HYPERBOLIC SYMMETRY

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Abstract—The use of computer graphics to create repeating patterns of the hyperbolic plane is a recent and natural development in the history of hyperbolic patterns. A program is described which generates such patterns in the Poincaré model of hyperbolic geometry. The program can easily be extended to create patterns with color symmetry.

1. INTRODUCTION

Few hyperbolic patterns have been created compared with the number of Euclidean plane patterns [1] and [2]. The main reason for this is the requirement for many precise geometric constructions or numerical calculations. A natural solution to this problem is to use a computer to perform the computations and a computer-driven output device to display the pattern. Several programs have been written to generate hyperbolic patterns and a new one will be described below. A sample of the output from this program is shown in Fig. 1. This new program allows for the creation of patterns with color symmetry, an advance over previous programs. Figure 1 provides an example of 2-color (black-white) symmetry; 3-color and 4-color symmetry are exhibited in Figs. 5(a), 13 and 14.

Most hyperbolic patterns are represented in the Poincaré model of hyperbolic geometry. This model and regular hyperbolic tessellations are described in Secs. 2 and 3. The symmetry group of a pattern is defined in Sec. 4. The computer-generation of hyperbolic patterns can be broken down into two steps: (1) creation of the basic subpattern and (2) replication of the subpattern. These steps are detailed in Secs. 5 and 6. Finally it is shown how the replication algorithm may be modified to incorporate color symmetry.

2. HYPERBOLIC GEOMETRY

Unlike the Euclidean plane and the sphere, the entire (i.e. complete) hyperbolic plane cannot be isometrically embedded in 3-dimensional Euclidean space. Thus, any model of hyperbolic geometry in Euclidean 3-space must distort distance. However, there are conformal models (i.e. where the hyperbolic measure of an angle is just its Euclidean measure).

The Poincaré circle model is conformal and has the additional property that it is represented in a bounded region of the Euclidean plane—this is useful when we desire to show an entire pattern. The points of this model are the interior points of the bounding circle. The hyperbolic lines are circular arcs orthogonal to the bounding circle, including diameters. For example, the backbones of the fish in Fig. 1 lie on hyperbolic lines. Also, all the fish in Fig. 1 are the same hyperbolic size, showing that equal hyperbolic distances are represented by decreasing Euclidean distances as one approaches the bounding circle.

3. REPEATING HYPERBOLIC PATTERNS

We will define a repeating hyperbolic pattern to be a pattern composed of hyperbolically congruent copies of a basic subpattern or motif. Either the right side or the left side of any one of the fish of Fig. 1 serves as a motif for that pattern if color is disregarded. (If color is taken into account, a motif may be formed from half of a black fish together with half of an adjoining white fish.)

Other important examples of repeating hyperbolic patterns include the regular tessellations \( \{p, q\} \) of the hyperbolic plane by regular \( p \)-sided polygons, or \( p \)-gons, meeting \( q \) at a vertex (see Coxeter and Moser[3, Chapter 5]). It is necessary that \( (p - 2)(q - 2) > 4 \) to obtain a
Fig. 1. The top view of flounder-like fish arranged in a repeating hyperbolic pattern in the style of M. C. Escher's picture "Circle Limit I."

Fig. 2. The tessellation {6, 4} (solid lines), the dual tessellation {4, 6} (dotted lines), and other lines (dashed) of reflective symmetry of the two tessellations.
Hyperbolic symmetry

Hyperbolic tessellation. Figure 2 shows the tessellation \( \{6, 4\} \) in solid lines and its dual tessellation, \( \{4, 6\} \), in dotted lines.

If \((p - 2)(q - 2) = 4\) or \((p - 2)(q - 2) < 4\), one obtains regular tessellations of the Euclidean plane or sphere, respectively.

4. SYMMETRY GROUPS

A symmetry of a pattern is a congruence or isometry (hyperbolic distance-preserving transformation) of the hyperbolic plane which transforms the pattern onto itself. Reflections across the backbone lines are symmetries of Fig. 1. All the solid, dotted, and dashed lines of Fig. 2 are lines of reflective symmetry of that pattern. Hyperbolic reflections are either Euclidean reflections across diameters of the bounding circle or inversions with respect to orthogonal circular arcs. A line of reflective symmetry, i.e. the fixed line of a reflection, is called a mirror. Disregarding color, other symmetries of Fig. 1 include quarter-turns (rotations by \( \pi/2 \)) about points where trailing edges of fin-tips meet, and translations by one fish-length along backbone lines. The symmetry group of a pattern is the set of all symmetries of that pattern.

The inradii and circumradii of \( \{p, q\} \) lie on mirrors and divide each \( p \)-gon into \( 2p \) right triangles with acute angles of \( \pi/p \) and \( \pi/q \). The symmetry group of \( \{p, q\} \), denoted \([p, q]\), can be generated by reflections across the three sides of any such triangle, i.e. all symmetries of \( \{p, q\} \) can be obtained by successively applying a finite number of those reflections (see Coxeter and Moser[3, page 54]). Thus, \([6, 4]\) is the symmetry group of the pattern of Fig. 2. Note that the groups \([p, q]\) and \([q, p]\) are isomorphic.

As mentioned above, we may have \((p - 2)(q - 2) = 4\), in which case we obtain the Euclidean groups \([4, 4] = p4m\) and \([3, 6] = p6m\). As pointed out in [3, Sec. 4.6 and Table 4], these groups contain all the 17 plane crystallographic groups as subgroups. Similarly, the groups \([2, q]\), \([3, 3]\), \([3, 4]\) and \([3, 5]\) contain all the discrete spherical groups as subgroups. Thus, the groups \([p, q]\) are quite general and have significance beyond hyperbolic geometry.

The orientation-preserving subgroup of index 2 in \([p, q]\) is denoted by \([p, q]^+\) and consists of all symmetries which can be obtained by successively applying an even number of the reflections which generate \([p, q]\). The group \([p, q]^+\) can be generated by any two of the rotations by \( 2\pi/p \), \( 2\pi/q \), or \( \pi \) about the vertices of the right triangle formed by the mirrors of the reflections which generate \([p, q]\). In fact, those rotations can be produced by successive applications of pairs of the reflections which generate \([p, q]\). This is because reflections across intersecting lines produce a rotation about the intersection point equal to twice the angle of intersection, just as in the Euclidean case. Figure 3 shows a pattern with symmetry group \([5, 5]^+]\)—all symmetries of that pattern are orientation-preserving since only the left sides of the fish show. In Fig. 4(a), we see a pattern with symmetry group \([5, 4]^+]\).

There is another subgroup, \([p^+, q]\) (\( q \) must be even), of index 2 in \([p, q]\), which contains rotations by \( 2\pi/p \) about the centers of the \( p \)-gons of \([p, q]\) and reflections across the sides of the \( p \)-gons (this is why \( q \) must be even). In fact, \([p^+, q]\) can be generated by a \( p \)-fold rotation about the center of any \( p \)-gon in \([p, q]\) and a reflection across the side of that \( p \)-gon. Figure 4(b) shows a pattern with symmetry group \([5^+, 4]\)—note that adjacent pinwheels spin in opposite directions. These groups have been studied by Coxeter[4] and Sinkov[5].

The Dutch artist M. C. Escher used two instances of these groups, \([3^+, 8]\) and \([4^+, 6]\), for his patterns “Circle Limit II” (if the differences in shading are disregarded) and “Circle Limit IV;” these patterns are reproduced in Figs. 5(a) and 5(b). If color is disregarded, Fig. 1 also has symmetry group \([4^+, 6]\). Notice that mirrors pass through the center of the bounding circle in Figs. 1 and 5, whereas Fig. 4(b) has a 5-fold center of rotation at the center of the bounding circle. For more about the groups \([p, q]\) and their subgroups, see Coxeter and Moser[3, Secs. 4.3 and 4.4].

5. CREATING A MOTIF

Having laid the mathematical foundation, we now turn to the algorithmic structure of the pattern-creating program. The design of the motif is the first step in the pattern-creation process;
the second step, replication, involves transforming copies of the motif about the hyperbolic plane and will be discussed in the next section.

In order to design the motif, we must choose a symmetry group. If the hyperbolic plane is covered without overlap by transformed copies of a connected set under elements of a symmetry group, that set is called a fundamental region for the symmetry group (copies of the fundamental region may overlap along boundaries). If there are reflections in the symmetry group, at least part of the boundary of the fundamental region must lie along mirrors of those reflections (the boundary certainly cannot cross a mirror). Thus, the only choice for a fundamental region for \([6, 4]\) is one of the right triangles of Fig. 2, for if the fundamental region did not fill out such a triangle, it would violate the condition that copies of it cover the hyperbolic plane.

Fig. 3. A pattern with symmetry group \([5, 5]^*\).

Fig. 4. (a) A pattern with symmetry group \([5, 4]^*\). (b) A pattern with symmetry group \([5^*, 4]\).
There is more flexibility in choosing a fundamental region for \([p^+, q]\). The boundary may be formed by (1) a curve from the center of a \(p\)-gon to the \(p\)-gon’s boundary, (2) a copy of that curve rotated by \(2\pi/p\) about the \(p\)-gon center (if the copy does not cross the original curve) and (3) that part of the \(p\)-gon’s boundary between the two curves. A sample fundamental region for \([4^+, 6]\) with a bold boundary is shown in Fig. 6. There is even more flexibility in choosing a fundamental region for \([p, q]^+\): the bounding curve must pass through one \(p\)-fold, one \(q\)-fold, and one 2-fold center of rotation (without crossing itself; it may pass through two \(p\)-fold or two \(q\)-fold centers of rotation).
In order to facilitate the replication process, coordinates are chosen so that a $p$-gon of the underlying $\{p, q\}$ is centered at the origin and the positive $x$-axis bisects a side of that $p$-gon. To fix notation, we will label the origin $P$ and the upper vertex of the side bisected by the positive $x$-axis will be labelled $Q$, as in Fig. 7(a). Thus, a fundamental region may always be chosen so that its boundary contains $P$ and $Q$. If it is desired to generate a pattern with symmetry group $[p^*, q]$ with its mirrors passing through the center of the bounding circle, coordinates are chosen so that a $p$-gon of the underlying $\{p, q\}$ has a vertex at the origin $P$, one of its edges along the positive $x$-axis and its center $Q$ above the $x$-axis (see Fig. 7(b)).

In order that copies of the motif not cross each other in the final pattern, the motif must be drawn within a fundamental region. In fact, the motif may fill out the fundamental region, thus creating a pattern of interlocking motifs—a characteristic feature of M. C. Escher's repeating patterns. In all the patterns of Figs. 1–6 except Figs. 5(a) and 5(b), the motif coincides with a fundamental region; in Figs. 5(a) and 5(b) the curved arrow motif lies strictly inside a fundamental region in each case.

The motif can be entered as a sequence of points by using an input device such as a cursor, thumbwheel-controlled crosshairs, or light pen. As each new point is entered, a line segment may be drawn from the previous point, or the line segment may be omitted if it is desired to start a new series of line segments. After a line segment has been entered, the computer program shows transformed copies of the segment in adjacent copies of the fundamental region. Thus, as the motif is being entered in a fundamental region, transformed copies of the motif are built up in adjacent copies of the fundamental region. This is the feature of the program that facilitates the creation of interlocking motifs. In Fig. 6, the original motif is outlined in bold lines and adjacent copies are numbered 1, 2, 3 and 4 (the motifs numbered 3 and 4 are not used in creating the interlocking motif since they are separated from the original motif by mirrors).

6. REPLICATING THE PATTERN

After the motif has been entered, copies of it may be transformed about the hyperbolic plane; this replication step generates the final pattern. The replication step takes place in two stages. In the first stage, all motifs within fundamental regions having the center of the bounding circle $P$ as a boundary point are combined to form a larger subpattern called the $p$-gon pattern. The $p$-gon pattern is formed by reflecting or rotating (depending on the symmetry group) the original motif about $P$, i.e. these are ordinary Euclidean reflections and rotations. The $p$-gon pattern of Figs. 3, 4(a) (and 4(b)), 5(a) and 5(b) are the group of five centered fish, the centered pinwheel, the central cross, and the three central devils, respectively.

The advantages of forming the $p$-gon pattern are simplification of the replication algorithm and reduction of the number of transformations needed for the second stage of the replication.
process. For patterns with symmetry groups \([p, q], [p, q]^+\), or \([p^+, q]\), the reduction factors are \(2p\), \(p\), and \(p\) respectively. If the motif forms an interlocking pattern, so will the \(p\)-gon pattern.

The algorithm for accomplishing the second stage of the replication step depends on the observation that the \(p\)-gons of the tessellation \([p, q]\) form layers, and therefore so do copies of the \(p\)-gon pattern. The first layer of \(p\)-gons is just the central \(p\)-gon. The \((k + 1)\)-st layer is defined inductively as the set of those \(p\)-gons not in any previous layer, but which share an edge or a vertex with the \(k\)-th layer. The first three layers of \([6, 4]\) are shown in Fig. 2 (solid lines), and the first five layers of \([7, 3]\) are shown in Fig. 8. We will build up the entire repeating pattern with layers of \(p\)-gon patterns in the same way that \([p, q]\) may be built up with layers of \(p\)-gons.

The first layer of the repeating pattern is formed by simply drawing the \(p\)-gon pattern. In order to extend the pattern from the \(k\)-th layer to the \((k + 1)\)-st layer, first recall that there is a vertex \(Q\) of the central \(p\)-gon of \([p, q]\) on the boundary of the fundamental region containing the original motif. Next, suppose that the \(p\)-gon pattern has been transformed to the \(k\)-th layer so that \(Q\) is transformed to a vertex \(Q'\) common to the \(k\)-th and \((k + 1)\)-st layers. Then the pattern may be extended (locally) to the \((k + 1)\)-st layer by successively rotating or reflecting the transformed \(p\)-gon pattern about the vertex \(Q'\) to all \(p\)-gon positions in the \((k + 1)\)-st layer (except that the last position will be covered by a transformation about the next vertex). Thus, all \(p\)-gon patterns of the \((k + 1)\)-st layer may be obtained by transforming suitable \(p\)-gon patterns of the \(k\)-th layer about vertices common to the two layers.

We now describe in more detail the algorithm for obtaining the entire repeating pattern from the \(p\)-gon pattern for the groups \([p, q]\) and \([p, q]^+\) when \(p > 3\) and \(q > 3\). The algorithm must be modified slightly to handle the group \([p^+, q]\) or the cases \(p = 3\) or \(q = 3\). The algorithm will be described using the Pascal programming language; the data structures describing the \(p\)-gon pattern and the transformations will be given after the algorithm. Following the description of the data structures, we will define a procedure \texttt{DrawPgonPattern} which draws a transformed \(p\)-gon pattern, given the transformation.

The first layer, i.e. the \(p\)-gon pattern itself, is drawn by:

\[
\text{DrawPgonPattern(Identity)}
\]
If the number of desired layers, \( n_{\text{Layers}} \), is greater than one, then a recursive procedure, \( \text{Replicate} \) (Fig. 9), may be used, as described above, to extend the pattern from one layer to the next. \( \text{Replicate} \) has three parameters: (1) the transformation which takes the \( p \)-gon pattern to the present position in the \( k \)-th layer, (2) the number of additional layers to be drawn after the \( k \)-th layer and (3) the manner in which the \( p \)-gon in the present position lies adjacent to the \((k - 1)\)-st layer of \( p \)-gons—it either shares an edge or a vertex with the previous layer (in the case \( q = 3 \)—not considered in this discussion—\( p \)-gons in one layer share either one or two edges with the previous layer of \( p \)-gons). In order to be precise when dealing with this third parameter, we make the following type definition:

\[
\text{AdjacencyType} = (\text{Edge}, \text{Vertex})
\]

The way \( \text{Replicate} \) works is that the \( p \)-gon pattern is first drawn in the present position (in the \( k \)-th layer). Then, if there are additional layers to be drawn, \( \text{Replicate} \) generates recursive calls to itself, after computing the transformations to those \( p \)-gon positions in the next layer which share an edge or vertex with the \( p \)-gon in the present position.

\( \text{Replicate} \) uses the constant counter-clockwise rotations, \( \text{RotateP} \) by \( 2\pi/p \) about \( P \) (the center of the bounding circle), and \( \text{RotateQ} \) by \( 2\pi/q \) about the \( p \)-gon vertex \( Q \) (see Fig. 7(a)). \( \text{Replicate} \) also uses the variable rotations \( \text{RotateCenter} \) about \( p \)-gon centers, and \( \text{RotateVertex} \) about \( p \)-gon vertices. All these transformations are represented by matrices.

```pascal
PROCEDURE Replicate(InitialTran: Transformation; LayersToDo: INTEGER;
Adjacency: AdjacencyType);
VAR i, j, ExposedEdges, VertexPgons: INTEGER;
RotateCenter, RotateVertex: Transformation;
BEGIN
  DrawPgonPattern(InitialTran);
  (If there are more layers to be drawn, compute the
   transformations to appropriate positions in the next
   layer and call Replicate with those transformations)
  IF LayersToDo > 0 THEN
    BEGIN
      CASE Adjacency OF
        Edge: BEGIN
          ExposedEdges := p - 3;
          MatrixMult(RotateCenter, InitialTran, Rotate3P)
        END;
        Vertex: BEGIN
          ExposedEdges := p - 2;
          MatrixMult(RotateCenter, InitialTran, Rotate2P)
        END
        (Where Rotate2P = RotateP * RotateP
         and Rotate3P = Rotate2P * RotateP)
        END (CASE);
      FOR i := 1 TO ExposedEdges DO BEGIN
        MatrixMult(RotateVertex, RotateCenter, RotateQ);
        Replicate(RotateVertex, LayersToDo - 1, Edge);
        IF i < ExposedEdges THEN
          VertexPgons := q - 3
        ELSE (IF i = ExposedEdges THEN
          VertexPgons := q - 4;
        FOR j := 1 TO VertexPgons DO BEGIN
          MatrixMult(RotateVertex, RotateVertex, RotateQ);
          Replicate(RotateVertex, LayersToDo - i, Vertex)
          END (FOR j);
        MatrixMult(RotateCenter, RotateCenter, RotateP)
      END (FOR i)
      END (IF LayersToDo > 0)
    END (Replicate)
```

Fig. 9. The recursive procedure \( \text{Replicate} \) which extends the pattern from one layer of \( p \)-gon patterns to the next.
The following Pascal code will generate the second and subsequent layers:

```pascal
RotateCenter := Identity;
FOR i := 1 TO p DO
BEGIN
  MatrixMult(RotateVertex, RotateCenter, RotateQ);
  Replicate(RotateVertex, nLayers - 2, Edge);
  FOR J := 1 TO q - 3 DO
  BEGIN
    MatrixMult(RotateVertex, RotateVertex, RotateQ);
    Replicate(RotateVertex, nLayers - 2, Vertex)
  END;
  MatrixMult(RotateCenter, RotateCenter, RotateP)
END
```

There are several observations to be made about Replicate (Fig. 9) and the Pascal code above. First, MatrixMult(C, A, B) is just a procedure to perform the matrix multiplication \( C \leftarrow AB \). Eventually we will represent the position of points by column vectors. Thus, \( TX \) will denote the transformed vector obtained by multiplying the original vector \( X \) by the matrix \( T \).

Next, suppose that \( T \) is a transformation taking the original fundamental region to a new position. Then the rotations \( T \, \text{RotateP} \, T^{-1} \) and \( T \, \text{RotateQ} \, T^{-1} \) about the transformed points \( TP \) and \( TQ \) are the analogs of \( \text{RotateP} \) and \( \text{RotateQ} \). So, if we apply the analog of \( \text{RotateP} \) to a transformed point \( TX \), we obtain

\[
(T \, \text{RotateP} \, T^{-1}) \, (TX) = T \, \text{RotateP} \, X,
\]

i.e. we obtain the same effect by applying \( \text{RotateP} \) and \( T \) in reverse order to the original point (which may be assumed to be in the original motif). Of course, the same observation holds for \( \text{RotateQ} \) and its analog. These observations are used to simplify Replicate (and the Pascal code above) by avoiding the computation of conjugates of \( \text{RotateP} \) and \( \text{RotateQ} \). Thus, we may think of \( \text{RotateP} \) and \( \text{RotateQ} \) as being their conjugated analogs if we read the effect of the matrix products from left to right (e.g. the effect of applying \( T \, \text{RotateP} \) to a point is the same as first applying \( T \) and then applying the analog of \( \text{RotateP} \)).

Finally, note that once the \( p \)-gon pattern for \([p, q]\) is formed (using reflections), the entire pattern can be built up from the \( p \)-gon pattern using \( \text{RotateP} \) and \( \text{RotateQ} \) alone.

In the replication algorithm it is convenient to use the \textit{Weierstrass model} of hyperbolic geometry whose points lie on the upper sheet of the hyperboloid \( x^2 + y^2 - z^2 = -1 \) (and \( z > 0 \)). The efficacy of this model arises from the fact that isometries can be represented by \( 3 \times 3 \) Lorenz matrices (Lorenz matrices preserve the quadratic form \( x^2 + y^2 - z^2 \)). Consequently, the composition of two symmetries is represented by the product of the matrices representing those two symmetries. The hyperbolic lines of this model are the branches of the hyperbolas formed by the intersections of the upper sheet of the hyperboloid with planes passing through the origin. For more on the Weierstrass model, see Faber[6, Chapter 7].

The Weierstrass model is related to the Poincaré circle model, embedded as the unit disc in the \( xy \)-plane, by stereographic projection toward the point \((0, 0, -1) \)' (the "South Pole" of the hyperboloid). Specifically, the projection is given by:

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} \rightarrow \frac{1}{1 + z} \begin{bmatrix}
x \\
y \\
0
\end{bmatrix}
\]

The inverse projection is given by:

\[
\begin{bmatrix}
x \\
y \\
0
\end{bmatrix} \rightarrow \frac{1}{1 - x^2 - y^2} \begin{bmatrix}
2x \\
2y \\
1 + x^2 + y^2
\end{bmatrix}
\]
The transformations RotateP and RotateQ are defined relative to the centered tessellation \( \{p, q\} \) described in Sec. 5 and shown in Fig. 7(a). Let ReflectP, ReflectQ and Reflect2 denote the reflections across (1) the side of the central \( p \)-gon bisected by the positive \( x \)-axis, (2) the \( x \)-axis, and (3) the circumradius \( PQ \), respectively, as shown in Fig. 10. The matrices representing ReflectP, ReflectQ, and Reflect2 are:

\[
\begin{bmatrix}
-cosh(2b) & 0 & sinh(2b) \\
0 & 1 & 0 \\
-sinh(2b) & 0 & cosh(2b)
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
cos(2\pi/p) & sin(2\pi/p) & 0 \\
sin(2\pi/p) & -cos(2\pi/p) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

respectively, where

\[
cosh(b) = \frac{\cos(\pi/q)}{\sin(\pi/p)}
\]

\[
cosh(2b) = 2 \cosh^2(b) - 1
\]

\[
sinh(2b) = \sqrt{\cosh^2(2b) - 1}
\]

The counterclockwise rotations, RotateP by \( 2\pi/p \) about \( P \) and RotateQ by \( 2\pi/q \) about \( Q \) are represented by the matrix products Reflect2 ReflectQ and ReflectP Reflect2, respectively.

A convenient data structure to represent the points of the hyperboloid version of the \( p \)-gon pattern is a record composed of a 3-vector and a pen attribute which is of the following type:

\[
\text{PenAttribute} = (\text{Up}, \text{Down}, \text{Red}, \text{Yellow}, \text{Blue})
\]
Then the \( p \)-gon pattern may be represented by a record consisting of the number of points in the \( p \)-gon pattern and an array of points. Pascal type declarations for these data structures and the procedure \texttt{DrawPgonPattern} are shown in Fig. 11 (in which \texttt{VectorMult(Y, M, X)} multiplies the vector \( X \) by the matrix \( M \) and places the result in \( Y: Y \leftarrow MX \)). Note that \texttt{PgonPattern} is a global variable to \texttt{DrawPgonPattern}.

In summary, the pattern may be replicated by (1) using the inverse of the stereographic projection to map the \( p \)-gon pattern up onto the hyperboloid, (2) transforming the \( p \)-gon pattern to a new position by a Lorenz matrix representing a symmetry of the pattern, (3) stereographically projecting the transformed \( p \)-gon down onto the unit disc and (4) repeating steps (2) and (3) until the desired number of layers have been completed. Figure 12 shows the pattern of Fig. 5(b) on the hyperboloid.

7. COLOR SYMMETRY

Suppose that each of the motifs of an uncolored pattern receives one of \( k \) colors. That colored pattern is said to have \emph{k-color symmetry}, or simply \emph{color symmetry}, if each symmetry in the symmetry group of the uncolored pattern maps all motifs of one color to motifs of another (possibly the same) color, i.e. the symmetries of the uncolored pattern cause permutations of the colors of the colored pattern. See Loeb\[7\], Senechal\[8\] and \[9\], Shubnikov and Koptsik\[10\], and Wieting\[11\] for more on color symmetry.

```pascal
TYPE PenAttribute = (Up, Down, Red, Yellow, Blue);

PointType = RECORD
  X: ARRAY [1..3] OF REAL;
  Pen: PenAttribute
END;

VAR PgonPattern: RECORD
  Npoints: 1..MaxPoints;
  Points: ARRAY [1..MaxPoints] OF PointType
END

PROCEDURE DrawPgonPattern(T: Transformation);
VAR i: INTEGER;
  Xtrans: ARRAY [1..3] OF REAL;
  u, v: REAL;
BEGIN
  WITH PgonPattern DO
    FOR i := 1 TO nPoints DO
      WITH Points[i] DO
      BEGIN
        (Compute position of transformed point)
        VectorMult(Xtrans, T, X);

        (Project Xtrans to the xy-plane)
        u := Xtrans[1] / (1 + Xtrans[3]);
        v := Xtrans[2] / (1 + Xtrans[3]);

        (Take appropriate pen action)
        CASE Pen OF
          Up: MoveTo(u, v);
          Down: DrawTo(u, v);
          Red: Color := Red;
          Yellow: Color := Yellow;
          Blue: Color := Blue
        END (CASE)
      END (WITH Points[i])
    END (WITH PgonPattern)
END
```

Fig. 11. Relevant type declarations and the procedure \texttt{DrawPgonPattern} which draws a transformed \( p \)-gon pattern, the transformation being a parameter of \texttt{DrawPgonPattern}.
Figure 1 and the checkerboard pattern are examples of 2-color symmetry. In Fig. 1, the colors black and white are interchanged by a quarter-turn about the meeting point of fin-tips or by a translation by one fish-length along a line of fish. However, reflection across a backbone line preserves the colors.

Figures 5(a) and 13 exhibit 3-color symmetry. The colors are represented by three different hatchings: dotted, dashed, and dot-dashed. In Fig. 5(a), rotations of $2\pi/3$ about meeting points of three crosses cyclically permute the three colors, whereas all reflections in the symmetry group preserve colors. On the other hand, some of the reflections in the symmetry group of Fig. 13 interchange two of the three colors.

In Fig. 14, we see an example of 4-color symmetry, the fourth color being represented by solid line hatching. Figures 13 and 14 are interesting in that they are examples of the only two kinds of hyperbolic color symmetry not having Euclidean or spherical analogs among patterns with 2-, 3-, and 4-color symmetry and (uncolored) symmetry groups $[p, q]$, $[p, q]^+$, and $[p^+, q]$. 

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Fig. 12. The pattern of Fig. 5(b) on the hyperboloid of the Weierstrass model of hyperbolic geometry.

Fig. 13. A pattern with 3-color symmetry (the three colors are represented by dotted, dashed, and dot-dashed hatchings).
The procedure DrawPgonPattern of Fig. 11 allows for the drawing of the original motif in three different colors, but all motifs will be colored exactly the same way—i.e. there will be no (non-trivial) color symmetry. To simplify the following discussion, we will assume that the original motif has been drawn in a single color from now on.

In order to produce a pattern with color symmetry, we must do the following: when a symmetry is applied to the original motif to bring it to a new position, the color permutation induced by that symmetry must be applied to the color of the original motif to obtain the color of the motif in the new position. We can achieve this by including the color permutation induced by a symmetry in the representation of that symmetry as a transformation.

Permutations are conveniently represented by arrays. For example, if Perm is the array representing the permutation Red → Blue, Yellow → Red, and Blue → Yellow, then Perm[Red] is Blue, Perm[Yellow] is Red, and Perm[Blue] is Yellow. Thus, we may represent symmetries as records containing a matrix part and a permutation part:

```
ColorType = Red..Blue;
Transformation = RECORD
  Matrix: ARRAY [1..3, 1..3] OF REAL;
  Perm: ARRAY [ColorType] OF ColorType
END
```

The only necessary modification to Replicate (Fig. 11) is to replace the calls to MatrixMult by calls to TranMult:

```
PROCEDURE TranMult(VAR C: Transformation; A, B: Transformation);
VAR Color: ColorType;
BEGIN
  MatrixMult(C.Matrix, A.Matrix, B.Matrix);
  FOR Color := Red TO Blue DO
    C.Perm[Color] := A.Perm[B.Perm[Color]]
  END
END
```
In DrawPgonPattern, we must replace T by its matrix part in the call to VectorMult:

\[
\text{VectorMult(Xtrans, T.Matrix, X)}
\]

and we must modify the color changes in the CASE statement to:

Red: \( \text{Color} := T.\text{Perm}[\text{Red}] \);  
Yellow: \( \text{Color} := T.\text{Perm}[\text{Yellow}] \);  
Blue: \( \text{Color} := T.\text{Perm}[\text{Blue}] \).

Also, when the p-gon pattern is being built up from the original motif, each copy of the motif must begin with a "point" indicating a pen action to select the appropriate (permuted) color.

It is easy to see how the above techniques could be extended to more than three colors. In the remainder of the discussion of color symmetry, we will use positive integers to represent colors (instead of their actual names) and the standard notation for permutations.

We need only specify color permutations for the generators of a symmetry group, since all other symmetries can be expressed in terms of the generators. However, the permutations must satisfy the same relations as the generators. For instance, the permutation part of a reflection must consist of disjoint transpositions: (1 2), (1 2)(3 4), (1 6)(2 3), etc. Similarly, the permutation part of a rotation must consist of disjoint cycles whose lengths divide the period of the rotation (the rotations of interest will all have finite periods). There may be additional restrictions imposed by other relations.

As an example, the generators for the group \([p^+, q]\) satisfy the relations:

\[
\begin{align*}
(\text{RotateP})^p &= \text{ReflectP}^q = \text{Identity} \\
(\text{RotateP}^{-1} \text{ReflectP} \text{RotateP} \text{ReflectP})^{\frac{q}{2}} &= \text{Identity}.
\end{align*}
\]

Thus, if \( p = 4 \) and \( q = 8 \), the permutations (1 2 3 4) and (1 2)(3 4) may be used to represent the permutation parts of RotateP and RotateQ, respectively. However, if we replace (1 2)(3 4) by (1 2), the third relation above is no longer satisfied.

8. CONCLUSION

The hyperbolic pattern-creating program has been used to draw many patterns, including the ones in this paper. An earlier, more specialized version of the program [12] was used to generate all four of M. C. Escher’s hyperbolic Circle Limit patterns. The present program has also been used in the classification of the types of 2-, 3-, and 4-color symmetry associated with the groups \([p, q], [p, q]^+\) and \([p^+, q]\).

Possible directions of further research include extending the program to draw patterns whose symmetry groups are not subgroups of \([p, q]\) or are other subgroups of \([p, q]\) besides \([p, q]^+\) and \([p^+, q]\) (Escher's "Circle Limit I" and "Circle Limit III" fall into this latter category; see Coxeter[13]). Another direction would be the investigation of hyperbolic \(k\)-color symmetry for \( k > 4 \).

REFERENCES

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