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Note

The Knowlton–Graham Partition Problem

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A set partition technique that is useful for identifying wires in cables can be recast in the language of 0–1 matrices, thereby resolving an open problem stated by R. L. Graham in Volume 1 of this journal. The proof involves a construction of 0–1 matrices having row and column sums without gaps. © 1996 Academic Press, Inc.

A long cable contains n indistinguishable wires. Two people, one at each end, want to label the wires consistently so that both ends of each wire receive the same label. An interesting way to achieve this was proposed by Knowlton [3]: Partition $\{1, ..., n\}$ into disjoint sets in two ways $A_1, ..., A_p$ and $B_1, ..., B_q$, subject to the condition that at most one element appears both in an A set of cardinality j and in a B set of cardinality k, for each j and k. We can then use the coordinates (j, k) to identify each element. Graham [2] proved that such partitioning schemes exist if and only if $n \neq 2$, 5, or 9.

By restating the problem in terms of 0–1 matrices, it is possible to prove Graham's theorem more simply and to sharpen the result of [2].

LEMMA 1. *Knowlton–Graham partitions for n exist if and only if there is a matrix of 0s and 1s having row sums* $(r_1, ..., r_m)$ *and column sums* $(c_1, ..., c_m)$ *such that* r_j *and* c_j *are multiples of* j *and* $r_1 + \cdots + r_m = c_1 + \cdots + c_m = n$.

Proof. If $A_1, ..., A_p$ and $B_1, ..., B_q$ are partitions of $\{1, ..., n\}$ with the Knowlton–Graham property, let a_{jk} be the number of elements that appear in an A set of cardinality j and a B set of cardinality k. Then a_{jk} is 0 or 1; and $r_j = \sum_k a_{jk}$ is j times the number of A sets of cardinality j, while $c_k = \sum_j a_{jk}$ is k times the number of B sets of cardinality k.

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Conversely, given such a matrix, we can use its rows to define $A_1, ..., A_p$ such that each 1 in row j is in an A set of cardinality j; similarly, its columns define $B_1, ..., B_q$ such that each 1 in column k is in a B set of cardinality k. ∎

For example, the symmetric matrix

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}$$

has row and column sums (2, 6, 3, 4, 5, 6) that satisfy the divisibility condition and sum to 26. To identify 26 wires, we can associate the 1s with arbitrary labels {a, ..., z},

```
· a · · · b
c d e f g h
· i · · j k
· l · m n o
· p q r s t
u v w x y z
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The person at one end of the cable labels the wires with {a, ..., z} arbitrarily and makes connections so that each element of row j is connected to exactly $j-1$ other elements of its row; for example, the connected components might be $A_1, ..., A_9$ = {a}, {b}, {c, d}, {e, f}, {g, h}, {i, j, k}, {l, m, n, o}, {p, q, r, s}, {u, v, w, x, y, z}. The person at the other end now uses properties of conductivity to tell what row each wire belongs to. The wires at that end can then be labeled {A, . . . , Z} in such a way that {A, B} = {a, b}, {C, D, E, F, G, H} = {c, d, e, f, g, h}, etc. Now the wires at the first end are disconnected, while at the other end they are connected so that each element of column k is connected to exactly $k-1$ other elements of its column. For example, the connected components might now be $B_1, ..., B_9$ = {C}, {U}, {A, D}, {I, L}, {P, V}, {E, Q, W}, {F, M, R, X}, {G, J, N, S, Y}, {B, H, K, O, T, Z}. Once this has been done, the people at both ends of the cable can give unique coordinates (j, k) to each wire, knowing its row and column.

Knowlton–Graham partitions are said to have order m if the largest cardinality of $A_1, ..., A_p$, $B_1, ..., B_q$ is m.

THEOREM 1. (Graham). *Knowlton–Graham partitions of n having order m are possible only if* $\binom{m+1}{2} \leqslant n \leqslant J(m)$, *where*

$$J(m) = \sum_{j=1}^{m} j\lfloor m/j \rfloor.$$

Proof. By Lemma 1, Knowlton–Graham partitions of order m imply the existence of an $m \times m$ matrix of 0s and 1s having row sums $(r_1, ..., r_m)$ and column sums $(c_1, ..., c_m)$, where both r_j and c_j are multiples of j for $1 \leqslant j \leqslant m$ and where $r_m + c_m > 0$. Clearly $r_j \leqslant m$; so r_j is at most $j\lfloor m/j \rfloor$, the largest multiple of j that does not exceed m. This establishes the upper bound $J(m)$.

If $r_m > 0$, we must have $r_m = m$; this implies $c_j > 0$ for all j; hence $c_m = m$. Similarly, $c_m > 0$ implies that $r_m = c_m = m$. So we must have $r_j > 0$ for all j; hence $r_j \geqslant j$ for all j; hence $n = \sum_{j=1}^{m} r_j \geqslant \sum_{j=1}^{m} j = \binom{m+1}{2}$. ∎

When $m = 1, 2, 3, 4$, Theorem 1 says that $1 \leqslant n \leqslant 1$, $3 \leqslant n \leqslant 4$, $6 \leqslant n \leqslant 8$, and $10 \leqslant n \leqslant 15$, respectively; this explains why the values $n = 2, 5, 9$ are impossible. For $m \geqslant 4$ we have $J(m) \geqslant \binom{m+2}{2}$, so there are no more gaps. In fact, as $m \to \infty$ we have

$$J(m) = m^2 - \sum_{j=1}^{m} (m \bmod j) = \frac{\pi^2}{12} m^2 + O(m \log m)$$

(see [4, Eq. 4.5.3–21]); therefore $J(m)/\binom{m+2}{2}$ approaches the limiting value $\pi^2/6 \approx 1.64$.

The main purpose of this note is to prove the converse of Theorem 1, namely that Knowlton–Graham partitions of order m do exist for all n in the range $\binom{m+1}{2} \leqslant n \leqslant J(m)$. This question was left open in [2], where Graham observed that it was not sufficient simply to represent n in the form $r_1 + \cdots + r_m$, where each r_j is a positive multiple of j. For example, there is no appropriate 0–1 matrix having row sums (1, 6, 6, 4, 5, 6). If there were, we would necessarily have $c_1 \leqslant 5$, $c_2 = 4$, $c_3 = 3$, $c_4 = 4$, $c_5 = 5$, $c_6 = 6$, and $c_1 + \cdots + c_6 < r_1 + \cdots + r_6$.

Gale [1] and Ryser [5] independently found an elegant necessary and sufficient condition for the existence of 0–1 matrices having given row and column sums, but their theorem does not seem to lead easily to the result needed here. Instead, we can use a direct recursive construction.

LEMMA 2. *Let* $(r_1, ..., r_m)$ *and* $(c_1, ..., c_m)$ *be integers satisfying the conditions*

$$r_1 + \cdots + r_m = c_1 + \cdots + c_m,$$

$$m \geqslant r_1 \geqslant \cdots \geqslant r_m \geqslant 0, \qquad m \geqslant c_1 \geqslant \cdots \geqslant c_m \geqslant 0,$$

$$r_{j+1} \geqslant r_j - 1, \qquad c_{j+1} \geqslant c_j - 1 \qquad \textit{for} \quad 1 \leqslant j < m.$$

Then there exists an $m \times m$ matrix of 0*s and* 1*s having row sums* $(r_1, ..., r_m)$ *and column sums* $(c_1, ..., c_m)$.

Proof. This is obvious when $m = 1$, so we may assume inductively that $m > 1$. Let $p = r_1$ and $q = c_1$ and consider the numbers

$$(r'_1, ..., r'_{m-1}) = (r_2 - 1, ..., r_q - 1, r_{q+1}, ..., r_m)$$
$$(c'_1, ..., c'_{m-1}) = (c_2 - 1, ..., c_p - 1, c_{p+1}, ..., c_m).$$

The lemma will be proved if we construct an $(m-1) \times (m-1)$ matrix of 0s and 1s having row sums $(r'_1, ..., r'_{m-1})$ and column sums $(c'_1, ..., c'_{m-1})$, because we can achieve the desired result by appending a new first row and a new first column. In fact it suffices, by row and column permutations, to construct a 0–1 matrix with row and column sums equal to the numbers $(r''_1, ..., r''_{m-1})$ and $(c''_1, ..., c''_{m-1})$ obtained by sorting $(r'_1, ..., r'_{m-1})$ and $(c'_1, ..., c'_{m-1})$ into nonincreasing order.

Since $r''_1 + \cdots + r''_{m-1} = r_1 + \cdots + r_m - p - q + 1 = c_1 + \cdots + c_m - p - q + 1 = c''_1 + \cdots + c''_{m-1}$, we can use the induction hypothesis if we verify that $r''_1 \leqslant m - 1$, $r''_{m-1} \geqslant 0$, $r''_{j+1} \geqslant r''_j - 1$; the similar inequalities for $(c''_1, ..., c''_{m-1})$ follow by symmetry.

Suppose $r''_1 = m$; this implies $r_{q+1} = m$ and $q < m$. Therefore, $r_q = \cdots = r_2 = r_1 = m$ and we have $(q+1)m \leqslant r_1 + \cdots + r_m = c_1 + \cdots + c_m \leqslant mc_1 = qm$, a contradiction.

Suppose $r''_{m-1} < 0$; this implies $r_q = 0$. Therefore $(q-1) + \cdots + 1 + 0 \geqslant r_1 + \cdots + r_m = c_1 + \cdots + c_m \geqslant q + (q-1) + \cdots + 1$, another contradiction.

Suppose finally that $r''_{j+1} < r''_j - 1$. This could happen only if $r''_j = r_k$ and $r''_{j+1} = r_l - 1$ for some k and l with $r_k > r_l$. But we would not decrease r_l unless we had also decreased r_k. ∎

The construction of Lemma 2 produces a symmetric matrix when $(r_1, ..., r_m) = (c_1, ..., c_m)$. Let us say that Knowlton–Graham partitions are *symmetric* if they correspond to a symmetric matrix. We are now ready to prove the main result.

THEOREM 2. *Symmetric Knowlton–Graham partitions of n having order m exist whenever* $\binom{m+1}{2} \leqslant n \leqslant J(m)$.

Proof. When n is in the stated range but not equal to $\binom{m+1}{2}$, there is a number $s \leqslant m/2$ such that we can write $n = t_1 + \cdots + t_m$, where

$$\begin{aligned} t_j &= j, && \text{for} \quad s < j \leqslant m; \\ t_s &= ks && \text{for some} \quad k,\ 1 < k \leqslant \lfloor m/s \rfloor, \\ t_j &= j\lfloor m/j \rfloor, && \text{for} \quad 1 < j < s; \\ m - s < t_1 &\leqslant m, && \text{if} \quad s > 1. \end{aligned}$$

When $n = J(m)$, this is true with $s = \lfloor m/2 \rfloor$, $k = \lfloor m/s \rfloor$, and $t_1 = m$. Otherwise we can find such a representation by first representing $n+1$ and subtracting 1 from t_1; then if $s > 1$ and $t_1 = m - s$, we replace t_1 by m and subtract s from t_s; finally, if $t_s = s$, we decrease s by 1.

The remaining case $n = \binom{m+1}{2}$ is simpler because we can write $n = t_1 + \cdots + t_m$, where $t_j = j$ for all j. This is a representation of essentially the same form but with $s = 0$.

Notice that t_j is a multiple of j, for $1 \leqslant j \leqslant m$. We can also verify that the set $\{t_1, ..., t_m\}$ consists simply of the consecutive elements $\{t_{s+1}, ..., t_m\} = \{s+1, ..., m\}$. For we have $t_s > s$ and $t_j > m - s \geqslant s$ for all $j < s$, because $j \lfloor m/j \rfloor = m - (m \bmod j)$.

Let $(r_1, ..., r_m)$ and $(c_1, ..., c_m)$ be the numbers $(t_1, ..., t_m)$ sorted into non-increasing order. Lemma 2 tells us how to construct a symmetric 0–1 matrix having these row and column sums. After an appropriate permutation of rows and columns, the row and column sums can be made equal to $(t_1, ..., t_m)$; this yields Knowlton–Graham partitions, by Lemma 1. ▮

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