples

Let $I \subset R$ denote an ideal with $c = \operatorname{height} I$. The following example shows that the isomorphism $\operatorname{Hom}_R(H_I^c(R), H_I^c(R)) \simeq R$ is not preserved by passing to the localization with respect to a prime ideal.

Example 7.1. Let k denote an algebraically closed field. Let $R = k[|a, b, c, d, e|]$ denote the formal power series ring in five variables. Let $I \subset R$ denote the prime ideal with the parametrization

$$a = su^2, \qquad b = stu, \qquad c = tu(t-u), \qquad d = t^2(t-u), \qquad e = u^3.$$

It is easy to see that $I = (ad - bc, a^2c + abe - b^2e, c^3 + cde - d^2e, ade - bde + ac^2)$. Moreover $\dim R/I = 3$, $n = 5$ and $H_I^i(R) = 0$ for all $i \neq 2, 3$ as follows from the Second Vanishing Theorem (see [6]). Clearly $V(I)$ is connected in codimension one because I is a prime ideal. So it follows (cf. Theorem 4.3) $\dim_k \operatorname{Ext}_R^3(k, H_I^2(R)) = 1$. Therefore the endomorphism ring $\operatorname{Hom}_R(H_I^2(R), H_I^2(R))$ is isomorphic to R (see Lemma 4.2).

Let $\mathfrak{p} = (a, b, c, d)R$. Then $\dim R_\mathfrak{p}/IR_\mathfrak{p} = 2$. The ideal $IR_\mathfrak{p}$ corresponds to the parametrization $(x, xy, y(y-1), y^2(y-1))$. It follows that $V(IR_\mathfrak{p}) \setminus \{\mathfrak{p}R_\mathfrak{p}\}$ is not formally connected (see [4, 3.4.2]). Therefore $V(I\widehat{R_\mathfrak{p}})$ has two connected components. Whence $\operatorname{Hom}_{R_\mathfrak{p}}(H_{IR_\mathfrak{p}}^2(R_\mathfrak{p}), H_{IR_\mathfrak{p}}^2(R_\mathfrak{p})) \simeq \widehat{R_\mathfrak{p}}^2$, because $\dim_{k(\mathfrak{p})} \operatorname{Ext}_{R_\mathfrak{p}}^2(k(\mathfrak{p}), H_{IR_\mathfrak{p}}^2(R_\mathfrak{p})) = 2$.

The next example (invented by Hochster) shows that the Bass numbers $\dim_k \operatorname{Ext}_R^i(k, H_I^c(R))$ depend upon the characteristic of the ground field k.

Example 7.2. Let $R = k[|x_1, \ldots, x_6|]$ denote the formal power series rings in six variables over the basic field k. Let I denote the ideal generated by the two-by-two minors of the matrix

$$M = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}.$$

Then R/I is a four-dimensional Cohen–Macaulay ring and $c = \operatorname{height} I = 2$. It follows that $H_I^i(R) = 0$ for all $i \neq 2$ provided k is a field of characteristic $p > 0$. Furthermore $H_I^i(R) = 0$ for all $i \neq 2, 3$ and $H_I^3(R) \neq 0$ in the case of k a field of characteristic zero. This is shown via the Reynolds operator. Clearly R/I has an isolated singularity, so that $\operatorname{Supp} H_I^3(R) \subseteq \{\mathfrak{m}\}$. By virtue of Corollary 5.4 and Theorem 4.1 it follows that $\operatorname{Ext}_R^i(k, H_I^2(R)) = 0$ for all $i < 4$ if k is of positive characteristic, while $\operatorname{Ext}_R^2(k, H_I^2(R)) \neq 0$ if k is of characteristic zero.

Clearly $H^0_{\mathfrak{m}}(H^3_I(R)) \simeq H^3_I(R)$, and therefore $H^3_I(R)$ is an injective R-module (see Theorem 4.1). Now we apply the derived functor $\mathrm{R}\operatorname{Hom}_R(k, \cdot)$ to the short exact sequence of the truncation complex (see Definition 5.2). Because $H^3(C^{\cdot}_R(I)) \simeq H^3_I(R)$ and $H^i(C^{\cdot}_R(I)) = 0$ for all $i \neq 3$ it induces an isomorphism $\operatorname{Ext}^2_R(k, H^2_I(R)) \simeq \operatorname{Hom}_R(k, H^3_I(R))$.

Finally Uli Walther (see [20, Example 6.1]) has computed that $H^3_I(R) \simeq E_R(k)$. Therefore $\dim \operatorname{Ext}^2_R(k, H^2_I(R)) = 1$ in the case of characteristic zero.

Example 7.3 shows that the number of maximal ideals of $B = \operatorname{Hom}_R(H^c_I(R), H^c_I(R))$ does not coincide with the rank of B as R-module.

Example 7.3. Let $\mathbb{Q}$ denote the field of rational numbers. Consider $\mathbb{Q}(i)$ denote the field extension of $\mathbb{Q}$ by the imaginary unit. Let $R = \mathbb{Q}[|w, x, y, z|]$ and $S = \mathbb{Q}(i)[|w, x, y, z|]$ denote the formal power series ring in four variables over $\mathbb{Q}$ and $\mathbb{Q}(i)$ respectively. Let $J = (w - ix, y - iz) \cap (w + ix, y + iz) \subset S$ and $I = J \cap R$. Then $I = (w^2 + x^2, y^2 + z^2, wy + xz, wz - xy)$ is a two-dimensional prime ideal. Therefore (cf. Theorem 4.4) $B_R = \operatorname{Hom}_R(H^2_I(R), H^2_I(R))$ is a local ring. Moreover

$$B_S = \operatorname{Hom}_S\big(H^2_J(S), H^2_J(S)\big) \simeq S/(w - ix, y - iz) \oplus S/(w + ix, y + iz)$$

as it follows by the Mayer–Vietoris sequence for local cohomology. It is easily seen that $B_R \simeq \mathbb{Q}[a]/(a^2 + 1)[|w, x, y, z|]$. In fact, $\dim \operatorname{Ext}^2_R(R/\mathfrak{m}, H^c_I(R)) = 2$.

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