Additive Commutators of Rational 2×2 Matrices*

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ABSTRACT

The expression of det $(AB\text{-}BA)$ as a norm in both $(Q(\sqrt{m})$ and $Q(\sqrt{n})$ for 2×2 rational matrices with characteristic roots in $Q(\sqrt{m})$, resp. $Q(\sqrt{n})$, is studied here further, see [1]. A necessary and sufficient condition for this element to be also a norm in $Q(\sqrt{m}, \sqrt{n})$ is obtained.

In a recent note [6] the following fact was proved:

THEOREM 1. *Let A,B be integral 2×2 matrices. Let the characteristic roots of A be α,α', assumed irrational. Then the determinant of $AB-BA$ is a negative norm in $Q(\alpha)$.*

This is now complemented by the following result, which concerns rational matrices. Although Theorem 1 dealt with integral matrices, the norm obtained is, in general, not the norm of an integer. However, the application of the whole investigation is of a number theoretic nature. For, expressing a rational number as a norm in a quadratic field leads to the solution of a ternary diophantine equation $ax^2+by^2+cz^2=0$, where a, b, c are integers.

THEOREM 2. *Every negative norm in a quadratic field $Q(\sqrt{m})$ can be represented as* $\det(AB-BA)$, *where A,B are 2×2 rational matrices and A has its characteristic roots in $Q(\sqrt{m})$.*

This result will be obtained by an extension of another theorem stated without proof in [6].

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THEOREM 3. *Let Z be the matrix* $\begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix}$, *where d is a rational number, not a square. If Z is expressed in the form* $Z = XY - YX$, *where* X, Y *are rational matrices,*[1] *then X can be so chosen that its characteristic roots lie in a field* $Q(\sqrt{M})$, *where M is an arbitrary norm in* $Q(\sqrt{d})$.

Proof of Theorem 3. Let

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \qquad Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix};$$

then

$$Z = XY - YX = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix},$$

where

$$z_{11} = -z_{22} = x_{12}y_{21} - x_{21}y_{12}, \tag{1}$$

$$z_{12} = (x_{11} - x_{22})y_{12} - (y_{11} - y_{22})x_{12}, \tag{2}$$

$$z_{21} = -(x_{11} - x_{22})y_{21} + (y_{11} - y_{22})x_{21}. \tag{3}$$

If $Z = \begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix}$, then

$$z_{11} = 0, \qquad z_{12} = 1, \qquad z_{21} = d. \tag{4}$$

The number z_{11} can be considered as the negative determinant of the last two linear equations defined by (4) with $x_{11} - x_{22}$, $y_{11} - y_{22}$ as unknowns. Since $z_{11} = 0$, we have

$$-dy_{12} = y_{21}, \qquad -dx_{12} = x_{21}. \tag{5}$$

It further follows that both $x_{11} - x_{22}$, x_{12} can be chosen arbitrarily. Hence the characteristic roots of X lie in the field generated by the square root of $(\operatorname{tr} X)^2 - 4\det X = (x_{11} - x_{22})^2 - 4dx_{12}^2$, which is an arbitrary norm in $Q(\sqrt{d})$. This proves Theorem 3. ∎

Proof of Theorem 2. Let d, m be given rational numbers, not squares, and let d be a norm in $Q(\sqrt{m})$. Express $\begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix}$ in the form $XY - YX$, where

[1] This is possible, by a theorem of Shoda [5].

by Theorem 3 the characteristic roots of X are arbitrary norms in $Q(\sqrt{d})$. Since d is a norm in $Q(\sqrt{m})$, it follows that m is a norm in $Q(\sqrt{d})$.[2] Hence the matrix X can be chosen with characteristic roots in $Q(\sqrt{m})$. (The rational parts of the roots of X are arbitrary too, since a rational scalar matrix may be added to X without changing $x_{11}-x_{22}, x_{12}$.) Denote this matrix by A, and any matrix Y which satisfies the equation $\begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix} = AY - YA$ by B. By Eq. (2) there is such a B. We then have

$$\det(AB-BA) = -d.$$

To summarize: In the equation

$$C = (AB-BA)$$

with characteristic roots of A in $Q(\sqrt{m})$, we have: $\det C$ is a negative norm in $Q(\sqrt{m})$ and m is a norm in $Q(\sqrt{-\det C})$. In particular, this also establishes a link between the fields $Q(\sqrt{d})$ and $Q(\sqrt{-d})$, for $-d$ is a norm in $Q(\sqrt{d})$. ■

Theorem 1 as stated here is only part of Theorem 1 in [6]. There it was shown that $\det(AB-BA)$ is simultaneously a negative norm in $Q(\sqrt{n})$ if B has its characteristic roots in $Q(\sqrt{n})$. The question then arises: if $AB-BA = \begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix}$ is given, as well as the field $Q(\sqrt{m})$ in which the characteristic roots of A lie, how arbitrary is the field $Q(\sqrt{n})$ in which the roots of B lie?

This question is answered by the following theorem:[3]

Theorem 4.[4] *The numbers m, n satisfy the relation*

$$4mn = r^2 - d, \qquad r \in Q.$$

[2] This follows by an easy computation and is expressed in one of the properties of Hilbert's norm residue symbol.

[3] In connection with Theorems 4 and 5 advice by J. Carroll, H. Kisilevsky, H. Zassenhaus and E. C. Dade was helpful.

[4] Here we have to be careful about the meaning of m and n. For when talking about $Q(\sqrt{m})$, the number m is given up to a square factor. What we need for m here is the discriminant of the characteristic polynomial of the matrix in question.

Hence mn is a norm in $Q(\sqrt{d})$ of a special nature, and d is a norm in $Q(\sqrt{mn})$.

Proof of Theorem 4. In the proof of Theorem 3 it is shown that

$$4m=(x_{11}-x_{22})^2-4dx_{12}^2=\text{norm}(\alpha) \qquad \text{with } \alpha\in Q(\sqrt{d}), \tag{6}$$

$$4n=(y_{11}-y_{22})^2-4dy_{12}^2=\text{norm}(\beta) \qquad \text{with } \beta\in Q(\sqrt{d}). \tag{7}$$

Further, via (4):

$$\alpha\beta=[(x_{11}-x_{22})y_{12}-(y_{11}-y_{22})x_{12}\lambda]\cdot 2\sqrt{d}+r=2\sqrt{d}+r, \tag{8}$$

where α,β are each chosen as one of the two conjugates of the two elements in (6), (7), and r is a rational number. Taking norms in (8) leads to the result. The number r can be obtained by using (6) and (7) again, leading to

$$r=(x_{11}-x_{22})(y_{11}-y_{22})-4dx_{12}y_{12}. \tag{9}$$

■

Among problems that remain is a more detailed study of the quadratic fields $Q(\sqrt{m})$, $Q(\sqrt{n})$ in which d can be norm simultaneously. It is known that such a number need not be a norm of the biquadratic field $Q(\sqrt{m},\sqrt{n})$ with respect to Q; for an example see [2]. See also [3] and [4] for a general study of such questions.

Here the following result will be obtained in the special case under consideration.

THEOREM 5. *Let $d=-\det(AB-BA)$, where A,B are 2×2 integral matrices, and where the characteristic polynomial of A is x^2-m and that of B is x^2-n, $m,n\in Z$, and neither a square. Then d is a norm in $Q(\sqrt{m},\sqrt{n})$ if and only if there is an ideal in the ideal class in $Z[\sqrt{m}]$ corresponding to the matrix A and an ideal in the ideal class in $Z[\sqrt{n}]$ corresponding to the matrix B such that the quotient of their norms is a relative norm of an element of $Q(\sqrt{m},\sqrt{n})$ with respect to $Q(\sqrt{m})$.*

Proof. For the definition of matrix classes and their correspondence with ideal classes see [7]. There it is shown that the principal class in $Z[\sqrt{m}]$ corresponds to the class of the companion matrix of x^2-m. Further it is shown that the matrix class corresponding to any ideal class is obtained from a similarity carried out on the companion matrix via an ideal matrix of an ideal in the ideal class in question. The determinant (later called Δ in this proof) of an ideal matrix is the norm of the ideal. (Since multiplication by unimodular matrices is permitted for ideal matrices, this determinant may actually be assumed positive.) Let M be the ideal matrix with determinant Δ_M which transforms A into its companion matrix, and let N be the ideal matrix with determinant Δ_N, which transforms B into its companion matrix. Replace A by $M^{-1}AM$ and B by $N^{-1}BN$ and then apply a similarity via M to AB-BA leading to

$$\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix} MN^{-1} \begin{pmatrix} 0 & 1 \\ n & 0 \end{pmatrix} NM^{-1} - MN^{-1} \begin{pmatrix} 0 & 1 \\ n & 0 \end{pmatrix} NM^{-1} \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}.$$

Let $NM^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

A straightforward computation gives

$$d = -\det(AB-BA) = X(\Delta_M/\Delta_N)^2 \left\{ \left[na^2 - c^2 - m(-nb^2 + d^2) \right]^2 - m(2nab - 2cd)^2 \right\} = X(\Delta_M/\Delta_N)^2 \operatorname{norm} \lambda,$$

where "norm" here means from $Q(\sqrt{m})$ to Q, and where

$$\lambda = c^2 + md^2 - na^2 - mnb^2 + \sqrt{m}\,(2cd - 2abn).$$

Put

$$\alpha = c + d\sqrt{m} + a\sqrt{n} + b\sqrt{mn}\,,$$

$$\bar{\alpha} = c + d\sqrt{m} - a\sqrt{n} - b\sqrt{mn}\,.$$

Then $\alpha\bar{\alpha} = \lambda$. Hence d is a norm in $Q(\sqrt{m}, \sqrt{n})$ with respect to Q if and only if Δ or $-\Delta$ is a relative norm from this field with respect to Q. This proves Theorem 5. ■

An analogous fact holds for $Q(\sqrt{n})$, $Q(\sqrt{mn})$.

REMARK. If $Z = XY - YX$ and X has rational roots, then these roots can be chosen arbitrarily. This is a special case of a theorem proved in [1].

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