Elementary operators and the Aluthge transform

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ABSTRACT

We characterize essential normality for certain elementary operators acting on the Hilbert-Schmidt class. We find the Aluthge transform of an elementary operator of length one. We show that the Aluthge transform of an elementary 2-isometry need not be a 2-isometry. We also characterize hermitian elementary operators of length two.

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1. Introduction

Let $\mathcal{H}$ denote a separable complex Hilbert space and $B(\mathcal{H})$ the bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, $T^*$ denotes the adjoint of $T$. The Hilbert-Schmidt Class, $C_2(\mathcal{H})$, is the vector space of all compact operators $S$ defined on a separable complex Hilbert space $\mathcal{H}$, satisfying the following condition, cf. [17]: If $\{\varphi_k : k \in J\}$ is an orthonormal subset of $\mathcal{H}$, then $\sum_{k \in J} |\langle S\varphi_k, \varphi_k \rangle|^2 < +\infty$. It is well known that $C_2(\mathcal{H})$ equipped with the inner product $\langle S, T \rangle = \text{tr}(ST^*)$, the trace of $ST^*$, is a Hilbert space, see [16]. If $A$ and $B$ are bounded operators on $\mathcal{H}$ then $\mathcal{L}(T) = ATB$ is a bounded operator on $C_2(\mathcal{H})$. Such an operator $\mathcal{L}$ is an example of an elementary operator with associated symbols $A$ and $B$. In this paper we investigate various structural properties of elementary operators on $C_2(\mathcal{H})$. The goal is to relate the properties of the operator to those of the symbols defining it. This program is
similar to what has been done for composition operators on analytic function spaces, see [7]. Using some powerful results of Fong and Sourour [8], we obtain a characterization of essential normality of elementary operators of length one. We also obtain characterizations of various generalizations of isometries and projections, see Propositions 2.1 and 2.2.

In Section 3, we find the polar representation of an elementary operator of length one. This allows us to explore the Aluthge transform of these elementary operators. For $T \in B(\mathcal{H})$, let the polar decomposition of $T$ be given as $T = U|T|$, where $|T| = (T^*T)^{1/2}$. The Aluthge transform for $T$ is given by $\tilde{T} = |T|^{1/2}U|T|^{1/2}$. This transform was introduced by Aluthge in [4] to study “p-hyponormal operators”. Recall that a bounded linear operator $T$ is $p$-hyponormal whenever $(TT^*)^p > (T^*T)^p$ for some $p \in (0, \infty)$. When $p = 1$, $T$ is said to be hyponormal. Also recall that an operator $T \in B(\mathcal{H})$ is a 2-isometry if $(T^*)^2T^2 - 2T^*T + I = 0$. In this section we give an example of an elementary operator of length one which is a 2-isometry but its Aluthge transform is not.

In the final section of the paper we address some properties of elementary operators of length two. In particular, we characterize the hermitian elementary operators of length two. Our work was motivated by a result in [14] where it is noted that the elementary operator $L(T) = ATB^* + A^*TB$ is hermitian for any $A \in B(\mathcal{H})$. Our characterization includes this operator as a special case.

We begin by formally defining an elementary operator of length $n$. Given two sequences of linearly independent and bounded operators on $\mathcal{H}$, $\{A_i\}_{i=1, \ldots, n}$ and $\{B_i\}_{i=1, \ldots, n}$, we define the elementary operator $L$, on $C_2(\mathcal{H})$, as follows:

$$L(T) = \sum_{i=1}^{n} A_iTB_i.$$ 

The integer $n$ is called the length of $L$ and the operators $A_1, \ldots, A_n, B_1, \ldots, B_n$ are the associated symbols. In particular, elementary operators of length one are of the form $L(T) = ATB$, with nonzero symbols $A$ and $B$.

It is a straightforward calculation to show that the adjoint $L^*$ on $C_2(\mathcal{H})$ is given by $L^*(T) = \sum_{i=1}^{n} A_i^*TB_i^*$.

In this paper, we study operator theoretic properties of elementary operators and how they relate to those of the associated symbols. The main tool in our proofs are two theorems due to Fong and Sourour, cf. [8].

**Theorem 1.1** (Fong and Sourour). Let $\Phi(T) = A_1TB_1 + A_2TB_2 + \cdots + A_mB_m$, with $T$ an operator in $C_2(\mathcal{H})$.

1. If $\{B_1, B_2, \ldots, B_n\}$ ($n \leq m$) is linearly independent, and $(c_{ij})$ denote constants for which $B_j = \sum_{k=1}^{n} c_{kj}B_k$ ($n + 1 \leq j \leq m$), then $\Phi \equiv 0$ if and only if $A_k = -\sum_{j=n+1}^{m} c_{kj}A_j$ ($1 \leq k \leq n$). If $m = n$, then $A_1, A_2, \ldots, A_m$ are equal to zero.

2. If $\{B_1, B_2, \ldots, B_n\}$ ($n \leq m$) is linearly independent, $(c_{ij})$ denote constants for which $B_j = \sum_{k=1}^{n} c_{kj}B_k$ ($n + 1 \leq j \leq m$) and $\Phi$ is compact, then $A_k + \sum_{j=n+1}^{m} c_{kj}A_j$ ($1 \leq k \leq n$) are compact operators. If $m = n$, then $A_1, A_2, \ldots, A_m$ are compact.

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**2. General properties of elementary operators of length one**

In this section we consider elementary operators of length one, i.e. $L(T) = ATB$, i.e. with $A$ and $B \in B(\mathcal{H})$, and $T \in C_2(\mathcal{H})$. We investigate how operator theoretic properties of $A$ and $B$ relate to those of $L$.

We begin by recalling the definition of generalized projection and generalized $q$-projection as given in [13].

**Definition 2.1** (cf. [13, 18]). A bounded operator $T$ on a Hilbert space is called a generalized projection if and only if $T^2 = T^*$. More generally, given an integer $q > 1$, $T$ is a generalized $q$-projection if and only if $T^q = T^*$. 

Proposition 2.1. Let \( q \) be an integer greater than 1 and \( \mathcal{L} \) an elementary operator of length one, defined by \( \mathcal{L}(T) = ATB \). The operator \( \mathcal{L} \) is a generalized \( q \)-projection if and only if \( A \frac{1}{\|A\|} \) and \( B \frac{1}{\|B\|} \) are generalized \( q \)-projections and \( \|A\|\|B\| = 1 \).

**Proof.** If \( \mathcal{L} \) is a generalized \( q \)-projection then \( A^qTB^q = A^*TB^* \), for every \( T \) in \( \mathcal{C}_2(\mathcal{H}) \). The adjoint of \( \mathcal{L} \), \( \mathcal{L}^* \), is given by \( \mathcal{L}^*(S) = A^*SB^* \) since

\[
\langle \mathcal{L}(T), S \rangle = \text{tr}(ATBS^*) = \text{tr}(TBS^*A) = \text{tr}(T(A^*SB^*)^*) = \langle T, A^*SB^* \rangle. 
\]

Fong–Sourour’s Theorem implies that there exists a nonzero scalar \( \lambda \) so that \( A^q = \lambda A^* \) and \( B^q = B^* \). This implies that \( A \) and \( B \) are normal operators, therefore \( \|A^q\| = \|A\|^q = |\lambda|\|A\| \) and \( |\lambda| = \|A\|^{q-1} = \frac{1}{\|B\|^{q-1}} \). This shows that \( A \frac{1}{\|A\|} \) and \( B \frac{1}{\|B\|} \) are generalized \( q \)-projections and \( \|A\|\|B\| = 1 \). Conversely, we have that \( A^qTB^q = \|A\|^{-1}A^*T\|B\|^{-1}B^* = A^*TB^* \). \( \square \)

We now consider generalizations of the concept of isometry introduced by Patel in [15]. This new concept is closely related to hyponormal operators. In [15], the author shows that every quasi-isometry of norm 1 is hyponormal.

**Definition 2.2** (cf. [15]). A bounded operator \( T \) on a Hilbert space is said to be a quasi-isometry if and only if \( T^*T \geq T^2 \). More generally, given a positive integer \( q \), greater than 1, \( T \) is said to be \( q \)-quasi-isometry if and only if \( T^qT^q = T^q \).

**Proposition 2.2.** Let \( q \) be an integer greater than 1 and \( \mathcal{L} \) an elementary operator of length one, defined by \( \mathcal{L}(T) = ATB \). The operator \( \mathcal{L} \) is a \( q \)-quasi-isometry if and only if there exists a positive real scalar \( \lambda \) so that \( \left( \frac{1}{\|A\|^{q-1}} \right) A \) and \( \left( \frac{1}{\|B\|^{q-2}} \right) B^* \) are \( q \)-quasi-isometries.

**Proof.** If \( \mathcal{L}(T) = ATB \) is a \( q \)-quasi-isometry, then \( A^qATB^qB^q = A^*ATBB^* \), for every \( T \). Fong–Sourour’s theorem implies that there exists a scalar \( \lambda \) so that

\[
A^q = \lambda A^* \quad \text{and} \quad B^q = BB^*. 
\]

This implies \( \lambda > 0 \). Therefore \( \left( \frac{1}{\|A\|^{q-1}} \right) A \) and \( \left( \frac{1}{\|B\|^{q-2}} \right) B^* \) are \( q \)-quasi-isometries. The converse implication is straightforward. \( \square \)

**Definition 2.3** (cf. [6] or [9]). A bounded operator \( T \) on a Hilbert space \( \mathcal{H} \) is said to be quasi-normal if and only if \( TT^*T = T^*T^2 \). The operator \( T \) is said to be essentially normal if and only if \( TT^* = T^*T \) is a compact operator on \( \mathcal{H} \).

In [14], Magajna characterizes quasi-normal elementary operators in terms of the associated symbols, as stated in the following proposition.

**Proposition 2.3** (Magajna). An elementary operator of length one, \( \mathcal{L}(T) = ATB \), is quasi-normal if and only if \( A \) and \( B \) are quasi-normal.

Magajna’s proposition motivates our generalization to essential quasi-normality of elementary operators. Elementary operators of length 2 are more difficult to investigate but similar techniques also show that \( \mathcal{L}(T) = ATB - T \) is quasi-normal if and only if \( A \) and \( B \) are quasi-normal.

**Proposition 2.4.** The elementary operator \( \mathcal{L}_1(T) = ATB \) (or \( \mathcal{L}_2(T) = ATB - T \)) is essentially normal if and only if one of the following conditions hold:
(1) $A^*A$ and $AA^*$ are compact.
(2) $B$ is normal and $A$ is essentially normal.
(3) $B^*B$ and $BB^*$ are compact.
(4) $A$ is normal and $B$ is essentially normal.

**Proof.** We give the proof for $\mathcal{L}_1$. The proof for $\mathcal{L}_2$ follows similarly. We first assume that $\mathcal{L}_1$ is essentially normal. If in addition, $\{BB^*, B^*B\}$ is linearly independent, then Fong–Sourour’s Theorem implies that $A^*A$ and $AA^*$ are compact. If $B^*B = \lambda BB^*$, then Theorem 1.1(2) implies that $A^*A - AA^*$ is a compact operator on $\mathcal{H}$. The condition $B^*B = \lambda BB^*$ also implies that $\lambda = 1$ and $B$ is normal. It is clear that the conditions listed in the statement are sufficient for the essential normality of $\mathcal{L}_1$. □

Recall that an operator $T$ on a Banach space $X$ is algebraic if there is a polynomial $p$ such that $p(T) = 0$. We observe that if an elementary operator of length one, $L(T) = ATB$, is algebraic then it follows from Theorem 1.1(1) that both operators $A$ and $B$ must be algebraic. The converse does not hold. We consider a nilpotent and nonzero operator $A$. We define $L(T) = ATA$. It is easy to see that $L$ is not algebraic, since any polynomial of $L$ is of the form $c_0Id + c_1L$, with $c_0$ and $c_1$ scalars. Since $\{A, Id\}$ is linearly independent, Fong–Sourour’s theorem implies that $c_0 = c_1 = 0$.

Shift operators in Hilbert space have long provided examples and counterexamples for various conjectures in Operator Theory. An operator $S$ is called a shift if it is an isometry and $\bigcap_{n=0}^{\infty} \text{range}(S^n) = \{0\}$. It is of interest to know whether an elementary operator on $\mathcal{L}_2(\mathcal{H})$ is a shift. It follows from Theorem 1.1(1) that the operator $L(T) = ATB$ is an isometry if and only if $A$ and $B^*$ are isometries.

**Proposition 2.5.** If the operator $L(T) = ATB$ is an isometry, then $L$ is a shift if and only if $A$ or $B^*$ is a shift.

**Proof.** If $L$ is an isometry and $A$ (or $B^*$) is a shift then $L$ is also a shift. In fact, if there exists $T \in \bigcap_{n=1}^{\infty} \text{range}(L^n)$ then, for every $n$, there exists $T_n$ so that $A^nT_nB^n = T$. We have $A^nT_nB^n = T$ (or $B^*A^nT_nB^n = T^*v$ respectively), for every $v \in \mathcal{H}$. This implies that $Tv = 0$ (or $T^*v = 0$). In either case we have that $T = 0$. If $A$ and $B^*$ are isometries but not shifts, this would imply the existence of $v_n$ and $\omega_n$ so that $A^n v_n = \omega \neq 0$ and $B^n\omega = \omega \neq 0$. We now consider $L^n(v_n \otimes \omega_n) = A^n(v_n \otimes \omega_n)B^n = v \otimes \omega \neq 0$. This shows that the operator $v \otimes \omega$ is in the intersection $\bigcap_{n=0}^{\infty} \text{range}(L^n)$. Hence $L$ is not a shift. □

3. Aluthge transforms of elementary operators of length one

We now investigate how standard properties of elementary operators of length one are preserved under the Aluthge transform, cf. [4]. In this study the polar factorization of the operator plays a crucial role. We start by determining a factorization of an elementary operator of length one into a product of a partial isometry and a positive operator. We recall the definition of partial isometry and we state Theorem 3.1 from [6, p. 244] (cf. [9, p. 54]) to be used in the proof of the forthcoming Theorem 3.2.

**Definition 3.1.** An operator $U$ on a Hilbert space $\mathcal{H}$ is said to be a partial isometry if there exists a closed subspace $M$ such that

$$\|Ux\| = \|x\| \text{ for any } x \in M, \text{ and } Ux = 0 \text{ for any } x \in M^\perp.$$  

Every bounded operator $T$ on a Hilbert space $\mathcal{H}$ can be written as $T = U|T|$, where $|T| = \sqrt{T^*T}$, $\text{range}(U) = \text{range}(|T|)$, and $U^*U|T| = |T|$. Moreover $U$ and $|T|$ are unique if $\ker(U) = \ker(|T|)$. An additional fact about partial isometries is the following result, (cf. [6] or [9]).

**Theorem 3.1.** If $U$ be an operator on a Hilbert space $\mathcal{H}$, then the following statements are mutually equivalent:

- $U$ is a partial isometry.
- $U^*U$ is a projection.
- $U|U| = |U|$.
(1) $U$ is a partial isometry.
(2) $U^*U$ is a partial isometry.
(3) $UU^*U = U$.

These results give us sufficient machinery to derive the polar representation of an elementary operator of length one. We denote by $L_S(T) = ST$ and $R_S(T) = TS$ the left and right multiplication operators.

**Theorem 3.2.** If $L$ is an elementary operator of length one on $c_2(\mathcal{H})$, given by $L(T) = ATB$, and $A = U|A|$ and $B^* = U_B^*|B^*|$ are the polar decompositions of $A$ and $B^*$ respectively, then the following holds:

1. $|L| = L\hat{A}_R|B^*|$ is a positive operator and $|L|^2 = L^*L$.
2. $U_L = L\hat{A}_R U^*_B$ is a partial isometry on the range of $|L|$.
3. $L = U_L|L|$ is the polar decomposition of $L$.

**Proof.** Let $A = U_A|A|$, $B^* = U_B^*|B^*|$ where $|A| = \sqrt{AA^*}$ and $|B^*| = \sqrt{BB^*}$, $\ker(U_A) = \ker(|A|)$ and $\ker(|B^*|) = \ker(U_B^*)$. We set $|L|(T) = |A|T|B^*|$. We see that $|L|^2(T) = L^*L(T) = A^*ATBB^* = |A|^2T|B^*|^2$. This allows us to conclude that $|L|(T) = |A|T|B^*|$ is a square root of $L^*L$. To see that $|L|(T) = |A|T|B^*|$ is a nonnegative operator, we define $S(T) = |A|^\frac{1}{2}T|B^*|^\frac{1}{2}$. Then clearly $S^* = S$. Since $|L| = S^2$ then $|L|$ is nonnegative. Therefore $S^*S(T) = |A|T|B^*|$. This shows the statement in item (1).

We set $U_L = L\hat{A}_R U^*_B$. Since $A_A$ and $U_B^*$ are partial isometries then Theorem 3.1(3) states that

$$U_AU^*_AU_A = U_A \quad \text{and} \quad U_B^*U_B^*U_B^* = U_B^*.$$ 

This implies that

$$U_LU^*_LU_L(T) = U_AU^*_AU_A Tu_B^*U_B^*U_B^* = U_L,$$

and so $U_L$ is a partial isometry, as claimed in item (2). We now show statement (3). We first observe that $L = U_L|L|$. It remains to show that such decomposition of $L$ into a product of a partial isometry with a positive operator is unique. This is equivalent to showing that $\ker(|L|) = \ker(U_L)$. To this end we suppose that $U_L(T) = U_A Tu_B^* = 0$. It follows that $T(\text{range}(U_B^*)) \subseteq \ker(U_A) = \ker(|A|)$. Since $U_B^*$ is a projection onto $\text{range}(|B^*|)$ then $|L|(T) = |A|T|B^*| = |A|Tu_B^*U_B^*|B^*| = 0$. Hence $\ker(U_L) \subseteq \ker(|L|)$. To see the other inclusion we suppose that $|L|(T) = |A|T|B^*| = 0$. It follows that $\ker(|L|) \subseteq \ker(|A|) = \ker(U_A)$. Therefore, $U_A T|B^*| = 0$. Hence $B^* = 0$. We have $0 = (U_A T|B^*|v, w) = (v, |B^*|^*U_A^*w)$. Therefore $|B^*|^*U_A^*w = 0$ and the range($T^*U_A^*$) $\subseteq \ker(|B^*|) = \ker(U_B^*)$. This implies that $U_B^*T^*U_A^*w = 0$ and hence $U_ATu_B^* = 0$. This shows that $\ker(|L|) \subseteq \ker(U_L)$.

**Corollary 3.1.** If $L(T) = U_L|L|(T)$ then the Aluthge transform $\tilde{L}$ is given by $\tilde{L}(T) = A\tilde{T}(B^*)^*$.

It has been observed by many authors that the Aluthge transform (see [4,11]) respects the properties of the operator. Furthermore, in some cases it has stronger properties than the original operator, cf. [10]. We will investigate the behavior of the Aluthge transform of a 2-isometry, a concept due to Agler, see [1,2,3]. An operator $T$ on a Hilbert space $\mathcal{H}$ is a 2-isometry if $T^*T^2 - 2T^*T + Id = 0$. We first state a characterization of 2-isometric elementary operators obtained by the authors in [5].

**Theorem 3.3 (cf. [5]).** If $A$ and $B$ are bounded operators on a Hilbert space $\mathcal{H}$ and $L$ is an operator on $c_2(\mathcal{H})$ given by $L(T) = ATB$, then $L$ is a 2-isometry if and only if one of the following two conditions holds:

1. There exists a scalar $\mu$ so that $A^*A = \mu Id$ and $\sqrt{\mu}B^*$ is a 2-isometry, or
2. There exists a scalar $\mu$ so that $BB^* = \mu Id$ and $\sqrt{\mu}A$ is a 2-isometry.
Example 3.1. Let us consider the weighted shift $A : \ell^2 \to \ell^2$, given by

$$A(x_1, x_2, x_3, \ldots) = (0, a_1x_1, a_2x_2, a_3x_3, \ldots).$$

We assume that the weights satisfy the conditions $1 \leq |a_i| < 2$ and $|a_i|^2 = 2 - \frac{1}{|a_i - 1|^2}$, as for example $|a_i| = \sqrt{\frac{1 + \sqrt{5}}{2}}$. These conditions are necessary and sufficient conditions for a weighted shift like $A$ to be a 2-isometry, see [5]. It is straightforward to compute the Aluthge transform of $A$. Let $\theta_i$ be chosen such that $e^{i\theta_i} |a_i| = a_i$. Then

$$\tilde{A}(x_1, x_2, x_3, \ldots) = (0, e^{i\theta_1} |a_1a_2|^{1/2} x_1, e^{i\theta_2} |a_2a_3|^{1/2} x_2, e^{i\theta_3} |a_3a_4|^{1/2} x_3, \ldots).$$

It is easy to see that $\tilde{A}$ is not a 2-isometry. Hence the elementary operator $L(T) = A T I$ is a 2-isometry but $\tilde{L}(T) = \tilde{A} T I$ is not.

The next example shows that the Aluthge transform can be an isometry (and hence a 2-isometry) with the original operator not a 2-isometry.

Example 3.2. Consider the weighted shift $A_1 : \ell^2 \to \ell^2$ given by

$$A_1(x_1, x_2, x_3, \ldots) = \left(0, \frac{1}{2}x_1, 2x_2, \frac{1}{2}x_3, 2x_4, \ldots\right).$$

Then $A_1$ is not a 2-isometry, yet $\tilde{A}_1(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$ is an isometry and hence a 2-isometry.

4. A characterization of Hermitian elementary operators of length two

In this section we consider operators on $c_2(\mathcal{H})$ of the form $L(T) = A_1 T B_1 + A_2 T B_2$, where $A_1, A_2$ and $\{B_1, B_2\}$ are two linearly independent sets of bounded operators on $\mathcal{H}$. As previously observed, the adjoint of $L$ is given by $L^* (T) = A_1^* T B_1^* + A_2^* T B_2^*$. Magajna noted that elementary operators of length two could be hermitian without the symbols being hermitian. In particular, $L(T) = A^* T B + A T B^*$ is hermitian independently of the operators $A$ and $B$ being hermitian. The next theorem gives a characterization of hermitian operators of length two, which includes Magajna’s example as a special case.

Theorem 4.1. Let $L$ be an elementary operator of length two, acting on $c_2(\mathcal{H})$, with symbols $A_1, A_2, B_1, B_2 \in B(\mathcal{H})$ such that $\{A_1, A_2\}$ and $\{B_1, B_2\}$ are two linearly independent subsets of $B(\mathcal{H})$. The operator $L$ is hermitian if and only if there exist scalars $\lambda$ and $\mu$ so that one of the following statements holds:

(i) $A_2^* = \lambda A_1^* + \mu A_1, B_1 = \tilde{\lambda} B_2 - \mu B_2^*$.

(ii) $A_1 = \lambda A_2^* + \mu A_2, B_2 = \tilde{\lambda} B_1 - \mu B_1$, and $|\lambda| = |\mu|$.

(iii) $A_1^* = \lambda A_1, A_2^* = \mu A_2, B_1 = \tilde{\lambda} B_1^*, B_2 = \mu B_2^*$ and $|\lambda| = |\mu| = 1$.

Proof. If $L$ is hermitian, then

$$A_1^* T B_1^* + A_2^* T B_2^* - A_1 T B_1 - A_2 T B_2 = 0 \text{ for all } T \in c_2(\mathcal{H}).$$

We first assume that $\{A_1, A_2\}$ is a maximal linearly independent subset of $\{A_1, A_2, A_1^*, A_2^*\}$. Fong–Sourour’s Theorem implies that there exist scalars $\alpha_1, \alpha_2, \beta_1, \beta_2$ so that
\[
\begin{align*}
A_1^* &= \alpha_1 A_1 + \alpha_2 A_2 \quad \text{(eq: a)} \\
A_2^* &= \beta_1 A_1 + \beta_2 A_2 \quad \text{(eq: b)}
\end{align*}
\]
\[
B_1 = \alpha_1 B_1^* + \beta_1 B_2^* \quad \text{(eq: c)}
\]
\[
B_2 = \alpha_2 B_1^* + \beta_2 B_2^* \quad \text{(eq: d)}
\]

From (eq:a) and (eq:b) we have
\[
A_1 = \alpha_1 [\alpha_1 A_1 + \alpha_2 A_2] + \alpha_2 [\beta_1 A_1 + \beta_2 A_2]
\]
and
\[
A_2 = \beta_1 [\alpha_1 A_1 + \alpha_2 A_2] + \beta_2 [\beta_1 A_1 + \beta_2 A_2].
\]

Therefore
\[
1 - |\alpha|^2 = \alpha_1 \alpha_2 + \alpha_2 \beta_2 = 0, \quad 1 - |\beta|^2 = \alpha_2 \beta_1 + \beta_1 \beta_2 = 0.
\]
These equations also imply that \(|\alpha|^2 = |\beta|^2|.

If we assume that \(\alpha \neq 0\), then Eqs. (eq:a) and (eq:d) become
\[
A_1^* = \frac{1}{\alpha_2} A_1^* - \frac{\alpha_1}{\alpha_2} A_1 \quad \text{and} \quad B_1^* = \frac{1}{\alpha_2} B_2 - \frac{\beta_2}{\alpha_2} B_2^*.
\]
We set \(\lambda = \frac{1}{\alpha_2}\) and \(\mu = -\frac{\alpha_1}{\alpha_2}\). Therefore \(B_1 = \lambda B_2 - \mu B_2^*\), as listed in (i).

If \(\alpha = 0\) then \(|\beta|^2 = |\alpha|^2 = 1\) and Eqs. (eq:a) and (eq:d) reduce to
\[
A_1^* = \alpha_1 A_1 \quad \text{and} \quad B_1^* = \beta_1 B_2^*.
\]
respectively. If, in addition, we assume \(\beta_1 \neq 0\), then Eqs. (eq:b) and (eq:c) become
\[
A_1^* = \alpha_1 A_2 - \frac{\alpha_1}{\beta_1} B_2 A_2^* \quad \text{and} \quad B_2^* = \frac{\beta_1}{B_2} A_1^* - \frac{\alpha_1}{\beta_1} B_2^*,
\]
respectively. We now set \(\lambda = \frac{1}{\beta_1}\) and \(\mu = -\frac{\beta_2}{\beta_1}\) to obtain the relations in (ii).

If \(\alpha = \beta_1 = 0\), then \(|\alpha|^2 = |\beta|^2 = 1\) and the system (1) reduces to
\[
A_2^* = \beta_2 A_2, \quad B_1 = \alpha_1 B_1^*, \quad A_1^* = \alpha_1 A_1 \quad \text{and} \quad B_2 = \beta_2 B_2^*.
\]
The equations in (iii) follow by setting \(\lambda = \alpha_1\) and \(\mu = \beta_2\).

Now we assume \([A_1, A_2, A_1^*, A_2^*] \) is a maximal linearly independent subset of \([A_1, A_2, A_1^*, A_2^*] \). Theorem 1.1(1) implies the existence of scalars \(\alpha_1\), \(\alpha_2\), and \(\alpha_3\) so that
\[
A_2^* = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_1^*, \quad \text{and} \quad B_2 = -\alpha_3 B_2^*.
\]
Therefore \(B_1 = \alpha_1 B_2^*, \quad B_2 = \alpha_2 B_2^* \quad \text{and} \quad B_1^* = -\alpha_3 B_2^*, \quad \text{then} \quad \{B_1, B_2\} \) is linearly dependent. This contradicts our initial assumption. Similar reasoning applies if \([A_1, A_2, A_2^*] \) is a maximal linearly independent subset of \([A_1, A_2, A_1^*, A_2^*] \). It is a straightforward computation to verify that those relations listed in any of the items (i)–(iii) imply that \(\mathcal{L} \) is a hermitian operator. This completes the proof.

We conclude the paper with the following result on essential normality of a special elementary operator of length two.

**Proposition 4.1.** If \(\mathcal{L}(T) = AT - TB\), with A and B nonzero and bounded operators on \(\mathcal{H}\), then \(\mathcal{L}\) is essentially normal if and if A and B are normal operators.

**Proof.** It is clear that A and B normal imply that \(\mathcal{L}^* \mathcal{L} - \mathcal{L} \mathcal{L}^* = 0\). Conversely, we have that \((\mathcal{L}^* \mathcal{L} - \mathcal{L} \mathcal{L}^*)(T) = (A^* A - AA^*)T + T(BB^* - B^* B)\). Theorem 1.1(1) asserts that there exists a scalar \(\alpha\) so that \(A^* A - AA^* = \alpha \mathcal{I} \) and \(\alpha \mathcal{I} = BB^* - B^* B\). Kleinecke’s theorem [12] implies that \(\alpha = 0\). This completes the proof.

**References**