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Unitarity in space–time noncommutative field theories

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Abstract

In noncommutative field theories conventional wisdom is that the unitarity is noncompatible with the perturbation analysis when time is involved in the noncommutative coordinates. However, as suggested by Bahns et al. recently, the root of the problem lies in the improper definition of the time-ordered product. In this article, functional formalism of S-matrix is explicitly constructed for the noncommutative ϕ^p scalar field theory using the field equation in the Heisenberg picture and proper definition of time-ordering. This S-matrix is manifestly unitary. Using the free spectral (Wightmann) function as the free field propagator, we demonstrate the perturbation obeys the unitarity, and present the exact two particle scattering amplitude for $(1+1)$-dimensional noncommutative nonlinear Schrödinger model.

1. Introduction

Quantum field theory on noncommutative spacetimes arises typically in restrictive phase space [1] and has some applications in condensed matter physics such as in quantum Hall effect [2]. This formalism has much more interesting features if the noncommuting coordinates involve time, i.e., noncommuting space–time. The framework of this noncommutative spaces can implement the possible deviations from the smoothness of spacetime at small distances and results in a modification of uncertainty relations for spacetime coordinates [3].

Despite this facinating possibility in space–time noncommutative field theories, in the perturbative field theories [4] it is asserted that the theories possess a serious problem, i.e., the lack of unitarity [5] and there are some attempts to cure this problem such as in the Hamiltonian picture [6].

Contrary to this view, Bahns et al. [7] recently pointed out that this unitarity problem is not inherent in the noncommutative field theories but rather due to the ill-defined time-ordered product expansion.

In this Letter we elaborate on this view. In Section 2, we present the S-matrix explicitly in the functional form and show how unitarity problems are cured. In terms of perturbative loop correction, the same result is presented in Section 3. As a further concrete example, we present exact 2-particle scattering amplitude for the noncommutative version of the integrable nonlinear Schrödinger model in $1+1$ dimension.

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2. *S*-matrix

Quantum field theory on the noncommutative space–time can be constructed into a nonlocal field theory on a commutative spacetime, using $\star$-product of fields. One of the convenient $\star$-product representations is the Moyal product,

$$f \star g(x) = e^{\frac{i}{2}\partial_x \wedge \partial_y} f(x) g(y)|_{y=x}, \tag{1}$$

where $a \wedge b = a_\mu \theta^{\mu\nu} b_\nu$. $\theta^{\mu\nu}$ is an antisymmetric c-number representing the space–time noncommutativeness, $i\theta^{\mu\nu} = [x^\mu, x^\nu]$. This Moyal product makes the kinetic term of the action the usual field theory, and allows the conventional perturbation with the proper vertex correction corresponding the nonlocal interaction [4].

We adopt a real scalar field theory for simplicity. The Lagrangian constitutes of the free part and interacting part. The interaction Lagrangian in $D-1$ space is given as

$$L_I(t) = -\frac{g}{p!} \int d^{D-1}x\, \phi_\star^p(x,t) \tag{2}$$

where g is a coupling constant. $\phi_\star^p = \phi \star \phi \star \cdots \star \phi$ is the noncommutative version of ϕ^p theory where p is a positive integer.

To construct the S-matrix, one assumes the out-going field satisfy the in-coming free field commutator relation

$$\left[\phi_{\rm in}(x), \phi_{\rm in}(0)\right] = i\Delta(x) \tag{3}$$

so that the in- and out- fields are related by

$$\phi_{\rm out} = S^{-1} \phi_{\rm in} S. \tag{4}$$

This relation is not, however, automatically satisfied. It is demonstrated in [8] that nonlocal field theories may not respect the assumption. The out-field commutator relation need be checked to be consistent.

We quantize the field using the Heisenberg picture [9]. The field at arbitrary time can be obtained from the field equation

$$\left(\Box + m^2\right)\phi(x) = \xi\left(\phi(x)\right), \tag{5}$$

where ξ is the functional of fields, derived from the interaction Lagrangian

$$\xi\left(\phi(x)\right) \equiv \frac{\delta}{\delta\phi(x)} \int dt\, L_I(t) = -\frac{g}{(p-1)!} \phi_\star^{p-1}(x). \tag{6}$$

Its solution is given using the retarded progator $\Delta_{\rm ret}(x) = -\theta(x^0)\Delta(x)$ (advanced propagator $\Delta_{\rm ad}(x) = \theta(-x^0)\Delta(x)$),

$$\begin{aligned} \phi(x) &= \phi_{\rm in}(x) + \Delta_{\rm ret} \circ \xi\left(\phi(x)\right) \\ &= \phi_{\rm out}(x) + \Delta_{\rm ad} \circ \xi\left(\phi(x)\right), \end{aligned} \tag{7}$$

where $\circ$ denotes the convolution, $\Delta_{\rm ret} \circ \xi(x) = \int d^D y\, \Delta_{\rm ret}(x-y)\xi(y)$.

Now the out-field can be put iteratively in terms of the in-field,

$$\phi_{\rm out}(x) = \phi_{\rm in}(x) - \Delta \circ \xi\left(\phi(x)\right), \tag{8}$$

if ϕ is written as $\phi = \phi_0 + \phi_1 + \phi_2 + \cdots$ where ϕ_n represents the order of g^n contribution. A few explicit solutions of ϕ_n's are given as

$$\phi_0(x) = \phi_{\rm in}(x),$$

$$\phi_1(x) = -\frac{g}{(p-1)!}\Delta_{\rm ret}\circ\phi_{0\star}^{(p-1)}(x),$$
$$\phi_2(x) = -\frac{g}{(p-1)!}\Delta_{\rm ret}\circ\left(\phi_1\star\phi_{0\star}^{(p-2)}+\phi_0\star\phi_1\star\phi_{0\star}^{(p-3)}+\cdots+\phi_{0\star}^{(p-2)}\star\phi_1\right)(x).$$

As $x^0\to\infty$ the fields $\phi(x)$ reduces to the out-field $\phi_{\rm out}$ and $\Delta_{\rm ret}(x)\to-\Delta$ in consistent with Eq. (8).

We have checked explicitly the commutator relation of the out-field $\phi_{\rm out}(x)$ in Eq. (8) up to the order of $O(g^4)$, at which order the unitarity problem arises in the nonlocalized QED and Yukawa coupling [8]. It turns out that as the free commutation relation holds for the out-field of an action without star-product, so does for the out-field of the action with $\star$-product. All the higher order terms cancel out independent of the $\star$-product. We expect the result holds for all orders. This justifies the assumption of the unitary S-matrix between in- and out-fields.

With the notation $S=e^{i\delta}$, the out-field would be written as

$$\phi_{\rm out}=S^{-1}\phi_{\rm in}S=\phi_{\rm in}+[\phi_{\rm in},i\delta]+\frac{1}{2}\big[[\phi_{\rm in},i\delta],i\delta\big]+\cdots. \tag{9}$$

The first order term in g results in the equation, $[\phi_{\rm in},i\delta]=-\Delta\circ\xi(\phi_{\rm in}(x))$, and determines δ to the first order in g as

$$\delta=\int_{-\infty}^{\infty}dt\,L_I\big(\phi_{\rm in}(t)\big)+O\big(g^2\big). \tag{10}$$

Higher order solutions require the time-ordering as in the ordinary field theory. However, the $\star$-product introduces a subtlety in the time-ordering and a consistent unitary S-matrix is given as

$$S=1+i\int_{-\infty}^{\infty}dt\,\mathcal{F}_1\big(V\big(\phi_{\rm in}(t)\big)\big)+i^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dt_1\,dt_2\,\mathcal{F}_{12}\big(\theta_{12}V\big(\phi_{\rm in}(t_1)\big)V\big(\phi_{\rm in}(t_2)\big)\big)\cdots$$
$$+\,i^n\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dt_1\cdots dt_n\,\mathcal{F}_{12\ldots n}\big(\theta_{12\ldots n}V\big(\phi_{\rm in}(t_1)\big)\cdots V\big(\phi_{\rm in}(t_n)\big)\big)+\cdots. \tag{11}$$

$V(\phi_{\rm in}(t))$ is interaction Lagrangian before $\star$-product,

$$V\big(\phi_{\rm in}(t)\big)\equiv-\frac{g}{p!}\int d^{D-1}x\,\phi_{\rm in}^p(x,t),$$

and the time-ordering is given in terms of the step function,

$$\theta_{12\ldots n}=\theta(t_1-t_2)\theta(t_2-t_3)\ldots\theta(t_{n-1}-t_n).$$

$\star$-operation $\mathcal{F}_{12\ldots n}\equiv\mathcal{F}_1\mathcal{F}_2\cdots\mathcal{F}_n$ introduces the $\star$-product to the actions

$$\mathcal{F}_{12\ldots n}\big(V(t_1)V(t_2)\cdots V(t_n)\big)=L_I(t_1)L_I(t_2)\cdots L_I(t_n), \tag{12}$$

whose operation is independent of the permutation of the action. In the presence of the step-function, we assume a minimal realization. For example, explicitly we put

$$\mathcal{F}_{xy}\big(\theta\big(x^0-y^0\big)\phi^p(x)\phi^p(y)\big)=\mathcal{F}_x\mathcal{F}_y\big(\theta\big(x^0-y^0\big)\phi(x_1)\cdots\phi(x_p)\phi(y_1)\cdots\phi(y_p)\big)\big|_{x_i=x,\,y_i=y},$$

where

$$\mathcal{F}_x\equiv\exp\left(\frac{i}{2}\big(\partial_{x_1}\wedge(\partial_{x_2}+\cdots+\partial_{x_p})+\partial_{x_2}\wedge(\partial_{x_3}+\cdots+\partial_{x_p})+\cdots+\partial_{x_{p-1}}\wedge\partial_{x_p}\big)\right)$$

and $\theta(x^0-y^0)$ is put to $\theta(x_i^0-y_j^0)$ in the presence of the spectral function $\Delta(x_i^0-y_j^0)$. We emphasize that our minimal realization assumption is that the time-ordering step function is used only once between two vertices. So, in the presence of many spectral functions which connect two vertices we have only one step function,

$$\theta\big(x^0-y^0\big)\prod_{i,j}\Delta(x_i-y_j)\to\theta\big(x_a^0-y_b^0\big)\prod_{i,j}\Delta(x_i-y_j),$$

where a (b) is just one of indices among i's (j's). This operation is done explicitly in Eqs. (16) and (21) below.

Introducing the time-ordering with $\star$-product,

$$T_\star\big\{V(t_1)V(t_2)\big\}=\mathcal{F}_{12}\big(\theta_{12}V(t_1)V(t_2)+\theta_{21}V(t_2)V(t_1)\big) \tag{13}$$

we can put the S-matrix as

$$S=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{-\infty}^{\infty}dt_1\cdots\int_{-\infty}^{\infty}dt_n\,T_\star\big\{V\big(\phi_{\rm in}(t_1)\big)\cdots V\big(\phi_{\rm in}(t_n)\big)\big\}\equiv T_\star\exp\left(i\int_{-\infty}^{\infty}dt\,V\big(\phi_{\rm in}(t)\big)\right). \tag{14}$$

One can check order by order that this S-matrix is unitary $S^{-1}=S^\dagger$ and reproduces the in- and out-field relation Eq. (8). We present here the sketch of the proof of unitarity of the S-matrix up to the order of g^2. The higher order proof goes similarly with the ordinary perturbation case since in this proof only the time-ordering matters irrespective of the $\star$-operation. The unitarity of the S-matrix in Eq. (11) is proved if the following identity is satisfied: $A_2+A_2^\dagger=A_1^\dagger A_1=A_1^2$ where

$$A_1=\int_{-\infty}^{\infty}dt_1\,\mathcal{F}_1(V_1),\qquad A_2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dt_1\,dt_2\,\mathcal{F}_{12}(\theta_{12}V_1V_2).$$

The proof goes as follows:

$$\begin{aligned}A_2+A_2^\dagger&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dt_1\,dt_2\,\mathcal{F}_{12}\big(\theta_{12}(V_1V_2+V_2V_1)\big)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dt_1\,dt_2\,\mathcal{F}_{12}\big((\theta_{12}+\theta_{21})V_1V_2\big)\\&=\int_{-\infty}^{\infty}dt_1\,\mathcal{F}_1(V_1)\int_{-\infty}^{\infty}dt_2\,\mathcal{F}_1(V_2)=A_1^\dagger A_1,\end{aligned} \tag{15}$$

where we use the change of variables to get the second identity and the identity $\theta_{12}+\theta_{21}=1$ for the last identity.

On the other hand, the out field is obtained from the S-matrix relation:

$$\begin{aligned}S^\dagger\phi_{\rm in}(x)S&=\phi_0(x)+i\int dy\,\big(\phi_0(x)A_1(y)-A_1(y)\phi_0(x)\big)\\&\quad+i^2\int dy_1\,dy_2\,\big(\phi_0(x)A_2(y_1,y_2)-A_1(y_1)^\dagger\phi_0A_1(y_2)+A_2(y)^\dagger\phi_0(x)\big)+O\big(g^3\big)\\&=\phi_0+i\int dy_1\,\mathcal{F}_1\big(\big[\phi_0(x),V(y_1)\big]\big)+i^2\int dy_1\,dy_2\,\mathcal{F}_{12}\big(\theta_{12}\big[[\phi_0(x),V(y_1)],V(y_2)\big]\big)+O\big(g^3\big).\end{aligned} \tag{16}$$

It is clear that the out field relation in Eq. (8) up to the order g^2 is reproduced in Eq. (16) if one uses the commutation of the fields $[[\phi_0(x),V(y_1)],V(y_2)]$ and the time-ordering step function θ_{12} before performing the $\star$-operation.

We give some comments on other approaches of finding the unitary S-matrix. First, one may start with the time-ordering outside the $\star$-operation as in [7], then one may add higher derivatives in order to reproduce the above

S-matrix Eq. (11). For example, we put $A_2 = a_2 + ic_2$ at the order g^2,

$$a_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \theta_{12}\mathcal{F}_{12}(V_1 V_2), \qquad ic_2 = -\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\theta_{12}\mathcal{F}_{12} - \mathcal{F}_{12}\theta_{12})\big([V_1, V_2]\big). \tag{17}$$

a_2 is the ordinary time-ordered one and $a_2 + a_2^\dagger = A_1^2$. The correction term c_2 satisfies the relation $c_2 = c_2^\dagger$ (note that the † operation is applied to the field ϕ not the time-ordering or $\star$-operation) and provides the higher derivative terms if one evaluates the commutator of the step function and the $\star$-product, which leaves the time derivatives of the fields as well as of the spectral functions. One sees the similar behavior for higher order terms, which will be published elsewhere.

Second, given the S-matrix of Eq. (11), the scattering amplitudes can be constructed as a perturbative series in the coupling constant. This S-matrix is obtained using the Lagrangian formalism in the Heisenberg picture. The equivalence of the Hamiltonian formalism such as in [6] is not easy to see since the symplectic structure is not simply tractable due to the explicit time dependence of fields in the interaction Lagrangian.

Third, suppose one tries to obtain an interaction field at time t from the in-field. In the ordinary interaction picture one defines the unitary transformation,

$$\phi_I(t) = U(t)^\dagger \phi_{\rm in}(t) U(t) \tag{18}$$

with $S = \lim_{t\to\infty} U(t)$. Requiring the dynamical evolution both for the in and interaction fields, $\dot\phi_{\rm in}(t) \equiv [-iL_0(\phi_{\rm in}), \phi_{\rm in}(t)]$ and $\dot\phi_I(t) \equiv [-iL(\phi_I), \phi_I(t)]$, one would obtain the dynamical equation for the unitary operator, $\dot U(t) = iL_I(\phi_{\rm in}(t))U(t)$, on the condition that

$$UL(\phi_I)U^\dagger = L(\phi_{\rm in}). \tag{19}$$

However, this condition is not compatible with the Eq. (18) due to the space–time noncommutative $\star$-product of the action. The unitary operator $U(t)$ does not transform the in-field action to interaction field action. The same conclusion also goes for Heisenberg picture. Nevertheless, the difficulty of constructing the unitary operator does not mean that one cannot construct S-matrix. The transformation between in- and out-field Eq. (4) is enough for the existence of S-matrix Eq. (11).

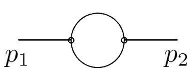
3. Propagator and unitarity

To illustrate the point described in Section 2 more concretely, we will consider ϕ^3 theory,

$$L_I(t) = -\frac{g}{3!}\int d^{D-1}x\, \frac{1}{2}\big(\phi_\star^3(x,t) + \text{h.c.}\big) \tag{20}$$

and calculate the one-loop contribution to the propagator in momentum space. The momentum space calculation will be complementary with the coordinate space representation given in Section 2.

The connected one loop contribution to the self-energy with external momentum p_1 and p_2 is given from the second term of S-matrix in Eq. (11), denoted as S_2 in the following:

$\langle p_1|S_2|p_2\rangle_c =$  $= -\frac{1}{2}\iint d^Dx\, d^Dy\, \langle p_1|T_\star\big(V\big(\phi_{\rm in}(t_1)\big)V\big(\phi_{\rm in}(t_2)\big)\big)|p_2\rangle_c,$

where $\langle\cdots\rangle_c$ refers to the one-particle irreducible function. Using the one particle representation, $\langle p|\phi_{\rm in}(x)|0\rangle = Ne^{ipx}$ with N a proper normalization constant, and the integration representation of the step function

$$\theta(t) = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi i}\frac{e^{-i\omega t}}{\omega + i\epsilon}$$

we have

$$\langle p_1|S_2|p_2\rangle_c = -\left(\frac{g}{3!}\right)^2 \iint d^D x\, d^D y \langle p_1|\mathcal{F}_{xy}\big(\theta\big(x^0 - y^0\big)\phi_0^3(x)\phi_0^3(y)\big)|p_2\rangle_c$$
$$= \left(\frac{g}{3!}\right)^2 \int \cdots \int \frac{d^D x\, d^D y\, d^D k\, d^D l\, d\omega}{(2\pi i)(2\pi)^{2D}(\omega + i\epsilon)} e^{ix(p_1-k-l-\omega)-iy(p_2-k-l-\omega)}$$
$$\times |N|^2 \tilde{\Delta}_+(k)\tilde{\Delta}_+(l) \sum_{\{a\}\{b\}} \cos\left(\frac{a_2 \wedge a_3}{2}\right)\cos\left(\frac{b_2 \wedge b_3}{2}\right) + p_1 \leftrightarrow p_2. \tag{21}$$

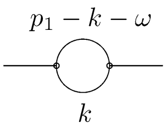
The summation is over the set of momenta, $\{a\}$ and $\{b\}$,

$$\big\{(a_1, a_2, a_3)\big\} = \big\{(p_1, -k, -l - \omega), (-k, p_1, -l - \omega), (-k, -l, p_1)\big\},$$
$$\big\{(b_1, b_2, b_3)\big\} = \big\{(-p_2, k, l + \omega), (k, -p_2, l + \omega), (k, l + \omega, -p_2), k \leftrightarrow l\big\}$$

and $\tilde{\Delta}_+(k) = 2\pi\delta(k^2 - m^2)\theta(k^0)$ is the Fourier transform of the free spectral function,

$$\Delta_+(x) = \langle 0|\phi_{\text{in}}(x)\phi_{\text{in}}(0)|0\rangle = \int \frac{d^D k}{(2\pi)^D} e^{-ikx} \tilde{\Delta}_+(k). \tag{22}$$

Integrating over coordinates x and y, we are left with the momentum representation,

$\langle p_1|S_2|p_2\rangle_c = p_1$  p_2

$$= \frac{g^2}{2}(2\pi)^D \delta^D(p_1 - p_2) \iint \frac{d^D k\, d^D l\, d\omega}{(2\pi)^{2D}(2\pi i)(\omega + i\epsilon)}$$
$$\times (2\pi)^D \delta^D(p_1 - k - l - \omega)|N|^2 \tilde{\Delta}_+(k)\tilde{\Delta}_+(l) \cos^2\left(\frac{p_1 \wedge l}{2}\right). \tag{23}$$

This result shows that the external energy-momentum is manifestly conserved. However, the internal momentum need not be conserved; there appears the spurious momentum ω in the internal vertex, which traces back to the noncommutativeness of space and time coordinates. One may avoid this unpleasant feature by introducing the retarded positive spectral function,

$$\theta(x^0)\Delta_+(x) = \int \frac{d^D k}{(2\pi)^D} e^{-ikx} \tilde{\Delta}_R(k), \quad \tilde{\Delta}_R(k) = \frac{i}{2\omega_k} \frac{1}{(k_0 - \omega_k + i\epsilon)},$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$. In terms of this retarded function, we have Eq. (23) as

$$\langle p_1|S_2|p_2\rangle_c = \frac{g^2}{2}(2\pi)^D \delta^D(p_1 - p_2) \int \frac{d^D k}{(2\pi)^{2D}} |N|^2 \tilde{\Delta}_R(k)\tilde{\Delta}_+(p - k) \cos^2\left(\frac{p_1 \wedge k}{2}\right). \tag{24}$$

The real part of the S-matrix is given as

$$\langle p_1|S_2 + S_2^\dagger|p_2\rangle_c = -(2\pi)^D \delta^D(p_1 - p_2) F_+(p_1), \tag{25}$$

where

$$F_+(p) = g^2 \int \frac{d^D k}{(2\pi)^D} |N|^2 \tilde{\Delta}_+(k)\tilde{\Delta}_+(p_1 - k) \cos^2\left(\frac{p_1 \wedge k}{2}\right)$$

due to the identity $\frac{1}{\omega+i\epsilon} = P(\frac{1}{\omega}) - i\pi\delta(\omega)$. On the other hand, $SS^\dagger$ of the order g^2 comes from the first term in the S-matrix Eq. (11):

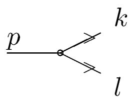
$$\begin{aligned}\langle p_1|S_1S_1^\dagger|p_2\rangle_c &= \frac{g^2}{2}\int\cdots\int \frac{d^Dxd^Dyd^Dkd^Dl}{(2\pi)^{2D}}|N|^2\tilde{\Delta}_+(k)\tilde{\Delta}_+(l) \\ &\quad\times e^{ix(p_1-k-l)-iy(p_2-k-l)}\cos^2\left(\frac{p_1\wedge k}{2}\right) + p_1 \leftrightarrow p_2 \\ &= (2\pi)^D\delta^D(p_1-p_2)F_+(p_1).\end{aligned} \tag{26}$$

This demonstrates the unitarity relation up to the one-loop order:

$$\langle p_1|S_2+S_2^\dagger|p_2\rangle_c + \langle p_1|S_1S_1^\dagger|p_2\rangle_c = 0. \tag{27}$$

In other words, the one-loop correction $F_+(p)$ is written in terms of on-shell particles only,

$$F_+(p) = \sum_{\substack{P l^0>0,\, l^2=m^2 \\ k^0>0,\, k^2=m^2}} \left| \text{[diagram]} \right|^2. \tag{28}$$



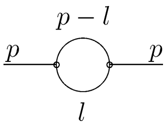
$F_+(p)$ gives a finite contribution when $p^2 > 4m^2$. In CM ($p^0 = E$, $\vec{p} = 0$), this gives

$$F_+(p) = (4\pi)^{2-D}\frac{(E^2-4m^2)^{(D-3)/2}}{2E}\int d\Omega\cos^2\left(\frac{p\wedge l}{2}\right). \tag{29}$$

One might think that using the property of the Feynman propagator $i\Delta_F(x) = \theta(x^0)\Delta_+(x) + \theta(-x^0)\Delta_-(x)$;

$$-\big(\Delta_F(x)\big)^2 = \theta\big(x^0\big)\big(\Delta_+(x)\big)^2 + \theta\big(-x^0\big)\big(\Delta_-(x)\big)^2, \tag{30}$$

the one-loop contribution Eq. (21) can be rewritten in terms of the Feynman propagator instead of the spectral function used in Eq. (23),



$$\begin{aligned}G(p) &= \text{[diagram]} \\ &= -\frac{g^2}{4}\delta^D(p_1-p_2)\iint d^Dk\,d^Dl\,\delta^D(p_1-k-l)|N|^2\tilde{\Delta}_F(k)\tilde{\Delta}_F(l)\cos^2\left(\frac{k\wedge l}{2}\right) \\ &= \frac{g^2}{4}\delta^D(p_1-p_2)\int d^Dl\frac{|N|^2\cos^2(\frac{p_1\wedge l}{2})}{((p-l)^2-m^2+i\varepsilon)(l^2-m^2+i\varepsilon)},\end{aligned} \tag{31}$$

as has been carried out in [5]. The two approaches are equivalent if the noncommutativeness involves in the space coordinates only ($\theta^{0i} = 0$). In this case the $\star$-operation and the time-ordering commutes with each other and Eq. (30) is allowed.

However, for the problematic space–time noncommutative case ($\theta^{0i} \neq 0$), two approaches are not the same anymore. In this case, the time ordering need to be done before $\star$-operation and Eq. (30) is not justified since

$$\begin{aligned}&-\Delta_F(x_1-y_1)\Delta_F(x_2-y_2) \\ &\quad\neq \theta\big(x_1^0-y_1^0\big)\Delta_+(x_1-y_1)\Delta_+(x_2-y_2) + \theta\big(-x_1^0+y_1^0\big)\Delta_-(x_1-y_1)\Delta_-(x_2-y_2), \\ &-\Delta_F(x_1-y_1)\Delta_F(x_2-y_2) \\ &\quad\neq \theta\big(x_1^0-y_1^0\big)\theta\big(x_2^0-y_2^0\big)\Delta_+(x_1-y_1)\Delta_+(x_2-y_2) \\ &\qquad+ \theta\big(-x_1^0+y_1^0\big)\theta\big(-x_2^0+y_2^0\big)\Delta_-(x_1-y_1)\Delta_-(x_2-y_2),\end{aligned}$$

and there are cross terms. Some of this step functions are ill-defined once the $\star$-operation is performed and the x_i's (y_i's) are identified as x (y), and some of the step functions provide additional contribution to the final result. From this behavior, it is not surprising to see that the Feynamn rule will not be the naive generalization such as in Eq. (31). In contrast to this, the use of the spectral function $\Delta_\pm$ with the appropriate time-ordering takes care of the subtleties and results in the correct unitarity condition.

The similar one-loop result can be used to check the unitarity of the scattering matrix in $\phi_\star^p$ theory. And one can perform higher loop calculation without any conceptual difficulty. We back up this idea further using an integrable field theory. In $1+1$ dimension, nonrelativistic nonlinear Schrödinger model is known to be integrable and its exact S-matrix is known [10]. Here, we give the exact two-particle scattering matrix for the noncommutative version of the model with $\theta^{01} = \theta\varepsilon^{01}$. This model is the $(1+1)$-dimensional version of the nonrelativistic ϕ^4 theory [11].

4. Non-relativistic nonlinear Schrödinger model in 1 + 1 dimension

The free Lagrangian of this model is the conventional Schrödinger one and the interaction Lagrangian is given as

$$L_I(t) = -\frac{v}{4}\int d\mathbf{x}\,\psi^\dagger \star \psi^\dagger \star \psi \star \psi(t,\mathbf{x}), \tag{32}$$

where we use the bold-face letter for spatial vector to distinguish from the 2-vector. The in-field ψ_{in} satisfies the commutation relation, $[\psi_{\text{in}}(\mathbf{x},t), \psi_{\text{in}}^\dagger(\mathbf{y},t)] = \delta(\mathbf{x}-\mathbf{y})$ and is given in momentum space,

$$\psi_{\text{in}}(x) = \int \frac{d^2k}{(2\pi)^2}\tilde{D}_+(k)a(\mathbf{k})e^{-ikx}, \qquad \psi_{\text{in}}^\dagger(x) = \int \frac{d^2k}{(2\pi)^2}\tilde{D}_+(k)a^\dagger(\mathbf{k})e^{ikx}, \tag{33}$$

with $[a(\mathbf{k}), a^\dagger(\mathbf{l})] = 2\pi\delta(\mathbf{k}-\mathbf{l})$ and $\tilde{D}_+(p) = 2\pi\delta(p^0 - \mathbf{p}^2/2)$. In this noncommutative case also, the particle number operator $\mathcal{N} = \int d\mathbf{x}\,\psi^\dagger\psi$ is conserved and this simplifies the perturbative calculation greatly. The propagator is given in terms of the positive spectral function,

$$D_+(x) = \langle 0|\psi_{\text{in}}(x)\psi_{\text{in}}^\dagger(0)|0\rangle = \int \frac{d^2p}{(2\pi)^2}e^{-ipx}\tilde{D}_+(p). \tag{34}$$

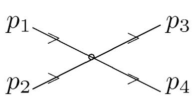
The time-ordering in the S-matrix is simplified due to the absence of anti-particles in this nonrelativistic case,

$$\begin{aligned} D_R(x) &= \theta\big(x^0\big)\langle 0|\psi_{\text{in}}(x)\psi_{\text{in}}^\dagger(0)|0\rangle \\ &= -\int_{-\infty}^{\infty}\frac{d\omega}{2\pi i}\frac{e^{-i\omega x^0}}{\omega + i\epsilon}\int\frac{d^2p}{(2\pi)^2}e^{-ipx}\tilde{D}_+(p) = \int\frac{d^2p}{(2\pi)^2}e^{-ipx}\tilde{D}_R(p) \end{aligned} \tag{35}$$

with $\tilde{D}_R(p) = i/(p^0 - \mathbf{p}^2/2 + i\epsilon)$.

The four point vertex is given as

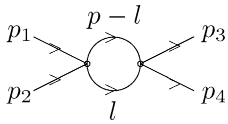
$$\Gamma_0(p_1, p_2; p_3, p_4) =$$



$$= -iv(2\pi)^2\delta^2(p_1 + p_2 - p_3 - p_4)\cos\left(\frac{p_1\wedge p_2}{2}\right)\cos\left(\frac{p_3\wedge p_4}{2}\right). \tag{36}$$

One-loop correction to the vertex is given as

$$\Gamma_1(p_1, p_2; p_3, p_4) =$$



$$= -\frac{v^2}{2}(2\pi)^2\delta^2(p_1 + p_2 - p_3 - p_4)\xi(p_1, p_2)\cos\left(\frac{p_1 \wedge p_2}{2}\right)\cos\left(\frac{p_3 \wedge p_4}{2}\right), \tag{37}$$

where ξ is defined as

$$\xi(p_1, p_2) = \int \frac{d^2 l}{(2\pi)^2}\tilde{D}_R(l)\tilde{D}_+(p - l)\cos^2\left(\frac{l \wedge p}{2}\right)$$

with $p = p_1 + p_2 = p_3 + p_4$. When p_1 and p_2 are on-shell, its value is given by

$$\xi(p_1, p_2) = \frac{1}{|\mathbf{p}_1 - \mathbf{p}_2|}\cos\left(\frac{\theta|\mathbf{p}_1||\mathbf{p}_2||\mathbf{p}_1 - \mathbf{p}_2|}{4}\right)e^{\frac{i\theta|\mathbf{p}_1||\mathbf{p}_2||\mathbf{p}_1 - \mathbf{p}_2|}{4}}. \tag{38}$$

Higher loop corrections are given in chained bubble diagrams and the complete loop corrections to the vertex are given in the geometric sum,

$$\Gamma(p_1, p_2; p_3, p_4) = \Gamma_0(p_1, p_2; p_3, p_4)\left(1 + \left(\frac{-iv\xi(p_1, p_2)}{2}\right) + \left(\frac{-iv\xi(p_1, p_2)}{2}\right)^2 \cdots\right)$$

$$= (2\pi)^2\delta^2(p_1 + p_2 - p_3 - p_4)\cos\left(\frac{p_1 \wedge p_2}{2}\right)\cos\left(\frac{p_3 \wedge p_4}{2}\right)\frac{-iv}{1 + i\frac{v}{2}\xi(p_1, p_2)}. \tag{39}$$

From this one obtains the on-shell 2-particle scattering amplitude,

$$\langle p_3, p_4|S|p_1, p_2\rangle_{(2,2)} = \big(\delta(\mathbf{p}_1 - \mathbf{p}_3)\delta(\mathbf{p}_2 - \mathbf{p}_4) + \delta(\mathbf{p}_1 - \mathbf{p}_4)\delta(\mathbf{p}_2 - \mathbf{p}_3)\big)S_{(2,2)},$$

$$S_{(2,2)} = 1 + \left(\frac{\xi(p_1, p_2) + \xi^*(p_1, p_2)}{2}\right)\left(\frac{-iv}{1 + i\frac{v}{2}\xi(p_1, p_2)}\right) = \frac{1 - i\frac{v}{2}\xi^*(p_1, p_2)}{1 + i\frac{v}{2}\xi(p_1, p_2)}. \tag{40}$$

This exact scattering matrix is manifestly unitary, $S^\dagger_{(2,2)} = S^{-1}_{(2,2)}$, and smoothly reduces to the commutative field theoretical value if we put the noncommutative parameter $\theta = 0$.

To summarize, we have demonstrated how the perturbative analysis in the space–time noncommutative field theories respects the unitarity if S-matrix is defined with the proper time-ordering and the free spectral function is used instead of the Feynman propagator.

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References

[1] S.M. Girvin, T. Jach, Phys. Rev. B 29 (1984) 5617;
G.V. Dunne, R. Jackiw, C.A. Trugenberger, Phys. Rev. D 41 (1990) 661;
N. Read, Phys. Rev. B 58 (1998) 16262;
D. Bigatti, L. Susskind, Phys. Rev. D 62 (2000) 066004.

[2] L. Susskind, hep-th/0101029;
B.H. Lee, K.S. Moon, C. Rim, Phys. Rev. D 64 (2001) 085014;
A.P. Polychronakos, J. High Energy Phys. 0104 (2001) 011.
[3] N. Seiberg, E. Witten, J. High Energy Phys. 9909 (1999) 032, and references therein.
[4] T. Filk, Phys. Lett. B 376 (1996) 53.
[5] J. Gomis, T. Mehen, Nucl. Phys. B 591 (2000) 265;
L. Alvarez-Gaume, J.L. Barbon, R. Zwicky, J. High Energy Phys. 0105 (2001) 057;
C.S. Chu, J. Lukierski, W.J. Zakrzewski, hep-th/0201144.
[6] J. Gomis, K. Kamimura, J. Llosa, Phys. Rev. D 63 (2001) 045003.
[7] D. Bahns, S. Doplicher, K. Fredenhagen, G. Piacitelli, Phys. Lett. B 533 (2002) 178.
[8] T. Imamura, S. Sunakawa, R. Utiyama, Prog. Theor. Phys. 11 (1954) 291;
R. Marnelius, Phys. Rev. D 10 (1974) 3411.
[9] C.N. Yang, D. Feldman, Phys. Rev. 79 (1950) 972.
[10] E.H. Lieb, W. Liniger, Phys. Rev. 130 (1963) 1605;
H.B. Thacker, Rev. Mod. Phys. 53 (1981) 253.
[11] D. Bak, S.K. Kim, K.-S. Soh, J.H. Yee, Phys. Rev. Lett. 85 (2000) 3087.